Electronic Colloquium on Computational Complexity, Report No. 141 (2013)

Boolean matrices with prescribed row/column sums and stable homogeneous polynomials: combinatorial and algorithmic applications

Leonid Gurvits
The City College of New York, New York, NY 1033
gurvits@cs.cccny.cuny.edu

October 5, 2013

Abstract

We prove a new efficiently computable lower bound on the coefficients of stable homogeneous polynomials and present its algorthmic and combinatorial applications. Our main application is the first poly-time deterministic algorithm which approximates the partition functions associated with boolean matrices with prescribed row and column sums within simply exponential multiplicative factor. This new algorithm is a particular instance of new polynomial time deterministic algorithms related to the multiple partial differentiation of polynomials given by evaluation oracles

1 Basic Definitions and Motivations

For given two integer vectors $\mathbf{r} = (r_1, ..., r_n)$ and $\mathbf{c} = (c_1, ..., c_m)$, we denote as $BM_{\mathbf{r},\mathbf{c}}$ the set of boolean $n \times m$ matrices with prescribed rows sums $\mathbf{r}$ and column sums $\mathbf{c}$.

ISSN 1433-8092

Next, we introduce an analogue of the permanent (a partition function associated with $BM_{\mathbf{r},\mathbf{c}}$):

$$PE_{\mathbf{r},\mathbf{c}}(A) =: \sum_{B \in BM(\mathbf{r},\mathbf{c})} \prod_{1 \leq i \leq n; 1 \leq j \leq m} A(i,j)^{B(i,j)}, \tag{1}$$

where A is $n \times m$ complex matrix. Note that if A is a $n \times n$ matrix; $\mathbf{r} = \mathbf{c} = e_n$, where e_n is n-dimensional vector of all ones, then the definition (1) reduces to the permanent: $PE_{e_n,e_n}(A) = per(A)$.
The main focus of this note is on bounds and **deterministic** algorithms for $PE_{\mathbf{r},\mathbf{c}}(A)$ in the non-negative case $A \geq 0$. To avoid messy formulas, we will mainly focus below on the uniform square case, i.e. $n = m$ and $r_i = c_j = r, 1 \leq i,j \leq n$ and use simplified notations: $BM_{\mathbf{re_n},\mathbf{re_n}} =: BM(r,n); PE_{\mathbf{re_n},\mathbf{re_n}}(A) =: PE(r,A)$.
Boolean matrices with prescribed row and column sums is one of the most classical and intensely studied topics in analytic combimatorics, with applications to many areas from applied statistics to the representation theory. We, as many other researchers, are interested in the counting aspect, i.e. in computing/bounding/approximating the partition function $PE_{\mathbf{r},\mathbf{c}}(A)$. It was known already to W.T.Tutte [17] that this partition function can be in poly-time reduced to the permanent. Therefore, if A is nonnegative the famous **FPRAS** [19] can be applied and this was already mentioned in [19] as one of the main applications. We are after deterministic poly-time algorithms. A. Barvinok initiated this, deterministic, line of algorithmic research in [14]. He also used the reduction to the permanent and the Van Der Waerden-Falikman-Egorychev (**VFE**) [10], [9] celebrated lower bound on the permanent of doubly-stochastic matrices:

$$per(A) \geq vdw(n) =: \frac{n!}{n^n}, A \in BM(i,n).$$

The techniques in [14] result in a deterministic poly-time algorithm approximating $PE(r,A)$ within multiplicative factor $(\Omega(\sqrt{n}))^n$ for any fixed r, even for $r = 1$. Such poor approximation is due the fact that the reduction to the permanent produces highly structured $n^2 \times n^2$ matrices. **VFE** bound is clearly a powerful algorithmic tool, as was recently effectively illustrated in [18]. Yet, neither **VFE** nor even more refined Schrijver's lower bound [2] are sharp enough for those structured matrices. This phenomenon was observed by A. Schrijver 30 years ago in [1]. The author introduced in [11] and [4] a new approach to lower bounds. We will give a brief description of the approach and refine it. The new lower bounds are asymptotically sharp and allow, for instance, to get a deterministic poly-time algorithm to approximate $PE(r, A)$ within multiplicative factor $f(r)^{-1}$ where

$$f(r) = (\frac{vdw(n)}{vdw(r)vdw(n-r)})^{n\frac{r-1}{r}} \frac{vdw(n)}{vdw(r)^{\frac{n}{r}}} \approx (\sqrt{2\pi \min(r, n-r)})^{-n}.$$

Besides, we show that algorithm from [14] actually approximates within(roughly) multiplicative factor $f(r)^2$. So, for fixed r or $n-r$ the new bounds give simply exponential factor. But, say for $r = \frac{n}{2}$, the current factor is not simply exponential. *Is there a deterministic Non-Approximability result for $PE(\frac{n}{2}, A)$?*
We also study the sparse case, i.e. when, say, the columns of matrix A have relatively small number of non-zero entries. In this direction we generalize, reprove, sharpen the results of A. Schrijver [1] on how many k-regular subgraphs $2k$-regular bipartite graph can have.
The main moral of this paper is that when one needs to deal with the permanent of highly structured matrices the only (and often painless) way to get sharp lower bounds is to use **stable polynomials approach**.Prior to [11] and [4] **VFE** was, essentially, the only general purpose non-trivial lower bound on the permanent. It is not true anymore.

1.1 Generating polynomials

The goal of this subsection is to represent $PE_{\mathbf{r},\mathbf{c}}(A)$ as a coefficient of some effectively computable polynomial.

1. The following natural representation in the case of unit weights, i.e $A(i,j) \equiv 1$, was already in [16], the general case of it was used in [14].

$$PE_{\mathbf{r},\mathbf{c}}(A) = [\prod_{1 \le i \le n} y_i^{r_i} \prod_{1 \le j \le m} x_j^{c_j}] \prod_{1 \le i \le n, 1 \le j \le m} (1 + A(i,j)x_j y_i), \qquad (2)$$

   i.e. $PE_{\mathbf{r},\mathbf{c}}(A)$ is the coefficient of the monomial $\prod_{1 \le i \le n} y_i^{r_i} \prod_{1 \le j \le m} x_j^{c_j}$ in the non-homogeneous polynomial $\prod_{1 \le i \le n, 1 \le j \le m}(1 + A(i,j)x_j y_i)$.
   It is easy to convert non-homogeneous formula (2) into a homogeneous one:

$$PE_{\mathbf{r},\mathbf{c}}(A) = [\prod_{1 \le j \le m} x_j^{c_j} \prod_{1 \le i \le n} z_i^{m-r_i}] \prod_{1 \le i \le n, 1 \le j \le m} (z_i + A(i,j)x_j). \qquad (3)$$

As the polynomial $\prod_{1\leq i\leq n, 1\leq j\leq m}(z_i + A(i,j)x_j)$ is a product of linear forms, the formula (3) allows to express $PE_{\mathbf{r},\mathbf{c}}(A)$ as the permanent of some $nm \times nm$ matrix, the fact essentially proved in a very different way in [17].The permanent also showed up, in a similar context of Eulerian Orientations, in [1].
Indeed, associate with any $k \times l$ matrix B the product polynomial

$$Prod_B(x_1, ..., x_l) =: \prod_{1\leq i\leq k} \sum_{1\leq j\leq l} B(i,j)x_j. \quad (4)$$

Then

$$[\prod_{1\leq j\leq l} x_j^{\omega_j}]Prod_B(x_1, ..., x_l) = per(B_{\omega_1,...,\omega_l}) \prod_{1\leq j\leq l} (\omega_j!)^{-1}, \quad (5)$$

where $k \times k$ matrix $B_{\omega_1,...,\omega_l}$ consists of ω_j copies of the jth column of B, $1 \leq j \leq l$.

2. We will use below the following equally natural representation. Recall the definition of standard symmetric functions:

$$S_k(x_1, ..., x_m) = \sum_{1\leq i_1<..<i_k\leq m} \prod_{1\leq j\leq k} x_{i_j},$$

and define the following homogeneous polynomial

$$ES_{\mathbf{r};A}(x_1, ..., x_m) = \prod_{1\leq i\leq n} S_{r_i}(A(i,1)x_1, ..., A(i,m)x_m). \quad (6)$$

Then

$$PE_{\mathbf{r},\mathbf{c}}(A) = [\prod_{1\leq j\leq n} x_j^{c_j}]ES_{\mathbf{r};A}(x_1, ..., x_m). \quad (7)$$

Remark 1.1: Note that in the square case $n = m$, the polynomial $ES_{e_n;A} = Prod_A$. The polynomial $ES_{\mathbf{r};A}$ is, of course, related to the polynomial
$TM(z_1, ..., z_n; x_1, ..., x_m) =: \prod_{1\leq i\leq n, 1\leq j\leq m}(z_i + A(i,j)x_j)$:

$$ES_{\mathbf{r};A}(x_1, ..., x_m) = const \prod_{1\leq i\leq n} \frac{\partial^{m-r_i}}{\partial z_i^{m-r_i}} TM(z_i = 0, 1 \leq i \leq n; x_1, ..., x_m). \quad (8)$$

∎

1.2 Exact algorithms

It is well known that the coefficient $[\prod_{1 \leq j \leq n} x_j^{c_j}] ES_{\mathbf{r},\mathbf{c};A}(x_1, ..., x_m)$ can be computed by evaluating the polynomial $ES_{\mathbf{r};A}$ at $\prod_{1 \leq j \leq n}(1 + c_j)$ points. Which gives(see Remark (1.1)) an exact algorithm for $PE_{\mathbf{r},\mathbf{c}}(A)$ of complexity

$$O\left(\min(\prod_{1 \leq j \leq n}(1 + c_j) nm \log(m), \prod_{1 \leq i \leq m}(1 + r_i) nm \log(n))\right).$$

Thus if $n > m$ and m is fixed then the exists a polynomial in n exact deterministic algorithm to compute $PE_{\mathbf{r},\mathbf{c}}(A)$.

1.3 Previous Work

Estimation of the cardinality $|BM_{\mathbf{r},\mathbf{c}}| = PE_{\mathbf{r},\mathbf{c}}(A)$, where $A = J_{n,m} = e_n e_m^T$ is a matrix of all ones, is one of classical topics in analytic combinatorics. The reader may consult Barvinok's paper [14] for references to most major results on the topic.
To avoid messy formulas, we will mainly focus below on the uniform square case, i.e. $n = m$ and $r_i = c_j = r, 1 \leq i, j \leq n$ and use simplified notations:

$$BM_{\mathbf{re_n},\mathbf{re_n}} =: BM(r, n); PE_{\mathbf{re_n},\mathbf{re_n}}(A) =: PE(r, A).$$

It is easy to see that $PE(r, A)$ is #P-Complete for all $1 \leq r < n$. The connection to the permanent implies that for non-negative matrices A there is **FPRAS** for $PE(r, A)$. We are interested in this paper in deterministic algorithms. We briefly recall the main idea behind Barvinok's algorithm from [14]:
Define

$$\alpha_{\mathbf{r},\mathbf{c}}(A) = \inf_{z_j, x_i > 0} \frac{\prod_{1 \leq i \leq n, 1 \leq j \leq m}(z_j + A(i,j)x_i)}{\prod_{1 \leq i \leq n} x_i^{r_i} \prod_{1 \leq j \leq m} z_j^{n - c_j}}.$$

Then

$$\alpha_{\mathbf{re_n},\mathbf{re_n}}(A) \geq PE(r, A) \geq \frac{vdw(n^2)}{(vdw(n-r)vdw(r))^n} \alpha(A), \tag{9}$$

where $vdw(k) =: \frac{k!}{k^k}$. As the number $\log(\alpha(A))$ can computed(approximated within small additive error) via the convex minimization, the bounds (9) give a poly-time deterministic algorithm to approximate $PE(r, A)$ within multiplicative factor $\gamma_n =: (\frac{vdw(n^2)}{(vdw(n-r)vdw(r))^n})^{-1}$. The factor γ_n is not simply exponential even for $r = 1$, indeed $(\gamma_n)^{\frac{1}{n}} \approx const(\sqrt{n})$ for a fixed r. The proof of (9) in [14] is based on the **Sinkhorm's Scaling** and the Van Der Waerden-Falikman-Egorychev lower bound on the permanent of doubly-stochastic matrices.

2 Our Results

We prove and apply in this paper an optimized version of our lower bounds on the coefficients of **H-Stable** polynomials [8]. The lower bounds in [8] were obtained by a "naive" application of the lower on the mixed derivative of **H-Stable** polynomials [4].
When applied to the polynomial $\prod_{1 \leq i \leq n, 1 \leq j \leq m}(z_j + A(i,j)x_i)$, the main result of the current paper implies the following bounds:

$$\alpha(A) \geq PE(r, A) \geq \left(\frac{vdw(n)}{vdw(n-r)vdw(r)}\right)^{2n-1} \alpha(A) \qquad (10)$$

I.e. for the fixed r the Barvinok's approach gives a deterministic algorithm to approximate $PE(r, A)$ within simply exponential factor $(e^r vdw(r))^{2n}$. We stress again that this result seems to be unprovable by using only Van Der Waerden-Falikman-Egorychev and alike purely permanental bounds, even the newest ones in [7].
When applied to the the polynomial $ES_{\mathbf{r};A}(x_1, ..., x_m)$, our new bounds imply the following general inequality

$$\mu(A) \geq PE_{\mathbf{r},\mathbf{c}}(A) \geq \prod_{2 \leq j \leq n} \left(\frac{vdw(n)}{vdw(n-c_j)vdw(c_j)}\right) \mu(A), \qquad (11)$$

where

$$\mu(A) =: \inf_{x_j > 0} \frac{ES_{\mathbf{r};A}(x_1, ..., x_m)}{\prod_{1 \leq j \leq m} x_j^{c_j}}$$

Note that

$$\log(\mu) = \inf_{\sum_{1 \leq j \leq m} y_j = 0} \log(ES_{\mathbf{r};A}(exp(\frac{y_1}{c_1}), ..., exp(\frac{y_m}{c_m}))),$$

and the function $\log(ES_{\mathbf{r},\mathbf{c};A}(exp(\frac{y_1}{c_1}),...,exp(\frac{y_m}{c_m})))$ is convex in ys. For the fixed $\mathbf{r} = re_n, \mathbf{c} = re_n$ this gives a deterministic poly-time algorithm to approximate $PE(r, A)$ within simply exponential factor $(e^r vdw(r))^n$. This paper does not give the detailed complexity analysis of this convex minimization, say based on the **Ellipsoid Algorithm**. We rather present a very simple converging algorithm, which does pretty well in practice.
In the sparse case we get a much better lower bound(not fully optimized yet):

$$\mu(A) \geq PE_{\mathbf{r},\mathbf{c}}(A) \geq \prod_{2 \leq j \leq m} \left(\frac{vdw(Cl_j)}{vdw(Cl_j - c_j)vdw(c_j)} \right) \mu(A), \tag{12}$$

where $Cl_j = \min(\sum_{1 \leq k \leq j} c_j, Col(j))$ and $Col(j)$ is the number of nonzero entries in the jth column of A.
In the uniform case, i.e. $m = n$, $\mathbf{r} = \mathbf{c} = re_n$ this gives the following bound

$$\mu(A) \geq PE(r, A) \geq \prod_{2 \leq j \leq m} \left(\frac{vdw(C_j)}{vdw(C_j - r)vdw(r)} \right) \mu(A), \tag{13}$$

where $C_j = \min(rj, Col(j))$.
Our final result is the following combinatorial lower bound: Let $A \in BM_{k\mathbf{r},k\mathbf{c}} \neq \emptyset$. Then

$$\inf_{x_j > 0} \frac{ES_{\mathbf{r},\mathbf{c};A}(x_1,...,x_m)}{\prod_{1 \leq j \leq n} x_j^{c_j}} = \prod_{1 \leq i \leq n} \binom{kr_i}{r_i} \tag{14}$$

and

$$PE_{\mathbf{r},\mathbf{c}}(A) \geq \prod_{1 \leq i \leq n} \binom{kr_i}{r_i} \prod_{2 \leq j \leq m} \frac{vdw(kc_j)}{vdw(kc_j - c_j)vdw(c_j)} \tag{15}$$

The formula (15) can be sligthly, i.e. by $const(k, t) > 1$, improved in the regular case. In particular, $const(2, t) = \left(\binom{2t}{t}\right)^{-1} 2^{2t}$.
Let $A \in BM_{kte_n,kte_n}$, where k, t are positive integers. Then

$$PE_{te_n,te_n}(A) \geq \binom{kt}{t}^n \left(\frac{vdw(kt)}{vdw((k-1)t)vdw(t)} \right)^{n-k} \frac{vdw(kt)}{vdw(t)^k}. \tag{16}$$

The inequalities (15, 16) generalize and improve results from [1].
All the inequalities in this section are fairly direct corollaries of Theorem(5.1) (see the main inequality (35)).

3 Stable Homogeneous Polynomials

3.1 Definitions, previous results and the naive approach

The next definition introduces key notations and notions.

Definition 3.1:

1. The linear space of homogeneous polynomials with real (complex) coefficients of degree n and in m variables is denoted $Hom_R(m,n)$ ($Hom_C(m,n)$). We denote as $Hom_+(m,n)$ the closed convex cone of polynomials $p \in Hom_R(m,n)$ with nonnegative coefficients.

2. For a polynomial $p \in Hom_+(n,n)$ we define its **Capacity** as

$$Cap(p) = \inf_{x_i > 0, \prod_{1 \leq i \leq n} x_i = 1} p(x_1, \dots, x_n) = \inf_{x_i > 0} \frac{p(x_1, \dots, x_n)}{\prod_{1 \leq i \leq n} x_i}. \qquad (17)$$

3. Consider a polynomial $p \in Hom_C(m,n)$,

$$p(x_1, \dots, x_m) = \sum_{(r_1, \dots, r_m)} a_{r_1, \dots, r_m} \prod_{1 \leq i \leq m} x_i^{r_i}.$$

   We define $Rank_p(S)$ as the maximal joint degree attained on the subset $S \subset \{1, \dots, m\}$:

$$Rank_p(S) = \max_{a_{r_1, \dots, r_m} \neq 0} \sum_{j \in S} r_j. \qquad (18)$$

   If $S = \{i\}$ is a singleton, we define $deg_p(i) = Rank_p(S)$.

4. A polynomial $p \in Hom_C(m,n)$ is called **H-Stable** if $p(Z) \neq 0$ provided $Re(Z) > 0$; is called **H-SStable** if $p(Z) \neq 0$ provided $Re(Z) \geq 0$ and $\sum_{1 \leq i \leq m} Re(z_i) > 0$.
   (**We coined the term "H-Stable" to stress two things: Homogeneity and Hurwitz' stability.**)

5. We define

$$vdw(i) = \frac{i!}{i^i}; G(i) = \frac{vdw(i)}{vdw(i-1)} = \left(\frac{i-1}{i}\right)^{i-1}, i > 1; G(1) = 1. \qquad (19)$$

   Note that $vdw(i)$ and $G(i)$ are strictly decreasing sequences.

∎

The main inequality in [4] was stated as the following theorem

Theorem 3.2: *Let $p \in Hom_+(n,n)$ be* **H-Stable** *polynomial. Then the following inequality holds*

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0,\dots,0) \geq \prod_{2 \leq i \leq n} G(\min(i, deg_p(i))) Cap(p). \tag{20}$$

So, if $p \in Hom_+(n,n)$ is **H-Stable** and $deg_p(i) \leq k \leq n$ for $k+1 \leq i \leq n$ then the following inequality holds:

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0,\dots,0) \geq G(k)^{n-k} \frac{k!}{k^k} Cap(p). \tag{21}$$

For $k = n$ we get the feneralized Van der Waerden-Falikman-Egorychev inequality:

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0,\dots,0) \geq \frac{n!}{n^n} Cap(p). \tag{22}$$

3.2 A naive generalization to the general monomials

Let $p \in Hom_+(m,n)$ and consider an integer vector

$$\mathbf{c} = (c_1, ..., c_m) \in R^m_+, \sum_{1 \leq i \leq m} c_i = n.$$

Define the following polynomial $q \in Hom_+(n,n)$:

$$q(y_{1,1}, ..., y_{1,c_1}; y_{2,1}, ..., y_{2,c_2}; ...; y_{m,1}, ..., y_{m,c_m}) = p(z_1, ..., z_m),$$

where $z_i = \frac{\sum_{1 \leq j \leq c_i} y_(i,j)}{c_i}, 1 \leq i \leq m$. *Note that if the polynomial p is* **H-Stable** *then q also is.*
Also, analogously to (17), let us define:

$$Cap_{\mathbf{c}}(p) =: Cap_{c_1,\dots,c_m}(p) =: \inf_{x_j > 0} \frac{p(x_1, ..., x_m)}{\prod_{1 \leq j \leq m} x_j^{c_j}}. \tag{23}$$

Fact 3.3:

1. $Cap_{c_1,\dots,c_m}(p) = Cap(q)$.

2. $[x_1^{c_1}...x_m^{c_m}]p = \prod_{1 \leq i \leq m} \frac{c_i!}{c_i^{c_i}} \frac{\partial^n}{\prod_{1\leq i \leq m} \prod_{1 \leq j \leq c_i} \partial y_{i,j}} q(0)$

Corollary 3.4: *Let $p \in Hom_+(m,n)$ be* **H-Stable** *and $deg_p(i) \leq k, 1 \leq i \leq m$. Then the following inequality holds:*

$$Cap_{c_1,...,c_m}(p) \geq [x_1^{c_1}...x_m^{c_m}]p \geq Cap_{c_1,...,c_m}(p)(\prod_{1 \leq i \leq m}(vdw(c_i))^{-1})G(k)^{n-k}vdw(k). \tag{24}$$

If $k = n$ then

$$Cap_{c_1,...,c_m}(p) \geq [x_1^{c_1}...x_m^{c_m}]p \geq Cap_{c_1,...,c_m}(p)(\prod_{1 \leq i \leq m}(vdw(c_i))^{-1})vdw(n). \tag{25}$$

Example 3.5:

1. Let $A = J_n$ be the $n \times n$ matrix of all ones and consider the **H-Stable** polynomial $ES_{re_n;J_n}(x_1,...,x_n) = \prod_{1 \leq i \leq n} S_r(x_1,...,x_n)$. Then

$$Cap_{re_n}(ES_{re_n;J_n}) = \binom{n}{r}^n ; deg_{ES_{re_n;J_n}}(i) = n, 1 \leq i \leq n.$$

Applying the inequality (24) we get that

$$|BM(r,n)| = [x_1^r....x_n^r]ES_{re_n;J_n} \geq \binom{n}{r}^n (\frac{r^r}{r!})^n (\frac{n-1}{n})^{(n-1)n(r-1)} \frac{n!}{n^n} \tag{26}$$

It was proved by Everrett and Stein in [15](their proof is rather involved) that

$$|BM(r,n)| = \frac{(rn)!}{(r!)^{2n}} exp(-\frac{1}{2}(r-1)^2)\beta(r,n), \tag{27}$$

where $\lim_{n \to \infty} \beta(r,n) = 1$ for any fixed integer number r. Our lower bound from from (26)

$$|BM(r,n)| \geq HYP(r,n) = \binom{n}{r}^n (\frac{r^r}{r!})^n (\frac{n-1}{n})^{(n-1)n(r-1)} \frac{n!}{n^n}$$

is valid for all values of r. Directly applying Stirling formula, we get that for the fixed r

$$\lim_{n \to \infty} \frac{HYP(r,n)}{|BM(r,n)|} = (\sqrt{r})^{-1}.$$

Not bad at all, considering how computationally and conceptually simple is our derivation of (26)!
The same reasoning applies to general non-negative $n \times n$ matrices:

$$Cap_{re_n}(ES_{re_n;A}) \geq PE(r,A) \geq Cap_{re_n}(ES_{re_n;A})(\frac{r^r}{r!})^n(\frac{n-1}{n})^{(n-1)n(r-1)}\frac{n!}{n^n}. \quad (28)$$

As $\log(Cap_{re_n}(ES_{re_n;A}))$ can be expressed in terms of convex minimization, for a fixed r the inequality (28) justifies a deterministic poly-time algorithm to approximate $PE(r,A)$ within simply exponential factor $(e^r vdw(r))^n$.
We stress that the current, say in [12], [13], very delicate, accurate and efficient estimates of $|BM_{\mathbf{r},\mathbf{c}}|$(valid for some ranges of $(\mathbf{r},\mathbf{c})$ can not be, at least directly, applied to estimate $PE_{\mathbf{r},\mathbf{c}}(A)$ even for boolean matrices A because of the #P-Hardness of $PE_{\mathbf{r},\mathbf{c}}(A)$.

2. Applying the inequality (24) to the polynomial $TM(z_1,...,z_n;x_1,...,x_m) = \prod_{1\leq i\leq n, 1\leq j\leq m}(z_j + A(i,j)x_i)$ we get that

$$\alpha \geq PE(r,A) \geq \alpha(vdw(r)(vdw(n-r))^{-n}\left(\frac{n-1}{n}\right)^{(n-1)n(n-1)}\frac{n!}{n^n}$$

It is better than (9), yet does not give a simply exponential factor.

3. Let $A \in BM_{t\mathbf{r},t\mathbf{c}}$. We are interested in a lower bound on $PE_{\mathbf{r},\mathbf{c}}(A)$. I.e. on how many $\frac{1}{t}$-shrunken copies of itself the boolean matrix A contains. Now, $ES_{\mathbf{r};A}(x_1,...,x_m) = \prod_{1\leq i\leq n} S_{r_i}(A(i,1)x_1,...,A(i,m)x_m)$ and $PE_{\mathbf{r},\mathbf{c}}(A) = [\prod_{1\leq j\leq n} x_j^{c_j}]ES_{\mathbf{r};A}(x_1,...,x_m)$. Easy computation: If $A \in BM_{t\mathbf{r},t\mathbf{c}}$ then $Cap_{\mathbf{c}}(ES_{\mathbf{r};A}) = \prod_{1\leq i\leq n}\binom{tr_i}{r_i}$
Let us consider the simplest non-trivial case of $A \in BM(2r,n)$. Applying the inequality (24), we get that

$$PE(r,A) \geq \binom{2r}{r}^n (vdw(r))^{-n}((\frac{2r-1}{2r})^{(2r-1)n(r-1)}\frac{r!}{r^r}.$$

Yet, the asymtotically correct bound from [1] is

$$PE(r,A) \geq \left(\binom{2r}{r}^2 (2)^{-2r}\right)^n. \quad (29)$$

Not only we will improve in the current paper the bound (29), but also show that if all columns of $n \times n$ nonnegative matrix A have at most tr non-zero entries then $PE(r, A)$ can be deterministically in poly-time approximated within the factor $(\frac{vdw(tr)}{vdw(r)vdw(r(t-1))})^{-n}$.

$\blacksquare$

4 New Observations, which were overlooked in [4]

Associate with a polynomial $p \in Hom_+(n, n)$ the following sequence of polynomials $q_i \in Hom_+(i, i)$:

$$q_n = p, q_i(x_1, \dots, x_i) = \frac{\partial^{n-i}}{\partial x_{i+1} \dots \partial x_n} p(x_1, \dots, x_i, 0, \dots, 0); 1 \leq i \leq n-1.$$

The inequality (20) is, actually, a corollary of the following inequality, which holds for **H-Stable** polynomials:

$$Cap(q_i) \geq Cap(q_{i-1}) \geq G(deg_{q_i}(i))Cap(q_i), n \geq i \geq 2. \tag{30}$$

As $Cap(q_1) = \frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0, \dots, 0)$, one gets that

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0, \dots, 0) \geq \prod_{2 \leq i \leq n} G(deg_{q_i}(i)))Cap(p). \tag{31}$$

The inequality (20) follows from (31) because $G(i)$ is decreasing and $deg_{q_i}(i) \leq \min(i, deg_p(i)$. There were several reasons why the inequality (20) was stated as the main result:

1. It is simpler to understand than more general one (31). It was sufficient for the killer application: a short, transparent proof of the (improved) Schrijver's lower bound on the number of perfect matchings in k-regular bipartite graphs.

2. For the most of natural polynomials, the gegrees $deg_{q_i}(i)$ are straightforward to compute. Moreover, if a polynomial p with integer coefficients is given as an evaluation oracle then $Rank_p(S)$ can be computed in polynomial time via the univariate interpolation. On the other hand,

if $i = n - [n^a], a > 0$ then even deciding whether $deg_{q_i}(i)$ is zero or not is **NP-HARD**. Indeed, consider, for instance, the following family of polynomials, essentially due to A. Barvinok:

$$p(x_1, ..., x_n) = Bar_A(x_n, ..., x_{n-[n^a]+1})(x_1 + ... + x_{n-[n^a]})^{n-[n^a]},$$

where $Bar_A(x_n, ..., x_{n-[n^a]}) = tr((Diag(x_n, ..., x_{n-[n^a]})A)^{[n^a]})$ and A is the adjacency matrix of an undirected graph. If the graph has a Hamiltonian cycle then $deg_{q_i}(i) = i$ and is zero otherwise.

4.1 New Structural Results

The following simple bound was overlooked in [4]:
$deg_{q_i}(i) \leq \min(Rank_p(\{i, ..., n\}) - n + i, deg_p(i))$. So, if

$$Rank_p(\{j, ..., n\}) - n + j \leq k : k + 1 \leq j \leq n \tag{32}$$

then

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0, \dots, 0) \geq Cap(p) G(k)^{n-k} vdw(k). \tag{33}$$

Example 4.1: Let A be $n \times n$ doubly-stochastic matrix with the following pentagon shaped support: $A(i, j) = 0 : j - i \geq n - k$. Then the product polynomial $Prod_A(x_1, ..., x_n) = \prod_{1 \leq i \leq n} \sum_{1 \leq j \leq n} A(i, j) x_j$ satisfies the inequalities (32) and $cap(Prod_A) = 1$. Therefore $per(A) \geq G(k)^{n-k} vdw(k)$. This lower bound for the permanent was proved by very different methods in [20], moreover it was shown there that it is sharp. Therefore, the more general bound (33) is sharp as well. ∎

We remind the following result(combination of results in [6] and [11]).

Theorem 4.2: *Let* $p \in Hom_+(m, n), p(x_1, ..., x_m) = \sum_{r_1 + ... + r_m = n} a_{r_1, ..., r_n} x_1^{r_1} ... x_m^{r_m}$ *be* **H-Stable**. *Then*

1. $$a_{r_1, ..., r_m} > 0 \Longleftrightarrow \sum_{j \in S} r_j \leq Rank_p(S) : S \subset \{1, \dots, m\}. \tag{34}$$

2. *The set function* $Rank_p(S)$ *is submodular.*

3. *As* $a_{r_1, ..., r_m} > 0$ *iff* $\min_{S \subset \{1, ..., m\}}(Rank_p(S) - \sum_{j \in S} r_j) \geq 0$ *hence given the evaluation oracle for* p *there is deterministic strongly polynomial algorithm to decide whether* $a_{r_1, ..., r_m} > 0$.

Lemma 4.3: *Let $p \in Hom_+(n,n)$ be* **H-Stable** *polynomial with integer coefficients given as an evaluation oracle.Then for any $i \geq 1$ there is a deterministic strongly polynomial algorithm to compute $deg_{q_i}(i)$, i.e. with the number of logical and arithmetic operations bounded by $poly(n)$.*

Proof: Associate with the number i and any polynomial $p \in Hom_+(n,n)$ the following polynomials $P_l(y_1, ..., y_n) = p(z_1, ..., z_n)$
where $0 \leq l \leq n-i-1$ and $z_j = y_1 + ... + y_l, 1 \leq j \leq i-1; z_i = y_{l+1} + ... + y_{n-i}; z_{i+k} = y_{i+k}, 1 \leq k \leq n-i$.
Then $deg_{q_i}(i) \geq n-i-l$ iff $\frac{\partial^n}{\partial y_1 ... \partial y_n} P(0, \ldots, 0) > 0$. Now, if the original polynomial p is **H-Stable** then the polynomials P_l are **H-Stable** as well, have integer coefficients and there are evaluation oracles for them. Therefore we can apply the submodular minimization algorithm from Theorem (4.2) to decide whether the monomial $y_1...y_n$ is in the support of P_l. Running this algorithm at most $i \leq n$ times will give us $deg_{q_i}(i)$. ∎

Example 4.4: [Gale-Ryser Inequalities]. Consider the following **H-Stable** polynomial
$GR_{\mathbf{r},\mathbf{c}}(x_1, ..., x_m) = \prod_{1 \leq i \leq n} S_{r_i}(x_1, ..., x_m); \sum_j c_j = \sum_i r_i$.
Clearly, the monomial $\prod_{1 \leq j \leq n} x_j^{c_j}$ is in the support iff the set $BM_{\mathbf{r},\mathbf{c}}$ is not empty, i.e. there exists a boolean matrix with column sums $\mathbf{c}$ and row sums $\mathbf{r}$. It is easy to see that $Rank_{GR_{\mathbf{r},\mathbf{c}}}(S) = \sum_{1 \leq i \leq n} \min(|S|, r_i)$. It follows from the characterization (34) that $BM_{\mathbf{r},\mathbf{c}}$ is not empty iff $\sum_{j \in S} c_j \leq \sum_{1 \leq i \leq n} \min(|S|, r_i)$ for all subsets $S \subset \{1, ..., m\}$. Equivalently, for the ordered column sums $c_{j_1} \geq c_{j_2} \geq ... \geq c_{j_m}$ the following inequalities hold:
$\sum_{1 \leq k \leq t} c_{j_k} \leq \sum_{1 \leq i \leq n} \min(t, r_i); 1 \leq t \leq m$.
These are the famous Gale-Ryser inequalities, albeit stated without Ferrers matrices. ∎

5 Main New Lower Bound

Let $p \in Hom_+(m,n)$ be a homogeneous polynomial in m variables, of degree n and with non-negative coefficients. We fix a monomial $\prod_{1 \leq j \leq m} x_j^{c_j}, \sum_{1 \leq j \leq m} c_j = n$ and assume WLOG that $c_j > 0, 1 \leq j \leq m$. Let $0 \leq a_{c_1,...,c_m} = [\prod_{1 \leq j \leq m} x_j^{c_j}]p$ be a coefficient of the monomial. Define $Cap_{c_1,...,c_m}(p) =: \inf_{x_j > 0} \frac{p(x_1,...,x_m)}{\prod_{1 \leq j \leq m} x_j^{c_j}}$. Clearly, $a_{c_1,...,c_m} \leq Cap_{c_1,...,c_m}(p)$.

Theorem 5.1: *Let $p \in Hom_+(d,m)$ be* **H-Stable**. *Define the following family of polynomials:*
$Q_m = p, Q_i \in Hom(i, n-(c_m+...+c_{i+1})), m-1 \geq i \geq 1$*:*

$$Q_i = \frac{\partial^{c_m+...+c_{i+1}}}{\partial x_{i+1}^{c_{i+1}} \dots \partial x_m^{c_m}} p(x_1, \dots, x_i, 0, \dots, 0); 1 \leq i \leq m-1.$$

Denote $dg(i) =: deg_{Q_i}(i)$. Then the following inequality holds

$$a_{c_1,...,c_m} \geq Cap_{c_1,...,c_m}(p) \prod_{2 \leq j \leq m} \frac{vdw(dg(j))}{vdw(dg(j))vdw(dg(j)-c_j)} \tag{35}$$

Remark 5.2: It is easy to see that for a fixed value of c the function $\frac{vdw(K)}{vdw(c)vdw(K-c)} = (vdw(c))^{-1}G(K)G(K-1)...G(K-c+1)$ is strictly decreasing in $K \geq c$. Also, the $dg(j) \leq \min(j, n-(c_m+...+c_{j+1}))$. ∎

Corollary 5.3: *Let $p \in Hom_+(m,n)$ be* **H-Stable**. *Then the following (non-optimized but easy to use) lower bound holds:*

$$a_{c_1,...,c_m} \geq Cap_{c_1,...,c_m}(p) \prod_{1 \leq j \leq m} \frac{vdw(deg_p(j))}{vdw(c_j)vdw(deg_p(j)-c_j)} \tag{36}$$

Our proof is, similarly to [4], by induction, which is based on the following bivariate lemma.

Lemma 5.4: *$p \in Hom_+(2,d)$ be* **H-Stable**, *i.e. $p(x_1,x_2) = \sum_{0 \leq i \leq d} a_i x_1^{d-i} x_2^i$ and $1 \leq c_2 < d$. Then*

$$a_{c_2} \geq Cap_{d-c_2,c_2}(p) \frac{vdw(d)}{vdw(c_2)vdw(d-c_2)}.$$

Proof: Define the following polynomial $P \in Hom_+(d,d)$:
$P(y_1,...,y_{d-c_2}; z_1,...,z_{c_2}) = p(\frac{1}{d-c_2}\sum_{1 \leq k \leq d-c_2} y_k, \frac{1}{c_2}\sum_{1 \leq i \leq c_2} z_i)$.
It follows from the standard AG inequality that $Cap_{d-c_2,c_2}(p) = Cap(P)$ and it is easy to see that P is **H-Stable**. Consider the following polynomial $R(z_1,...,z_{c_2}) =: \prod_{1 \leq k \leq d-c_2} \frac{\partial}{\partial y_k} P(y_k = 0, 1 \leq k \leq d-c_2; z_1,...,z_{c_2})$. First, it follows from (30) that $Cap(R) \geq G(d)...G(c_2+1)Cap(P)$. By the direct inspection, $R(z_1,...,z_{c_2}) = a_{c_2} vdw(d-c_2)(\frac{1}{c_2}\sum_{1 \leq i \leq c_2} z_i)^{c_2}$. Therefore $Cap(R) = a_{c_2} vdw(d-c_2)$.
Putting things together gives that

$a_{c_2} \geq \frac{G(d)...G(c_2+1)}{vdw(d-c_2)} Cap_{d-c_2,c_2}(p) = \frac{vdw(d)}{vdw(c_2)vdw(d-c_2)} Cap_{d-c_2,c_2}(p)$.
∎

Proof: [Proof of Theorem (5.1)]. Let $p \in Hom_+(m,n)$ be **H-Stable**. Expand it in the last variable:
$p(x_1,...,x_m) = \sum_{0 \leq i \leq deg_p(m)} x_m^i T_i(x_1,...,x_{m-1})$. Our goal is to prove that

$$Cap_{c_1,...,c_{m-1}}(T_{c_m}) \geq \frac{vdw(d)}{vdw(c_m)vdw(d-c_m)} Cap_{c_1,...,c_{m-1},c_m}(p). \tag{37}$$

Fix positive numbers $(y_1,...,y_{m-1})$ and consider the following bivariate polynomial: $W(t,x_m) = p(ty_1,...,ty_{m-1},x_m)$. The polynomial W is of degree n and **H-Stable**. Note that $W(t,x_m) \geq Cap_{c_1,...,c_m}(p) t^{d-c_m} x_m^{c_m} \prod_{1 \leq i \leq m-1} y_i^{c_j}$. It follows from Lemma(5.4) that
$T_{c_m} i(y_1,...,y_{m-1}) \geq \frac{vdw(d)}{vdw(c_m)vdw(d-c_m)} Cap_{c_1,...,c_m}(p) \prod_{1 \leq i \leq m-1} y_i^{c_j}$, which proves the inequality (37). Now the polynomial $T_{c_m} \in Hom_+(m-1, n-c_m)$ is also **H-Stable** [4]. Thus we can apply the same argument to the polynomial $T_{c_m}(x_1,...,x_{m-1})$ and so on until only the first variable x_1 remains. ∎

6 Algorithms to compute parameters of Theorem (5.1)

To compute the lower bound in (35) we need the degrees $deg_{Q_i}(i)$ and the capacity $Cap_{c_1,...,c_m}(p)$. We assume that the polynomial p has rational coefficients and is given an evaluation oracle, say, on the integer vectors. Essentially this model allows to do, besides logic, only low-dimensional interpolations. Lemma (4.3) allows to compute $deg_{Q_i}(i)$ in this model in $poly(n)$ operations, i.e. with now dependence on the bit-wise complexity of the coefficients.

6.1 A practical algorithm to approximate $Cap_{c_1,...,c_m}(p)$

Associate with a polynomial $p \in Hom_+(m,n)$ and a non-negative vector $(c_1,...,c_m), \sum_{1 \leq i \leq m} c_i = n$ the following maps:

$$F(x_1,...,x_m) = (y_1,...,y_m); y_i = \frac{x_i}{a_i}, G(X) = F(X)(\prod_{1 \leq i \leq m} a_i^{c_i})^{\frac{1}{n}}$$

where $a_i = \frac{x_i \frac{\partial}{\partial x_i} p(X)}{c_i p(X)}$. In other words $y_i = \frac{\alpha}{\frac{\partial}{\partial x_i} p(X)}$, α is a normalizing constant. Note that it follows from the Euler's identity for homogeneous functions that

$$mul(X) =: \prod_{1 \le i \le m} a_i^{c_i} \le \left(\sum_{1 \le i \le m} \frac{x_i \frac{\partial}{\partial x_i} p(X)}{np(X)} \right)^n = 1, \tag{38}$$

and the map G preserves the product of powers $\prod_{1 \le i \le m} x_i^{c_i}$.

Lemma 6.1: *Suppose that the polynomial $p \in Hom_+(m,n)$ is log-concave on R^m_{++}. Then the following inequality holds:*

$$p(G(X)) \le mul(X) p(X).$$

Proof: The log-concavity gives the following inequality

$$\log(p(y_1, ..., y_m)) \le \log(p(x_1, ..., x_m)) + \sum_{1 \le i \le m} \frac{\frac{\partial}{\partial x_i} p(X)}{p(X)} (y_i - x_i).$$

So if $Y = F(X), X \in R^m_{++}$ then the Euler's identity gives the following inequality

$$\log(p(y_1, ..., y_m)) \le \log(p(x_1, ..., x_m)) + \sum_{1 \le i \le m} (c_i - \frac{x_i \frac{\partial}{\partial x_i} p(X)}{p(X)} = \log(p(x_1, ..., x_m)).$$

Finally, $p(G(X)) = mul(X) p(F(X)) \le mul(X) p(X)$. ∎We suggest the following algorithm to approximate $Cap_{c_1,...,c_m}(p)$:
Start with $x_{i,0} = 1, 1 \le i \le m$ and recursively compute the following vector sequence:

$$X_{k+1} = G(X_k)), k \ge 1.$$

Stop if $mul(X_k) \ge 1 - \epsilon$, where $\epsilon << 1$ and output $Cap_{c_1,...,c_m}(p) \approx p(X_k)$.
This algorithm does not work for general polynomials in $Hom_+(m,n)$ (just consider $p(x_1, x_2) = 2(x_1)^2 + (x_2)^2$ and $(c_1, c_2) = (1,1)$). But Lemma (6.1) essentially proves that if $Cap_{c_1,...,c_m}(p) > 0$ and $p \in Hom_+(m,n)$ is log-concave then the algorithm converges. In fact, it is a generalization of the famous(and efficient in practice) Sinkhorn's scaling algorithm to the product of symmetric functions $ES_{\mathbf{r};A}(x_1, ..., x_m)$. Sinkhorn's scaling algorithm corresponds to the product of linear forms.
As any **H-Stable** polynomial $p \in Hom_+(m,n)$ is log-concave, we can apply

Lemma(6.1). In practice it usually takes just a few steps of the Sinkhorn's scaling to get a very good approximation. We expect the same from our generalization. Note that each step of the algorithm for $ES_{\mathbf{r};A}$ boils down to computing n symmetric functions of m variables(the evaluation of the polynomial) and nm symmetric functions of $m-1$ variables(the evaluation of the gradient).

7 Applications of Theorem (5.1)

Example 7.1:

1. The polynomial from [14] $TM(z_1,...,z_n;x_1,...,x_m) =: \prod_{1\leq i\leq n, 1\leq j\leq m}(z_i + A(i,j)x_j)$. Consider, just for the illustration, the square uniform case: $n=m$, $\mathbf{c}=(n-r,...,n-r;r,...,r)$. Note that the degrees of all variable are bounded by n. Using non-optimized lower bound (36) we get that the coefficient

$$a_{n-r,...,n-r;r,...,r} \geq Cap_{n-r,...,n-r;r,...,r}(TZ)\left(\frac{vdw(n)}{vdw(r)vdw(n-r)}\right)^{2n}$$

2. We give a better lower bound on $|BM(r,n)|$. The polynomial is $Sym_{r,n}(\mathbf{x}) =: (S_r(x_1,...,x_n))^n$. Degree of each variable is n. $Cap_{r,...,r}(Sym_{r,n}) = \binom{n}{r}^n$. The slightly optimized lower bound is, assuming that r divides n, as follows

$$|BM(r,n)| \geq \binom{n}{r}^n \left(\frac{vdw(n)}{vdw(r)vdw(n-r)}\right)^{\frac{n(r-1)}{r}} vdw(n)(vdw(r)^{-\frac{n}{r}}). \tag{39}$$

For the fixed r this new bound is asymptotically equal to the "old" bound (26), but, say, for $r=\Omega(n)$ the bound (39) is exponentially in n greater than the bound (26).

3. Let $A \in BM(tr,n)$. We are interested in a lower bound on $PE(r,A)$. I.e. on how many $\frac{1}{t}$-shrunken copies of itself the boolean matrix A contains.
Now, the polynomial is $ES_{\mathbf{r};A}(x_1,...,x_m) = \prod_{1\leq i\leq n} S_{r_i}(A(i,1)x_1,...,A(i,m)x_m)$ and we are after a lower bound on $[\prod_{1\leq j\leq n} x_j^{c_j}]ES_{\mathbf{r};A}(x_1,...,x_m)$.
Easy computation: $Cap_{\mathbf{c}}(ES_{\mathbf{r};A}) = (\binom{tr}{r})^n$ if $A \in BM(tr,n)$.

Theorem (5.1) and Remark (5.2) give the following lower bound:

$$PE(r,A) \geq \left(\binom{tr}{r}\right)^n \left(\frac{vdw(tr)}{vdw(r)vdw(r(t-1))}\right)^{n-t} vdw(rt)(vdw(r))^{-t},$$

improving (and reproving) the bounds from [1]. In the same way we can get bounds on $PE(r,A)$ for matrices A with r-sparse columns.

∎

8 An analogue of Van Der Waerden Conjecture

It is well known(a fairly direct corollary of the famous Edmond's result that the intersection of two matroidal polytopes is equal to the polytope of the the intersection of the matroids) that the convex hull $CO(BM(r,n)) = \{A : 0 \leq A(i,j) \leq 1, Ae = A^T e = re.$
Note that for $r = 1$ the polytope $CO(BM(r,n))$ is equal to the polytope Ω_n of doubly-stochastic $n \times n$ matrices and $Pe(1,A) = Per(A)$.
The natural question, an analogue of the Van Der Waerden Conjecture($r = 1$) is to compute the following minimum and maximum:

$$NMin(r,n) =: \min_{A \in CO(BM(r,n))} PE(r,A), NMax(r,n) =: \max_{A \in CO(BM(r,n))} PE(r,A)$$

Remark 8.1: Recall that $NMin(1,n) = \frac{n!}{n^n}$ and it took more than 50 years to prove. On the other hand, the equality $\max_{A \in CO(BM(r,n))} Pe(1,A) = \max_{A \in \Omega_n} Per(A)$ is fairly trivial. ∎

First of all, any matrix $A \in BM(r,n)$ is a local maximum. Therefore $NMin(r,n) < 1$. One would guess that the minimum is attained at the "center" $Cen_{r,n} =: \frac{r}{n} J_n, J_n = ee^T$.
The Everett-Stein asymptotically exact estimate and our easily proved lower bound give that:

$$PE(r, \frac{r}{n} J_n) \geq const(r) \left(\frac{r}{n}\right)^{rn} \frac{(rn)!}{(r!)^{2n}} \geq const(r) \left(\frac{r^{2r}}{e^r (r!)^2}\right)^n.$$

Define $n(r) =: \frac{r^{2r}}{e^r (r!)^2}$. By the direct inspection $n(1) < 1, n(2) < 1$ but $n(i) > 1$ for all $i \geq 3$.

So, there is a possibility that $NMin(2,n) = PE(2, Cen_{2,n})$. On the other hand, for $r \geq 3$ the "center" is not a minimum, at least for sufficiently large n.
The following (preliminary) result provides some lower and upper bounds on $NMin(r,n)$ and $NMax(r,n)$.

Proposition 8.2: *Let A, B be nonnegative $n \times m$ matrices, $0 < a < 1$. Then for all positive vectors $(x_1, ..., x_m)$ the following inequality holds*

$$ES_{\mathbf{r};aA+(1-a)B}(x_1, ..., x_m) \geq (ES_{\mathbf{r};A}(x_1, ..., x_m))^a (ES_{\mathbf{r};B}(x_1, ..., x_m))^{1-a}.$$

Proof: Follows from the well known fact that the symmetric functions are log-concave on the positive orthant. ∎

Corollary 8.3: *The functional $G(A) =: \log(Cap_{c_1,...,c_m}(ES_{\mathbf{r};A})$ is concave on the convex cone of non-negative matrices.*
Therefore, $Cap_{r,...,r}(ES_{re_n;A}) \geq 1$ if $A \in CO(BM(r,n))$.

It is as easy to prove the upper bound:

$$Cap_{r,...,r}(ES_{re_n;A}) \leq Cap_{r,...,r}(ES_{re_n;\frac{r}{n}J_n}) = (\binom{n}{r})^n (\frac{r}{n})^{rn}, A \in CO(BM(r,n)). \tag{40}$$

Indeed, it follows from Newton's inequalities that $Sym_r(x_1, ..., x_n) \leq (\frac{x_1+...+x_n}{n})^r \binom{n}{r}$. Therefore, if $A \in CO(BM(r,n)$ then

$$Cap_{r,...,r}(ES_{re_n;A}) \leq ES_{re_n;A}(1, ..., 1) \leq (\binom{n}{r})^n (\frac{r}{n})^{rn}.$$

We put these observation in the following statement about our current knowledge on the range of $PE(r,A), A \in CO(BM(r,n)$.

Lemma 8.4: *Let $A \in CO(BM(r,n)$ then*

$$\min_{A \in CO(BM(r,n))} Cap_{r,...,r}(ES_{re_n;A}) = 1, \max_{A \in CO(BM(r,n))} Cap_{r,...,r}(ES_{re_n;A}) = (\binom{n}{r})^n (\frac{r}{n})^{rn}.$$

And, assuming that r divides n, the following inequalities hold

$$(\binom{n}{r})^n (\frac{r}{n})^{rn} \geq PE(r,A) \geq (\frac{vdw(n)}{vdw(r)vdw(n-r)})^{n^{\frac{r-1}{r}}} \frac{vdw(n)}{vdw(r)^{n/r}}.$$

9 Acknowledgements

The author acknowledges the support of NSF grant 116143.

References

[1] Schrijver, A. Bounds on permanents, and the number of 1-factors and 1-factorizations of bipartite graphs. Surveys in combinatorics (Southampton, 1983), 107134, London Math. Soc. Lecture Note Ser., 82, Cambridge Univ. Press, Cambridge, 1983.

[2] A. Schrijver, Counting 1-factors in regular bipartite graphs, Journal of Combinatorial Theory, Series B 72 (1998) 122–135.

[3] M. Laurent and A. Schrijver, On Leonid Gurvits' proof for permanents, Amer. Math. Monthly 117 (2010), no. 10, 903-911.

[4] L. Gurvits, Van der Waerden/Schrijver-Valiant like conjectures and stable (aka hyperbolic) homogeneous polynomials: one theorem for all, Electronic Journal of Combinatorics 15 (2008).

[5] L. Gurvits, A polynomial-time algorithm to approximate the mixed volume within a simply exponential factor. *Discrete Comput. Geom.* 41 (2009), no. 4, 533–555.

[6] L. Gurvits, Combinatorial and algorithmic aspects of hyperbolic polynomials, 2004; available at http://xxx.lanl.gov/abs/math.CO/0404474.

[7] L. Gurvits, Unleashing the power of Schrijver's permanental inequality with the help of the Bethe Approximation,2011, available at http://arxiv.org/abs/1106.2844.

[8] L. Gurvits, On multivariate Newton-like inequalities. Advances in combinatorial mathematics, 61-78, Springer, Berlin, 2009; available at http://arxiv.org/pdf/0812.3687v3.pdf.

[9] G.P. Egorychev, The solution of van der Waerden's problem for permanents, *Advances in Math.*, 42, 299-305, 1981.

[10] D. I. Falikman, Proof of the van der Waerden's conjecture on the permanent of a doubly stochastic matrix, *Mat. Zametki* 29, 6: 931-938, 957, 1981, (in Russian).

[11] L. Gurvits, Hyperbolic polynomials approach to Van der Waerden/Schrijver-Valiant like conjectures: sharper bounds, simpler proofs and algorithmic applications, *Proc. 38 ACM Symp. on Theory of Computing (StOC-2006)*,417-426, ACM, New York, 2006.

[12] C. Greenhill, B.D. McKay, and X. Wang, Asymptotic enumeration of sparse 0-1 ma- trices with irregular row and column sums, Journal of Combinatorial Theory. Series A 113 (2006), 291324.

[13] C. Greenhill and B.D. McKay, Random dense bipartite graphs and directed graphs with specified degrees, Random Structures and Algorithms 35 (2009), 222249.

[14] A. Barvinok, On the number of matrices and a random matrix with prescribed row and column sums and 01 entries, Adv. Math. 224 (2010), no. 1, 316339.

[15] Everett, C. J. and Stein, P. R.; The asymptotic number of integer stochastic matrices, Discrete Math. 1 (1971/72), no. 1, 55-72.

[16] B.D. McKay, Asymptotics for 0-1 matrices with prescribed line sums, Enumeration and Design, Academic Press, Canada (1984), pp. 225-238

[17] W.T. Tutte, A short proof of the factor theorem for finite graphs, Canad. J. Math., 6, 347-352, 1954.

[18] N.K. Vishnoi: A Permanent Approach to the Traveling Salesman Problem. FOCS 2012: 76-80

[19] M. Jerrum, A. Sinclair, and E. Vigoda, A polynomial-time approximation algorithm for the permanent of a matrix with nonnegative entries, Journal of the ACM 51 (2004), 671697.

[20] Suk Geun Hwang, MATRIX POLYTOPE AND SPEECH SECURITY SYSTEMS, Korean J. CAM. Vol. 2(1995), No. 2, pp. 3 - 12.

ECCC ISSN 1433-8092

http://eccc.hpi-web.de