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Communication is bounded by root of rank

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Abstract

We prove that any total boolean function of rank r can be computed by a deterministic communication protocol of complexity $O(\sqrt{r} \cdot \log(r))$. Equivalently, any graph whose adjacency matrix has rank r has chromatic number at most $2^{O(\sqrt{r} \cdot \log(r))}$. This gives a nearly quadratic improvement in the dependence on the rank over previous results.

1 Introduction

The *log-rank conjecture* proposed by Lovász and Saks [8] suggests that for any boolean function $f : X \times Y \to \{-1, 1\}$ its deterministic communication complexity $\mathrm{CC}^{\mathrm{det}}(f)$ is polynomially related to the logarithm of the rank of the associated matrix. Validity of this conjecture is one of the most fundamental open problems in communication complexity. Very little progress has been made towards resolving it. The best known bounds are

$$(\log \mathrm{rank}(f))^{\log_3 6} \leq \mathrm{CC}^{\mathrm{det}}(f) \leq \log(4/3) \cdot \mathrm{rank}(f),$$

where the lower bound is due to Kushilevitz (unpublished, cf. [9]) and the upper bound is due to Kotlov [3]. A conditional improvement was made by Ben-Sasson et al. [1], who showed that the polynomial Freiman-Ruzsa conjecture from additive combinatorics implied $\mathrm{CC}^{\mathrm{det}}(f) \leq O(\mathrm{rank}(f)/\log \mathrm{rank}(f))$. Recently, Gavinsky and the author [2] showed that in order to prove the log-rank conjecture, it suffices to prove it for weaker notions of protocols (specifically, randomized protocol, low information-cost protocols, or zero-communication protocols). We build upon their work and achieve a near quadratic improvement in the dependence on the rank in upper bound.

Theorem 1.1. *Let* $f : X \times Y \to \{-1, 1\}$ *be a boolean function with* $\mathrm{rank}(f) = r$. *Then* $\mathrm{CC}^{\mathrm{det}}(f) \leq O(\sqrt{r} \cdot \log r)$. *Equivlanetly, any graph whose adjacency matrix has rank* r *has chromatic number at most* $2^{O(\sqrt{r} \cdot \log r)}$.

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The proof is based on analyzing the discrepancy of boolean functions. The discrepancy of f is given by

$$\mathrm{disc}(f) = \min_{\mu} \max_{R} \left| \sum_{(x,y)\in R} f(x,y)\mu(x,y) \right|$$

where μ ranges over all distributions over $X \times Y$ and R ranges over all rectangles, e.g. $R = A \times B$ for $A \subset X, B \subset Y$. Discrepancy is a well-studied property in the context of communication complexity lower bounds, see e.g. [7] for an excellent survey. It is known that low-rank matrices have noticeable discrepancy.

Theorem 1.2 ([5, 6])**.** *Let $f : X \times Y \to \{-1, 1\}$ be a boolean function with $\mathrm{rank}(f) = r$. Then $\mathrm{disc}(f) \geq \frac{1}{8\sqrt{r}}$.*

Discrepancy can be used to prove upper bounds as well. Linial et al. [5] showed that functions of discrepancy δ have randomized (or quantum) protocols of complexity $O(1/\delta^2)$. Unfortunately, this does not give any improved bounds in general, as there is always a trivial protocol using r bits. We show that the combination of high discrepancy and low rank implies an improved bound. Our main technical theorem is the following.

Theorem 1.3. *Let $f : X \times Y \to \{-1, 1\}$ be a function with $\mathrm{rank}(f) = r$ and $\mathrm{disc}(f) = \delta$. Then f has a monochromatic rectangle R of size*

$$|R| \geq 2^{-O(\log(r)/\delta)} |X \times Y|.$$

Setting $\delta \geq 1/(8\sqrt{r})$ we obtain the following corollary.

Corollary 1.4. *Let $f : X \times Y \to \{-1, 1\}$ be a function with $\mathrm{rank}(f) = r$. Then f has a monochromatic rectangle R of size*

$$|R| \geq 2^{-O(\sqrt{r} \cdot \log(r))} |X \times Y|.$$

Theorem 1.1 follows from Theorem 1.3 combined with the following result of Nisan and Wigderson [9]. They show that the ability to find large monochromatic rectangles is sufficient in order to establish existence of efficient deterministic protocols. The original proof of [9] analyzed the case where the monochromatic rectangles guaranteed are of inverse-quasi-polynomial relative size. Here we extend the analysis to general bounds on the rectangle size.

Theorem 1.5 ([9])**.** *Assume that for any function $f : X \times Y \to \{-1, 1\}$ of $\mathrm{rank}(f) = r$ there exists a monochromatic rectangle of size $|R| \geq 2^{-c(r)} |X \times Y|$. Then any boolean function of rank r is computable by a deterministic protocol of complexity $O(\log^2 r + \sum_{i=0}^{\log r} c(r/2^i))$. In particular, setting $c(r) = O(\sqrt{r} \cdot \log(r))$ implies a protocol of complexity $O(\sqrt{r} \cdot \log(r))$.*

Paper organization. We give preliminary definitions in Section 2. We prove Theorem 1.3 in Section 3. We prove Theorem 1.5 is Section 4. We discuss further research in Section 5.

2 Preliminaries

For standard definitions in communication complexity we refer the reader to [4]. We give here only the basic definitions we would require.

Let $f : X \times Y \to \{-1, 1\}$ be a total boolean function, where X and Y are finite sets. The *rank* of f is the rank of its associated $X \times Y$ matrix. The set of rectangles is $\mathcal{R} \stackrel{\text{def}}{=} \{A \times B : A \subset X, B \subset Y\}$. A rectangle R is monochromatic if the value of f is constant over R, e.g. all 1 or all -1. For a rectangle $R \in \mathcal{R}$, the average of f over R is defined as

$$\mathbb{E}_R[f] \stackrel{\text{def}}{=} \frac{1}{|R|} \sum_{(x,y) \in R} f(x, y).$$

We shorthand $\mathbb{E}[f] \stackrel{\text{def}}{=} \mathbb{E}_{X \times Y}[f]$ for the average of f.

The discrepancy of f with respect to a distribution μ on $X \times Y$ is the maximal bias achieved by a rectangle,

$$\mathrm{disc}_\mu(f) \stackrel{\text{def}}{=} \max_{R \in \mathcal{R}} \left| \sum_{(x,y) \in R} \mu(x, y) f(x, y) \right|.$$

The discrepancy of f is the minimal discrepancy possible over all possible distributions μ,

$$\mathrm{disc}(f) \stackrel{\text{def}}{=} \min_\mu \mathrm{disc}_\mu(f).$$

Note that discrepancy is an hereditary property. That is, if R is a rectangle then the discrepancy of f restricted to R is at least the original discrepancy of f. Similarly, low rank is a hereditary property, as ranks of sub-matrices cannot exceed the rank of the original matrix.

3 Obtaining a monochromatic rectangle

We prove the following formal version of Theorem 1.3 from in the introduction.

Theorem 3.1. *Let $f : X \times Y \to \{-1, 1\}$ be a function with $\mathrm{disc}(f) = \delta$. Then f has a monochromatic rectangle R of size*

$$|R| \geq (1/8) \cdot 2^{-6 \cdot \log(4r)/\delta} |X \times Y|.$$

We prove Theorem 3.1 in a sequence of lemmas. First, we show we can find a rectangle which is slightly more biased than the full matrix.

Lemma 3.2. *Let $f : X \times Y \to \{-1, 1\}$ be a function with $\mathbb{E}[f] = \alpha \geq 0$ and $\mathrm{disc}(f) = 3\delta$. Then there exists a rectangle R such that*

$$\mathbb{E}_R[f] \geq \alpha + \delta(1 - \alpha^2) \frac{|X \times Y|}{|R|}.$$

Proof. Let $F_+ = \{(x,y) \in X \times Y : f(x,y) = 1\}$ and $F_- = \{(x,y) \in X \times Y : f(x,y) = -1\}$ be the subset of elements mapped to either 1 or -1 by f. Let $N = |X \times Y|$. By assumption, $|F_+| = \frac{1+\alpha}{2}N$ and $|F_-| = \frac{1-\alpha}{2}N$.

Let μ be a distribution over $X \times Y$ giving equal weight to elements in each set, such that the total probability of each of F_+ and F_- is $1/2$. That is,

$$\mu(x,y) = \begin{cases} \frac{1}{(1+\alpha)N} & \text{if } (x,y) \in F_+ \\ \frac{1}{(1-\alpha)N} & \text{if } (x,y) \in F_- \end{cases}$$

Note that by construction, $\sum_{x \in X, y \in Y} \mu(x,y) f(x,y) = 0$.

Since $\mathrm{disc}_\mu(f) \geq \mathrm{disc}(f) = 3\delta$, there exists a rectangle $R_1 = A \times B$ such that

$$\left| \sum_{x \in A, y \in B} \mu(x,y) f(x,y) \right| \geq 3\delta. \tag{1}$$

Let $A' = X \setminus A$ and $B' = Y \setminus B$ be the complements of A, B, and consider the four rectangles

$$R_1 = A \times B, R_2 = A' \times B, R_3 = A \times B', R_4 = A' \times B'.$$

We will show that for one of the rectangles $R_1, \ldots, R_4$ satisfies the requirements of the lemma. Since

$$\sum_{i=1}^{4} \sum_{(x,y) \in R_i} \mu(x,y) f(x,y) = \sum_{x \in X, y \in Y} \mu(x,y) f(x,y) = 0,$$

there must be $R \in \{R_1, \ldots, R_4\}$ such that

$$\sum_{(x,y) \in R} \mu(x,y) f(x,y) \geq \delta. \tag{2}$$

Note that for the rectangle R we have

$$N \cdot \sum_{(x,y) \in R} \mu(x,y) f(x,y) = \frac{|F_+ \cap R|}{1+\alpha} - \frac{|F_- \cap R|}{1-\alpha} = \frac{1}{1-\alpha^2} \sum_{(x,y) \in R} f(x,y) - \frac{\alpha}{1-\alpha^2}|R|.$$

Hence,

$$\sum_{(x,y) \in R} f(x,y) = \alpha|R| + N(1-\alpha^2) \sum_{(x,y) \in R} \mu(x,y) f(x,y) \geq \alpha|R| + N(1-\alpha^2)\delta.$$

The lemma follows by dividing both sides by $|R|$. □

We next apply Lemma 3.2 iteratively. It will be more convenient to denote momentarily $\mathbb{E}[f] = 1 - \beta$ for $0 \leq \beta \leq 1$. We will need the following simple technical claim.

Claim 3.3. *Let $x_1, \ldots, x_n \geq 0$ be positive numbers such that $\sum x_i \leq r$. Then $\prod x_i \leq 2^r$.*

Proof. The arithmetic-geometric mean inequality implies that

$$\prod_{i=1}^{n} x_i \leq (r/n)^n \leq ((r/n)^{n/r})^r.$$

One can verify that $y^{1/y} \leq 2$ for all $y \geq 0$ and the claim follows. $\square$

Lemma 3.4. *Let $f : X \times Y \to \{-1, 1\}$ be a function with $\mathbb{E}[f] = 1 - \beta \geq 0$ and $\mathrm{disc}(f) = 3\delta$. Then there exists a rectangle R such that*

$$|\mathbb{E}_R[f]| \geq 1 - \beta/2$$

and

$$\frac{|X \times Y|}{|R|} \leq 2^{2/\delta}.$$

Proof. We apply Lemma 3.2 iteratively until we reach the required rectangle. Let $R_0 \stackrel{\text{def}}{=} X \times Y$, and for $i \geq 1$ define R_i iteratively by applying Lemma 3.2 to f restricted to R_{i-1}. We stop at the first iteration t for which $\mathbb{E}_{R_t}[f] \geq 1 - \beta/2$. We need to lower bound the size of R_t when this occurs.

Let $\alpha_i \stackrel{\text{def}}{=} \mathbb{E}_{R_i}[f]$ and $\gamma_i \stackrel{\text{def}}{=} |R_{i-1}|/|R_i| \geq 1$. Note that $|X \times Y|/|R_t| = \prod_{i=1}^{t} \gamma_i$ is the quantity we need to upper bound. We have by Lemma 3.2 that

$$\alpha_i \geq \alpha_{i-1} + \delta(1 - \alpha_{i-1}^2)\gamma_i. \tag{3}$$

Recall that we stop the process when $\alpha_t \geq 1 - \beta/2$, hence for $i < t$ we have

$$1 - \alpha_{i-1}^2 \geq 1 - (1 - \beta/2)^2 = \beta - \beta^2/4 \geq \beta/2.$$

Hence, for all $i \leq t$ we have

$$\alpha_i \geq \alpha_{i-1} + \delta \cdot (\beta/2) \cdot \gamma_i. \tag{4}$$

We will apply (4) to bound $\prod_{i=1}^{t} \gamma_i$. We have

$$\beta \geq \alpha_t - \alpha_0 \geq \delta \cdot (\beta/2) \cdot \sum_{i=1}^{t} \gamma_i.$$

This implies that $\sum_{i=1}^{t} \gamma_i \leq 2/\delta$. Applying Claim 3.3 we conclude that

$$\frac{|X \times Y|}{|R|} = \prod_{i=1}^{t} \gamma_i \leq 2^{2/\delta}.$$

$\square$

Corollary 3.5. *Let $f : X \times Y \to \{-1, 1\}$ be a function with* $\mathrm{disc}(f) = 3\delta$*. Then for any $\varepsilon > 0$ there exists a rectangle R such that*

$$|\mathbb{E}_R[f]| \geq 1 - \varepsilon$$

and

$$\frac{|X \times Y|}{|R|} \leq 2^{2 \cdot \log(2/\varepsilon)/\delta}.$$

Proof. We can assume without loss of generality that $\mathbb{E}[f] \geq 0$ as otherwise we apply the theorem to $-f$. Let k be minimal such that $\varepsilon \geq 2^{-k}$. Apply Lemma 3.4 iteratively $k \leq \log(2/\varepsilon)$ times. □

We will also need the following lemma from [2].

Lemma 3.6 ([2])**.** *Let $f : X \times Y \to \{-1, 1\}$ be a function with* $\mathrm{rank}(f) = r$ *and* $\mathbb{E}[f] \geq 1 - \varepsilon$ *for $\varepsilon \leq 1/2r$. Then f has a monochromatic rectangle R size $|R| \geq |X \times Y|/8$.*

We include for completeness its proof.

Proof. Since f is a sign matrix, the condition $\mathbb{E}[f] \geq 1 - 1/2r$ implies that $f(x, y) = -1$ for at most $1/4r$ fraction of the inputs. Let $A \subset X$ be the set of rows for which at most $1/2r$ fraction of the elements are -1,

$$A = \left\{x \in X : |\{y \in Y : f(x, y) = -1\}| \leq |Y|/2r\right\}.$$

By Markov inequality, $|A| \geq |X|/2$. Let $x_1, \dots, x_r \in A$ be indices so that their rows span $A \times Y$. Let

$$B = \{y \in Y : f(x_1, y) = \ldots = f(x_r, y) = 1\}.$$

Since each of the rows $x_1, \dots, x_r$ contain at most $1/2r$ fraction of elements which are -1 we have $|B| \geq |Y|/2$. Now, this implies that all rows in $A \times B$ are either the all one or all minus one. Choosing the largest half gives the required rectangle. □

The proof of Theorem 3.1 follows immediately from Corollary 3.5 and Lemma 3.6.

Proof of Theorem 3.1. Apply Corollary 3.5 with $\varepsilon = 1/2r$, followed by Lemma 3.6. □

4 From monochromatic rectangles to protocols

We prove Theorem 1.5 in this section. The proof follows the original proof of [9] except that we analyze the case of general bounds for the monochromatic rectangles.

To recall, we assume that for any function $f : X \times Y \to \{-1, 1\}$ of $\mathrm{rank}(f) = r$ there exists a monochromatic rectangle of size $|R| \geq 2^{-c(r)}|X \times Y|$. Let f be such a function, and consider the partition of its corresponding matrix as

$$\begin{pmatrix} R & S \\ P & Q \end{pmatrix}$$

As R is monochromatic, $\mathrm{rank}(R) = 1$. Hence, $\mathrm{rank}(S) + \mathrm{rank}(P) \leq r + 1$. Assume w.l.o.g that $\mathrm{rank}(S) \leq r/2 + 1$ (otherwise, exchange the role of the rows and columns player). The row player sends one bit, indicating whether their input x is in the top or bottom half of the matrix. If it is in the top half the rank decreases to $\leq r/2 + 1$. If it is in the bottom half, the size of the matrix reduces to at most $(1 - 2^{-c(r)})|X \times Y|$. Iterating this process defines a protocol tree. We next bound the number of leaves of the protocol. By standard techniques, any protocol tree can be balanced so that the communication complexity is logarithmic in the number of leaves (cf. [4, Chapter 2, Lemma 2.8]).

Consider the protocol which stops once the rank drops to $r/2$. The protocol tree in this case has at most $O(2^{c(r)} \cdot \log(m))$ leaves, and hence can be simulated by a protocol sending only $O(c(r) + \log\log(m))$ bits. Note that since we can assume f has no repeated rows or columns, $m \leq 2^{2r}$ and hence $\log\log(m) \leq \log(r) + 1$. Next, consider the phase where the protocol continues until the rank drops to $r/4$. Again, this protocol can be simulated by $O(c(r/2) + \log(r))$ bits of communication. Summing over $r/2^i$ for $i = 0, \ldots, \log(r)$ gives the bound.

5 Further research

We provide a bound on the communication complexity that is near to linear in the discrepancy. This seem to be tight for our proof technique. One possible approach to improve the result is to improve the dependency of the discrepancy on the rank. Another interesting direction is to combine our current approach with the additive combinatorics approach of [1].

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