

Fourier Concentration from Shrinkage

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Abstract

For Boolean functions computed by de Morgan formulas of subquadratic size or read-once de Morgan formulas, we prove a sharp concentration of the Fourier mass on “small-degree” coefficients. For a Boolean function $f : \{0, 1\}^n \rightarrow \{1, -1\}$ computable by a de Morgan formula of size s , we show that

$$\sum_{A \subseteq [n] : |A| > s^{1/\Gamma + \epsilon}} \hat{f}(A)^2 \leq \exp(-s^{\epsilon/3}),$$

where Γ is the shrinkage exponent for the corresponding class of formulas: $\Gamma = 2$ for de Morgan formulas, and $\Gamma = 1/\log_2(\sqrt{5} - 1) \approx 3.27$ for read-once de Morgan formulas. We prove that this Fourier concentration is essentially optimal.

As an application, we get that subquadratic-size de Morgan formulas have negligible correlation with parity, and are learnable under the uniform distribution, and also lossily compressible, in subexponential time. Finally, we establish the tight $\Theta(s^{1/\Gamma})$ bound on the average sensitivity of read-once formulas of size s ; this mirrors the known tight bound $\Theta(\sqrt{s})$ on the average sensitivity of general de Morgan formulas of size s .

1 Introduction

Over the past thirty years, there have been a number of striking examples of interplay between complexity and algorithms. We know that computationally hard problems are useful for building secure cryptosystems [BM84, Yao82, HILL99], and derandomization [NW94, BFNW93, IW97, Uma03]. On the other hand, circuit lower bounds are implied by non-trivial algorithms for SAT [KL82, Kan82, Wil10, Wil11] or Polynomial Identity Testing [KI04]. It has also been observed that *techniques* used to prove existing circuit lower bounds are often useful for designing learning algorithms [LMN93], SAT algorithms [Zan98, San10, ST12, IMP12, BIS12, CKK⁺13, CKS13], and pseudorandom generators (PRGs) [Bra10, IMZ12, GMR⁺12, TX13] for the same class of circuits. In particular, the method of *random restrictions*, useful for proving lower bounds against AC^0 circuits [FSS84, Yao85, Hås86] and de Morgan formulas [Sub61, And87, Hås98, San10, KR13, KRT13], turns out to be also useful for designing such algorithms for the same circuit class.

We give another example of the connection between random restrictions and algorithms for small de Morgan formulas. We show tight Fourier concentration for small de Morgan formulas, which is similar to the Fourier concentration for AC^0 circuits shown in the celebrated paper by Linial, Mansour, and Nisan [LMN93]. More precisely, we use concentrated shrinkage of de Morgan

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formulas under random restrictions to show that most of the Fourier mass of such formulas lies on low-weight coefficients. Here *concentrated* shrinkage means that a formula shrinks in size *with high probability* when hit by a random restriction. Such concentrated shrinkage is implicitly proved by [IMZ12] (which considered the case of *pseudorandom* restrictions), building upon the earlier “shrinkage in expectation” results by [HRY95, Hås98].

As an immediate consequence of this Fourier concentration, we obtain, similarly to [LMN93], strong correlation lower bounds against parity, learning algorithms under the uniform distribution, and average sensitivity bounds for both general de Morgan formulas and read-once de Morgan formulas (with better parameters for read-once formulas). We provide more details next.

1.1 Our results

1.1.1 Fourier concentration and correlation bounds

The Fourier transform of a Boolean function $f : \{0, 1\}^n \rightarrow \{1, -1\}$ is a way to express f in the orthogonal basis of functions $\chi_S(x_1, \dots, x_n) = (-1)^{\sum_{i \in S} x_i}$, over all subsets $S \subseteq [n]$. Intuitively, the coefficient of f at the basis function χ_S , denoted $\hat{f}(S)$, measures the correlation between f and the parity function on the inputs x_i , for $i \in S$. Thus, one would expect that the classes of circuits for which the parity function is hard to compute would not have much weight on high-degree Fourier coefficients $\hat{f}(S)$ for large sets S , i.e., that such circuits would exhibit *concentration* of the Fourier spectrum over low-degree coefficients.

The first such connection between complexity of computing parity and Fourier concentration was shown by Linial, Mansour, and Nisan [LMN93], based on the strong average-case lower bounds for AC^0 circuits against the parity function [Hås86]. We extend the approach of [LMN93] to the case of subquadratic-size de Morgan formulas, which cannot compute the parity function in the worst case [Khr71], or even on average (as follows from the work in the quantum setting [BBC⁺01, Rei11]).

Our main result is the following.

Theorem 1.1. *Let $f : \{0, 1\}^n \rightarrow \{1, -1\}$ be a Boolean function computable by a de Morgan formula of size s . Then, for any constant $0 < \epsilon < 1/2$, and any sufficiently large s ,*

$$\sum_{A \subseteq [n] : |A| > s^{1/\Gamma + \epsilon}} \hat{f}(A)^2 \leq \exp(-s^{\epsilon/3}),$$

where Γ is the shrinkage exponent for the corresponding class of formulas: $\Gamma = 2$ for de Morgan formulas, and $\Gamma = 1/\log_2(\sqrt{5} - 1) \approx 3.27$ for read-once de Morgan formulas.

In words, this means that we have a very good approximation (in the ℓ_2 norm) to a de Morgan formula of size s with a polynomial of degree about $s^{1/\Gamma}$, where Γ is the shrinkage exponent for the class of formulas. We also show our Fourier concentration is essentially optimal (Lemma 5.3).

As an immediate corollary of Theorem 1.1, we get that the parity on n bits cannot be computed correctly on more than $1/2 + \exp(-s^{\epsilon/3})$ fraction of inputs by any de Morgan formula of size $s < n^{2/(1+2\epsilon)} \leq n^{2(1-\epsilon)}$, for any constant $0 < \epsilon < 1/2$.

1.1.2 Learning and compression

As a consequence of such Fourier concentration, we get, similarly to [LMN93], that the class of de Morgan formulas of size s is *learnable* in time $n^{s^{1/\Gamma + \epsilon}}$ to within error $\exp(-s^{\Omega(\epsilon)})$, over the uniform distribution, where $\Gamma = 2$ for general de Morgan formulas, and $\Gamma \approx 3.27$ for read-once de Morgan formulas (see Theorem 7.2).

This class of formulas is also *lossily compressible* in the sense of [CKK⁺13]. That is, given the truth table of a Boolean function $f : \{0, 1\}^n \rightarrow \{1, -1\}$ which is promised to be computable by an unknown de Morgan (read-once) formula of size s , we can compute in deterministic time $2^{O(n)}$, a Boolean circuit C of size about $n^{s^{1/\Gamma+\epsilon}}$, where Γ is the corresponding shrinkage exponent, such that C agrees with f on all but $\exp(-n^{\Omega(1)})$ fraction of n -bit inputs.

1.1.3 Average sensitivity

Informally, the average sensitivity of a Boolean function $f : \{0, 1\}^n \rightarrow \{1, -1\}$, denoted $AS(f)$, measures the number of influential coordinates in a typical input $x \in \{0, 1\}^n$, where a coordinate $i \in [n]$ is influential if flipping the i th bit in x flips the value $f(x)$; we give a more formal definition below. The Fourier concentration we show immediately yields the upper bound $s^{1/\Gamma+o(1)}$ on the average sensitivity of read-once de Morgan formulas of size s , where $\Gamma \approx 3.27$ is the shrinkage exponent for read-once formulas. However, we show (thanks to a personal communication by Nitin Saurabh) that the stronger upper bound $O(s^{1/\Gamma})$ can be obtained from [Bop89]. We then demonstrate the matching lower bound $\Omega(s^{1/\Gamma})$. Combined with known results for general de Morgan formulas, this yields the following tight connection between the shrinkage exponent and the average sensitivity for the class of (general and read-once) de Morgan formulas.

Corollary 1.2. *Let $f : \{0, 1\}^n \rightarrow \{1, -1\}$ be a Boolean function computable by a de Morgan formula of size s . Then $AS(f) \leq O(s^{1/\Gamma})$, where Γ is the shrinkage exponent for the corresponding class of formulas: $\Gamma = 2$ for de Morgan formulas, and $\Gamma = 1/\log_2(\sqrt{5} - 1) \approx 3.27$ for read-once de Morgan formulas. The average sensitivity $\Omega(s^{1/\Gamma})$ can be achieved with size s de Morgan formulas for $\Gamma = 2$, and read-once formulas for $\Gamma = 1/\log_2(\sqrt{5} - 1) \approx 3.27$.*

1.2 Our techniques

Our starting point is the result from [LMN93] which relates the Fourier spectrum of a given Boolean function f for “large” Fourier coefficients to the *expected* Fourier spectrum of the corresponding “large” Fourier coefficients for a *random restriction* of the function f ; here a random restriction is obtained by first deciding, with probability p for each variable, whether to restrict it, and then assigning randomly each selected variable either 0 or 1. If a random restriction is likely to have fewer than t variables (for some parameter t), then all Fourier coefficients of degree at least t are zero (since it’s impossible to have a nonzero correlation with the parity function on t variables if your function depends on fewer than t variables). Thus, if we have a “high-probability” shrinkage result for a given class of formulas under random restrictions (showing that a random restriction is likely to shrink the size of a given formula), we immediately get a corresponding Fourier concentration result, where the error bound of the concentration result is the same as the error bound for the shrinkage result.

This approach works directly for the case of *read-once* de Morgan formulas, which are known to shrink with high probability under “pseudorandom” restrictions [IMZ12], and the same analysis of [IMZ12] can be used also for the case of truly random restrictions, yielding an exponentially small error, as shown in our Theorem 4.2 below.

However, for the case of *general* de Morgan formulas, such a “high-probability” shrinkage result is simply not true. The problem is posed by the presence of “heavy” variables, the variables that occur too often in a given formula. The notion of a random restriction needs to be modified so that the heavy variables are always restricted, while each of the remaining light variables is chosen

to be restricted with some probability p . We adapt the result of [LMN93] mentioned above to the setting of such modified restrictions.

Still, in order to get strong Fourier concentration, one needs the parameter p of a random restriction to be quite small (e.g., n^ϵ/n), while the known shrinkage result of [IMZ12] applies only to relatively large values of p (e.g., $p > n^{-1/8}$). The solution is to apply a number of restrictions recursively, each with a relatively large value of p_i , so that the product of the p_i 's is as small as we want. Fortunately, the connection between the Fourier spectrum of the original function and of its appropriate random restriction fits in well with such a recursive argument.

To prove the optimality of our Fourier concentration, we exhibit a family of small de Morgan formulas that have non-trivial correlation with the parity function. Roughly, the constructed formula computes the AND of parities of small disjoint subsets of the input variables (see Lemma 5.3).

The learning and compression results follow immediately from Fourier concentration, using standard methods (cf. [LMN93]). For the *lower bound* on the average sensitivity of read-once formulas, we use an explicit family of read-once formulas constructed by [PZ93] (building on [Val84b]), which are known to be shrinkage-resistant. We show that the same read-once formulas of size s have average sensitivity $\Omega(s^{1/\Gamma})$ (see Theorem 7.5).

1.3 Related work

For size s de Morgan formulas, Ganor, Komargodski, and Raz [GKR12] proved the upper bound $\sum_{|A| > s^{1/2}/\epsilon} \hat{f}(A)^2 \leq O(\epsilon)$, which is tight for constant $\epsilon > 0$. Actually, as pointed out to us by one of the referees, a stronger result (matching our Theorem 1.1 for de Morgan formulas) can be deduced from some of the quantum-based results mentioned by [GKR12]. In particular, the results in quantum query complexity [FGG08, ACR⁺07, RŠ08, Rei09] imply that every de Morgan formula F of size s can be approximated by a polynomial of degree $D \leq O(t \cdot s^{1/2+o(1)})$ with point-wise error at most 2^{-t} , and hence also in the ℓ_2 -norm with the same error 2^{-t} . This is easily seen to imply that the Fourier spectrum of F above the degree D is at most 2^{-t} . Setting $t = s^\epsilon$ yields the required Fourier concentration as in Theorem 1.1.

In contrast, our Theorem 1.1 is proved using classical (non-quantum) arguments, and exhibits a deep connection between the Fourier concentration parameters for a class of formulas and the shrinkage exponent for the same class of formulas; the above-mentioned quantum results (based on point-wise polynomial approximations) do not distinguish between read-once and general de Morgan formulas.

Lee [Lee09] shows (based on a long line of work in the quantum setting [BBC⁺01, FGG08, ACR⁺07, RŠ08, Rei09]) that every de Morgan formula of size s can be computed as the *sign*(p) for a multilinear polynomial p of degree $O(\sqrt{s})$. In particular, this completely resolves a conjecture by O'Donnell and Servedio made in the conference version of [OS10], which implies that Boolean functions computable by size s de Morgan formulas are PAC-learnable in time $O(n^{\sqrt{s}})$. For general de Morgan formulas, this is stronger than our uniform-distribution learning result as it holds for *any* distribution. On the other hand, our proof is completely classical, and we also get a better running time for the case of read-once formulas.

The tight average sensitivity bound $\Theta(\sqrt{s})$ for general de Morgan formulas of size s follows from the work of [Shi00, Lee09] (using the quantum approach); an alternative (classical) proof is also given by [GKR12]. For read-once formulas of size s , the upper bound $O(s^{1/\Gamma})$ on the average sensitivity, where Γ is the corresponding shrinkage exponent for read-once formulas, is implicit in the work of [Bop89]. This observation was made by Nitin Saurabh [personal communication, 2013], and we include his argument, with his permission, in Section C of the appendix.

As observed in [KRT13], the quantum-setting work of [BBC⁺01, Rei11] implies that any de Morgan formula of size $o((n/\log(1/\epsilon))^2)$ has correlation at most $1/2 + \epsilon$ with the n -bit parity. This is comparable to the correlation bound implied by our Fourier concentration result (Corollary 7.1).

Remainder of the paper. We state the basics in Section 2, and show how to adapt the approach of [LMN93] in Section 3. We prove the required concentrated shrinkage results for general and read-once de Morgan formulas in Section 4. We derive the Fourier concentration result for general de Morgan formulas in Section 5, and for read-once formulas in Section 6. In Section 7 we give the applications of the Fourier concentration result to correlation with parity, learning, compression, and average sensitivity for de Morgan formulas. We state some open questions in Section 8. The Appendix contains some proofs omitted from the main body of the paper.

2 Preliminaries

2.1 Notation

We denote by $[n]$ the set $\{1, 2, \dots, n\}$. We use $\exp(a)$ to denote the exponential function 2^a , where a is some numerical expression. All logarithms are base 2 unless explicitly stated otherwise.

2.2 Formulas

A *de Morgan formula* F on n variables x_1, \dots, x_n is a binary tree whose leaves are labeled by variables or their negations, and whose internal nodes are labeled by the logical operations AND or OR. The *size* of a formula F , denoted by $L(F)$, is the number of leaves in the tree.

A de Morgan formula is called *read-once* if every variable appears at most once in the tree. Note that the size of a read-once formula on n variables is at most n .

2.3 Fourier transform

We review some basics of Fourier analysis of Boolean functions (see, e.g., [Wol08] for a survey). We think of an n -variate *Boolean* function as $\{-1, 1\}$ -valued, i.e., as $f : \{0, 1\}^n \rightarrow \{-1, 1\}$. For a subset $A \subseteq [n]$, we denote by χ_A the Boolean function mapping $x_1, \dots, x_n \in \{0, 1\}^n$ to the parity $(-1)^{\sum_{i \in A} x_i}$. Let $f : \{0, 1\}^n \rightarrow \mathbb{R}$ be any function. The *Fourier coefficient* of f at A is defined as $\hat{f}(A) := \mathbf{Exp}_{x \in \{0, 1\}^n}[f(x) \cdot \chi_A(x)]$. Note that $\hat{f}(A)$ is exactly the *advantage* of f at computing χ_A , the parity of the inputs from A .

The *Parseval identity* is $\sum_{A \subseteq [n]} \hat{f}(A)^2 = \mathbf{Exp}_{x \in \{0, 1\}^n}[f(x)^2]$. Note that for a Boolean function $f : \{0, 1\}^n \rightarrow \{-1, 1\}$, we get $\sum_{A \subseteq [n]} \hat{f}(A)^2 = 1$.

2.4 Random restrictions

For $0 < p < 1$, we define a p -restriction ρ of the set of n variables x_1, \dots, x_n as follows: for each $i \in [n]$, with probability p assign x_i the value $*$ (i.e., leave x_i unrestricted), and otherwise assign x_i uniformly at random a value 0 or 1. We denote by R_p the class of p -restrictions.

For a Boolean function $f(x_1, \dots, x_n)$ and a random restriction ρ , f_ρ denotes the restricted function obtained from f using ρ ; f_ρ is a function of the variables left unrestricted by ρ . For a Boolean function f , a subset S of variables, and a string $r \in \{0, 1\}^{|S|}$, the notation $f_{S \leftarrow r}$ means the restriction of f where the variables in S are assigned the values given in r . We can combine

different restrictions. For example, $f_{S \leftarrow r, \rho}$ means the restriction of f where we assign the values r to the variables in S , and then apply a restriction ρ to the resulting function in variables $[n] \setminus S$.

2.5 Chernoff-Hoeffding bound

We will use the following version of the Chernoff-Hoeffding bound [Che52, Hoe63].

Lemma 2.1 (Chernoff-Hoeffding). *Let $X = \sum_{i=1}^t X_i$ be the sum of independent random variables such that each X_i is in the range $0..s$, and $\mathbf{Exp}[X] < E$, for $s, E \geq 1$. Then $\mathbf{Pr}[X > 8E] < 2^{-E/s}$.*

3 Fourier concentration via random restrictions

3.1 The starting point

We use the following result of [LMN93]; for completeness, we prove it in Section A of the Appendix.

Theorem 3.1 ([LMN93]). *For arbitrary n -variate Boolean function f , integer $t > 0$ and a real number $0 < p < 1$ such that $pt \geq 8$,*

$$\sum_{|A| > t} \hat{f}(A)^2 \leq 2 \cdot \mathbf{Exp}_{\rho \in R_p} \left[\sum_{B : |B| > pt/2} \hat{f}_\rho(B)^2 \right].$$

3.2 Intuition for de Morgan formulas

Imagine we had a “dream version” of the concentrated shrinkage result for de Morgan formulas: For any $0 < p < 1$, a given de Morgan formula F on n variables of size s will shrink to size $s' \leq p^2 s$ with probability $1 - \gamma$, for some “small” γ . Let us pick p so that $p^2 s < n$.

Note that a formula of size s' has at most s' variables, and hence, all its Fourier coefficients for the sets of size greater than s' are 0. In the notation of Theorem 3.1, every p -restriction ρ , such that the formula size of F_ρ is less than $pt/2$, contributes 0 to the overall expectation; every other restriction ρ (where the formula doesn't shrink) contributes at most 1 (by the Parseval equality). Equating $p^2 s$ and $pt/2$, we get for $t = 2ps$, $\sum_{|A| > t} \hat{F}(A)^2 \leq 2 \cdot \gamma$. Since $t < n$ and $p \geq 1/n$, this also forces $s < n^2/2$. For $s \leq n^{2-2\epsilon}$, we can set $p = n^\epsilon/n$, getting $t = n^{1-\epsilon}$. (Note that since we need $pt \geq 8$, we get $p \geq \Omega(1/\sqrt{s})$, and thus, $t \geq \Omega(\sqrt{s})$.)

In reality, we don't have such concentrated shrinkage. First, it is not true because a formula may have “heavy” variables (those that appear too frequently in the formula), and if such a heavy variable is missed (assigned $*$) by a p -random restriction, no substantial shrinkage of the formula size will occur. Thus we need to ensure that the heavy variables are always restricted.

Secondly, the best known concentrated shrinkage results of [IMZ12, KRT13] do not work for very small p . The way around it is to apply a number of random restrictions one after the other, for appropriately chosen p_1, p_2, \dots, p_k , thereby simulating a single restriction with the parameter $p = \prod_{i=1}^k p_i$; such a workaround was already used in [IMZ12] and [KRT13].

The following lemma will handle heavy variables. Intuitively, it says that each variable restricted increases the effective degree of where the Fourier coefficients could be large by at most 1.

Lemma 3.2. *Let f be a Boolean function, and x a variable for f . Let f_0 be f with x set to 0, f_1 with x set to 1. For any $\delta \geq 0$, if both $\sum_{A, |A| \geq t} \hat{f}_0(A)^2 \leq \delta$ and $\sum_{A, |A| \geq t} \hat{f}_1(A)^2 \leq \delta$, then $\sum_{A, |A| \geq t+1} \hat{f}(A)^2 \leq \delta$.*

Proof. For $y := 1 - 2x$, we can write $f = 1/2(1+y)f_0 + 1/2(1-y)f_1 = 1/2(f_0 + f_1) + (1/2)(f_0 - f_1)y$. Then, for any set A not containing x ,

$$\begin{aligned}\hat{f}(A)^2 + \hat{f}(x \cup A)^2 &= (1/2(\hat{f}_0(A) + \hat{f}_1(A)))^2 + (1/2(\hat{f}_0(A) - \hat{f}_1(A)))^2 \\ &= 1/2 \cdot \hat{f}_0(A)^2 + 1/2 \cdot \hat{f}_1(A)^2.\end{aligned}$$

Summing this over all A with $|A| \geq t$ yields at most δ by the assumptions for the restricted functions. Every B of size $\geq t + 1$ (containing x or not) is included in this sum. \square

So to upperbound the Fourier mass of the coefficients for sets A with $|A| > t$, the idea is to set all “heavy” variables (say, z of them), and upperbound the Fourier mass for each restricted function over the coefficients for sets B with $|B| > t - z$. If we can bound the Fourier mass of each restricted function by some δ , then, by Lemma 3.2, we get the same upper bound for the Fourier mass of the original function over the sets of size greater than $(t - z) + z = t$, as required.

4 Concentrated shrinkage of de Morgan formulas

Here we prove the following shrinkage results for general and read-once de Morgan formulas, implicit in [IMZ12].

Theorem 4.1 (Shrinkage of general de Morgan formulas). *There exists a $c > 0$ such that, for every L and every de Morgan formula F with $L(F) \leq L$ on n variables that does not have any variable appearing more than h times, and for every $0 < p < 1$,*

$$\Pr_{\rho \in R_p} \left[L(F_\rho) \geq c \cdot p^2 \cdot \log^{3/2}(1/p) \cdot L \right] \leq L(F) \cdot \exp(-p^6 \cdot L/h).$$

Theorem 4.2 (Shrinkage of read-once de Morgan formulas). *For any read-once de Morgan formula $F(x_1, \dots, x_n)$ and $p = (n^\epsilon/n)^{1/\Gamma}$, for some $\epsilon \in [0, 1]$, we have*

$$\Pr_{\rho \in R_p} \left[L(F_\rho) > 2^b \cdot p^\Gamma \cdot n \right] \leq \exp(-\Omega(n^{\epsilon/2})),$$

for some $b = O(1/\epsilon)$, where $\Gamma = 1/\log(\sqrt{5} - 1) \approx 3.27$.

Both of these results are proved using the well-known “shrinkage in expectation” results for the corresponding classes of formulas [Hås98, HRY95, DZ94]. The proof idea is to decompose a given formula into a few batches of independent subformulas (with some extra conditions) and apply “shrinkage in expectation” to each subformula. Since the subformulas in each batch are independent, we can use the Chernoff-Hoeffding inequality to argue that the shrinkage occurs with high probability in each batch, and hence, by the union bound, also for the entire original formula.

We provide more details below. First, in Section 4.1, we give arguments common for the proofs of both these results. Then we prove Theorem 4.1 in Section 4.2, and Theorem 4.2 in Section 4.3.

4.1 Preliminary arguments

We will be using the following “shrinkage in expectation” results. Håstad [Hås98] showed that the shrinkage exponent for de Morgan formulas is 2 (see also [Tal14] for an alternative, tighter proof).

Theorem 4.3 ([Hås98]). *There exists a $c > 0$ such that, for every de Morgan formula F on n variables and for every $0 < p < 1$,*

$$\mathbf{Exp}_{\rho \in R_p} [L(F_\rho)] \leq c \cdot \left(p^2 \cdot \mu(p, L(F)) \cdot L(F) + p \cdot \sqrt{L(F)} \right),$$

where $\mu(p, L(F)) = 1 + \log^{3/2} \min\{1/p, L(F)\}$.

Håstad, Razborov, and Yao [HRY95] settled the shrinkage exponent for read-once formulas; their result was tightened by Dubiner and Zwick [DZ94].

Theorem 4.4 ([HRY95, DZ94]). *For every read-once formula $F(x_1, \dots, x_n)$ and a parameter $0 < p < 1$,*

$$\mathbf{Exp}_{\rho \in R_p} [L(F_\rho)] \leq O \left(p^\Gamma \cdot n + p \cdot n^{1/\Gamma} \right),$$

where $\Gamma = 1/\log(\sqrt{5} - 1) \approx 3.27$.

Next, we decompose a given (general or read-once) de Morgan formula as follows.

Lemma 4.5 ([IMZ12]). *For every positive s and any de Morgan formula F on the set X of variables with $L(F) \geq s$, there exist m de Morgan formulas G_1, \dots, G_m for $m \leq O(L(F)/s)$, such that*

1. $L(G_i) \leq s$, for all $1 \leq i \leq m$,
2. for each $1 \leq i \leq m$, G_i has at most 2 occurrences of “special” variables outside of X (different variables for different G_i ’s), and
3. for any restriction ρ of the variables X , $L(F_\rho) \leq \sum_{i=1}^m L((G_i)_{\rho'})$ where $\rho'(x) = \rho(x)$ for $x \in X$ and $\rho'(x) = *$ otherwise.

Moreover, if F is a read-once formula, then so is every formula G_i in the collection.

Proof sketch. Find a subformula of size between $s/2$ and s ; a maximal subformula of size at most s has size at least $s/2$. Replace the subformula with a new variable, called a subtree variable. Repeatedly find either a subformula with exactly 2 subtree variables and of size less than s , or a subformula with at most 1 subtree variable and of size between $s/2$ and s . (Take a minimal subformula with size greater than $s/2$. If it has more than 2 subtree variables, take a minimal subformula with at least 2 such variables; since each of its child formulas has at most 1 subtree variable, it must have exactly 2.) Since each time, we either remove s nodes and create at most 1 new subtree variable, or reduce the number of subtree variables by one, we get a partition of the formula into $O(L(F)/s)$ subformulas each of size at most s and with at most 2 subtree variables. \square

The special variables correspond to the inputs which are outputs of some other subformulas. We want to analyze the effect of a random restriction on F by using the upper bound of item (3) of Lemma 4.5. To this end, we need to handle random restrictions that leave some specified variables (the “special” variables in our case) unrestricted.

The idea is to take each subformula G_i and construct a new subformula G'_i by replacing each special variable in G_i with a restriction-resistant formula (on new variables, different for different special variables); here we call a formula “restriction-resistant” if, for at least $3/4$ of random restrictions, the resulting restricted formula remains a non-constant function. Then we upperbound the expected size $\mathbf{Exp}_{\rho'} [L((G_i)_{\rho'})]$, for ρ' that leaves special variables unrestricted, by twice the

expected size $\mathbf{Exp}_\rho[L((G'_i)_\rho)]$, for a standard random restriction ρ . The latter expectation can be upperbounded using the above-mentioned “shrinkage in expectation” results.

For general de Morgan formulas, the parity function on k inputs is not likely to simplify to a constant function for most p -restrictions, where $pk \gg 1$; the size of such a de Morgan formula is $O(k^2)$. For read-once de Morgan formulas, the existence of restriction-resistant formulas follows from the work by Valiant [Val84a]; see Section B in the appendix for the proof of the following.

Lemma 4.6 ([IMZ12]). *For every $0 < p < 1$, there exists a read-once de Morgan formula H of size $O(1/p^4)$ such that, for all but at most $1/4$ of p -restrictions ρ , we have*

$$H_\rho(\vec{0}) = 0 \quad \text{and} \quad H_\rho(\vec{1}) = 1, \quad (1)$$

where $\vec{0}$ and $\vec{1}$ denote the inputs of all 0's and all 1's, respectively.

Now we can analyze the expected shrinkage of de Morgan formulas under p -restrictions that leave some specified variables unrestricted. Let G_i be any formula in the decomposition of Lemma 4.5, with at most two occurrences of special variables. Let H be a shrinkage-resistant formula in the sense that, for all but at most $1/4$ fraction of p -restrictions σ , the restricted formula H_σ is not a constant function. Let G'_i be obtained from G_i by replacing the special variables in G_i by independent copies of the formula H on new, disjoint sets of variables. Let ρ' be a p -restriction on the variables of G_i such that the special variables are assigned $*$. Let ρ be a p -restriction on all variables of G'_i which agrees with ρ' on all variables of G_i .

We have the following.

Claim 4.7. $\mathbf{Exp}_{\rho'}[L((G_i)_{\rho'})] \leq 2 \cdot \mathbf{Exp}_\rho[L((G'_i)_\rho)]$.

Proof of Claim 4.7. Let A be the event that a random p -restriction on the variables of two copies of H leaves both these formulas non-constant. By the union bound, the probability of A is at least $1/2$. Conditioned on A , we have $L((G_i)_{\rho'}) \leq L((G'_i)_\rho)$. Thus, for a fixed ρ' , and for a random ρ extending ρ' , we get

$$\mathbf{Exp}_\rho[L((G'_i)_\rho)] \geq (1/2) \cdot L((G_i)_{\rho'}).$$

Taking the expectation over ρ' on both sides of this inequality yields the desired claim. \square

Now we are ready to prove our concentrated shrinkage results.

4.2 Proof of Theorem 4.1

Let $s = c_0 p^{-2}$ for some constant c_0 . Using Lemma 4.5, decompose a given formula F into $O(L(F)/s)$ subformulas G_i 's.

Let H be a de Morgan formula on $2/p$ fresh variables that computes the parity function. Each such de Morgan formula for parity on $2/p$ variables has size $O(1/p^2)$. The probability that each of $2/p$ variables is assigned (0 or 1) by a random p -restriction is $(1-p)^{2/p} \leq e^{-2} \leq 1/4$. Thus H is shrinkage-resistant.

Form G'_i by replacing special variables in G_i by independent copies of the formula H . Since each G_i is of size at most $s = c_0/p^2$ and the size of H is $O(1/p^2)$, we get that each G'_i has size c'_0/p^2 , for some constant c'_0 . By Claim 4.7 and Håstad's Theorem 4.3, we get, for each G'_i ,

$$\mathbf{Exp}[L((G_i)_{\rho'})] \leq 2 \cdot \mathbf{Exp}_\rho[L((G'_i)_\rho)] \leq c_1 \cdot \log^{3/2} s, \quad (2)$$

for some constant c_1 , where ρ' is a p -restriction on the variables of G_i excluding the special variables, and ρ is a p -restriction extending ρ' to all variables of G'_i .

Thus, we have a collection of $O(L(F)/s)$ formulas G^i , each of size at most s , such that no variable appears in more than h of the G^j 's, and such that $L(F_\rho) \leq \sum L((G_i)_{\rho'})$. So our lemma reduces to showing concentration for the latter sum of random variables whose expectations are upperbounded by Eq. (2).

Since each G_i shares any variables with at most sh other G_j 's, we can partition G_i 's into $O(sh)$ batches, each of at most $O(L(F)/(s^2h))$ formulas, so that the formulas in each batch are totally independent, having no variables in common. By Eq. (2), the expected total formula size within each batch is $O(L(F)(\log^{3/2} s)/(s^2h))$. As a random variable, this is the sum of independent random variables in the range 0 to s . By the Chernoff-Hoeffding bound of Lemma 2.1, the probability that the sum of the formula sizes in any batch is larger than $c_3L(F)(\log^{3/2} s)/(s^2h)$ is less than $2^{-\Omega(L(F)(\log^{3/2} s)/s^3h)}$. There are strictly less than $L(F) \leq L$ batches, so a union bound gives the probability that all batches are of size $O(L(F)(\log^{3/2} s)/(s^2h))$ except with probability $L \cdot \exp(-\Omega(L(F)/(s^3h))) = L \cdot \exp(-\Omega(p^6L(F)/h))$. If they are, then summing up over the at most $O(sh)$ batches, $L(F_\rho) \leq O(L(F)(\log^{3/2} s)/s) = O(p^2 \cdot L(F) \cdot \log^{3/2}(1/p))$.

4.3 Proof of Theorem 4.2

First we prove a shrinkage result useful for relatively large parameters p .

Theorem 4.8. *There exist constants $d, d' > 0$ such that the following holds for any read-once de Morgan formula $F(x_1, \dots, x_n)$ and $0 < p < 1$:*

$$\Pr_{\rho \in R_p} [L(F_\rho) \geq d \cdot p^\Gamma \cdot n] \leq \exp(-d' \cdot p^8 \cdot n),$$

where $\Gamma = 1/\log(\sqrt{5} - 1) \approx 3.27$.

Proof. Set $s = c/p^4$, for a constant c to be determined. Using Lemma 4.5, partition a given formula F (of size n) into $O(n/s)$ subformulas G_1, \dots, G_m of size at most s each.

Let H be a shrinkage-resistant read-once formula from Lemma 4.6. Define G'_i to be G_i with special variables in G_i replaced by independent copies of H . Note that $L(G'_i) \leq L(G_i) + O(1/p^4)$, which can be made at most $2 \cdot L(G_i)$ by choosing $p \geq (c/L(G))^{1/4}$ for a sufficiently large constant $c > 0$. By Claim 4.7 and Theorem 4.4, we get for each G_i that

$$\mathbf{Exp}_{\rho'} [L((G_i)_{\rho'})] \leq c' \cdot p^\Gamma \cdot s, \tag{3}$$

for some constant c' , where ρ' is a p -restriction over the variables of G_i excluding the special variables.

By Lemma 4.5, we have $L(F_\rho) \leq \sum_i L((G_i)_{\rho'})$. Note that the latter is the sum of independent random variables, as different G_i 's have no variables in common (due to F being read-once). Each of these random variables is in the range $0..s$, with expectation upperbounded by Eq. (3). Hence, the expectation of the sum of these random variables is at most $c''np^\Gamma$, for some constant c'' . By the Chernoff-Hoeffding bound of Lemma 2.1, the probability that $L(F_\rho)$ is greater than $8c''np^\Gamma$ is less than $2^{-c''np^\Gamma/s} \leq \exp(-d' \cdot p^8 \cdot n)$, for some constant d' . \square

Now we prove Theorem 4.2, by recursively applying Theorem 4.8.

Proof of Theorem 4.2. Set $q := n^{-\epsilon/16}$, and $k := 16(1 - \epsilon)/(\Gamma \cdot \epsilon)$. We will apply k random q -restrictions to our original formula F . Let F_i be the formula F after i restrictions are applied to F , and let s_i be the size of F_i ; we have $F_0 = F$ and $s_0 = n$.

Consider stage i , for $1 \leq i \leq k$. If $s_{i-1} \leq p^\Gamma n = n^\epsilon$, then s_i will also be less than n^ϵ with probability 1. Assuming $s_{i-1} \geq n^\epsilon$, we get by Theorem 4.8 that $\Pr_{\rho \in R_q}[s_i \geq d \cdot q^\Gamma \cdot s_{i-1}] \leq \exp(-d' \cdot q^8 \cdot n^\epsilon) \leq \exp(-d' n^{\epsilon/2})$. It follows that with probability at least $1 - k \cdot \exp(-d' n^{\epsilon/2})$, we have $s_k \leq d^k q^{k\Gamma} n \leq d^k \cdot p^\Gamma \cdot n$, as claimed. \square

5 Fourier concentration of de Morgan formulas

The main result of this section is the following.

Theorem 5.1. *Let $f : \{0, 1\}^n \rightarrow \{-1, 1\}$ be a Boolean function computed by a de Morgan formula $F(x_1, \dots, x_n)$ of size at most s . Then, for any constant $0 < \epsilon < 1/2$, and any sufficiently large s ,*

$$\sum_{|A| > s^{1/2+\epsilon}} \hat{f}(A)^2 \leq \exp(-s^{\epsilon/3}).$$

5.1 Proof of the main result

Theorem 5.1 is implied by the following quantitative version (by equating ϵ of Theorem 5.1 with $\epsilon_k/2$ of Theorem 5.2).

Theorem 5.2. *For each integer $k \geq 0$ there is a constant b so that, for any Boolean function f computed by a de Morgan formula of size at most s , and for $\epsilon_k = (11/12)^k$ and $t = s^{(1+\epsilon_k)/2}$,*

$$\sum_{|A| > t} \hat{f}(A)^2 \leq ks \cdot \exp\left(-\Omega\left(\frac{s^{\epsilon_k/4}}{\log^b s}\right)\right),$$

for s sufficiently large.

Proof. The proof is by induction on k . The base case $k = 0$ is trivial, since the formula depends on at most s variables, and the sum is over sets of size larger than s .

Assume the theorem holds for $k \geq 0$. We will prove it for $k+1$. Let $t = s^{(1+\epsilon_{k+1})/2}$. Let $h = \sqrt{s}$. There are at most \sqrt{s} variables that are more than h -heavy (i.e., have more than h occurrences) in a minimal formula for f . Let f' be any restriction of f assigning values to the heavy variables. We will show that each f' has

$$\sum_{|A| \geq t'} \hat{f}'(A)^2 \leq ks \cdot \exp(-s^{\Omega(\epsilon_k)}),$$

where $t' = t/2$. The claim will then follow from Lemma 3.2, since $t > t' + \sqrt{s}$ when $s > 4^{(12/11)^k}$.

For each such f' , consider a random restriction $\rho \in R_p$ for $p = s^{-1/24}/(c(\log s)^a)$, for constants a and c to be chosen later. By Theorem 3.1, we get

$$\sum_{A: |A| \geq t'} \hat{f}'(A)^2 \leq 2 \cdot \mathbf{Exp}_\rho \left[\sum_{B: |B| \geq pt'/2} \hat{f}'_\rho(B)^2 \right].$$

By Theorem 4.1, except with probability $s \cdot \exp(-p^6 s/h) = s \cdot \exp(-\Omega(s^{1/4}/\log^{6a} s))$, the function f'_ρ has formula size at most $s'' = c_1 p^2 s \log^{3/2} s = (c_1/c^2) s^{11/12} (\log s)^{3/2-2a} = s^{11/12} (\log s)^{3/2-2a}$, where we set $c := \sqrt{c_1}$.

Let t'' be the inductive Fourier coefficient size for the theorem with k and the formula size s'' , i.e.,

$$\begin{aligned} t'' &= (s'')^{(1+\epsilon_k)/2} \\ &= s^{(11/24)(1+\epsilon_k)} \cdot (\log s)^{(3/4-a)(1+\epsilon_k)} \\ &= s^{11/24+\epsilon_{k+1}/2} \cdot (\log s)^{(3/4-a)(1+\epsilon_k)}. \end{aligned}$$

We claim we can pick a so that $pt'/2$ (the crucial value in Theorem 3.1) is greater than t'' . First,

$$\begin{aligned} pt'/2 &= s^{(1+\epsilon_{k+1})/2} \cdot s^{-1/24} / (4c(\log s)^a) \\ &= s^{11/24+\epsilon_{k+1}/2} / (4c(\log s)^a) \\ &= (t''/4c)(\log s)^{a\epsilon_k - (3/4)(1+\epsilon_k)}. \end{aligned}$$

For any $a > (3/4)(1+\epsilon_k)/\epsilon_k$, the exponent of $\log s$ will be greater than zero, so the whole expression will be larger than t'' for sufficiently large s .

Thus, except for the $s \cdot \exp(-s^{1/4}/\log^{6a} s)$ fraction of ρ 's where shrinkage fails, we have

$$\begin{aligned} \sum_{B: |B| \geq pt'/2} \hat{f}'_{\rho}(B)^2 &\leq ks'' \cdot \exp\left(-\Omega\left(\frac{(s'')^{\epsilon_k/4}}{\text{poly log } s''}\right)\right) \\ &\leq ks \cdot \exp\left(-\Omega\left(\frac{s^{\epsilon_{k+1}/4}}{\text{poly log } s}\right)\right). \end{aligned}$$

Since the sum of squares of Fourier coefficients is bounded by 1 (by Parseval's identity), we can add the chance of failure, getting the bound $(k+1)s \cdot \exp(-\Omega(s^{\epsilon_{k+1}/4}/\text{poly log } s))$, as required. \square

5.2 Optimality of the Fourier concentration for de Morgan formulas

Let $f: \{0, 1\}^n \rightarrow \{1, -1\}$ be a Boolean function computed by a de Morgan formula of size s . Since the parity of m bits can be computed by a size $O(m^2)$ de Morgan formula, we have that $\hat{f}(A) = 1$ for a set $A \subseteq [n]$ of size $|A| = O(\sqrt{s})$. Thus, in order to get a non-trivial upper-bound on the Fourier spectrum $\sum_{|A| > t} \hat{f}(A)^2$, we need to set $t > \sqrt{s}$.

In fact, as we shall argue, in order to get the upper bound 2^{-n^γ} , for some $\gamma > 0$, we need to set $t > \sqrt{s} \cdot n^{\gamma/2}$. This shows that our Fourier concentration for sub-quadratic size de Morgan formulas, Theorem 5.1, is tight, up to a constant factor in front of the parameter ϵ .

Lemma 5.3. *For any $\gamma > 0$ and $t \leq n$, there is a de Morgan formula on n inputs of size $O(t^2/n^\gamma)$ that computes the parity on t bits with advantage 2^{-n^γ} .*

Proof. Consider the following formula $F(x_1, \dots, x_n)$. Set $m = \lfloor n^\gamma \rfloor$. Without loss of generality assume that m is odd; otherwise take $m - 1$. Divide x_1, \dots, x_t into m disjoint blocks of size t/m each. Compute the parity of each block, using a de Morgan formula of size $O(t^2/m^2)$, and output the AND of the results over all blocks. The overall formula size of F is $O((t^2/m^2) \cdot m) = O(t^2/m)$.

Next we argue that F has advantage 2^{-m} in computing the parity of x_1, \dots, x_t . Note that F is correct when all m blocks have odd parity, which happens with probability 2^{-m} . If not all blocks have odd parity, our formula always outputs 0, which is correct for exactly $1/2$ of the inputs. \square

By Lemma 5.3, a function f computed by a de Morgan formula of size s may have $\hat{f}(A) \geq 2^{-n^\gamma}$ for a set A of size $|A| \leq t$ for t satisfying $O(t^2/n^\gamma) = s$, i.e., for $t = O(\sqrt{s} \cdot n^{\gamma/2})$. It follows that in order to achieve $\sum_{|A| > t} \hat{f}(A)^2 < 2^{-n^{\epsilon/6}}$, one needs to set $t > \sqrt{s} \cdot n^{\epsilon/12}$.

6 Fourier concentration of read-once de Morgan formulas

The main result of this section is the following.

Theorem 6.1. *Let $F(x_1, \dots, x_n)$ be any read-once de Morgan formula computing the Boolean function $f : \{0, 1\}^n \rightarrow \{1, -1\}$. Then for every $0 < \epsilon < 1$ such that $\epsilon > \Omega(1/\sqrt{\log n})$,*

$$\sum_{|A| > n^{1/\Gamma + \epsilon}} \hat{f}(A)^2 \leq \exp(-n^{\epsilon/3}),$$

where $\Gamma = 1/\log(\sqrt{5} - 1) \approx 3.27$.

Proof. For $p = (n^\epsilon/n)^{1/\Gamma}$, we get by Theorem 4.2 that for all but $\gamma := \exp(-\Omega(n^{\epsilon/2}))$ fraction of p -restrictions shrink the formula F to size at most $2^b \cdot n^\epsilon$, for some $b = O(1/\epsilon)$. Set t so that $2^b \cdot n^\epsilon = pt/2$. We get $t = n^{1/\Gamma + \epsilon(1-1/\Gamma)} \cdot 2^{O(1/\epsilon)} \leq n^{1/\Gamma + \epsilon}$, for ϵ such that $2^{O(1/\epsilon)} \leq n^{\epsilon/3}$, which holds for $\epsilon > \Omega(1/\sqrt{\log n})$. By Theorem 3.1, we conclude that $\sum_{|A| > t} \hat{f}(A)^2 \leq 2\gamma \leq \exp(-\Omega(n^{\epsilon/2})) \leq \exp(-n^{\epsilon/3})$, as required. \square

The bound in Theorem 6.1 is close to optimal (see Remark 7.6 below).

7 Applications

7.1 Correlation with Parity

Subquadratic-size de Morgan formula have exponentially small correlation with the parity function.

Corollary 7.1. *Every de Morgan formula of size at most $s = n^{2-2\epsilon}$, for some $0 < \epsilon < 1/2$, agrees with the parity function on n bits on at most $1/2 + \exp(-s^{\epsilon/3})$ fraction of inputs.*

Proof. Recall that the Fourier coefficient $\hat{f}(S)$ for a subset $S \subseteq [n]$ measures the correlation of f with the parity function on the positions in S . The result follows immediately from Theorem 5.1. \square

By Lemma 5.3, this correlation bound is essentially optimal.

7.2 Learning

As in [LMN93], the Fourier concentration result yields a learning algorithm for Boolean functions f computable by small de Morgan formulas, where the learner is given labeled examples $(x, f(x))$ for the uniform distribution over inputs x . The learning algorithm produces a function g such that g agrees with f on almost all inputs x . The error of the learning algorithm (i.e., the fraction of inputs where g and f disagree) and its running time depend on the parameters of the Fourier concentration result for the corresponding model of computation.

Theorem 7.2. *Under the uniform distribution, one can learn, to within error $\exp(-n^{\Omega(\epsilon)})$ for any $0 < \epsilon < 1/2$, Boolean functions $f : \{0, 1\}^n \rightarrow \{1, -1\}$ computable by formulas of size s in time $\exp(s^{1/\Gamma + \epsilon})$, where $\Gamma = 2$ for general de Morgan formulas and $\Gamma \approx 3.27$ for read-once formulas.*

Proof sketch. The proof mimics the analogous result in [LMN93] for AC⁰ circuits. Namely, for the case of a de Morgan formula of size s on n inputs that computes a Boolean function f , we approximate f with the Fourier expansion truncated at the degree $d := s^{1/2 + \epsilon}$, denoted \tilde{f} . The normalized squared ℓ_2 -norm of the difference $\|f - \tilde{f}\|^2/2^n$ is by Theorem 5.1 at most $\gamma := \exp(-s^{\epsilon/3})$.

It is easy to see that the function $\text{sign}(\tilde{f})$, the sign of \tilde{f} , agrees with f on all but at most γ fraction of inputs in $\{0, 1\}^n$. We can learn \tilde{f} by estimating all Fourier coefficients $\hat{f}(A)$, for $|A| \leq d$, through random sampling (assuming the uniform distribution on the samples $(x, f(x))$). It follows that one can learn a function $\text{sign}(p)$, for some degree d multilinear polynomial p , that agrees with f in all but γ fraction of inputs, where the learning algorithm takes time polynomial in the number of Fourier coefficients of degree at most d , i.e., at most $\text{poly}(n^d) = \exp(s^{1/2+\epsilon} \cdot \log n)$.

For read-once de Morgan formulas, we proceed in the same way as above, but using Theorem 6.1. This gives us degree $d = n^{1/\Gamma+\epsilon}$, and hence, the running time of the learning algorithm $\text{poly}(n^d) = \exp(n^{1/\Gamma+\epsilon} \cdot \log n)$, as required. \square

As mentioned earlier, using the quantum-setting results on the sign degree of de Morgan formulas [Lee09], one gets a PAC-learning algorithm for size s de Morgan formulas that runs in time $n^{O(\sqrt{s})}$. This is better than our uniform-distribution learning algorithm of Theorem 7.2; however, our result is proved in the classical setting. On the other hand, for read-once de Morgan formulas, our learning algorithm appears to be the fastest known. Previously, a comparable running time $\approx \exp(n^{1/3})$ was known for (read-once) DNFs, albeit in the stronger setting of PAC learning [KS04].

7.3 Compression

Given the truth table of a function f computable by a de Morgan formula of size s on n inputs, we can compute in time $\text{poly}(2^n)$ all Fourier coefficients of f , and then define the approximation \tilde{f} that is the truncated version of the Fourier expansion of f of degree at most $d = s^{1/2+\epsilon}$. As above, the function $g := \text{sign}(\tilde{f})$ is the Boolean function that agrees with f on all but at most $\exp(-n^{\Omega(\epsilon)})$ fraction of inputs. The size of the circuit computing g is at most $\text{poly}(n^d)$, which is less than $2^n/n$ for $s < n^{2-2\epsilon}$, for $0 < \epsilon < 1/2$.

In the language of [KK13, CKK⁺13], this means that we have a deterministic *lossy-compression* algorithm for the class of de Morgan formulas of sub-quadratic size.¹ Similarly, we get, for read-once de Morgan formulas on n inputs, a lossy-compression algorithm producing a circuit of size at most $\exp(n^{1/3+\epsilon})$ which agrees with the formula on all but at most $\exp(-n^{\Omega(\epsilon)})$ fraction of inputs.

7.4 Average sensitivity

Recall that for a Boolean function $f : \{0, 1\}^n \rightarrow \{1, -1\}$ and a string $w \in \{0, 1\}^n$, the *sensitivity* of f at w is the number of Hamming neighbors w' of w such that $f(w) \neq f(w')$. The *average sensitivity* of f , denoted by $AS(f)$, is the average over all $w \in \{0, 1\}^n$ of the sensitivity of f at w . It is shown by [KKL88] that

$$AS(f) = \sum_{A \subseteq [n]} |A| \cdot \hat{f}(A)^2. \quad (4)$$

The parity function on m bits has average sensitivity m . Since a de Morgan formula of size s can compute the parity on $\Omega(\sqrt{s})$ bits, we get a lower bound $\Omega(\sqrt{s})$ on the average sensitivity of de Morgan formulas of size s . Combining the result of [Lee09] on the approximate degree of size s de Morgan formulas being $O(\sqrt{s})$, with the result of Shi [Shi00] that approximate degree upperbounds average sensitivity, we immediately get the matching $O(\sqrt{s})$ upper bound on the average sensitivity of size s de Morgan formulas. Ganor et al. [GKR12] give an alternative proof of this upper bound, using completely classical (non-quantum) arguments.

¹Using the quantum results for de Morgan formulas [Lee09], one gets a *lossless* compression algorithm for size s de Morgan formulas that produces a circuit of size $\exp(\sqrt{s} \log n)$, agreeing with the de Morgan formula on all inputs.

For *read-once* formulas of size s , Eq. (4) and Theorem 6.1 readily imply the upper bound $s^{1/\Gamma+o(1)}$ on average sensitivity, where $\Gamma = 1/\log_2(\sqrt{5} - 1) \approx 3.27$ is the shrinkage exponent for read-once formulas. However, a stronger upper bound can be shown; see Section C for the proof.

Theorem 7.3 (implicit in [Bop89]). *Let $f : \{0, 1\}^n \rightarrow \{1, -1\}$ be a Boolean function computed by a read-once de Morgan formula. Then $AS(f) \leq n^{1/\Gamma}$.*

Next we argue that the average sensitivity bound for read-once formulas in Theorem 7.3 is tight. We will need a variant of Valiant's read-once formula defined in [PZ93] (where it was used to argue that the shrinkage exponent for read-once formulas can be at most Γ). The formula is defined as follows. Let $\{r_n\}$ be a sequence of bits. Set $F_{1,0} = x_1 \wedge x_2$, and $F_{1,1} = x_1 \vee x_2$. For $n > 1$, define $F_{n,b} = F_{n-1,1-b}$ NAND F_{n-1,r_n} , for $b \in \{0, 1\}$. Here we use disjoint sets of variables for all of our subformulas.

Let $p_{n,b}$ denote the probability that $F_{n,b} = 1$ on a uniformly random input.

Lemma 7.4 ([PZ93]). *There exists a sequence $\{r_n\}$ such that*

1. $p_{n,0}, p_{n,1} \rightarrow \psi := \frac{\sqrt{5}-1}{2} \approx 0.62$, as $n \rightarrow \infty$,
2. $p_{n,0} < \psi < p_{n,1}$ for every $n \geq 1$, and
3. there exists a constant $c > 0$ such that for every $n \geq 1$, $p_{n,1} - p_{n,0} < c \cdot \psi^n$.

For $N = 2^n$, we define $F_N(x_1, \dots, x_N) := F_{n,0}(x_1, \dots, x_N)$, for $F_{n,0}$ given by Lemma 7.4. We have the following.

Theorem 7.5. *For all large enough $N = 2^n$, the read-once formula $F_N(x_1, \dots, x_N)$ is such that $AS(f_N) \geq \Omega(N^{1/\Gamma})$, where f_N is the $\{1, -1\}$ -valued Boolean function computed by the formula F_N .*

Proof. For an N -variate Boolean function f , $AS(f) = \mathbf{Exp}_{w \in \{0,1\}^N} \left[\sum_{i=1}^N \chi(i, w) \right]$, where $\chi(i, w) = 1$ if $f(w_1, \dots, w_{i-1}, x_i, w_{i+1}, \dots, w_N)$ depends on x_i , and 0 otherwise. By the linearity of expectation, we have

$$AS(f) = \sum_{i=1}^N \mathbf{Pr}_{w \in \{0,1\}^N} [f(w_1, \dots, w_{i-1}, x_i, w_{i+1}, \dots, w_N) \text{ depends on } x_i],$$

where the i th probability expression in the summation above is also known as the influence of coordinate i on f , denoted $\mathbf{Inf}_i[f]$. We will show for our read-once formulas F_N computing the Boolean functions f_N that, for each $1 \leq i \leq N$, $\mathbf{Inf}_i[f_N] \geq \Omega(N^{1/\Gamma-1})$, concluding the proof.

Consider an arbitrary leaf x_i , $1 \leq i \leq N$, of F_N . For the path in the formula from the leaf x_i to the root, all non-leaf nodes v_1, \dots, v_n are NAND gates, except for the gate v_1 closest to the leaf x_i which may be either AND or OR. For each node v_j , let G_{v_j} be the subformula of F_N corresponding to the node u feeding into the gate v_j such that u is not on the path v_1, \dots, v_n .

For a given assignment w to the $N-1$ variables $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N$, the restricted function is non-constant (i.e., depends on x_i) iff each formula G_{v_j} , for $2 \leq j \leq n$, evaluates to 1 under the assignment w , while G_{v_1} evaluates to 1 if v_1 is an AND gate, or to 0 if v_1 is an OR gate. Since these formulas G_{v_j} 's depend on disjoint sets of variables, we get that the probability over w that the variable x_j survives w is $(1/2) \cdot p_2 \cdots p_n$, where, for $2 \leq j \leq n$, $p_j = \mathbf{Pr}_z[G_{v_j}(z) = 1]$, for z is a uniformly random assignment to the variables of G_{v_j} .

For $2 \leq j \leq n$, we have by construction that $G_{v_j} = F_{j-1, b_j}$ for some $b_j \in \{0, 1\}$. Hence, for each $2 \leq j \leq n$, we get by Lemma 7.4 that $p_j \geq p_{j-1, 0}$. Thus, x_i survives a random assignment w with probability at least $(1/2) \prod_{j=1}^{n-1} p_{j, 0}$.

Let $d \in \mathbb{N}$ be a smallest constant such that $c \cdot \psi^d \leq 1/2$, where c and ψ are as in Lemma 7.4. Note that, for each $j > d$, we have by Lemma 7.4 that $p_{j, 0} > \psi - c\psi^j = \psi(1 - c\psi^{j-1})$. Thus, we get for some constant $\alpha > 0$ that

$$\begin{aligned} (1/2) \prod_{j=1}^{n-1} p_{j, 0} &= \alpha \cdot \prod_{j=d+1}^{n-1} p_{j, 0} \\ &\geq \alpha \cdot \prod_{j=d+1}^{n-1} \psi(1 - c \cdot \psi^{j-1}) \\ &= (\alpha/\psi^{d+1}) \cdot \psi^n \cdot \prod_{j=d+1}^{n-1} (1 - c \cdot \psi^{j-1}). \end{aligned}$$

Using the inequality $1 - y \geq e^{-2y}$ valid for all $0 \leq y \leq 1/2$, we get

$$\prod_{j=d+1}^{n-1} (1 - c \cdot \psi^{j-1}) \geq e^{-2c \sum_{j=d}^{n-2} \psi^j} \geq e^{-2c\psi^d/(1-\psi)},$$

which is some positive constant. Hence, the probability that x_i survives a random assignment w to the other variables of F_N is at least $\beta \cdot \psi^n$ for some constant $\beta > 0$. Finally, we have $\psi^n = 2^{n \log_2 \psi} = N^{\log_2(2\psi)-1} = N^{1/\Gamma-1}$, which concludes the proof. \square

Remark 7.6. As a consequence of Theorem 7.5, to achieve $\sum_{|A|>t} \hat{f}(A)^2 \leq 1/n$, for a Boolean function $f : \{0, 1\}^n \rightarrow \{1, -1\}$ computable by some read-once de Morgan formula, one needs $t \geq \Omega(n^{1/\Gamma})$. Thus, our Fourier concentration bound in Theorem 6.1 is close to optimal.

8 Open questions

Does k -wise independence ϵ -fool read-once formulas of size n for $k = O((\log 1/\epsilon) \cdot n^{1/\Gamma})$ where Γ is the shrinkage exponent for read-once formulas? Note that for general de Morgan formulas of size n , the corresponding statement follows from the quantum results on the approximate degree $O(\sqrt{s})$ [Lee09]. Observe that the approximate degree for read-once formulas of size n must be at least $n^{1/2}$ (the same as that for general de Morgan formulas of size n), and so one needs a different argument for showing such a k -wise independence result for read-once formulas.

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A Proof of Theorem 3.1

We first re-state Theorem 3.1.

Theorem A.1 ([LMN93]). *For arbitrary n -variate Boolean function f , integer $t > 0$ and a real number $0 < p < 1$ such that $pt \geq 8$,*

$$\sum_{|A|>t} \hat{f}(A)^2 \leq 2 \cdot \mathbf{Exp}_{\rho \in R_p} \left[\sum_{|B|>pt/2} \hat{f}_\rho(B)^2 \right].$$

Proof. We have

$$\sum_{|A|>t} \hat{f}(A)^2 \leq 2 \cdot \mathbf{Exp}_S \left[\sum_{A: |A \cap S| > pt/2} \hat{f}(A)^2 \right] \quad (5)$$

$$= 2 \cdot \mathbf{Exp}_{S, r \in \{0,1\}^{|S^c|}} \left[\sum_{B: |B| > pt/2} \hat{f}_{S^c \leftarrow r}(B)^2 \right] \quad (6)$$

$$= 2 \cdot \mathbf{Exp}_{\rho \in R_p} \left[\sum_{B: |B| > pt/2} \hat{f}_\rho(B)^2 \right], \quad (7)$$

where the first expectation is over random sets S obtained by choosing each item $i \in [n]$, independently, with probability p ; the second expectation is over S as before, and over uniformly random assignment r (for the variables outside of S).

The last equality, Eq. (7), is by definition. The second equality, Eq. (6), is proved in Lemma A.2 below. It remains to argue the first inequality, Eq. (5).

Consider any set A of size greater than t . It will contribute $\hat{f}(A)^2$ to the expectation over S for every random set S that intersects A in more than $pt/2$ locations. The expected intersection size between S and A (where each element $i \in [n]$ is put into S with probability p) is $p|A| > pt$. By Chernoff, almost all sets S will intersect the set A in at least half the expected number of places; by requiring that $pt \geq 8$, we get that this holds for at least half of all random sets S . Multiplying this expectation by 2 ensures that each $\hat{f}(A)^2$ is counted at least once. \square

Lemma A.2 ([LMN93]). *For a Boolean function f on n variables, an arbitrary subset $S \subseteq [n]$, and an integer k , we have*

$$\sum_{A: |A \cap S| > k} \hat{f}(A)^2 = \mathbf{Exp}_{r \in \{0,1\}^{|S^c|}} \left[\sum_{|B| > k} \hat{f}_{S^c \leftarrow r}(B)^2 \right]. \quad (8)$$

Proof. We start by re-writing the left-hand side of Eq. (8):

$$\sum_{A: |A \cap S| > k} \hat{f}(A)^2 = \sum_{B \subseteq S: |B| > k} \sum_{D \subseteq S^c} \hat{f}(B \cup D)^2. \quad (9)$$

For all sets $B \subseteq S$ and $D \subseteq S^c$, we have

$$\begin{aligned} \hat{f}(B \cup D) &= \mathbf{Exp}_{x \in \{0,1\}^n} [f(x) \cdot \chi_{B \cup D}(x)] \\ &= \mathbf{Exp}_{r \in \{0,1\}^{|S^c|}, r' \in \{0,1\}^{|S|}} [f_{S^c \leftarrow r}(r') \cdot \chi_{(B \cup D) \cap S}(r') \cdot \chi_{(B \cup D) \cap S^c}(r)] \\ &= \mathbf{Exp}_{r \in \{0,1\}^{|S^c|}} \left[\chi_D(r) \cdot \mathbf{Exp}_{r' \in \{0,1\}^{|S|}} [f_{S^c \leftarrow r}(r') \cdot \chi_B(r')] \right] \\ &= \mathbf{Exp}_{r \in \{0,1\}^{|S^c|}} \left[\chi_D(r) \cdot \hat{f}_{S^c \leftarrow r}(B) \right]. \end{aligned}$$

Therefore, for every fixed $B \subseteq S$, we get

$$\begin{aligned} \sum_{D \subseteq S^c} \hat{f}(B \cup D)^2 &= \sum_D \left(2^{-|S^c|} \cdot \sum_{r \in \{0,1\}^{|S^c|}} \chi_D(r) \cdot \hat{f}_{S^c \leftarrow r}(B) \right)^2 \\ &= 2^{-2|S^c|} \cdot \sum_{r_1, r_2 \in \{0,1\}^{|S^c|}} \hat{f}_{S^c \leftarrow r_1}(B) \hat{f}_{S^c \leftarrow r_2}(B) \cdot \sum_D \chi_D(r_1 \oplus r_2), \end{aligned}$$

where $r_1 \oplus r_2$ denotes the bit-wise XOR of the two strings. Observing that

$$\sum_{D \subseteq S^c} \chi_D(r) = \begin{cases} 2^{|S^c|} & \text{if } r \text{ is an all-zero string} \\ 0 & \text{otherwise} \end{cases},$$

we can continue the above sequence of equalities, getting the following:

$$\begin{aligned} \sum_{D \subseteq S^c} \hat{f}(B \cup D)^2 &= 2^{-|S^c|} \cdot \sum_{r \in \{0,1\}^{|S^c|}} \hat{f}_{S^c \leftarrow r}(B)^2 \\ &= \mathbf{Exp}_{r \in \{0,1\}^{|S^c|}} \left[\hat{f}_{S^c \leftarrow r}(B)^2 \right]. \end{aligned}$$

Finally, plugging in the last expression into the right-hand side of Eq. (9), we conclude

$$\begin{aligned} \sum_{A: |A \cap S| > k} \hat{f}(A)^2 &= \sum_{B \subseteq S: |B| > k} \mathbf{Exp}_{r \in \{0,1\}^{|S^c|}} \left[\hat{f}_{S^c \leftarrow r}(B)^2 \right] \\ &= \mathbf{Exp}_{r \in \{0,1\}^{|S^c|}} \left[\sum_{|B| > k} \hat{f}_{S^c \leftarrow r}(B)^2 \right], \end{aligned}$$

as required. □

B Proof of Lemma 4.6

We re-state Lemma 4.6.

Lemma B.1 ([IMZ12]). *For every $0 < p < 1$, there exists a read-once de Morgan formula H of size $O(1/p^4)$ such that, for all but at most $1/4$ of p -restrictions ρ , we have*

$$H_\rho(\vec{0}) = 0 \quad \text{and} \quad H_\rho(\vec{1}) = 1,$$

where $\vec{0}$ and $\vec{1}$ denote the inputs of all 0's and all 1's, respectively.

For a Boolean function $f(x_1, \dots, x_n)$ and a parameter $p \in [0, 1]$, Boppana [Bop89] defined the *amplification function* $A_f(p) := \mathbf{Pr}_{x_1, \dots, x_n} [f(x_1, \dots, x_n) = 1]$, where each x_i is chosen independently at random to be 1 with probability p and 0 otherwise. Boppana [Bop89] also observed that Valiant [Val84a] implicitly proved the following.²

Theorem B.2 ([Val84a]). *Let T_k be a complete binary tree of depth $2k$ whose root is labeled with OR, the next layer of nodes with AND, the next layer with OR, and so on in the alternating fashion for all layers but the leaves. Let F_k be the read-once formula computed by T_k on 2^{2k} variables. Then for $\psi = (\sqrt{5} - 1)/2$ and any $p \in [0, 1]$,*

$$A_{F_k}(\psi - (1 - \psi)p) < 1/8 \quad \text{and} \quad A_{F_k}(\psi + (1 - \psi)p) > 7/8,$$

for $2k = \log_{2\psi} \frac{\psi - 1/\sqrt{3}}{(1-\psi)p} + O(1) = \log_{2\psi}(1/p) + O(1)$. The size of F_k is $2^{2k} = O(1/p^{1/\log_2 2\psi}) = O(1/p^\Gamma)$, for $\Gamma = 1/\log_2(\sqrt{5} - 1) \approx 3.27$.

²See also the lecture notes by Uri Zwick, www.cs.tau.ac.il/~zwick/circ-comp-new/six.ps, for an explicit proof.

Proof of Lemma B.1. We use Theorem B.2 to argue the existence of the required read-once formula H . Consider the following distribution D_k on read-once formulas:

Take T_k . Independently, assign each leaf of T_k the value 1 with probability $2\psi - 1$, and $*$ otherwise. Label the $*$ leaves with distinct variables x_i 's. Output the resulting read-once formula in the variables x_i 's.

Let F be a random read-once formula sampled according to D_k . Let ρ be a random p -restriction on the variables of F . Consider $F_\rho(\vec{1})$. This restricted formula on the all-one input string induces the probability distribution on the leaves of T_k where each leaf, independently, gets value 1 with probability $2\psi - 1 + 2(1 - \psi)p + 2(1 - \psi)(1 - p)/2 = \psi + (1 - \psi)p$. Using Theorem B.2, we get

$$\Pr_{F \in D_k, \rho \in R_p}[F_\rho(\vec{1}) = 1] = A_{F_k}(\psi + (1 - \psi)p) > 7/8. \quad (10)$$

Now consider $F_\rho(\vec{0})$. It induces the probability distribution on the leaves of T_k where each leaf, independently, is 1 with probability $2\psi - 1 + 2(1 - \psi)(1 - p)/2 = \psi - (1 - \psi)p$, and 0 otherwise. Using Theorem B.2, we get

$$\Pr_{F \in D_k, \rho \in R_p}[F_\rho(\vec{0}) = 1] = A_{F_k}(\psi - (1 - \psi)p) < 1/8. \quad (11)$$

Using Eqs. (10) and (11), we get by the union bound that

$$\Pr_{F \in D_k, \rho \in R_p}[F_\rho(\vec{1}) = 0 \text{ or } F_\rho(\vec{0}) = 1] < 1/8 + 1/8 = 1/4.$$

Finally, by averaging, there exists a particular read-once formula $H \in D_k$ such that, for all but less than $1/4$ of random p -restrictions ρ , we have $H_\rho(\vec{0}) = 0$ and $H_\rho(\vec{1}) = 1$. The size of this formula H is at most that of F_k , which is $O(1/p^\Gamma) \leq O(1/p^4)$. \square

C Upper bound on the average sensitivity of read-once formulas

Here we present the proof of Theorem 7.3. We first re-state the theorem for $\{0, 1\}$ -valued Boolean functions; clearly this does not affect the average sensitivity.

Theorem C.1 (implicit in [Bop89]). *Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function computed by a read-once de Morgan formula. Then $AS(f) \leq n^{1/\Gamma}$.*

We again use Boppana's amplification function, A_f , mentioned in Section B. Here we use a slightly more general definition of A_f : for a Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ and parameters $p_1, \dots, p_n \in [0, 1]$, define the *amplification function* $A_f(p_1, \dots, p_n) := \Pr_{x_1, \dots, x_n}[f(x_1, \dots, x_n) = 1]$, where each x_i is chosen independently at random to be 1 with probability p_i , and 0 with probability $1 - p_i$. For $p \in [0, 1]$, define $A_f(p) := A_f(p, \dots, p)$.

Boppana [Bop89, Theorem 2.1] proved the following upper bound on the *derivative* of A_f .

Theorem C.2 ([Bop89]). *For any read-once formula f of size n and any $0 < p < 1$,*

$$A'_f(p) \leq n^{1/\Gamma} \cdot \frac{H(A_f(p))}{H(p)},$$

where $H(p) := -p \log_2 p - (1 - p) \log_2 (1 - p)$ is the binary entropy function, and $\Gamma = 1/\log_2(\sqrt{5} - 1)$.

Lemma C.3 (N. Saurabh, personal communication). *For every monotone n -variate Boolean function f , we have $AS(f) = A'_f(1/2)$.*

Proof. Observe that

$$A'_f(1/2) = \sum_{i=1}^n \frac{\partial A_f(p_1, \dots, p_n)}{\partial p_i} \Big|_{(1/2, \dots, 1/2)}.$$

On the other hand, using monotonicity of f , we will show that each i th summand on the right-hand side of the above formula is exactly equal to $\mathbf{Inf}_i[f]$, the influence of coordinate i on f . Since $AS(f) = \sum_{i=1}^n \mathbf{Inf}_i[f]$, the lemma will follow.

We have

$$\mathbf{Inf}_i[f] = \sum_{x \in \{0,1\}^n : (f(x)=1) \wedge (f(x^i)=0)} \frac{1}{2^{n-1}},$$

where x^i denotes x with the i th coordinate flipped. Write $A_f(p_1, \dots, p_n) = \sum_{x \in \{0,1\}^n : f(x)=1} P_x$, where for $x = (x_1, \dots, x_n)$, $P_x := \prod_{i=1}^n p_i^{x_i} (1-p_i)^{1-x_i}$ is the probability mass contributed by the point x . Observe that, for points x and x^i , the partial derivatives of P_x and P_{x^i} with respect to p_i cancel each other. Thus, the points x and x^i such that $f(x) = f(x^i) = 1$ contribute 0 to the partial derivative of A_f with respect to p_i . Each x such that $f(x) = 1$ but $f(x^i) = 0$ must have its i th coordinate $x_i = 1$ by the monotonicity of f . Hence, each such x will contribute $(1/p_i) \cdot \prod_{j=1}^n p_j^{x_j} (1-p_j)^{1-x_j}$ to the partial derivative of A_f with respect to p_i . When all $p_j = 1/2$, this contribution is exactly $1/2^{n-1}$. \square

We can now finish the proof of Theorem C.1.

Proof of Theorem C.1. Without loss of generality, a given read-once Boolean function f can be assumed monotone: we can always remove negations from any negative literals in the read-once formula f , without changing $AS(f)$. By Theorem C.2 and Lemma C.3, we get $AS(f) \leq n^{1/\Gamma}$. $H(A_f(1/2)) \leq n^{1/\Gamma}$. \square