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En Route to the log-rank Conjecture: New Reductions and Equivalent Formulations

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Abstract

We prove that several measures in communication complexity are equivalent, up to polynomial factors in the logarithm of the rank of the associated matrix: deterministic communication complexity, randomized communication complexity, information cost and zero-communication cost. This shows that in order to prove the log-rank conjecture, it suffices to show that low-rank matrices have efficient protocols in any of the aforementioned measures.

Furthermore, we show that the notion of zero-communication complexity is equivalent to an extension of the common discrepancy bound. Linial et al. [Combinatorica, 2007] showed that the discrepancy of a sign matrix is lower-bounded by an inverse polynomial in the logarithm of the associated matrix. We show that if these results can be generalized to the extended discrepancy, this will imply the log-rank conjecture.

1 Introduction

The *log-rank conjecture* proposed by Lovász and Saks [8] suggested that for any function $f : X \times Y \to \{0,1\}$ its deterministic communication complexity $\mathrm{CC}^{\mathrm{det}}(f)$ is polynomially related to the logarithm of the rank of the associated matrix $M_f \stackrel{\mathrm{def}}{=} (f(x,y))_{x,y}$. Validity of this conjecture is one of the most fundamental open problems in communication complexity. Very little progress has been made towards resolving it. The best known bounds are

$$\Omega\Big(\log^{\log_3 6} \mathrm{rank}(M_f)\Big) \leq \mathrm{CC}^{\mathrm{det}}(f) \leq \log(4/3)\mathrm{rank}(M_f), \tag{1}$$

where the lower bound is due to Kushilevitz (unpublished, cf. [9]) and the upper bound is due to Kotlov [4]. Recently a conditional improvement has been made by Ben-Sasson et al. [1], who showed that the polynomial Freiman-Ruzsa conjecture from additive combinatorics implied $\mathrm{CC}^{\mathrm{det}}(f) \leq O(\mathrm{rank}(M_f)/\log \mathrm{rank}(M_f))$.

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In this work we study the relation of the log-rank conjecture to several communication complexity measures. Besides those ones that directly correspond to natural communication models (e.g., *randomized communication cost* or *non-deterministic communication cost*), there is a number of "auxiliary" complexity measures that are mostly used as technical tools for studying communication complexity. Probably, the best known and the most useful one is *discrepancy.* More recently a number of other measures have been introduced, including *partition bound*, *rectangle bound*, *smooth discrepancy*, *smooth rectangle bound*, *relaxed partition bound*, γ_2 *norm*, etc. (cf. [2, 3]). Most of the currently known structural separations and concrete lower bound proofs in communication complexity can be viewed as analyzing one of the auxiliary complexity measures with respect to a specific communication problem.

1.1 Our contribution

We show that several conjectures, seemingly weaker than the log-rank conjecture, are in fact equivalent to it. We do so by showing that several natural communication complexity measures are equivalent, up to a polynomial in the logarithm of the rank of the associated matrix.

Theorem 1.1 (Main result, informal)**.** *Let* $f : X \times Y \to \{0, 1\}$ *be a function and* M_f *its associated matrix. The following communication complexity costs are equivalent, up to* $\operatorname{poly}(\log \operatorname{rank}(M_f))$ *factors:*

- *The deterministic communication cost of* f
- *The randomized communication cost of* f *(with public randomness)*
- *The information cost of* f
- *The zero-communication cost of* f *(a slight variant of the relaxed partition bound)*

Regardless of $\operatorname{rank}(M_f)$, the last three measures are small whenever a short deterministic protocol exists. On the other hand, an assumption that any of those measures (or even all three of them) is small does not, in general, imply existence of a efficient deterministic protocol. We have the following immediate corollary.

Corollary 1.2. *To prove the log-rank conjecture, it suffices to show that any low-rank function has an efficient randomized protocol, or a protocol with low information cost, or a protocol with low zero-communication complexity.*

In the second part of this work we investigate the weakest notion of communication complexity mentioned in Theorem 1.1, namely that of zero-communication cost. We use linear-programming duality to show that it is equivalent to the following notion, which extends the usual definition of discrepancy: Let $F : X \times Y \to \{\pm 1\}$ be a sign matrix, then for $0 < \alpha < 1/3$ the α*-extended discrepancy* of F is

$$\begin{aligned} \operatorname{disc}_\alpha(F) &\stackrel{\text{def}}{=} \max K, \\ \textit{subject to:}\quad &\sigma \text{ is a distribution on } X \times Y; \\ &\forall R \in \mathcal{R} : \left| \sum_{(x,y)\in R} \sigma(x,y) \cdot F(x,y) \right| \le \frac{1}{K} + \alpha \cdot \sigma(R). \end{aligned}$$

The case of $\alpha = 0$ corresponds to the usual discrepancy bound. We show that for $\alpha > \Omega(1)$, the α-extended discrepancy is equivalent to zero-communication cost.

Corollary 1.3. *To prove the log-rank conjecture, it suffices to show that* $\mathrm{disc}_\alpha(F) \leq \mathrm{poly}(\log(rank(F)))$ *for any* $\alpha \geq \Omega(1)$.

Interestingly, Linial et al. [6, 7] showed that $\mathrm{disc}(F) \leq O(\sqrt{\mathrm{rank}(F)})$, which allows us to conclude that certain "slightly relaxed" version of our equivalent formulation in terms of extended discrepancy is already known to hold.

Finally, if we set $\alpha = O(1/\sqrt{\mathrm{rank}(F)})$ it immediately implies that $\mathrm{disc}_\alpha(F) \leq O(\sqrt{\mathrm{rank}(F)})$. This allows us to derive another interesting corollary.

Corollary 1.4. *Let F be a sign matrix with* $\mathrm{rank}(F) = r$ *and* $\mathrm{disc}(F) = d$. *Then F has a deterministic protocol of complexity $O(d^2 \log(d) \log^2(r))$.*

In particular, if for a matrix F one has a discrepancy bound that is better than the one guaranteed by [6, 7] by only a poly-logarithmic factor, that already implies existence of a shorter deterministic protocol than what is guaranteed by (1).

2 Preliminaries

Let $f : X \times Y \to \{0, 1\}$ be a total[1] Boolean function, where X and Y are finite sets. The *rank* of f is the rank of its associated $\{0, 1\}$-matrix. We review standard definitions in communication complexity (see, e.g., [5] for more definitions and discussion).

Unless stated otherwise, we let a randomized communication protocol use *shared randomness*. We will say that a protocol *computes f with respect to the input distribution μ* if it produces the right answer to $f(X, Y)$ with probability at least $2/3$ when $(X, Y) \sim \mu$. We will also say that a protocol *computes* f if it produces the right answer to $f(X, Y)$ for every $(X, Y) \in X \times Y$ with probability at least $2/3$. The *deterministic communication cost* of f, denoted $\mathrm{CC}^{\mathrm{det}}(f)$, is the maximal number of bits sent by an optimal deterministic protocol that computes f. The *randomized communication cost* of f, denoted $\mathrm{CC}^{\mathrm{rand}}(f)$, is the maximal number of bits sent by an optimal randomized protocol computing f. The *information cost* of a function, denoted $\mathrm{IC}(f)$, is the infimum of the total amount of information revealed by a randomized protocol computing f to each player about the other player's input, maximized over all choices of the input distribution.

Let $\mathcal{R} \stackrel{\mathrm{def}}{=} \{A \times B \mid A \subset X, B \subset Y\}$, and call the elements of $\mathcal{R}$ *rectangles.* A labeled rectangle is a pair (R, z) with $R \in \mathcal{R}$ and $z \in \{0, 1\}$. We will also need a somewhat less common notion of *zero-communication cost* of a function, which we define next.

Definition 1. (Zero-communication cost) *The zero communication cost of f with error ε, denoted $\mathrm{CC}^{\mathrm{zero}}_\varepsilon(f)$, is the minimal c such that the following holds. There exists a distribution ρ on labeled rectangles (R, z) such that for any $(x, y) \in X \times Y$,*

1. $\Pr_{(R,z)\sim\rho}[(x, y) \in R] \geq 2^{-c}$.

[1] We always assume that the communication task is a total function, as that is all we need in the context of the log-rank conjecture.

2. $\Pr_{(R,z)\sim\rho}[f(x,y)=z|(x,y)\in R]\geq 1-\varepsilon$.

We abbreviate $\mathrm{CC}^{\mathrm{zero}}(f)=\mathrm{CC}^{\mathrm{zero}}_{1/3}(f)$.

For all the notions of communication cost that we have defined, a protocol is considered *efficient* with respect to that specific cost if the latter is bounded by $\mathrm{poly}(\log n)$.

Claim 2.1. *For any function* f,

$$\mathrm{CC}^{\mathrm{det}}(f)\geq \mathrm{CC}^{\mathrm{rand}}(f)\geq \mathrm{IC}(f)\geq \Omega(\mathrm{CC}^{\mathrm{zero}}(f)).$$

Proof. The first two inequalities follow immediately from the definitions. The fact that $\mathrm{CC}^{\mathrm{zero}}(f)\leq O(\mathrm{IC}(f))$ has been established recently by Kerenidis et al. [3] (Theorem 1.1).[2] ∎

3 Zero-error protocols reduce to deterministic protocols for low-rank functions

We prove the following theorem in this section.

Theorem 3.1. *Let* $f : X\times Y\to\{0,1\}$ *be a Boolean function. Then* $\mathrm{CC}^{\mathrm{det}}(f)\leq O\big(\mathrm{CC}^{\mathrm{zero}}(f)\cdot(\log(\mathrm{rank}(f)))^2\big)$.

We fix the function f and prove Theorem 3.1 in the remainder of this section. A rectangle $R\subset X\times Y$ is called *monochromatic* if the value of f is constant on R (i.e., all zero or all one). We will use the following theorem of Nisan and Wigderson [9] which shows that to establish that a low-rank function has small deterministic communication cost, it suffices to show that any rectangle contains a large monochromatic sub-rectangle. We denote by $|R|$ the number of elements in a rectangle.

Lemma 3.2 ([9])**.** *Let* $f : X\times Y\to\{0,1\}$ *be a Boolean function. Assume that for any rectangle* $R_1\subset X\times Y$ *there exists a sub-rectangle* $R_2\subset R_1$ *such that* R_2 *is monochromatic and* $|R_2|\geq\delta|R_1|$*. Then* $\mathrm{CC}^{\mathrm{det}}(f)\leq O(\log(1/\delta)\cdot\log\mathrm{rank}(f))$.

Thus, we reduced proving the theorem to the task of showing that any rectangle contains a large monochromatic sub-rectangle. We next show that this follows given a zero-communication protocol with small enough error.

Lemma 3.3. *Let* $f : X\times Y\to\{0,1\}$ *be a Boolean function with* $\mathrm{rank}(f)=r$*. Set* $\varepsilon=1/8r$ *and assume that* $\mathrm{CC}^{\mathrm{zero}}_{\varepsilon}(f)=c$*. Then any rectangle* $R_1\subset X\times Y$ *contains a monochromatic sub-rectangle* $R_2\subset R_1$ *with* $|R_2|\geq(1/16)2^{-c}|R_1|$.

[2] The definition of zero-communication protocol given in [3] (or more accurately, that of a relaxed partition bound) is somewhat different than the definition we are using here. Our definition is less restricting (and probably, somewhat more natural), and therefore $\mathrm{CC}^{\mathrm{zero}}(f)\leq O(\mathrm{IC}(f))$ holds. The main reason for choosing our definition is that it allows more straightforward error reduction by repetition, which we will need later.

Proof. We first establish that there exists a sub-rectangle $R' \subset R_1$ which is nearly monochromatic, and then use the low-rank property of f to establish the existence of a monochromatic rectangle $R_2 \subset R'$.

Let ρ be the distribution on labeled rectangles guaranteed by the zero-communication cost assumption, and let $(R, z) \sim \rho$ and $R' = R \cap R_1$. Let $\mathcal{E} = \{(x, y) \in R_1 : f(x, y) \neq z\}$ be the set of inputs whose value disagrees with z. We will show that $|R'| \geq (1/2)2^{-c}|R_1|$ and $|R' \cap \mathcal{E}| \leq 2\varepsilon|R'|$ with non-zero probability.

Note that we can reformulate the definition of a zero-communication protocol as

$$\Pr[(x, y) \in R] \geq 2^{-c}; \quad \Pr[(x, y) \in R \text{ and } f(x, y) \neq z] \leq \varepsilon \Pr[(x, y) \in R],$$

where the probabilities are taken over $(R, z) \sim \rho$. So,

$$\Pr[(x, y) \in R] - (1/2\varepsilon) \Pr[(x, y) \in R \text{ and } f(x, y) \neq z] \geq (1/2)2^{-c}.$$

Summing over all $(x, y) \in R_1$ gives

$$\mathbb{E}[|R'| - 1/(2\varepsilon) \cdot |R' \cap \mathcal{E}|] \geq (1/2)2^{-c}|R_1|.$$

Thus, there must exist a choice for R' exceeding the average. In particular, it satisfies $|R'| \geq (1/2)2^{-c}|R_1|$ and $|R' \cap \mathcal{E}| \leq 2\varepsilon|R'|$.

Next we establish existence of a large monochromatic rectangle $R_2 \subset R'$. Assume that $R' = A' \times B'$. Let $A'' \subset A'$ be the set of rows having at most 4ε-fraction of elements disagreeing with z,

$$A'' = \{x \in A' : |\{y \in B' : f(x, y) \neq z\}| \leq 4\varepsilon|B'|\}.$$

By Markov's inequality we have $|A''| \geq |A'|/2$. We now apply the low-rank property of f. Consider the matrix generated by restricting f to $A'' \times B'$. Its rank is also at most $r = \text{rank}(f)$. Hence, there exist r elements $x_1, \ldots, x_r \in A''$ whose corresponding rows span all rows in A''. Define for $1 \leq i \leq r$ the sets of inputs taking the "wrong" value on row x_i,

$$B_i = \{y \in B' : f(x_i, y) \neq z\}.$$

We know by assumption that $|B_i| \leq 4\varepsilon|B'|$. Let $B'' = B' \setminus \cup_{i=1}^{r} B_i$. Then $|B''| \geq |B'|(1 - 4\varepsilon r) \geq |B'|/2$. Consider now the matrix restricted to $A'' \times B''$. It is spanned by rows which are all equal to z, hence all of its rows are constant! We take R_2 to be the set of rows taking the majority value. To conclude, we found a monochromatic rectangle $R_2 \subset R_1$ of size

$$|R_2| \geq (1/2)|A''||B''| \geq (1/8)|A'||B'| = (1/16)2^{-c}|R_1|,$$

as required.

□

We are nearly done, it remains to argue that the error in zero-communication protocols can be reduced efficiently.

Claim 3.4. *For any $1/2 > \lambda > \varepsilon > 0$,*

$$\mathrm{CC}^{\mathrm{zero}}_{\varepsilon}(f) \leq O\left(\mathrm{CC}^{\mathrm{zero}}_{\lambda}(f) \cdot \frac{\log(1/\varepsilon)}{(1/2-\lambda)^2}\right).$$

In particular, $\mathrm{CC}^{\mathrm{zero}}_{\varepsilon}(f) \leq O(\mathrm{CC}^{\mathrm{zero}}(f) \cdot \log(1/\varepsilon))$.

Proof. Denote $c \stackrel{\mathrm{def}}{=} \mathrm{CC}^{\mathrm{zero}}_{\lambda}(f)$, and let ρ be a distribution over labeled rectangles (R, z) satisfying

1. $\Pr_{(R,z)\sim\rho}[(x, y) \in R] \geq 2^{-c}$.
2. $\Pr_{(R,z)\sim\rho}[f(x, y) = z|(x, y) \in R] \geq 1 - \lambda$.

Let $t = O\Big(\frac{\log(1/\varepsilon)}{(1/2-\lambda)^2}\Big)$, and sample $(R_1, z_1), \ldots, (R_t, z_t) \sim \rho$ independently. We define R^* to be the intersection of $R_1, \ldots, R_t$ and z^* to be the majority value of $z_1, \ldots, z_t$. We claim that the resulting distribution of (R^*, z^*) gives a zero-communication protocol with error ε and cost ct.

In order to see this, fix $(x, y) \in X \times Y$. First, we verify that $(x, y) \in R^*$ frequently enough,

$$\Pr[(x, y) \in R^*] = \Pr[(x, y) \in R_1, \ldots, (x, y) \in R_t] = \prod_{i=1}^{t} \Pr[(x, y) \in R_i] \geq 2^{-ct}.$$

It remains to verify that $\Pr[f(x, y) = z^*|(x, y) \in R^*] \geq 1 - \varepsilon$. Let $p \stackrel{\mathrm{def}}{=} \Pr[f(x, y) = z|(x, y) \in R]$; for any $v_1, \ldots, v_t \in \{0, 1\}$ such that $|\{i \mid v_i = f(x, y)\}| = m$,

$$\Pr[z_1 = v_1, \ldots, z_t = v_t|(x, y) \in R^*] = \prod_{i=1}^{r} \Pr[z_i = v_i|(x, y) \in R_i] = p^m(1 - p)^{t-m}.$$

So, the probability that $z^* = f(x, y)$ conditioned on $(x, y) \in R^*$ is given by summing over all values $v_1, \ldots, v_t$ whose majority is equal to $f(x, y)$. This equals the probability that a binomial distribution with t trials and success probability $p \geq 1 - \lambda$ has at least $t/2$ successes:

$$\Pr[f(x, y) = z^*|(x, y) \in R^*] \geq \Pr[\mathrm{Bin}(t, 1 - \lambda) \geq t/2] \geq 1 - \alpha^{(1/2-\lambda)^2 \cdot t}$$

for some constant $0 < \alpha < 1$. Choosing $t = O\Big(\frac{\log(1/\varepsilon)}{(1/2-\lambda)^2}\Big)$ large enough gives the required bound. ■

4 Equivalent formulations

Recall the original log-rank conjecture of Lovász and Saks [8]:

Conjecture 4.1. (**log-rank**, [8]) *For every $\{0, 1\}$-valued matrix M,*

$$\mathrm{CC}^{\mathrm{det}}(M) \leq \mathrm{poly}(\log(\mathrm{rank}(M))).$$

We will present several equivalent formulations of the conjecture.

First, we give a version that "looks weaker" in the following sense: While the original conjecture can be phrased as "low rank implies an efficient deterministic protocol", this formulation only requires existence of an efficient protocol of one of several more powerful types.

Theorem 4.2. (**log-rank conjecture**, equivalent formulations) *The following statements are equivalent:*

1. *The log-rank conjecture (Conjecture 4.1).*
2. *For every* $\{0,1\}$*-valued* M*,* $\mathrm{CC}^{\mathrm{rand}}(M) \leq \mathrm{poly}(\log(\mathrm{rank}(M)))$*.*
3. *For every* $\{0,1\}$*-valued* M*,* $\mathrm{IC}(M) \leq \mathrm{poly}(\log(\mathrm{rank}(M)))$*.*
4. *For every* $\{0,1\}$*-valued* M*,* $\mathrm{CC}^{\mathrm{zero}}(M) \leq \mathrm{poly}(\log(\mathrm{rank}(M)))$*.*

Proof. From Claim 2.1 and Theorem 3.1 it follows that $\mathrm{CC}^{\mathrm{det}}(M)$, $\mathrm{CC}^{\mathrm{rand}}(M)$, $\mathrm{IC}(M)$ and $\mathrm{CC}^{\mathrm{zero}}(M)$ are equal, up to the factor of $\mathrm{poly}(\log(\mathrm{rank}(M)))$. ∎

4.1 Extended discrepancy: extrapolating between discrepancy and zero-communication cost

Let us denote $F(x,y) \stackrel{\mathrm{def}}{=} (-1)^{M_{x,y}}$ and $\mathrm{CC}^{\mathrm{zero}}_{\varepsilon}(F) = \mathrm{CC}^{\mathrm{zero}}_{\varepsilon}(M)$. Define for every $\{\pm 1\}$-valued matrix F its α*-extended discrepancy* as

$$
\begin{aligned}
& \mathrm{disc}_{\alpha}(F) \stackrel{\mathrm{def}}{=} \max K, \\
\textit{subject to:} \quad & \sigma \text{ is a distribution on } X \times Y; \\
& \forall R \in \mathcal{R} : \left| \sum_{(x,y)\in R} \sigma(x,y) \cdot F(x,y) \right| \leq \frac{1}{K} + \alpha \cdot \sigma(R).
\end{aligned}
$$

If we choose $\alpha = 0$ then $\mathrm{disc}_{\alpha}(F)$ equals the *discrepancy* of F, which is one of the most commonly used tools for proving lower bounds on $\mathrm{CC}^{\mathrm{rand}}(M)$.

Claim 4.3. *For every* $\{\pm 1\}$*-valued matrix* F *and every constant* $0 < \alpha < \frac{1}{3}$*,*

$$2^{\mathrm{CC}^{\mathrm{zero}}(F)} = \Theta(\mathrm{disc}_{\alpha}(F)).$$

Proof. Let us express $\mathrm{CC}^{\mathrm{zero}}(M)$ as the optimal value of a linear program,

$$
\begin{aligned}
& 2^{\mathrm{CC}^{\mathrm{zero}}_{\varepsilon}(M)} = \min \sum_{R\in\mathcal{R}} \left(w_{R,0} + w_{R,1}\right), \\
\textit{subject to:} \quad & \forall R \in \mathcal{R} : w_{R,0} \geq 0, w_{R,1} \geq 0; \\
& \forall (x,y) \in X \times Y : \sum_{R:(x,y)\in R} \left(w_{R,0} + w_{R,1}\right) \geq 1; \\
& \forall (x,y) \in X \times Y : \sum_{R:(x,y)\in R} \left(w_{R,M_{x,y}} - \frac{1-\varepsilon}{\varepsilon} \cdot w_{R,1-M_{x,y}}\right) \geq 0.
\end{aligned}
$$

Its dual can be written as

$$
\begin{aligned}
&2^{\mathrm{CC}_\varepsilon^{\mathrm{zero}}(M)} = \max K,\\
&\textit{subject to:}\quad \forall (x,y)\in X\times Y : C_{x,y}\geq 0;\ \mu \text{ is a distribution on } X\times Y;\\
&\qquad \forall R\in\mathcal{R}, z\in\{0,1\} : \mu(R) \leq \frac{1}{K} + \frac{1-\varepsilon}{\varepsilon}\cdot \sum_{\substack{(x,y)\in R\\ M_{x,y}=z}} C_{x,y} - \sum_{\substack{(x,y)\in R\\ M_{x,y}=1-z}} C_{x,y}.
\end{aligned}
$$

We can rewrite it as

$$
\begin{aligned}
&2^{\mathrm{CC}_\varepsilon^{\mathrm{zero}}(F)} = \max K,\\
&\textit{subject to:}\quad \sigma : X\times Y\to\mathbb{R}^+;\ \mu \text{ is a distribution on } X\times Y;\\
&\qquad \forall R\in\mathcal{R} : \left|\sum_{(x,y)\in R}\sigma(x,y)F(x,y)\right| - (1-2\varepsilon)\cdot\sigma(R) + \mu(R) \leq \frac{1}{K}.
\end{aligned} \tag{2}
$$

First, we show that $2^{\mathrm{CC}^{\mathrm{zero}}(F)} \geq \Omega(\mathrm{disc}_\alpha(F))$. For $\alpha < \frac{1}{3}$, let μ be a distribution on $X\times Y$, such that $\forall R\in\mathcal{R}$:

$$
\left|\sum_{(x,y)\in R}\mu(x,y)F(x,y)\right| \leq \frac{1}{\mathrm{disc}_\alpha(F)} + \alpha\mu(R).
$$

Then for $t \stackrel{\text{def}}{=} \frac{3\alpha}{1-3\alpha}$,

$$
\frac{t}{\alpha}\cdot\left|\sum_{R}\mu(x,y)F(x,y)\right| - (t+1)\cdot\mu(R) + \mu(R) \leq \frac{t}{\alpha\mathrm{disc}_\alpha(F)},
$$

and therefore,

$$
2^{\mathrm{CC}^{\mathrm{zero}}(F)} \geq \frac{1-3\alpha}{3}\cdot\mathrm{disc}_\alpha(F).
$$

For the other direction of our proof, let $\sigma : X\times Y\to\mathbb{R}^+$ and μ be a distribution on $X\times Y$, such that $\forall R\in\mathcal{R}$:

$$
\left|\sum_{(x,y)\in R}\sigma(x,y)F(x,y)\right| - (1-2\varepsilon)\cdot\sigma(R) + \mu(R) \leq \frac{1}{2^{\mathrm{CC}_\varepsilon^{\mathrm{zero}}(F)}}, \tag{3}
$$

which implies

$$
\left|\sum_{(x,y)\in R}\sigma(x,y)F(x,y)\right| \leq \frac{1}{2^{\mathrm{CC}_\varepsilon^{\mathrm{zero}}(F)}} + (1-2\varepsilon)\cdot\sigma(R).
$$

For $R = X\times Y$, (3) implies

$$
\sigma(X\times Y) \geq \left(1 - \frac{1}{2^{\mathrm{CC}_\varepsilon^{\mathrm{zero}}(F)}}\right)\cdot\frac{1}{1-2\varepsilon} \geq \frac{1}{2},
$$

as long as $\mathrm{CC}_\varepsilon^{\mathrm{zero}}(F) \geq 1$. Then for the distribution $\sigma' \stackrel{\text{def}}{=} \frac{\sigma}{\sigma(X\times Y)}$,

$$\left|\sum_{(x,y)\in R} \sigma'(x,y)F(x,y)\right| \leq \frac{2}{2^{\mathrm{CC}_\varepsilon^{\mathrm{zero}}(F)}} + (1-2\varepsilon)\cdot\sigma'(R).$$

Accordingly,

$$\mathrm{disc}_{1-2\varepsilon}(F) \geq \frac{2^{\mathrm{CC}_\varepsilon^{\mathrm{zero}}(F)}}{2}. \tag{4}$$

As long as $\alpha > 0$, we can use the error-reducing technique given by Claim 3.4, and therefore $2^{\mathrm{CC}^{\mathrm{zero}}(F)} \leq O(\mathrm{disc}_\alpha(F))$, as required. ■

The following equivalent formulations of the log-rank conjecture is immediate from Theorem 4.2 and Claim 4.3.

Theorem 4.4. (**log-rank conjecture**, an equivalent formulation) *The log-rank conjecture (Conjecture 4.1) is true if and only if the following holds for some $\alpha_0 \geq \Omega(1)$: For every $\{\pm 1\}$-valued matrix M and probability distribution μ on $X \times Y$, there exists a rectangle $R \in \mathcal{R}$ such that*

$$\left|\sum_{(x,y)\in R} \sigma(x,y)\cdot M_{x,y}\right| \geq \frac{1}{\mathrm{qpoly}(\mathrm{rank}(M))} + \alpha_0\cdot\sigma(R),$$

where $\mathrm{qpoly}(x) \stackrel{\text{def}}{=} \exp(\mathrm{poly}(\log x))$.

Interestingly, it was shown by Linial et al. [6, 7] that $\mathrm{disc}(M) \leq O\left(\sqrt{\mathrm{rank}(M)}\right)$. In other words,

Fact 4.5. ([6, 7]) *For every $\{\pm 1\}$-valued matrix M and probability distribution μ on $X \times Y$, there exists a rectangle $R \in \mathcal{R}$ such that*

$$\left|\sum_{(x,y)\in R} \sigma(x,y)\cdot M_{x,y}\right| \geq \Omega\left(\frac{1}{\sqrt{\mathrm{rank}(M)}}\right).$$

Note that the above statement can be viewed as a version of the equivalent formulation given in Theorem 4.4, relaxed by letting "$\alpha_0 = 1\Big/\sqrt{\mathrm{rank}(M)}$".

Finally, our techniques can be used to derive a polynomial upper bound on $\mathrm{CC}^{\mathrm{det}}(F)$ in terms of $\mathrm{disc}(F)$ and $\log(\mathrm{rank}(F))$.

Claim 4.6. *For a $\{\pm 1\}$-valued matrix F, let $d = \mathrm{disc}(F)$. Then*

$$\mathrm{CC}^{\mathrm{det}}(F) \leq O\big(d^2\cdot(\log(\mathrm{rank}(F)))^2\cdot\log d\big).$$

Proof. It is immediate from the definition that $\frac{1}{\mathrm{disc}(F)}$-extended discrepancy is equivalent to $\mathrm{disc}(F)$ up to a constant multiplicative factor, and therefore

$$\mathrm{disc}_{1/d}(F) \leq O(d).$$

In the proof of Claim 4.3 we have shown (cf. (4)) that

$$2^{\mathrm{CC}_{\varepsilon}^{\mathrm{zero}}(F)} \leq O(\mathrm{disc}_{1-2\varepsilon}(F))$$

holds for every $\varepsilon > 0$, and therefore

$$\mathrm{CC}^{\mathrm{zero}}_{\frac{1}{2}-\frac{1}{2d}}(F) \leq \log d + O(1).$$

By Claim 3.4,

$$\mathrm{CC}^{\mathrm{zero}}(F) \leq O\big(d^2 \log d\big),$$

and the result follows by Theorem 3.1. ■

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