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Boolean matrices with prescribed row/column sums and stable homogeneous polynomials: combinatorial and algorithmic applications

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Abstract

We prove a new efficiently computable lower bound on the coefficients of stable homogeneous polynomials and present its algorithmic and combinatorial applications. Our main application is the first poly-time deterministic algorithm which approximates the partition functions associated with boolean matrices with prescribed row and column sums within simply exponential multiplicative factor. This new algorithm is a particular instance of new polynomial time deterministic algorithms related to the multiple partial differentiation of polynomials given by evaluation oracles.

1 Basic Definitions and Motivations

For given two integer vectors $\mathbf{r} = (r_1, ..., r_n)$ and $\mathbf{c} = (c_1, ..., c_m)$, we denote as $BM_{\mathbf{r},\mathbf{c}}$ the set of boolean $n \times m$ matrices with prescribed row sums $\mathbf{r}$ and column sums $\mathbf{c}$.

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Next, we introduce an analogue of the permanent (a partition function associated with $BM_{\mathbf{r},\mathbf{c}}$):

$$PE_{\mathbf{r},\mathbf{c}}(A) =: \sum_{B \in BM(\mathbf{r},\mathbf{c})} \prod_{1 \leq i \leq n; 1 \leq j \leq m} A(i,j)^{B(i,j)}, \tag{1}$$

where A is an $n \times m$ complex matrix. We suppose that $0^0 = 1$. Note that if A is an $n \times n$ matrix; $\mathbf{r} = \mathbf{c} = e_n$, where e_n is an n-dimensional vector of all ones, then the definition (1) reduces to the permanent: $PE_{e_n,e_n}(A) = \mathrm{per}(A)$.

The main focus of this note is on bounds and **deterministic** algorithms for $PE_{\mathbf{r},\mathbf{c}}(A)$ in the case that the matrix A is real and non-negative. To avoid messy formulas, we will mainly focus below on the uniform square case, i.e. $n = m$ and $r_i = c_j = r, 1 \leq i,j \leq n$ and use simplified notations:

$$BM_{\mathbf{re_n},\mathbf{re_n}} =: BM(r,n); PE_{\mathbf{re_n},\mathbf{re_n}}(A) =: PE(r,A),$$

where e_n denotes the vector $(1,\ldots,1)$. Boolean matrices with prescribed row and column sums are one of the most classical and intensely studied topics in analytic combinatorics, with applications to many areas from applied statistics to the representation theory. We, and many other researchers, are interested in the counting aspect, i.e. in computing/bounding/approximating the partition function $PE_{\mathbf{r},\mathbf{c}}(A)$. It was known already to W.T.Tutte [20] that this partition function can be in poly-time reduced to the permanent. Therefore, if A is nonnegative the famous **FPRAS** [14] can be applied, as was already mentioned in [14] as one of the main applications. We are after deterministic poly-time algorithms. A. Barvinok initiated this, deterministic, line of algorithmic research in [2]. He also used the reduction to the permanent and the van der Waerden-Falikman-Egorychev (**VFE**) [5], [3] celebrated lower bound on the permanent of doubly-stochastic matrices:

$$\mathrm{per}(A) \geq \mathrm{vdw}(n) =: \frac{n!}{n^n}.$$

The techniques in [2] result in a deterministic poly-time algorithm approximating $PE(r,A)$ within multiplicative factor $(\Omega(\sqrt{n}))^n$ for any fixed r, even for $r = 1$. Such poor approximation is due the fact that the reduction to the permanent produces highly structured $n^2 \times n^2$ matrices. **VFE** bound is clearly a powerful algorithmic tool, as was recently effectively illustrated in [21]. Yet, neither **VFE** nor even more refined Schrijver's lower bound [18] are sharp enough for those structured matrices. This phenomenon was observed by A. Schrijver 30 years ago in [17]. The author introduced in [13] and [8] a new approach to lower bounds. We will give a brief description of the approach and refine it. The new lower bounds are asymptotically sharp and allow, for instance, to get a deterministic poly-time algorithm to approximate $PE(r, A)$ within multiplicative factor $f(r)^{-1}$ where

$$f(r) = \left(\frac{\mathrm{vdw}(n)}{\mathrm{vdw}(r)\mathrm{vdw}(n-r)}\right)^{n\frac{r-1}{r}} \frac{\mathrm{vdw}(n)}{\mathrm{vdw}(r)^{\frac{n}{r}}} \approx \left(\sqrt{2\pi \min(r, n-r)}\right)^{-n}.$$

We use the symbol $\approx$ to ignore all sub-exponential terms.

Besides, we show the algorithm from [2] actually approximates within(roughly) multiplicative factor $f(r)^{-2}$. So, for fixed r or $n-r$ the new bounds give simply exponential factor. But, say for $r = \frac{n}{2}$, the current factor is not simply exponential. *Is there a deterministic Non-Approximability result for $PE(\frac{n}{2}, A)$?*

We also study the sparse case, i.e. when, say, the columns of matrix A have relatively few non-zero entries. In this direction we generalize, reprove, sharpen the results of A. Schrijver [17] on how many k-regular subgraphs $2k$-regular bipartite graph can have.

The main message of this paper is that when one needs to deal with the permanent of highly structured matrices the only (and often painless) way to get sharp lower bounds is to use **stable polynomials approach**. Prior to [13] and [8] **VFE** this was, essentially, the only general purpose non-trivial lower bound on the permanent. It is not the case anymore.

1.1 Generating polynomials

The goal of this subsection is to represent $PE_{\mathbf{r},\mathbf{c}}(A)$ as a coefficient of some effectively computable polynomial.

1. The following natural representation in the case of unit weights, i.e $A(i,j) \equiv 1$, was already in [16], the general case of it was used in [2].

$$PE_{\mathbf{r},\mathbf{c}}(A) = \left[\prod_{1\le i\le n} y_i^{r_i} \prod_{1\le j\le m} x_j^{c_j}\right] \prod_{1\le i\le n, 1\le j\le m} (1 + A(i,j)x_j y_i), \qquad (2)$$

   i.e. $PE_{\mathbf{r},\mathbf{c}}(A)$ is the coefficient of the monomial $\prod_{1\le i\le n} y_i^{r_i} \prod_{1\le j\le m} x_j^{c_j}$ in the non-homogeneous polynomial $\prod_{1\le i\le n,1\le j\le m}(1 + A(i,j)x_j y_i)$.

It is easy to convert the non-homogeneous formula (2) into a homogeneous one:

$$PE_{\mathbf{r},\mathbf{c}}(A) = \left[\prod_{1\le j\le m} x_j^{c_j} \prod_{1\le i\le n} z_i^{m-r_i}\right] \prod_{1\le i\le n, 1\le j\le m} (z_i + A(i,j)x_j). \quad (3)$$

As the polynomial $\prod_{1\le i\le n,1\le j\le m}(z_i + A(i,j)x_j)$ is a product of linear forms, the formula (3) allows us to express $PE_{\mathbf{r},\mathbf{c}}(A)$ as the permanent of some $nm \times nm$ matrix, the fact essentially proved in a very different way in [20]. The permanent also showed up, in a similar context of Eulerian Orientations, in [17].

Indeed, associate with any $k \times l$ matrix B the product polynomial

$$Prod_B(x_1, ..., x_l) =: \prod_{1\le i\le k} \sum_{1\le j\le l} B(i,j)x_j. \quad (4)$$

Then

$$\left[\prod_{1\le j\le l} x_j^{\omega_j}\right] Prod_B(x_1, ..., x_l) = \mathrm{per}(B_{\omega_1,...,\omega_l}) \prod_{1\le j\le l} (\omega_j!)^{-1}, \quad (5)$$

where the $k \times k$ matrix $B_{\omega_1,...,\omega_l}$ consists of ω_j copies of the jth column of B, $1 \le j \le l$. We remind the reader that well-known formula (4) easily follows from the following obvious identity:

$$Prod_B(x_1, ..., x_l) = (k!)^{-1}\mathrm{per}([x_1B^{(1)}+\cdots+x_lB^{(l)}, \ldots, x_1B^{(1)}+\cdots+x_lB^{(l)}]),$$

where $B^{(j)}$ is the jth column of B.

2. We will use below the following equally natural representation. Recall the definition of standard symmetric functions:

$$S_k(x_1, ..., x_m) = \sum_{1\le i_1<..<i_k\le m} \prod_{1\le j\le k} x_{i_j},$$

and define the following homogeneous polynomial

$$ES_{\mathbf{r};A}(x_1, ..., x_m) = \prod_{1\le i\le n} S_{r_i}\left(A(i,1)x_1, ..., A(i,m)x_m\right). \quad (6)$$

Then

$$PE_{\mathbf{r},\mathbf{c}}(A) = \left[\prod_{1\le j\le n} x_j^{c_j}\right] ES_{\mathbf{r};A}(x_1, ..., x_m). \quad (7)$$

Remark 1.1: Note that in the square case $n = m$, the polynomial $ES_{e_n;A} = Prod_A$. The polynomial $ES_{\mathbf{r};A}$ is, of course, related to the polynomial

$$TM(z_1, ..., z_n; x_1, ..., x_m) =: \prod_{1 \leq i \leq n, 1 \leq j \leq m} (z_i + A(i, j)x_j) :$$

$$ES_{\mathbf{r};A}(x_1, ..., x_m) = \prod_{1 \leq i \leq n} \frac{1}{(m - r_i)!} \prod_{1 \leq i \leq n} \frac{\partial^{m-r_i}}{\partial z_i^{m-r_i}} TM(z_i = 0, 1 \leq i \leq n; x_1, ..., x_m). \tag{8}$$

∎

1.2 Exact algorithms

It is well known that the coefficient $\left[\prod_{1 \leq j \leq n} x_j^{c_j}\right] ES_{\mathbf{r},\mathbf{c};A}(x_1, ..., x_m)$ can be computed by evaluating the polynomial $ES_{\mathbf{r};A}$ at $\prod_{1 \leq j \leq n}(1 + c_j)$ points. Which gives(see Remark (1.1)) an exact algorithm for $PE_{\mathbf{r},\mathbf{c}}(A)$ of complexity

$$O\left(\min\left(\prod_{1 \leq j \leq n}(1 + c_j)nm\log(m), \prod_{1 \leq i \leq m}(1 + r_i)nm\log(n)\right)\right).$$

Thus if $n > m$ and m is fixed then there exists a polynomial in n exact deterministic algorithm to compute $PE_{\mathbf{r},\mathbf{c}}(A)$.

1.3 Previous Work

Estimation of the cardinality $|BM_{\mathbf{r},\mathbf{c}}| = PE_{\mathbf{r},\mathbf{c}}(A)$, where $A = J_{n,m} = e_n e_m^T$ is a matrix of all ones, is one of the classical topics in analytic combinatorics. The reader may consult Barvinok's paper [2] for references to most major results on the topic. We note that, in the dense case, the most general asymptotically exact formula for the number of boolean matrices with prescribed row and column sums is presented in [1].

To avoid messy formulas, we will mainly focus below on the uniform square case, i.e. $n = m$ and $r_i = c_j = r, 1 \leq i, j \leq n$ and use simplified notations:

$$BM_{\mathbf{re_n},\mathbf{re_n}} =: BM(r, n); PE_{\mathbf{re_n},\mathbf{re_n}}(A) =: PE(r, A).$$

It is easy to see that $PE(r, A)$ is #P-Complete for all $1 \leq r < n$. The connection to the permanent implies that for non-negative matrices A there is **FPRAS** for $PE(r, A)$. In this paper we are interested in deterministic algorithms. We briefly recall the Barvinok's algorithm from [2]:

Define

$$\alpha(A) = \inf_{z_j, x_i > 0} \frac{\prod_{1 \leq i \leq n, 1 \leq j \leq n} (z_j + A(i,j)x_i)}{\prod_{1 \leq i \leq n} x_i^r \prod_{1 \leq j \leq n} z_j^{n-r}}.$$

Then

$$\alpha \geq PE(r, A) \geq \frac{\mathrm{vdw}(n^2)}{(\mathrm{vdw}(n-r)\mathrm{vdw}(r))^n} \alpha(A), \tag{9}$$

where $\mathrm{vdw}(k) =: \frac{k!}{k^k}$. As the number $\log(\alpha(A))$ can computed(approximated within small additive error) via the convex minimization, the bounds (9) give a poly-time deterministic algorithm to approximate $PE(r, A)$ within multiplicative factor $\gamma_n =: \left(\frac{\mathrm{vdw}(n^2)}{(\mathrm{vdw}(n-r)\mathrm{vdw}(r))^n}\right)^{-1}$. The factor γ_n is not simply exponential even for $r = 1$; indeed $(\gamma_n)^{\frac{1}{n}} \approx \mathit{const}\,(\sqrt{n})$ for a fixed r. The proof of (9) in [2] is based on the **Sinkhorm's Scaling** and the van der Waerden-Falikman-Egorychev lower bound on the permanent of doubly-stochastic matrices.

2 Our Results

We prove and apply in this paper an optimized version of our lower bounds on the coefficients of **H-Stable** polynomials [12]. The lower bounds in [12] were obtained by a "naive" application of the lower on the mixed derivative of **H-Stable** polynomials [8].

When applied to the polynomial $\prod_{1 \leq i \leq n, 1 \leq j \leq n} (z_j + A(i,j)x_i)$, the main result of the current paper implies the following bounds:

$$\alpha(A) \geq PE(r, A) \geq \left(\frac{\mathrm{vdw}(n)}{\mathrm{vdw}(n-r)\mathrm{vdw}(r)}\right)^{2n-1} \alpha(A) \tag{10}$$

i.e. for the fixed r the Barvinok's approach gives a deterministic algorithm to approximate $PE(r, A)$ within simply exponential factor $(e^r \mathrm{vdw}(r))^{2n}$. We stress again that this result seems to be unprovable by using only van der Waerden-Falikman-Egorychev and similar purely permanental bounds, even the newest ones in [11].

When applied to the the polynomial $ES_{\mathbf{r};A}(x_1,...,x_m)$, our new bounds imply the following general inequality

$$\mu(A) \geq PE_{\mathbf{r},\mathbf{c}}(A) \geq \prod_{2\leq j\leq n}\left(\frac{\mathrm{vdw}(n)}{\mathrm{vdw}(n-c_j)\mathrm{vdw}(c_j)}\right)\mu(A), \tag{11}$$

where

$$\mu(A) =: \inf_{x_j>0}\frac{ES_{\mathbf{r};A}(x_1,...,x_m)}{\prod_{1\leq j\leq m} x_j^{c_j}}$$

Note that

$$\log(\mu(A)) = \inf_{\sum_{1\leq j\leq m} y_j=0}\log\left(ES_{\mathbf{r};A}\left(exp\left(\frac{y_1}{c_1}\right),...,exp\left(\frac{y_m}{c_m}\right)\right)\right),$$

and the function $\log\left(ES_{\mathbf{r},\mathbf{c};A}\left(exp\left(\frac{y_1}{c_1}\right),...,exp\left(\frac{y_m}{c_m}\right)\right)\right)$ is convex in y's. For the fixed $\mathbf{r} = re_n, \mathbf{c} = re_n$ this gives a deterministic poly-time algorithm to approximate $PE(r,A)$ within the simply exponential factor $(e^r\mathrm{vdw}(r))^n$. This paper does not give the detailed complexity analysis of this convex minimization, say based on the **Ellipsoid Algorithm**. We rather present a very simple converging algorithm, which does pretty well in practice.
In the sparse case we get a much better lower bound:

$$\mu(A) \geq PE_{\mathbf{r},\mathbf{c}}(A) \geq \prod_{2\leq j\leq m}\left(\frac{\mathrm{vdw}(Cl_j)}{\mathrm{vdw}(Cl_j-c_j)\mathrm{vdw}(c_j)}\right)\mu(A), \tag{12}$$

where $Cl_j = \min(\sum_{1\leq k\leq j} c_k, Col(j))$ and $Col(j)$ is the number of nonzero entries in the jth column of A.
In the uniform case, i.e. $m = n$, $\mathbf{r} = \mathbf{c} = re_n$ this gives the following bound

$$\mu(A) \geq PE(r,A) \geq \prod_{2\leq j\leq m}\left(\frac{\mathrm{vdw}(C_j)}{\mathrm{vdw}(C_j-r)\mathrm{vdw}(r)}\right)\mu(A), \tag{13}$$

where $C_j = \min(r, Col(j))$.

Our final result is the following combinatorial lower bound: Let $A \in BM_{k\mathbf{r},k\mathbf{c}} \neq \emptyset$. Then

$$\inf_{x_j>0}\frac{ES_{\mathbf{r},\mathbf{c};A}(x_1,...,x_m)}{\prod_{1\leq j\leq n} x_j^{c_j}} = \prod_{1\leq i\leq n}\binom{kr_i}{r_i} \tag{14}$$

and

$$PE_{\mathbf{r},\mathbf{c}}(A) \geq \prod_{1 \leq i \leq n} \binom{kr_i}{r_i} \prod_{2 \leq j \leq m} \frac{\mathrm{vdw}(kc_j)}{\mathrm{vdw}(kc_j - c_j)\mathrm{vdw}(c_j)} \tag{15}$$

The formula (15) can be slightly, i.e. by $const(k,t) > 1$, improved in the regular case. In particular, $const(2,t) = \left(\binom{2t}{t}\right)^{-1} 2^{2t}$.

Let $A \in BM_{kte_n,kte_n}$, where k,t are positive integers. Then

$$PE_{te_n,te_n}(A) \geq \binom{kt}{t}^n \left(\frac{\mathrm{vdw}(kt)}{\mathrm{vdw}((k-1)t)\mathrm{vdw}(t)}\right)^{n-k} \frac{\mathrm{vdw}(kt)}{\mathrm{vdw}(t)^k}. \tag{16}$$

The inequalities (15, 16) generalize and improve results from [17].

All the inequalities in this section are fairly direct corollaries of Theorem(5.1) (see the main inequality (35)).

3 Stable Homogeneous Polynomials

3.1 Definitions, previous results and the naive approach

The next definition introduces key notations and notions.

Definition 3.1:

1. The linear space of homogeneous polynomials with real (complex) coefficients of degree n and in m variables is denoted $Hom_R(m,n)$ ($Hom_C(m,n)$).

   We denote as $Hom_+(m,n)$ the closed convex cone of polynomials $p \in Hom_R(m,n)$ with nonnegative coefficients.

2. For a polynomial $p \in Hom_+(n,n)$ we define its **Capacity** as

$$Cap(p) = \inf_{x_i > 0, \prod_{1 \leq i \leq n} x_i = 1} p(x_1, \dots, x_n) = \inf_{x_i > 0} \frac{p(x_1, \dots, x_n)}{\prod_{1 \leq i \leq n} x_i}. \tag{17}$$

3. Consider a polynomial $p \in Hom_C(m,n)$,

$$p(x_1, \dots, x_m) = \sum_{(r_1, \dots, r_m)} a_{r_1, \dots, r_m} \prod_{1 \leq i \leq m} x_i^{r_i}.$$

   We define $Rank_p(S)$ as the maximal joint degree attained on the subset

$$S \subset \{1, \dots, m\} :$$

$$Rank_p(S) = \max_{a_{r_1,\dots,r_m} \neq 0} \sum_{j \in S} r_j. \tag{18}$$

If $S = \{i\}$ is a singleton, we define $deg_p(i) = Rank_p(S)$.

4. A polynomial $p \in Hom_C(m, n)$ is called **H-Stable** if $p(Z) \neq 0$ provided $Re(Z) > 0$; it is called **H-SStable** if $p(Z) \neq 0$ provided $Re(Z) \geq 0$ and $\sum_{1 \leq i \leq m} Re(z_i) > 0$.

   (**We coined the term "H-Stable" to stress two things: Homogeneity and Hurwitz' stability.**)

5. We define

$$G(i) = \frac{\mathrm{vdw}(i)}{\mathrm{vdw}(i-1)} = \left(\frac{i-1}{i}\right)^{i-1}, i > 1; G(1) = 1. \tag{19}$$

   Note that vdw(i) and $G(i)$ are strictly decreasing sequences.

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The main inequality in [8] was stated as the following theorem:

Theorem 3.2: *Let $p \in Hom_+(n, n)$ be* **H-Stable** *polynomial. Then the following inequality holds*

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0, \dots, 0) \geq \prod_{2 \leq i \leq n} G\big(\min(i, deg_p(i))\big) Cap(p). \tag{20}$$

So, if $p \in Hom_+(n, n)$ is **H-Stable** and $deg_p(i) \leq k \leq n$ for $k + 1 \leq i \leq n$ then the following inequality holds:

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0, \dots, 0) \geq G(k)^{n-k} \frac{k!}{k^k} Cap(p). \tag{21}$$

For $k = n$ we get the generalized van der Waerden-Falikman-Egorychev inequality:

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0, \dots, 0) \geq \frac{n!}{n^n} Cap(p). \tag{22}$$

3.2 A naive generalization to the general monomials

Let $p \in Hom_+(m,n)$ and consider an integer vector

$$\mathbf{c} = (c_1, ..., c_m) \in R_+^m, \sum_{1 \le i \le m} c_i = n.$$

Define the following polynomial $q \in Hom_+(n,n)$:

$$q(y_{1,1}, ..., y_{1,c_1}; y_{2,1}, ..., y_{2,c_2}; ...; y_{m,1}, ..., y_{m,c_m}) = p(z_1, ..., z_m),$$

where

$$z_i = \frac{\sum_{1 \le j \le c_i} y_{i,j}}{c_i},$$

$1 \le i \le m$. *Note that if the polynomial p is* **H-Stable** *then q is as well.*

Also, analogously to (17), let us define:

$$Cap_{\mathbf{c}}(p) =: Cap_{c_1,...,c_m}(p) =: \inf_{x_j > 0} \frac{p(x_1, ..., x_m)}{\prod_{1 \le j \le m} x_j^{c_j}}. \tag{23}$$

Fact 3.3: It holds that:

$$Cap_{c_1,...,c_m}(p) = Cap(q),$$

and

$$[x_1^{c_1} ... x_m^{c_m}]\, p = \prod_{1 \le i \le m} \frac{c_i^{c_i}}{c_i!} \frac{\partial^n}{\prod_{1 \le i \le m} \prod_{1 \le j \le c_i} \partial y_{i,j}} q(0)$$

Corollary 3.4: *Let $p \in Hom_+(m,n)$ be* **H-Stable** *and $deg_p(i) \le k, 1 \le i \le m$. Then the following inequality holds:*

$$Cap_{c_1,...,c_m}(p) \ge [x_1^{c_1} ... x_m^{c_m}]\, p \ge Cap_{c_1,...,c_m}(p) \left(\prod_{1 \le i \le m} (\mathrm{vdw}(c_i))^{-1} \right) G(k)^{n-k} \mathrm{vdw}(k). \tag{24}$$

If $k = n$ then

$$Cap_{c_1,...,c_m}(p) \ge [x_1^{c_1} ... x_m^{c_m}]\, p \ge Cap_{c_1,...,c_m}(p) \left(\prod_{1 \le i \le m} (\mathrm{vdw}(c_i))^{-1} \right) \mathrm{vdw}(n). \tag{25}$$

Example 3.5:

1. Let $A = J_n$ be the $n \times n$ matrix of all ones and consider the **H-Stable** polynomial $ES_{re_n;J_n}(x_1, ..., x_n) = \prod_{1 \le i \le n} S_r(x_1, ..., x_n)$. Then

$$Cap_{re_n}(ES_{re_n;J_n}) = \binom{n}{r}^n ; deg_{ES_{re_n;J_n}}(i) = n, 1 \le i \le n.$$

Applying the inequality (24) we get that

$$|BM(r,n)| = [x_1^r....x_n^r]\, ES_{re_n;J_n} \ge \binom{n}{r}^n \left(\frac{r^r}{r!}\right)^n \left(\frac{n-1}{n}\right)^{(n-1)n(r-1)} \frac{n!}{n^n} \tag{26}$$

It was proved by Everett and Stein in [4](their proof is rather involved) that

$$|BM(r,n)| = \frac{(rn)!}{(r!)^{2n}} exp\left(-\frac{1}{2}(r-1)^2\right)\beta(r,n), \tag{27}$$

where $\lim_{n\to\infty} \beta(r,n) = 1$ for any fixed integer number r. Our lower bound from from (26)

$$|BM(r,n)| \ge HYP(r,n) =: \binom{n}{r}^n \left(\frac{r^r}{r!}\right)^n \left(\frac{n-1}{n}\right)^{(n-1)n(r-1)} \frac{n!}{n^n}$$

is valid for all values of r. Directly applying the Stirling formula, we get that for the fixed r

$$\lim_{n\to\infty} \frac{HYP(r,n)}{|BM(r,n)|} = \left(\sqrt{r}\right)^{-1}.$$

Not bad at all, considering how computationally and conceptually simple our derivation of (26) is!

The same reasoning applies to general non-negative $n \times n$ matrices:

$$\begin{aligned} &Cap_{re_n}(ES_{re_n;A}) \ge PE(r,A) \\ &PE(r,A) \ge Cap_{re_n}(ES_{re_n;A}) \left(\frac{r^r}{r!}\right)^n \left(\frac{n-1}{n}\right)^{(n-1)n(r-1)} \frac{n!}{n^n}. \end{aligned} \tag{28}$$

As $\log(Cap_{re_n}(ES_{re_n;A}))$ can be expressed in terms of convex minimization, for a fixed r the inequality (28) justifies a deterministic poly-time algorithm to approximate $PE(r, A)$ within a simply exponential factor $(e^r \mathrm{vdw}(r))^n$.

We stress that the current, say in [6], [7], very delicate, accurate and efficient estimates of $|BM_{\mathbf{r},\mathbf{c}}|$(valid for some ranges of $(\mathbf{r}, \mathbf{c})$) can not be, at least directly, applied to estimate $PE_{\mathbf{r},\mathbf{c}}(A)$ even for boolean matrices A because of the #P-Completeness of $PE_{\mathbf{r},\mathbf{c}}(A)$.

2. Applying the inequality (24) to the polynomial

$$TM(z_1, ..., z_n; x_1, ..., x_n) = \prod_{1 \le i \le n, 1 \le j \le n} (z_j + A(i,j)x_i)$$

we get that

$$\alpha(A) \ge PE(r, A) \ge \alpha(A)(\mathrm{vdw}(r)(\mathrm{vdw}(n-r))^{-n} \left(\frac{n-1}{n}\right)^{(n(n-1)} \frac{n!}{n^n}.$$

It is better than (9), yet does not give a simply exponential factor.

3. Let $A \in BM_{t\mathbf{r},t\mathbf{c}}$. We are interested in a lower bound on $PE_{\mathbf{r},\mathbf{c}}(A)$. Now,

$$ES_{\mathbf{r};A}(x_1, ..., x_m) = \prod_{1 \le i \le n} S_{r_i}(A(i,1)x_1, ..., A(i,m)x_m)$$

and

$$PE_{\mathbf{r},\mathbf{c}}(A) = \left[\prod_{1 \le j \le n} x_j^{c_j}\right] ES_{\mathbf{r};A}(x_1, ..., x_m).$$

Easy computation: If $A \in BM_{t\mathbf{r},t\mathbf{c}}$, then $Cap_{\mathbf{c}}(ES_{\mathbf{r};A}) = \prod_{1 \le i \le n} \binom{tr_i}{r_i}$.

Let us consider the simplest non-trivial case of $A \in BM(2r, n)$. Applying the inequality (24) with $k = 2r$, we get that

$$PE(r, A) \ge \binom{2r}{r}^n (\mathrm{vdw}(r))^{-n} \left(\frac{2r-1}{2r}\right)^{(2r-1)(nr-2r)} \frac{(2r)!}{2r^{2r}}.$$

Yet, the asymptotically correct bound from [17] is

$$PE(r, A) \ge \left(\binom{2r}{r}^2 (2)^{-2r}\right)^n. \qquad (29)$$

Not only we will improve in the current paper the bound (29), but we will also show that if all columns of an $n \times n$ nonnegative matrix A have at most tr non-zero entries, then $PE(r, A)$ can be deterministically in poly-time approximated within the factor $\left(\frac{\mathrm{vdw}(tr)}{\mathrm{vdw}(r)\mathrm{vdw}(r(t-1))}\right)^{-n}$.

∎

4 New Observations, which were overlooked in [8]

Associate with a polynomial $p \in Hom_+(n, n)$ the following sequence of polynomials $q_i \in Hom_+(i, i)$:

$$q_n = p, q_i(x_1, \ldots, x_i) = \frac{\partial^{n-i}}{\partial x_{i+1} \ldots \partial x_n} p(x_1, \ldots, x_i, 0, \ldots, 0); 1 \leq i \leq n-1.$$

The inequality (20) is, actually, a corollary of the following inequality, which holds for **H-Stable** polynomials:

$$Cap(q_i) \geq Cap(q_{i-1}) \geq G(deg_{q_i}(i))Cap(q_i), n \geq i \geq 2. \tag{30}$$

As $Cap(q_1) = \frac{\partial^n}{\partial x_1 \ldots \partial x_n} p(0, \ldots, 0)$, one gets that

$$\frac{\partial^n}{\partial x_1 \ldots \partial x_n} p(0, \ldots, 0) \geq \prod_{2 \leq i \leq n} G\big(deg_{q_i}(i)\big)Cap(p). \tag{31}$$

The inequality (20) follows from (31) because $G(i)$ is decreasing and $deg_{q_i}(i) \leq \min(i, deg_p(i)$. There were several reasons why the inequality (20) was stated as the main result:

1. It is simpler to understand than more the general one (31). It was sufficient for the killer application: a short, transparent proof of the (improved) Schrijver's lower bound on the number of perfect matchings in k-regular bipartite graphs.

2. For most of the natural polynomials, the degrees $deg_{q_i}(i)$ are straightforward to compute. Moreover, if a polynomial p with integer coefficients is given as an evaluation oracle then $Rank_p(S)$ can be computed in

polynomial time via the univariate interpolation. On the other hand, if $i = n - [n^a], a \in (0,1)$, then even deciding whether $deg_{q_i}(i)$ is zero or not is **NP-HARD**. Indeed, consider, for instance, the following family of polynomials, essentially due to A. Barvinok:

$$p(x_1, ..., x_n) = Bar_A\left(x_n, ..., x_{n-[n^a]+1}\right)\left(x_1 + ... + x_{n-[n^a]}\right)^{n-[n^a]},$$

where $Bar_A\left(x_n, ..., x_{n-[n^a]}\right) = tr\left(\left(Diag(x_n, ..., x_{n-[n^a]})A\right)^{[n^a]}\right)$ and A is the adjacency matrix of an undirected graph. If the graph has a Hamiltonian cycle, then $deg_{q_i}(i) = i$ and is zero otherwise.

4.1 New Structural Results

The following simple bound was overlooked in [8]:

$$deg_{q_i}(i) \leq \min\left(Rank_p\left(\{i, ..., n\}\right) - n + i, deg_p(i)\right).$$

So, if

$$Rank_p(\{j, ..., n\}) - n + j \leq k : \text{ for } k+1 \leq j \leq n \tag{32}$$

then

$$\frac{\partial^n}{\partial x_1 \ldots \partial x_n} p(0, \ldots, 0) \geq Cap(p)G(k)^{n-k}\mathrm{vdw}(k). \tag{33}$$

Example 4.1: Let A be $n \times n$ doubly-stochastic matrix with the following pentagon shaped support: $A(i,j) = 0 : j - i \geq k$. Then the product polynomial $Prod_A(x_1, ..., x_n) = \prod_{1 \leq i \leq n} \sum_{1 \leq j \leq n} A(i,j)x_j$ satisfies the inequalities (32) and $Cap(Prod_A) = 1$. Therefore $\mathrm{per}(A) \geq G(k)^{n-k}\mathrm{vdw}(k)$. This lower bound for the permanent was proved by very different methods in [19]. Moreover, it was shown there that it is sharp. Therefore, the more general bound (33) is sharp as well. ∎

We recall the following result(combination of results in [10] and [13]).

Theorem 4.2: *Let* $p \in Hom_+(m,n), p(x_1, ..., x_m) = \sum_{r_1+...+r_m=n} a_{r_1,...,r_m} x_1^{r_1}...x_m^{r_m}$ *be* **H-Stable**. *Then*

1. *The set function* $Rank_p(S)$ *is submodular.*

2.

$$a_{r_1,\ldots,r_m} > 0 \Longleftrightarrow \sum_{j\in S} r_j \le Rank_p(S): \text{ for all } S \subset \{1,\ldots,m\}. \quad (34)$$

In other words, $a_{r_1,\ldots,r_m} > 0$ *iff* $0 \le \min_{S\subset\{1,\ldots,m\}} Rank_p(S) - \sum_{j\in S} r_j$.

3. *Note that the set function* $Rank_p(S) - \sum_{j\in S} r_j$ *is submodular. Therefore, given the evaluation oracle for p, there is deterministic strongly polynomial algorithm to decide whether* $a_{r_1,\ldots,r_m} > 0$.

Lemma 4.3: *Let* $p \in Hom_+(n,n)$ *be* **H-Stable** *polynomial with integer coefficients given as an evaluation oracle.Then for any* $i \ge 1$ *there is a deterministic strongly polynomial algorithm to compute* $deg_{q_i}(i)$*, i.e. with the number of logical and arithmetic operations bounded by* $poly(n)$.

Proof: The statement is essentially a corollary of part 3 of Theorem (4.2). If $i = 1$ then $deg_{q_1}(1) = 1$ if the monomial $x_1...x_n$ is in the support of p and equal zero otherwise. If $i = n$ then $deg_{q_n}(n) = deg_p(n)$. We can compute $deg_p(n)$ by interpolating univariate polynomial $p(1,...,1,t)$.
So, we can assume that $2 \le i < n$. Associate with any such number i and any polynomial $p \in Hom_+(n,n)$ the following polynomials $P_l(y_1,...,y_n) = p(z_1,....,z_n)$ where $1 \le l < i$ and $z_j = y_{l+1} + ... + y_i, 1 \le j \le i-1; z_i = y_1 + ... + y_l; z_{i+k} = y_{i+k}, 1 \le k \le n-i$.
If $l = i$ then $z_j = 0, 1 \le j \le i-1; z_i = y_1 + ... + y_i; z_{i+k} = y_{i+k}, 1 \le k \le n-i$.
It is easy to see that $deg_{q_i}(i) \ge l$ iff $\frac{\partial^n}{\partial y_1 ... \partial y_n} P_l(0,\ldots,0) > 0$. Now, if the original polynomial p is **H-Stable** then the polynomials P_l are **H-Stable** as well. In addition, they have integer coefficients and there are evaluation oracles for them. Therefore we can apply the submodular minimization algorithm from Theorem (4.2) to decide whether the monomial $y_1...y_n$ is in the support of P_l.
We start with $l = i$ and run this algorithm. The largest $l \ge 1$ such that the monomial $y_1...y_n$ is in the support of P_l is $deg_{q_i}(i)$. If none exists then $deg_{q_i}(i) = 0$. ∎

Example 4.4: [Gale-Ryser Inequalities]. Consider the following **H-Stable** polynomial

$$GR_{\mathbf{r},\mathbf{c}}(x_1,...,x_m) = \prod_{1\le i\le n} S_{r_i}(x_1,...,x_m); \sum_j c_j = \sum_i r_i.$$

Clearly, the monomial $\prod_{1 \le j \le n} x_j^{c_j}$ is in the support iff the set $BM_{\mathbf{r},\mathbf{c}}$ is not empty, i.e. there exists a boolean matrix with column sums $\mathbf{c}$ and row sums $\mathbf{r}$. It is easy to see that $Rank_{GR_{\mathbf{r},\mathbf{c}}}(S) = \sum_{1 \le i \le n} \min(|S|, r_i)$. It follows from the characterization (34) that $BM_{\mathbf{r},\mathbf{c}}$ is not empty iff $\sum_{j \in S} c_j \le \sum_{1 \le i \le n} \min(|S|, r_i)$ for all subsets $S \subset \{1, ..., m\}$. Equivalently, for the ordered column sums $c_{j_1} \ge c_{j_2} \ge ... \ge c_{j_m}$ the following inequalities hold:

$$\sum_{1 \le k \le t} c_{j_k} \le \sum_{1 \le i \le n} \min(t, r_i); 1 \le t \le m.$$

These are the famous Gale-Ryser inequalities, albeit stated without Ferrers matrices. ∎

5 Main New Lower Bound

Let $p \in Hom_+(m, n)$ be a homogeneous polynomial in m variables, of degree n and with non-negative coefficients. We fix a monomial $\prod_{1 \le j \le m} x_j^{c_j}, \sum_{1 \le j \le m} c_j = n$ and assume WLOG that $c_j > 0$ for all $1 \le j \le m$. Let $0 \le a_{c_1,...,c_m} = \left[\prod_{1 \le j \le m} x_j^{c_j}\right] p$ be a coefficient of the monomial. Define

$$Cap_{c_1,...,c_m}(p) =: \inf_{x_j > 0} \frac{p(x_1, ..., x_m)}{\prod_{1 \le j \le m} x_j^{c_j}}.$$

Clearly, $a_{c_1,...,c_m} \le Cap_{c_1,...,c_m}(p)$.

Theorem 5.1: *Let $p \in Hom_+(d, m)$ be* **H-Stable**. *Define the following family of polynomials:*

$$Q_m = p, Q_i \in Hom(i, n - (c_m + ... + c_{i+1})), m - 1 \ge i \ge 1 :$$

$$Q_i = \frac{\partial^{c_m + ... + c_{i+1}}}{\partial x_{i+1}^{c_{i+1}} \dots \partial x_m^{c_m}} p(x_1, \dots, x_i, 0, \dots, 0); 1 \le i \le m - 1.$$

Denote $\mathrm{dg}(i) =: deg_{Q_i}(i)$. *Then the following inequality holds*

$$a_{c_1,...,c_m} \ge Cap_{c_1,...,c_m}(p) \prod_{2 \le j \le m} \frac{\mathrm{vdw}(\mathrm{dg}(j))}{\mathrm{vdw}(c_j)\mathrm{vdw}(\mathrm{dg}(j) - c_j)} \tag{35}$$

Remark 5.2: It is easy to see that for a fixed value of c the function $\frac{\mathrm{vdw}(K)}{\mathrm{vdw}(c)\mathrm{vdw}(K-c)} = (\mathrm{vdw}(c))^{-1}G(K)G(K-1)...G(K-c+1)$ is strictly decreasing in $K \geq c$. Also, the $\mathrm{dg}(j) \leq \min(j, n-(c_m+...+c_{j+1}))$. ▮

Corollary 5.3: *Let $p \in Hom_+(m,n)$ be* **H-Stable**. *Then the following lower bound holds:*

$$a_{c_1,...,c_m} \geq Cap_{c_1,...,c_m}(p) \prod_{1 \leq j \leq m} \frac{\mathrm{vdw}(deg_p(j))}{\mathrm{vdw}(c_j)\mathrm{vdw}(deg_p(j)-c_j)} \tag{36}$$

Our proof is, similarly to [8], by induction, which is based on the following bivariate lemma:

Lemma 5.4: *Let $p \in Hom_+(2,d)$ be* **H-Stable**, *i.e. $p(x_1,x_2) = \sum_{0 \leq i \leq d} a_i x_1^{d-i} x_2^i$ and $1 \leq c_2 < d$. Then*

$$a_{c_2} \geq Cap_{d-c_2,c_2}(p) \frac{\mathrm{vdw}(d)}{\mathrm{vdw}(c_2)\mathrm{vdw}(d-c_2)}.$$

Proof: Define the following polynomial $P \in Hom_+(d,d)$:

$$P(y_1,...,y_{d-c_2}; z_1,...,z_{c_2}) = p\left(\frac{1}{d-c_2}\sum_{1 \leq k \leq d-c_2} y_k, \frac{1}{c_2}\sum_{1 \leq i \leq c_2} z_i\right).$$

It follows from the standard AG inequality that $Cap_{d-c_2,c_2}(p) = Cap(P)$ and it is easy to see that P is **H-Stable**. Consider the following polynomial

$$R(z_1,...,z_{c_2}) =: \prod_{1 \leq k \leq d-c_2} \frac{\partial}{\partial y_k} P(y_k = 0, 1 \leq k \leq d-c_2; z_1,...,z_{c_2}).$$

First, it follows from (30) that $Cap(R) \geq G(d)...G(c_2+1)Cap(P)$. By the direct inspection,

$$R(z_1,...,z_{c_2}) = a_{c_2}\mathrm{vdw}(d-c_2)\left(\frac{1}{c_2}\sum_{1 \leq i \leq c_2} z_i\right)^{c_2}.$$

Therefore $Cap(R) = a_{c_2}\mathrm{vdw}(d-c_2)$. Putting things together gives that

$$a_{c_2} \geq \frac{G(d)...G(c_2+1)}{\mathrm{vdw}(d-c_2)} Cap_{d-c_2,c_2}(p) = \frac{\mathrm{vdw}(d)}{\mathrm{vdw}(c_2)\mathrm{vdw}(d-c_2)} Cap_{d-c_2,c_2}(p).$$

▮

Proof of Theorem (5.1): Let $p \in Hom_+(m,n)$ be **H-Stable**. Define $d = deg_p(m)$ and expand it in the last variable:

$$p(x_1, ..., x_m) = \sum_{0 \le i \le d} x_m^i T_i(x_1, ..., x_{m-1}).$$

Our goal is to prove that

$$Cap_{c_1,...,c_{m-1}}(T_{c_m}) \ge \frac{\mathrm{vdw}(d)}{\mathrm{vdw}(c_m)\mathrm{vdw}(d-c_m)} Cap_{c_1,...,c_{m-1},c_m}(p). \tag{37}$$

Fix positive numbers $(y_1, ..., y_{m-1})$ and consider the following bivariate polynomial: $W(t, x_m) = p(ty_1, ..., ty_{m-1}, x_m)$. The polynomial W is of degree n and **H-Stable**. Note that

$$W(t, x_m) \ge Cap_{c_1,...,c_m}(p) t^{d-c_m} x_m^{c_m} \prod_{1 \le i \le m-1} y_i^{c_j}.$$

It follows from Lemma(5.4) that

$$T_{c_m}(y_1, ..., y_{m-1}) \ge \frac{\mathrm{vdw}(d)}{\mathrm{vdw}(c_m)\mathrm{vdw}(d-c_m)} Cap_{c_1,...,c_m}(p) \prod_{1 \le i \le m-1} y_i^{c_j},$$

which proves the inequality (37). Now the polynomial $T_{c_m} \in Hom_+(m-1, n-c_m)$ is also **H-Stable** [8]. Thus we can apply the same argument to the polynomial $T_{c_m}(x_1, ..., x_{m-1})$ and so on until only the first variable x_1 remains. ▮

6 Algorithms to compute parameters of Theorem (5.1)

To compute the lower bound in (35) we need the degrees $deg_{Q_i}(i)$ and the capacity $Cap_{c_1,...,c_m}(p)$. We assume that the polynomial p has rational coefficients and is given by an evaluation oracle, say, on the integer vectors. Essentially this model allows us to do only low-dimensional interpolations. Lemma (4.3) allows us to compute $deg_{Q_i}(i)$ in this model in $poly(n)$ operations, i.e. with no dependence on the bit-wise complexity of the coefficients.

6.1 A practical algorithm to approximate $Cap_{c_1,...,c_m}(p)$

Associate with a polynomial $p \in Hom_+(m,n)$ and a non-negative vector $(c_1,...,c_m), \sum_{1 \le i \le m} c_i = n$ the following maps:

$$F(x_1,...,x_m) = (y_1,...,y_m); y_i = \frac{x_i}{a_i}, G(X) = F(X)\left(\prod_{1 \le i \le m} a_i^{c_i}\right)^{\frac{1}{n}}$$

where

$$a_i = \frac{x_i \frac{\partial}{\partial x_i} p(X)}{c_i p(X)}.$$

In other words

$$y_i = \frac{\alpha}{\frac{\partial}{\partial x_i} p(X)},$$

where α is a normalizing constant.

Note that it follows from the Euler's identity for homogeneous functions that

$$mul(X) =: \prod_{1 \le i \le m} a_i^{c_i} \le \left(\sum_{1 \le i \le m} \frac{x_i \frac{\partial}{\partial x_i} p(X)}{np(X)}\right)^n = 1, \tag{38}$$

and the map G preserves the product of powers $\prod_{1 \le i \le m} x_i^{c_i}$.

Lemma 6.1: *Suppose that the polynomial $p \in Hom_+(m,n)$ is log-concave on the positive orthant R^m_{++}. Then the following inequality holds:*

$$p(G(X)) \le mul(X)p(X).$$

Proof: The log-concavity gives the following inequality

$$\log(p(y_1,...,y_m)) \le \log(p(x_1,...,x_m)) + \sum_{1 \le i \le m} \frac{\frac{\partial}{\partial x_i} p(X)}{p(X)}(y_i - x_i).$$

So if $Y = F(X), X \in R^m_{++}$ then the Euler's identity gives the following inequality

$$\log(p(y_1,...,y_m)) \le \log(p(x_1,...,x_m)) + \sum_{1 \le i \le m} \left(c_i - \frac{x_i \frac{\partial}{\partial x_i} p(X)}{p(X)}\right) = \log(p(x_1,...,x_m)).$$

Finally, $p(G(X)) = mul(X)p(F(X)) \le mul(X)p(X)$. ∎

We suggest the following algorithm to approximate $Cap_{c_1,...,c_m}(p)$:

Start with $x_{i,0} = 1, 1 \leq i \leq m$ and recursively compute the following vector sequence:

$$X_{k+1} = G(X_k)), k \geq 1.$$

Stop if $mul(X_k) \geq 1 - \epsilon$, where $\epsilon << 1$ and output $Cap_{c_1,...,c_m}(p) \approx p(X_k)$.

This algorithm does not work for general polynomials in $Hom_+(m,n)$ (just consider $p(x_1,x_2) = 2(x_1)^2 + (x_2)^2$ and $(c_1,c_2) = (1,1)$). But Lemma (6.1) essentially proves that if $Cap_{c_1,...,c_m}(p) > 0$ and $p \in Hom_+(m,n)$ is log-concave then the algorithm converges. In fact, it is a generalization of the famous (and efficient in practice) Sinkhorn's scaling algorithm to the product of symmetric functions $ES_{\mathbf{r};A}(x_1,...,x_m)$. Sinkhorn's scaling algorithm corresponds to the product of linear forms.

As any **H-Stable** polynomial $p \in Hom_+(m,n)$ is log-concave, we can apply Lemma (6.1). In practice it usually takes just a few steps of the Sinkhorn's scaling to get a very good approximation. We expect the same from our generalization. Note that each step of the algorithm for $ES_{\mathbf{r};A}$ boils down to computing n symmetric functions of m variables (the evaluation of the polynomial) and nm symmetric functions of $m-1$ variables (the evaluation of the gradient).

7 Applications of Theorem (5.1)

Example 7.1:

1. The polynomial from [2]

$$TM(z_1,...,z_n;x_1,...,x_m) =: \prod_{1 \leq i \leq n, 1 \leq j \leq m} (z_i + A(i,j)x_j).$$

Consider, just for the illustration, the square uniform case: $n = m$, $\mathbf{c} = (n-r,...,n-r;r,...,r)$. Note that the degrees of all variables are bounded by n. Using non-optimized lower bound (36) we get that the coefficient

$$a_{n-r,...,n-r;r,...,r} \geq Cap_{n-r,...,n-r;r,...,r}(TZ) \left(\frac{\mathrm{vdw}(n)}{\mathrm{vdw}(r)\mathrm{vdw}(n-r)} \right)^{2n}$$

2. We give a better lower bound on $|BM(r,n)|$. The polynomial is

$$Sym_{r,n}(\mathbf{x}) =: (S_r(x_1,...,x_n))^n.$$

The degree of each variable is n and $Cap_{r,...,r}(Sym_{r,n}) = \binom{n}{r}^n$. The slightly optimized lower bound is, assuming that r divides n, as follows:

$$|BM(r,n)| \geq \binom{n}{r}^n \left(\frac{\mathrm{vdw}(n)}{\mathrm{vdw}(r)\mathrm{vdw}(n-r)}\right)^{\frac{n(r-1)}{r}} \mathrm{vdw}(n)(\mathrm{vdw}(r)^{-\frac{n}{r}}). \tag{39}$$

For the fixed r this new bound is asymptotically equal to the "old" bound (26), but, say, for $r = \Omega(n)$ the bound (39) is exponentially in n greater than the bound (26).

3. Let $A \in BM_{t\mathbf{r},\mathbf{tc}}$, where $t > 1$ is a rational number such that the vectors $t\mathbf{r}, \mathbf{tc}$ are integral. We are interested in a lower bound on $PE_{\mathbf{r},\mathbf{c}}(A)$. Now,

$$ES_{\mathbf{r};A}(x_1,...,x_m) = \prod_{1\leq i\leq n} S_{r_i}(A(i,1)x_1,...,A(i,m)x_m)$$

and

$$PE_{\mathbf{r},\mathbf{c}}(A) = \left[\prod_{1\leq j\leq n} x_j^{c_j}\right] ES_{\mathbf{r};A}(x_1,...,x_m).$$

Easy computation: If $A \in BM_{t\mathbf{r},\mathbf{tc}}$ then $Cap_{\mathbf{c}}(ES_{\mathbf{r};A}) = \prod_{1\leq i\leq n}\binom{tr_i}{r_i}$, which gives the following non-trivial corollary Theorem (5.1):

Corollary 7.2: *If* $A \in BM_{t\mathbf{r},\mathbf{tc}}$ *then*

$$PE_{\mathbf{r},\mathbf{c}}(A) \geq L =: \prod_{1\leq i\leq n}\binom{tr_i}{r_i}\prod_{1\leq j\leq m}\frac{\mathrm{vdw}(tc_j)}{\mathrm{vdw}(c_j)\mathrm{vdw}((t-1)c_j)}$$

And $L = \left(\frac{(t-1)^{t-1}}{t^t}\right)^{\sum_{1\leq i\leq n} r_i} \prod_{1\leq i\leq n}\binom{tr_i}{r_i}\prod_{1\leq j\leq m}\binom{tc_j}{c_j}$.

We can get more precise lower bounds in the uniform case, i.e. when $A \in BM(tr,n)$ and t is integer. It follows that

$$Cap_{\mathbf{c}}(ES_{\mathbf{r};A}) = \binom{tr}{r}^n$$

if $A \in BM(tr,n)$. Theorem (5.1) and Remark (5.2) give the following lower bound:

$$PE(r,A) \geq \binom{tr}{r}^n \left(\frac{\mathrm{vdw}(tr)}{\mathrm{vdw}(r)\mathrm{vdw}(r(t-1))}\right)^{n-t} \mathrm{vdw}(rt)(\mathrm{vdw}(r))^{-t},$$

improving (and reproving) the bounds from [17]. For instance, for $t = 2$ we get the following lower bound for boolean matrices $A \in BM(2r,n)$

$$PE(r,A) \geq \binom{2r}{r}^{(2n-1)} 4^{-r(n-1)}.$$

In the same way we can get bounds on $PE(r,A)$ for matrices A with r-sparse columns.

∎

8 An analogue of van der Waerden Conjecture

It is well known(a fairly direct corollary of the famous Edmond's result that the polytope spanned by the indicators of the bases in the intersection of two matroids on the same ground set is the intersection of two polytopes, each spanned by the indicators of the bases in the corresponding matroid) that the convex hull $CO(BM(r,n)) = \{A : 0 \leq A(i,j) \leq 1, Ae = A^T e = re\}$.

Note that for $r = 1$ the polytope $CO(BM(r,n))$ is equal to the polytope Ω_n of doubly-stochastic $n \times n$ matrices and $Pe(1,A) = Per(A)$.

The natural question, an analogue of the van der Waerden Conjecture ($r = 1$) is to compute the following minimum and maximum:

$$NMin(r,n) =: \min_{A \in CO(BM(r,n))} PE(r,A),$$

$$NMax(r,n) =: \max_{A \in CO(BM(r,n))} PE(r,A).$$

Remark 8.1: Recall that $NMin(1,n) = \frac{n!}{n^n}$ and it took more than 50 years to prove. On the other hand, the equality

$$\max_{A \in CO(BM(1,n))} Pe(1,A) = \max_{A \in \Omega_n} Per(A) = 1$$

is fairly trivial. ∎

First of all, any matrix $A \in BM(r,n)$ is a local maximum and $PE(r,A) = 1$. Therefore $NMin(r,n) < 1$. One would guess that the minimum is attained, as in the proved van der Waerden Conjecture, at the "center" $Cen_{r,n} =: \frac{r}{n}J_n, J_n = ee^T$.

The Everett-Stein asymptotically exact estimate (or our easily proved lower bound) give that:

$$PE(r, \frac{r}{n}J_n) \geq const(r) \left(\frac{r}{n}\right)^{rn} \frac{(rn)!}{(r!)^{2n}} \geq const(r) \left(\frac{r^{2r}}{e^r (r!)^2}\right)^n .$$

Define

$$n(r) =: \frac{r^{2r}}{e^r (r!)^2}.$$

By the direct inspection $n(1) < 1, n(2) < 1$ but $n(i) > 1$ for all $i \geq 3$.

So, there is a possibility that $NMin(2,n) = PE(2, Cen_{2,n})$. On the other hand, for $r \geq 3$ the "center" is not a minimum, at least for sufficiently large n.

The following (preliminary) result provides some lower and upper bounds on $NMin(r,n)$ and $NMax(r,n)$.

Proposition 8.2: *Let A, B be nonnegative $n \times m$ matrices, $0 < a < 1$. Then for all positive vectors $(x_1, ..., x_m)$ the following inequality holds*

$$ES_{\mathbf{r};aA+(1-a)B}(x_1, ..., x_m) \geq (ES_{\mathbf{r};A}(x_1, ..., x_m))^a (ES_{\mathbf{r};B}(x_1, ..., x_m))^{1-a}. \quad (40)$$

Proof: This follows from the well known fact that the symmetric functions are log-concave on the positive orthant. ∎

Corollary 8.3: *As a direct corollary of (40), the functional $G(A) =: \log(Cap_{c_1,...,c_m}(ES_{\mathbf{r};A})$ is concave on the convex cone of non-negative matrices. Therefore,*

$$Cap_{r,...,r}(ES_{re_n;A}) \geq 1$$

if $A \in CO(BM(r,n))$.

It is as easy to prove the upper bound:

$$Cap_{r,...,r}(ES_{re_n;A}) \leq Cap_{r,...,r}(ES_{re_n;\frac{r}{n}J_n}) = \binom{n}{r}^n \left(\frac{r}{n}\right)^{rn}, A \in CO(BM(r,n)). \quad (41)$$

Indeed, it follows from Newton's inequalities that

$$Sym_r(x_1, ..., x_n) \leq \left(\frac{x_1 + ... + x_n}{n}\right)^r \binom{n}{r}.$$

Therefore, if $A \in CO(BM(r, n)$ then

$$Cap_{r,...,r}(ES_{re_n;A}) \leq ES_{re_n;A}(1, ..., 1) \leq \binom{n}{r}^n \left(\frac{r}{n}\right)^{rn}.$$

We put these observations in the following statement about our current knowledge on the range of $PE(r, A), A \in CO(BM(r, n)$.

Lemma 8.4: *Let $A \in CO(BM(r, n)$ then*

$$\min_{A \in CO(BM(r,n))} Cap_{r,...,r}(ES_{re_n;A}) = 1, \max_{A \in CO(BM(r,n))} Cap_{r,...,r}(ES_{re_n;A}) = \binom{n}{r}^n \left(\frac{r}{n}\right)^{rn}.$$

As for any non-negative matrix $PE(r, A) \leq Cap_{r,...,r}(ES_{re_n;A})$ we get the following upper bound:

$$\binom{n}{r}^n \left(\frac{r}{n}\right)^{rn} \geq PE(r, A).$$

Assuming that r divides n, the following lower bound follows from our main inequality (35):

$$PE(r, A) \geq \left(\frac{\mathrm{vdw}(n)}{\mathrm{vdw}(r)\mathrm{vdw}(n - r)}\right)^{n\frac{r-1}{r}} \frac{\mathrm{vdw}(n)}{\mathrm{vdw}(r)^{n/r}}.$$

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