

On r -Simple k -Path

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Abstract

An r -simple k -path is a path in the graph of length k that passes through each vertex at most r times. The r -SIMPLE k -PATH problem, given a graph G as input, asks whether there exists an r -simple k -path in G . We first show that this problem is NP-Complete. We then show that there is a graph G that contains an r -simple k -path and no simple path of length greater than $4 \log k / \log r$. So this, in a sense, motivates this problem especially when one's goal is to find a short path that visits many vertices in the graph while bounding the number of visits at each vertex.

We then give a randomized algorithm that runs in time

$$\text{poly}(n) \cdot 2^{O(k \cdot \log r / r)}$$

that solves the r -SIMPLE k -PATH on a graph with n vertices with one-sided error. We also show that a randomized algorithm with running time $\text{poly}(n) \cdot 2^{(c/2)k/r}$ with $c < 1$ gives a randomized algorithm with running time $\text{poly}(n) \cdot 2^{cn}$ for the Hamiltonian path problem in a directed graph - an outstanding open problem. So in a sense our algorithm is optimal up to an $O(\log r)$ factor.

1 Introduction

Let G be a directed graph on n vertices. A path ρ is called *simple* if all the vertices in the path are distinct. The SIMPLE k -PATH problem, given G as input, asks whether there exists a simple path in G of length k . This is a generalization of the well known HAMILTONIAN-PATH problem that asks whether there is a simple path passing through *all* vertices, i.e., a simple path of length n in G . As HAMILTONIAN-PATH is NP-complete, we do not expect to find polynomial time algorithms

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for SIMPLE k -PATH for general k . Moreover, we do not even expect to find good approximation algorithms for the corresponding optimization problem: the *longest path problem*, where we ask what is the length of the longest simple path in G . This is because Björklund et al. [6] showed that the longest path problem cannot be approximated in polynomial time to within a multiplicative factor of $n^{1-\epsilon}$, for any constant $\epsilon > 0$, unless $P=NP$. This motivated finding algorithms for SIMPLE k -PATH with running time whose dependence on k is as small as possible. The first result in this venue by Monien [16] achieved a running time of $k! \cdot \text{poly}(n)$. Since then, there has been extensive research on constructing algorithms for SIMPLE k -PATH running in time $f(k) \cdot \text{poly}(n)$, for a function $f(k)$ as small as possible [5, 2, 15, 10, 13]. The current state of the art is $2^k \cdot \text{poly}(n)$ by Williams [18] for directed graphs and $O(1.657^k) \cdot \text{poly}(n)$ by Björklund [7] for undirected graphs.

1.1 Our results

In this paper we look at a further generalization of SIMPLE k -PATH which we call r -SIMPLE k -PATH. In this problem instead of insisting on ρ being a simple path, we allow ρ to visit any vertex a fixed number of times. We now formally define the problem r -SIMPLE k -PATH.

Definition 1. Fix integers $r \leq k$. Let G be a directed graph.

- We say a path ρ in G is r -simple, if each vertex of G appears in ρ at most r times. Obviously, ρ is a simple path if and only if it is a 1-simple path.
- The r -SIMPLE k -PATH problem, given G as input, asks whether there exists an r -simple path in G of length k .

At first, one may wonder whether for some fixed $r > 1$, r -SIMPLE k -PATH always has a polynomial time algorithm. We show this is unlikely by showing that for any r , for some k r -SIMPLE k -PATH is NP-complete. See Theorem 4 in Section 3 for a formal statement and proof of this. Thus, as in the case of SIMPLE k -PATH, one may ask what is the best dependency of the running time on r and k that can be obtained in an algorithm for r -SIMPLE k -PATH.

Our main result is

Theorem 2. Fix any integers r, k with $2 \leq r \leq k$. There is a randomized algorithm running in time

$$\text{poly}(n) \cdot O\left(r^{\frac{2k}{r} + O(1)}\right) = \text{poly}(n) \cdot 2^{O(k \cdot \log r/r)}$$

solving r -SIMPLE k -PATH on a graph with n vertices with one-sided error.

One may ask how far from optimal is the dependency on k and r in Theorem 2. Theorem 4 implies that a running time of $\text{poly}(n) \cdot 2^{o(k/r)}$ would give an algorithm with running time $2^{o(n)}$ for HAMILTONIAN-PATH. Moreover, even a running time of $\text{poly}(n) \cdot 2^{c \cdot k/r}$, for a small enough constant $c < 1/2$, would

imply a better algorithm for HAMILTONIAN-PATH than those of [18, 7] which are the best currently known. So, in a sense our algorithm is optimal up to an $O(\log r)$ factor. We find closing this $O(\log r)$ gap, e.g. by a better reduction to HAMILTONIAN-PATH, or a better algorithm for r -SIMPLE k -PATH, to be an interesting open problem.

1.2 Finding a path with many distinct vertices

We give more motivation for the r -SIMPLE k -PATH problem. Suppose we are in a situation where we wish to find a relatively short path passing through many distinct vertices. Note that an r -simple path of length k must pass through at least k/r distinct vertices. Thus, in case, for example, a 2-simple path of length k exists, our algorithm for 2-SIMPLE k -PATH can be used to find a path of length k with at least $k/2$ distinct vertices in time $\text{poly}(n) \cdot 2^{k/2}$. One may ask how this would compare to the number of distinct vertices returned by the algorithms for SIMPLE k -PATH. We show there can be a large gap. Specifically, for any given k , we show there is a graph G where all simple paths are of length less than $4 \cdot \log k$, but G contains a 2-simple path of length k . See Theorem 7 for a precise statement.

2 Overview of the proof of Theorem 2

We give an informal sketch of Theorem 2. We are given a directed graph G on n vertices, and integers $r \leq k$. We wish to decide if G contains an r -simple path of length k . There are two main stages in our algorithm. The first is to reduce the task to another one concerning multivariate polynomials. This part, described below, is very similar to [1].

Reduction to a question about polynomials We want to associate our graph G with a certain multivariate polynomial p_G .

We associate with the i 'th vertex a variable x_i . The monomials of the polynomial will correspond to the paths of length k in G . So we have

$$p_G(\mathbf{x}) = \sum_{i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_k \in G} x_{i_1} \cdots x_{i_k},$$

where $i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_k \in G$ means that i_1, i_2, \dots, i_k is a directed path in G . An important issue is *over what field \mathbb{F} is p_G defined?* A central part of the algorithm is indeed choosing the appropriate field to work over. Another issue is how efficiently p_G can be evaluated? (Note that it potentially contains n^k different monomials.) Williams shows in [18] that using the adjacency matrix of G it can be computed in $\text{poly}(n)$ -time. See Section 5. For now, think of p_G as defined over \mathbb{Q} , i.e., having integer coefficients. It is easy to see that G contains an r -simple path of length k if and only if p_G contains a monomial such that the individual degrees of all variables are at most r . Let us call such a monomial

an r -monomial. Thus our task is reduced to checking whether a homogenous polynomial of degree k contains an r -monomial.

Checking whether p_G contains an r -monomial Let us assume in this overview for simplicity that $p = r + 1$ is prime. Let us view p_G as a polynomial over \mathbb{F}_p . One problem with doing this is that if we have p directed paths of length k passing through the same vertices in different order, this translates in p_G to p copies of the same monomial summing up to 0. To avoid this we need to look at a variant of p_G that contains auxiliary variables that prevent this cancelation. For details on this issue see [1] and Section 5. For this overview let us assume this does not happen. Recall that we have the equality $a^p = a$ for any $a \in \mathbb{F}_p$. Let us look at a monomial that is *not* an r -monomial, say $x_1^{r+1} \cdot x_2 = x_1^p \cdot x_2$. The equality mentioned implies this monomial is equivalent as a function from \mathbb{F}_p^n to \mathbb{F}_p to the monomial $x_1 \cdot x_2$. By the same argument, any monomial that is *not* an r -monomial will be ‘equivalent’ to one of smaller degree. More generally, p_G that is homogenous of degree k over \mathbb{Q} will be equivalent to a polynomial of degree smaller than k as a function from \mathbb{F}_p^n to \mathbb{F}_p if and only if it *does not* contain an r -monomial. Thus, we have reduced our task to the problem of *low-degree testing*. In this context, this problem is as follows: Given black-box access to a function $f : \mathbb{F}_p^n \rightarrow \mathbb{F}_p$ of degree at most k , determine whether it has degree exactly k or less than k , using few queries to the function. Here, for a function $f : \mathbb{F}_p^n \rightarrow \mathbb{F}_p$, by its degree we mean the total degree of the lowest-degree polynomial $p \in \mathbb{F}_p[x_1, \dots, x_n]$ representing it. Haramati, Shpilka and Sudan [12] gave an optimal solution (in terms of the number of queries) to this problem for prime fields. A little work is required to determine the exact running time of the test of [12] (in addition to the bound on the number of queries to f). See Section 6 for details. For details on dealing with the case that $r + 1$ is not prime, see Section 7.

3 Definitions and Preliminary Results

In this section we give some definitions and preliminary results that will be used throughout this paper.

Let $G(V, E)$ be a directed graph where V is the set of vertices and $E \subseteq V \times V$ the set of edges. We denote by $n = |V|$ the number of vertices in the graph and by $m = |E|$ the number of edges in the graph. A k -path or a *path of length k* is a sequence $\rho = v_1, \dots, v_k$ such that (v_i, v_{i+1}) is an edge in G for all $i = 1, \dots, k - 1$. A *path* is a k -path for some integer $k > 0$. A path ρ is called *simple* if all the vertices in the path are distinct. We say that a path ρ in G is r -*simple*, if each vertex of G appears in ρ at most r times. Obviously, a simple path is a 1-simple path.

Given as input a directed graph G on n vertices, the r -SIMPLE k -PATH problem asks for a given G whether it contains an r -simple path of length k . When $r = 1$ then the problem is called SIMPLE k -PATH. The r -SIMPLE PATH problem asks for a given G and integer k whether G contains an r -simple k -path of length k . The problem SIMPLE PATH is 1-SIMPLE PATH.

In this paper we will study the above problems.

The following result gives a reduction from r -SIMPLE k -PATH to SIMPLE k -PATH.

Lemma 3. *If r -SIMPLE k -PATH can be solved in time $T(r, k, n, m)$ then sr -SIMPLE k -PATH can be solved in time $T(r, k, sn, s^2m)$. In particular, If SIMPLE k -PATH can be solved in time $T(k, n, m)$ then r -SIMPLE k -PATH can be solved in time $T(k, rn, r^2m)$.*

Proof. Let G be a directed graph. Define the graph $G' = G \odot I_s$ where each vertex v in G is replaced with an independent set I_s of size s in G' with the vertices $v^{(1)}, \dots, v^{(s)}$. Each edge (u, v) in G is replaced by the edges $(u^{(i)}, v^{(j)})$, $1 \leq i, j \leq s$.

It is easy to see that there is a rs -simple k -path in G if and only if there is a r -simple k -path in G' . \square

We now show that r -SIMPLE PATH is NP-complete.

Theorem 4. *For any r the decision problem r -SIMPLE PATH is NP-complete.*

Proof. We will reduce deciding HAMILTONIAN-PATH on a graph of n vertices, to deciding r -SIMPLE $(2rn - n + 2)$ -PATH on a graph of $2 \cdot n$ vertices.

Given an input graph $G = (V, E)$ to HAMILTONIAN-PATH, we define a new graph $G' = (V', E')$ as follows. We let $V' = V \cup \bar{V}$, where $\bar{V} = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n\}$ and $E' = E \cup \bar{E}$ where

$$\bar{E} = \{(\bar{v}_i, v_i), (v_i, \bar{v}_i) | i \in [n]\}.$$

For $j \in [n]$, it will be convenient to denote by ρ_j , the path of length $2r - 1$ that begins at v_j , goes back and forth from v_j to \bar{v}_j and ends in v_j , i.e., $\rho_j \triangleq (v_j, \bar{v}_j, \dots, v_j, \bar{v}_j, v_j)$.

We make 2 observations.

1. If a vertex $\bar{v}_j \in \bar{V}$ appears r times in an r -simple path ρ then it must be the start or end vertex of ρ : To see this, note that by construction of G' , if \bar{v}_j is *not* the start or end vertex of ρ , visiting it r times requires visiting v_j $r + 1$ times.
2. Suppose ρ is an r -simple path that begins and ends in a vertex of V . If ρ visits a vertex $\bar{v}_j \in \bar{V}$ $r - 1$ times, then it must contain ρ_j as a subpath: To see this, note that as ρ does not start in \bar{v}_j , any visit to \bar{v}_j must have a visit to v_j before and after. The only way this would sum up to at most r visits in v_j is if these visits were continuous. In other words, only if ρ contains ρ_j .

We want to show that G contains a Hamiltonian path if and only if G' contains an r -simple path of length $2rn - n + 2$. Assume first that G contains a Hamiltonian path $v_{i_1} \cdot v_{i_2} \cdots v_{i_n}$. Define the path $\rho = \bar{v}_{i_1} \cdot \rho_{i_1} \cdot \rho_{i_2} \cdots \rho_{i_n} \cdot \bar{v}_{i_n}$. It is of length

$$n \cdot (2r - 1) + 2 = 2rn - n + 2,$$

and it is r -simple.

Now assume that we have an r -simple path ρ in G' of length $2nr - n + 2$. We first claim that ρ must start and end with a vertex from \bar{V} : Otherwise, using the first observation above, ρ contains at most $n + 1$ vertices appearing r times, and thus has length at most

$$(n + 1) \cdot r + (n - 1) \cdot (r - 1) = 2rn - n + 1.$$

Let ρ' be the path ρ with the first and last vertex deleted. So ρ' has length $2rn - n$ and begins and ends in a vertex of V . Note that by the first observation ρ' visits all vertices of \bar{V} at most $r - 1$ times. We now claim that for every $j \in [n]$, ρ' must contain ρ_j as a subpath. Otherwise, by the second observation, ρ' visits *some* vertex of \bar{V} *less* than $r - 1$ times. In this case it has length less than $n \cdot r + n \cdot (r - 1) = 2nr - n$. A contradiction. Thus ρ' contains every ρ_j as a subpath. It cannot contain anything else ‘between’ the ρ_j ’s, as then it would visit some vertex of V *more* than r times. So

$$\rho' = \rho_{i_1} \cdots \rho_{i_n},$$

for some ordering i_1, \dots, i_n of $[n]$. It follows that $v_{i_1} \cdots v_{i_n}$ is a Hamiltonian path in G . \square

The above result implies

Corollary 5. *If r -SIMPLE k -PATH can be solved in $T(r, k, n, m)$ time then HAMILTONIAN-PATH can be solved in $T(r, 2rn - n + 2, 2n, m + 2n)$.*

In particular, if there is an algorithm for r -SIMPLE k -PATH that runs in time $\text{poly}(n) \cdot 2^{(c/2)(k/r)}$ then there is an algorithm for HAMILTONIAN-PATH that runs in time $\text{poly}(n) \cdot 2^{cn}$.

4 Gap

In this section we show that the gap between the longest simple path and the longest r -simple path can be exponentially large even for $r = 2$.

We first give the following lower bound for the gap

Theorem 6. *If G contains an r -simple path of length k then G contains a simple path of length $\lfloor \frac{\log k}{\log r} \rfloor$.*

Proof. Let $t = \lfloor \frac{\log k - 1}{\log r} \rfloor$. Let ρ be an r -simple path whose first vertex is v_0 . We will use ρ to construct a simple path $\bar{\rho}$ of length $\lfloor \frac{\log k}{\log r} \rfloor$. We denote $\rho_0 = \rho$. As v_0 appears at most r times in ρ_0 , there must be a subpath ρ_1 of ρ_0 of length at least $(k - r)/r$ where v_0 does not appear. Let v_1 be the first vertex of ρ_1 . Similarly, for $1 < i \leq t$, we define the subpath ρ_i of ρ_{i-1} to be a subpath of length at least

$$(k - r - \dots - r^i)/r^i \geq (k - r^{i+1})/r^i,$$

where v_1, \dots, v_{i-1} do not appear, and define v_i to be the first vertex of ρ_i . Note that we can always assume there is an edge from v_{i-1} to v_i as we can start ρ_i just after an appearance of v_{i-1} in ρ_{i-1} . Note that for $1 \leq i \leq t$, such a v_i as defined indeed exists as $(k - r^{i+1})/r^i \geq 1$ when

$$k \geq 2 \cdot r^{i+1} \leftrightarrow i + 1 \leq (\log k - 1) / \log r$$

Thus, $v_0 \cdots v_{t-1}$ is a simple path of the desired length. \square

Before we give the upper bound we give the following definition. A *full r -tree* is a tree where each vertex has r children and all the leaves of the tree are in the same level. The root is on level 1.

Theorem 7. *There is a graph G that contains an r -simple path of length k and no simple path of length greater than $4 \log k / \log r$.*

Proof. We first give the proof for $r \geq 3$. Consider a full $(r-1)$ -tree of depth $\lceil \log n / \log(r-1) \rceil$. Remove vertices from the lowest level (leaves) so the number of vertices in the graph is n . Obviously there is an r -simple path of length $k \geq n$. Any simple tour in this tree must change level at each step and if it changes from level ℓ to level $\ell + 1$ it cannot go back in the following step to level ℓ . So the longest possible simple path is $2 \lceil \log n / \log(r-1) \rceil - 2 \leq 3.17(\log k / \log r)$.

For $r = 2$ we take a full binary tree (2-tree) and add an edge between every two children of the same vertex. The 2-simple path starts from the root v , recursively makes a tour in the left tree of v then moves to the root of the right tree of v (via the edge that we added) then recursively makes a tour in the right tree of v and then visit v again. Obviously this is a 2-simple path of length $k > n$. A simple tour in this graph can stay in the same level only twice, can move to a higher level or can move to a lower level. Again here if it moves from level ℓ to $\ell + 1$ it cannot go back in the following step to level ℓ . Therefore the longest simple path is of length at most $4 \log n \leq 4 \log k$. \square

5 From r -Simple k -Path to Multivariate Polynomial

The purpose of this section is to reduce the question of whether a graph G contains an r -simple k -path, to that of whether a certain multivariate polynomial *contains an r -monomial*, as defined below.

Definition 8 (*r -monomial*). *Fix a field \mathbb{F} . Fix a monomial $M = M(\mathbf{z}) = z_1^{i_1} \cdots z_t^{i_t}$.*

- *We say M is an r -monomial if no variable appears with degree larger than r in M . That is, for all $1 \leq j \leq t$, $i_j \leq r$.*
- *Let $f(\mathbf{z})$ be a multivariate polynomial over \mathbb{F} . We say f contains an r -monomial, if there is an r -monomial $M(\mathbf{z})$ appearing with a nonzero coefficient $c \in \mathbb{F}$ in f .*

We now describe this reduction.

Let $G(V, E)$ be a directed graph where $V = \{1, 2, \dots, n\}$. Let A be the adjacency matrix and B be the $n \times n$ matrix such that $B_{i,j} = x_i \cdot A_{i,j}$ where $x_i, i = 1, \dots, n$ are indeterminates. Let $\mathbf{1}$ be the row n -vector of 1s and $\mathbf{x} = (x_1, \dots, x_n)^T$. Consider the polynomial $p_G(\mathbf{x}) = \mathbf{1} \cdot B^{k-1} \cdot \mathbf{x}$. It is easy to see

$$p_G(\mathbf{x}) = \sum_{i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_k \in G} x_{i_1} \cdots x_{i_k}$$

where $i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_k \in G$ means that i_1, i_2, \dots, i_k is a directed path in G .

Obviously, for field of characteristic zero there is an r -simple k -path if and only if $p_G(\mathbf{x})$ contains an r -monomial. For other fields the later statement is not true. For example, in undirected graph, $k = 2$, and $r = 1$ if $(1, 2) \in E$ and the field is of characteristic 2 then the monomial $x_1 x_2$ occurs twice and will vanish in $p_G(\mathbf{x})$. We solve the problem as follows.

Let $B^{(m)}$ be an $n \times n$ matrices, $m = 2, \dots, k$, such that $B_{i,j}^{(m)} = x_i \cdot y_{m,i} \cdot A_{i,j}$ where x_i and $y_{m,i}$ are indeterminates. Let, $\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_k)$ and $\mathbf{y}_m = (y_{m,1}, \dots, y_{m,n})$. Let $\mathbf{x} \cdot \mathbf{y} = (x_1 y_{1,1}, \dots, x_n y_{1,n})$. Consider the polynomial $P_G(\mathbf{x}, \mathbf{y}) = \mathbf{1} B^{(k)} B^{(k-1)} \dots B^{(2)}(\mathbf{x} \cdot \mathbf{y})$. It is easy to see that

$$P_G(\mathbf{x}, \mathbf{y}) = \sum_{i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_k \in G} x_{i_1} \cdots x_{i_k} y_{1,i_1} \cdots y_{k,i_k}$$

Obviously, no two paths have the same monomial in P_G . Note that as P_G contains only $\{0, 1\}$ coefficients, we can define it over any field \mathbb{F} . It will actually be convenient to view it as a polynomial $P_G(\mathbf{x})$ whose coefficients are in the field of rational functions $\mathbb{F}(\mathbf{y})$. Therefore, for any field, there is an r -simple k -path if and only if $P_G(\mathbf{x}, \mathbf{y})$ contains an r -monomial in \mathbf{x} . We record this fact in the lemma below.

Lemma 9. *Fix any field \mathbb{F} . The graph G contains an r -simple k -path if and only if the polynomial P_G , defined over $\mathbb{F}(\mathbf{y})$, contains an r -monomial $M(\mathbf{x})$.*

6 Low Degree Tester

In this section we present a tester that determines whether a function $f : \mathbb{F}_p^n \rightarrow \mathbb{F}_p$ of degree *at most* d has, in fact, degree *less than* d . The important point is that the tester will be able to do this using few black-box queries to f . The results of this section essentially follow from the work of Haramaty, Sudan and Shpilka [12].

First, let us say precisely what we mean by the *degree* of a function $f : \mathbb{F}_p^n \rightarrow \mathbb{F}_p$. We define this to be the degree of the lowest degree polynomial $f' \in \mathbb{F}_p[\mathbf{x}]$ that agrees with f as a function from \mathbb{F}_p^n to \mathbb{F}_p . It is known from the theory of finite fields that there is a unique such f' , and that the individual degrees of all variables in f' are smaller than p . Moreover, given any polynomial $g \in \mathbb{F}_p[\mathbf{x}]$ agreeing with f as

a function from \mathbb{F}_p^n to \mathbb{F}_p , f' can be derived from g by replacing, for any $1 \leq i \leq n$, occurrences of x_i^t with $x_i^t \bmod x_i^p - x_i$ (i.e., $x_i^{((t-1) \bmod (p-1))+1}$ when $t \neq 0$). We do not prove these basic facts formally here. They essentially follow from the fact that $a^p = a$ for $a \in \mathbb{F}_p$.

This motivates the following definition.

Definition 10. Fix positive integers n, d and a prime p . Let $f \in \mathbb{F}_p[\mathbf{x}] = \mathbb{F}_p[x_1, \dots, x_n]$. We define $\deg_p(f)$ to be the degree of the polynomial f when replacing, for $1 \leq i \leq n$, x_i^t by $(x_i^t \bmod x_i^p - x_i)$. More formally, $\deg_p(f) \triangleq \deg(f')$ where

$$f'(x_1, \dots, x_n) \triangleq f(x_1, \dots, x_n) \bmod x_1^p - x_1, \dots, \bmod x_n^p - x_n.$$

Moreover, for a function $g : V \rightarrow \mathbb{F}_p$ where $V \subseteq \mathbb{F}_p^n$ is a subspace of dimension k , we define $\deg_p(g) = \min_f \deg_p(f)$ where $f \in \mathbb{F}_p[x_1, \dots, x_n]$ and $f|_V = g$. Here g can be regarded as a function in $\mathbb{F}_p[x_1, \dots, x_k]$.

We note that this notion of degree is affine invariant, i.e. does change after affine transformations. In addition it has the property that for any affine subspace V , $\deg_p(f|_V) \leq \deg_p(f)$.

We now present the main result of this section.

Lemma 11. There is a randomized algorithm A running in time $\text{poly}(n) \cdot p^{\lceil \frac{d}{p-1} \rceil + 1}$ that determines with constant one-sided error whether a function f of degree at most d has degree less than d . More precisely, given black-box access to a function $f : \mathbb{F}_p^n \rightarrow \mathbb{F}_p$ with $\deg_p(f) \leq d$,

- If $\deg_p(f) = d$, A accepts with probability at least $99/100$.
- If $\deg_p(f) < d$, A rejects with probability one.

The result essentially follows from the work of Haramaty, Shpilka and Sudan [12]. A technicality is to analyze the precise running time, and not just the query complexity as in [12]

Before proving Lemma 11, we state some required preliminary lemmas.

Lemma 12. Suppose we have black-box access to a function $f : \mathbb{F}_p^t \rightarrow \mathbb{F}_p$. Then we can determine in deterministic time $O(p^t)$ whether $\deg_p(f) \geq (p-1) \cdot t$.

Proof. Consider the algorithm that yields a positive answer if and only if $\sum_{\mathbf{a} \in \mathbb{F}_p^t} f(\mathbf{a}) = 0$. It is clear that the running time is indeed $O(p^t)$. Let us now show correctness. As in Definition 10, define

$$f'(x_1, \dots, x_n) \triangleq f(x_1, \dots, x_n) \bmod x_1^p - x_1, \dots, \bmod x_n^p - x_n,$$

so that $\deg_p(f) = \deg(f')$. We show that

1. The only monomial of degree $\geq t(p-1)$ in f' is $M_{\max} \triangleq \prod_{i=1}^t x_i^{p-1}$ and
2. the coefficient of M_{\max} in f' , is $(-1)^t \cdot \sum_{\mathbf{a} \in \mathbb{F}_p^t} f(\mathbf{a})$.

From these two items, it is clear that indeed $\deg_p(f) = \deg(f') \geq t \cdot (p-1)$ if and only if $\sum_{\mathbf{a} \in \mathbb{F}_p^t} f(\mathbf{a}) \neq 0$.

The first item is obvious, as the individual degrees in f' are at most $p-1$.

For the second item, let us calculate the coefficient of M_{\max} in f' . For every $\mathbf{a} \in \mathbb{F}_p^t$, consider the function $g_{\mathbf{a}} : \mathbb{F}_p^t \rightarrow \mathbb{F}_p$ that is one on \mathbf{a} and zero elsewhere. One can verify that $g_{\mathbf{a}}(\mathbf{x}) = \prod_{i=1}^t \frac{\prod_{\alpha \in \mathbb{F}_p \setminus \{\mathbf{a}_i\}} (\mathbf{x}_i - \alpha)}{\prod_{\beta \in \mathbb{F}_p \setminus \{0\}} \beta}$. Clearly, the coefficient of M_{\max} in $g_{\mathbf{a}}$ is $(\prod_{\beta \in \mathbb{F}_p \setminus \{0\}} \beta)^{-t} = (-1)^t$. Note that in $g_{\mathbf{a}}$, all individual degrees are smaller than p . Hence, $f' = \sum_{\mathbf{a} \in \mathbb{F}_p^t} f(\mathbf{a}) \cdot g_{\mathbf{a}}$ and the coefficient of M_{\max} in f' is $(-1)^t \cdot \sum_{\mathbf{a} \in \mathbb{F}_p^t} f(\mathbf{a})$. \square

The algorithm for Lemma 11 checks the degree of the function only on a small subspace. The key for its correctness is to show that when you restrict the function to a subspace (even for $n-1$ dimensional subspace) the degree does not decrease with high probability. The Lemma appeared in [12]. We give a proof sketch here for completeness

Lemma 13 (Theorem 1.5 in [12]). *Let \mathbb{F}_p be a field of prime size p and $f : \mathbb{F}_p^n \rightarrow \mathbb{F}_p$ be a function with $\deg_p(f) = t(p-1)$. The number of hyperplanes H such that $\deg_p(f|_H) < t(p-1)$ is at most p^{t+1}*

Proof sketch. We will assume w.l.o.g that f has the monomial $\prod_{i=1}^t x_i^{p-1}$. One can show that for any degree $t(p-1)$ polynomial f there is linear transformation A such that $f(Ax)$ has the monomial $\prod_{i=1}^t x_i^{p-1}$. So it will be enough to prove the lemma for the suitable transformation of f .

We will assume for simplicity that all the hyperplanes are of the form of $H_{\alpha} = \{x \mid x_1 = \sum_{i=2}^n \alpha_i x_i + \alpha_0\}$ for some $\alpha_2, \dots, \alpha_n$. Indeed, there are few more hyperplanes that does not depend on the first coordinate, but they don't contribute much to the upper bound.

To prove the lemma we will show that for any of the p^t possible values for $\alpha_2, \dots, \alpha_t, \alpha_0$ there are $< p$ possibilities for $\alpha_{t+1}, \dots, \alpha_n$ such that $\deg(f|_{H_{\alpha}}) < t(p-1)$. Fix $\alpha_2, \dots, \alpha_t, \alpha_0$. For simplicity we assume they are all zero, but the same bound goes for any $\alpha_2, \dots, \alpha_t, \alpha_0$ (one can reduce the general case to the zero case by some affine transformation).

Now consider all the monomials M in f with the following properties: (1) M divides $\prod_{i=2}^t x_i^{p-1}$ and (2) $\deg(M) = t(p-1)$. We can write the sum of all those monomials as $\prod_{i=2}^t x_i^{p-1} g(x_1, x_{t+1}, \dots, x_n)$. By definition, g is homogenous polynomial of degree $p-1$. Because $\prod_{i=1}^t x_i^{p-1}$ is a monomial of f , x_1^{p-1} is a monomial of g .

Because the hyperplanes does not depend on the variables x_2, \dots, x_t (recall, we assumed $\alpha_2 = \dots = \alpha_t = 0$) the degree of f can decrease on H_{α} only if the degree of g decreases on H_{α} . Because g is homogenous of degree $p-1$ and we consider only linear hyperplanes of the form $x_1 = L(x_{t+1}, \dots, x_n)$,

then $g|_{x_1=L}$ is still homogenous of degree $p-1$, so if the degree $\deg(g|_{x_1=L}) < p-1$ then $g|_{x_1=L} \equiv 0$. Now consider g as an univariate polynomial in x_1 over the field of rational functions in x_{t+1}, \dots, x_n . In this view our question is: how many field elements $L \in \mathbb{F}_p(x_{t+1}, \dots, x_n)$ are there such that $g(L) = 0$. From the fundamental theorem of the algebra the answer is $p-1$ and we are done. \square

From Lemma 13 we get the following corollary.

Corollary 14. *Let $n > t$ and $f : \mathbb{F}_p^n \rightarrow \mathbb{F}_p$ be a polynomial such that $\deg_p(f) = t(p-1)$. Then $\Pr_V [\deg_p(f|_V) = t(p-1)] \geq \frac{1}{p+1} \prod_{k=1}^{n-t-1} (1-p^{-k}) = \Omega(\frac{1}{p})$, where V is a random t -dimensional affine subspace.*

Proof. We proceed by induction on n . Consider first the base case, where $n = t+1$. In this case the number of t -dimensional affine subspaces $V \subseteq \mathbb{F}_p^{t+1}$ is $\frac{p^{t+2}-1}{p-1} > p^{t+1} + p^t$. By Lemma 13 on at most p^{t+1} of them $\deg(f|_V) < t(p-1)$ so the probability that $\deg(f|_V) = t(p-1)$ is $\frac{1}{p+1}$ as required.

Now assume the claim is true for $n-1$, and consider the following way of choosing a random t -dimensional affine subspace V . First choose a random hyperplane $H \subseteq \mathbb{F}_p^n$ and then choose a random t -dimensional affine subspace $V \subseteq H$. There are more than p^n hyperplanes $H \subseteq \mathbb{F}_p^n$, so by Lemma 13 the probability that $\deg_p(f|_H) = t(p-1)$ is at least $1 - p^{t+1-n}$. Moreover, in the event that $\deg_p(f|_H) = t(p-1)$, we can apply the induction hypothesis to $f|_H$. Hence,

$$\begin{aligned} \Pr [\deg_p(f|_V) = t(p-1)] &= \Pr [\deg_p((f|_H)|_V) = t(p-1) \mid \deg_p(f|_H) = t(p-1)] \cdot \Pr [\deg_p(f|_H) = t(p-1)] \\ &\leq \frac{1}{p+1} \prod_{k=1}^{(n-1)-t-1} (1-p^{-k}) \cdot (1-p^{t+1-n}) = \frac{1}{p+1} \prod_{k=1}^{n-t-1} (1-p^{-k}) \end{aligned}$$

\square

We are now ready to prove Lemma 11.

Proof of Lemma 11. Let $t = \lceil \frac{d}{p-1} \rceil$. We assume without loss of generality that $d = t(p-1)$: Otherwise, let $a = t(p-1) - d$ and consider the function $f'(x_0, x_1, \dots, x_n) \triangleq x_0^a \cdot f(x_1, \dots, x_n)$. It is easily checked that $\deg_p(f') \leq t(p-1)$. Also $\deg_p(f') = t(p-1)$ if and only if $\deg_p(f) = d$.

We will present an algorithm for the problem with one sided error probability $1 - \Omega(\frac{1}{p})$ that runs in time $\text{poly}(n) \cdot O(p^t)$. By repeating it $O(p)$ times, we can get down to error probability $1/100$ in running time $\text{poly}(n) \cdot O(p^{t+1})$ as required.

Consider the following algorithm. Choose a random t -dimensional affine subspace V . Accept if and only if the $\deg_p(f|_V) < t(p-1)$. Assume first that $\deg_p(f) < t(p-1)$. Then for any affine subspace V ,

$\deg_p(f|_V) \leq \deg_p(f) < t(p-1)$. On the other hand, if $\deg_p(f|_V) = t(p-1)$, Corollary 14 implies we will accept with probability at least $\Omega(\frac{1}{p})$.

We conclude by analyzing the running time. Choosing V can be done in $\text{poly}(n)$ -time. For checking whether $\deg_p(f|_V) = t(p-1)$, Lemma 12 gives running $O(p^t)$ assuming black-box access to $f|_V$. Given black-box access to f , we can compute $f|_V(\mathbf{a})$ for $\mathbf{a} \in \mathbb{F}_p^t$ in $\text{poly}(n)$ -time. The claimed running time of $\text{poly}(n) \cdot O(p^t)$ follows. \square

7 Testing if P_G contains an r -monomial

In this section we present a method for testing whether the polynomial P_G , described in Section 5, contains an r -monomial. This is done using the low-degree tester from the previous section.

As stated in Lemma 9, this is precisely equivalent to whether G contains an r -simple k -path. Recall we viewed P_G as a polynomial over a field of rational functions $\mathbb{F}_p(\mathbf{y})$. To obtain efficient algorithms, we first reduce the question to checking whether a different polynomial defined over \mathbb{F}_p rather than $\mathbb{F}_p(\mathbf{y})$ contains an r -monomial. It is important in the next Lemma that we are able to do this reduction for *any* p , in particular a ‘small’ one.

Lemma 15. *Fix any integers r, k , with $r \leq k$. Let p be any prime and $t = \lceil \log_p 10k \rceil$. Let G be a directed graph on n vertices. Given an adjacency matrix A_G for G , we can return in $\text{poly}(n)$ -time $\text{poly}(n)$ -size circuits computing polynomials $f_G^1, \dots, f_G^t : \mathbb{F}_p^n \rightarrow \mathbb{F}_p$ on inputs in \mathbb{F}_p^n such that*

- For $1 \leq i \leq t$, f_G^i is (either the zero polynomial or) homogenous of degree k .
- If G contains an r -simple k -path then with probability at least $9/10$, for some $1 \leq i \leq t$, f_G^i contains an r -monomial.
- If G does not contain an r -simple k -path, for all $1 \leq i \leq t$, f_G^i does not contain an r -monomial.

Proof. Note that the discussion in Section 5 implies we can compute P_G in $\text{poly}(n)$ -time over inputs in \mathbb{F}_p^{2n} . We choose random $\mathbf{b} \in \mathbb{F}_p^n$ and let

$$f_G(\mathbf{x}) \triangleq P_G(\mathbf{x}, \mathbf{b}).$$

Suppose P_G , as a polynomial over $\mathbb{F}(\mathbf{y})$, contains an r -monomial $M'(\mathbf{x})$. The coefficient $c_{M'}(\mathbf{y})$ of M' in P_G is a nonzero polynomial of degree k . So, by the Schwartz-Zippel Lemma, $c_{M'}(\mathbf{b}) = 0$ with probability at most $k/p^t \leq 1/10$. In the event that $c_{M'}(\mathbf{b}) \neq 0$, $f_G(\mathbf{x})$ is a homogenous polynomial of degree k in $\mathbb{F}_{p^t}[\mathbf{x}]$ containing an r -monomial. Let us assume from now on, we chose a \mathbf{b} such that indeed $a_{M'} \triangleq c_{M'}(\mathbf{b}) \neq 0$. We now discuss how to end up with polynomials having coefficients in \mathbb{F}_p rather than \mathbb{F}_{p^t} .

Let $T_1, \dots, T_t : \mathbb{F}_p^t \rightarrow \mathbb{F}_p$ be independent \mathbb{F}_p -linear maps. Suppose $f_G = \sum_M a_M \cdot M(\mathbf{x})$. For $1 \leq i \leq t$, define a polynomial $f_G^i \in \mathbb{F}_p[\mathbf{x}]$ by

$$f_G^i(\mathbf{x}) \triangleq \sum_M T_i(a_M) \cdot M(\mathbf{x}).$$

Note that for all $1 \leq i \leq t$, f_G^i is the zero polynomial or homogenous of degree k . As $a_{M'} \neq 0$, for some i , $T_i(a_{M'}) \neq 0$. For this i , f_G^i is homogenous of degree k and contains an r -monomial, specifically, the r -monomial $a_{M'} \cdot M'(\mathbf{x})$. We claim that for all $1 \leq i \leq t$, f_G^i can be computed by a $\text{poly}(n)$ -size circuit on inputs $\mathbf{a} \in \mathbb{F}_p^n$. This is because f_G and T_i are efficiently computable, and because for $\mathbf{a} \in \mathbb{F}_p^n$,

$$T_i(f_G(\mathbf{a})) = T_i\left(\sum_M a_M \cdot M(\mathbf{a})\right) = \sum_M T_i(a_M) \cdot M(\mathbf{a}) = f_G^i(\mathbf{a}),$$

where the second equality is due to the \mathbb{F}_p -linearity of T_i . □

The above lemma implies

Corollary 16. *Fix any prime p . Suppose that given black-box access to a polynomial $g \in \mathbb{F}_p[\mathbf{x}]$ that is homogenous of degree k , we can determine in time $\text{poly}(n) \cdot S$ if it contains an r -monomial. Then we can also determine in time $\text{poly}(n) \cdot S$ whether P_G as a polynomial over $\mathbb{F}_p(\mathbf{y})$ contains an r monomial.*

Our reduction to low-degree testing is based on the following simple observation that, for the right p and for homogenous polynomials, containing an r -monomial is equivalent to having a certain \deg_p -degree.

Lemma 17. *Suppose $g \in \mathbb{F}_p[\mathbf{x}]$ is a homogenous polynomial of degree k . Suppose $r = p - 1$. Then $\deg_p(g) = k$ if and only if g contains an r -monomial.*

Proof. If g contains an r -monomial M then, as $r < p$, $\deg_p(M) = k$, which implies that $\deg_p(g) = k$. If g does not contain an r -monomial, then for every monomial M in g there is an $i \in [n]$ such that the degree of x_i in M is at least $r + 1 = p$. So replacing x_i^p by x_i will reduce the degree of M and therefore $\deg_p(M) < k$. Since this happens for all monomials of g , $\deg_p(g) < k$. □

We introduce another element on notation that will be convenient in the rest of this section.

Definition 18. *Fix integers n, d and prime p . Let $f \in \mathbb{F}_p[\mathbf{x}]$ be an n -variate polynomial of degree at most d . We define $LDT(f, n, d, p)$ to be 1 if $\deg_p(f) = d$, and 0 otherwise.*

Before proceeding, we note that the results of Section 6 imply that given n, d, p and black-box access to f , $LDT(f, n, d, p)$ can be computed in time $\text{poly}(n) \cdot O(p^{\lceil d/(p-1) \rceil + 1})$. In particular, if given $\mathbf{a} \in \mathbb{F}_p^n$, we can compute $f(\mathbf{a})$ in $\text{poly}(n)$ -time, then we can compute $LDT(f, n, d, p)$ in time $\text{poly}(n) \cdot O(p^{\lceil d/(p-1) \rceil + 1})$. The following lemma is an easy corollary of Lemma 17.

Lemma 19. Fix integers r, k with $r \leq k$. Suppose $p = r + 1$ is prime. Let $g \in \mathbb{F}_p[\mathbf{x}]$ be homogenous of degree k and computable in $\text{poly}(n)$ -time. There is a randomized algorithm running in time

$$\text{poly}(n) \cdot O((r + 1)^{\lceil \frac{k}{r} \rceil + 1})$$

determining whether g contains an r -monomial.

Proof. The algorithm simply returns $LDT(g, n, d = k, p = r + 1)$. The running time follows from the discussion above. The correctness follows from Lemma 17. \square

We wish to have a similar result when $r + 1$ is not a prime.

Lemma 20. Fix integers r, k with $r \leq k$. Let p be the smallest prime such that $\frac{p-1}{r} \in \mathbb{Z}$. Let $g \in \mathbb{F}_p[\mathbf{x}]$ be homogenous of degree k and computable by a $\text{poly}(n)$ -size circuit. There is a randomized algorithm running in time $\text{poly}(n) \cdot O(p^{\lceil \frac{k}{r} \rceil + 1})$ determining whether g contains an r -monomial.

Proof. Denote $l \triangleq \frac{p-1}{r}$ and define

$$h(x_1, x_2, \dots, x_n) := g(x_1^l, x_2^l, \dots, x_n^l).$$

The algorithm returns $LDT(h, n, d = k \cdot l, p)$.

Note that h is homogenous of degree $k \cdot l$. Note also that h contains an $r \cdot l$ -monomial if and only if g contains an r -monomial. As $r \cdot l + 1 = p$ correctness now follows from Lemma 17. \square

The best known bound for the smallest prime number p that satisfies $r|p-1$ is $r^{5.5}$ due to Heath-Brown [17]. This gives a randomized algorithm running in time

$$\text{poly}(n) \cdot O(r^{\frac{5.5k}{r} + O(1)}).$$

Schinzel, Sierpinski, and Kanold have conjectured the value to be 2 [17]. In the following Theorem we give a better bound. We first give the following

Lemma 21. Fix integers r, k with $r \leq k$. Let p be the smallest prime such that there is an $l \in \mathbb{Z}$ for which $r \cdot l \leq p - 1$ and $(r + 1) \cdot l > p - 1$. Let $g \in \mathbb{F}_p[\mathbf{x}]$ be homogenous of degree k and computable by a $\text{poly}(n)$ -size circuit. There is a randomized algorithm running in time

$$\text{poly}(n) \cdot O\left(p^{\lceil \frac{l \cdot k}{p-1} \rceil + 1}\right)$$

determining whether g contains an r -monomial.

Proof. As in the proof of Lemma 20, we define $h(x_1, x_2, \dots, x_n) \triangleq g(x_1^l, x_2^l, \dots, x_n^l)$. The algorithm returns $LDT(h, n, d = k \cdot l, p)$. As in the proof of Lemma 20, h is homogenous of degree $k \cdot l$ and contains an $(r \cdot l)$ -monomial if and only if g contains an r -monomial. Furthermore, as $r \cdot l \leq p - 1$ and $(r + 1) \cdot l \geq p$, h contains a $(p - 1)$ -monomial if and only if g contains an r -monomial. Correctness now follows from Lemma 17. □

The main result of this section contains two results. The first is unconditional. The second is true if Cramer's conjecture is true. Cramer's conjecture states that the gap between two consecutive primes $p_{n+1} - p_n = O(\log^2 p_n)$, [9].

Theorem 22. (*Unconditional Result*) *Fix any integers r, k with $2 \leq r \leq k$. Let $g \in \mathbb{F}_p[\mathbf{x}]$ be homogenous of degree k and computable by a $\text{poly}(n)$ -size circuit. There is a randomized algorithm running in time*

$$\text{poly}(n) \cdot O\left(r^{\frac{2k}{r} + O(1)}\right)$$

determining whether g contains an r -monomial.

(*Conditional Result*) *If Cramer's Conjecture is true then the time complexity of the algorithm is*

$$\text{poly}(n) \cdot O\left(r^{\frac{k}{r} + o(\frac{k}{r})}\right).$$

Proof. We will find p and l as required in Lemma 21. Fix a prime p such that $r^2 + r + 1 < p < 2r^2 + 2r \leq 3r^2$. (This can be done as for any positive integer $t > 3$, there is always a prime between t and $2t$.)

Define $l \triangleq \lfloor \frac{p-1}{r} \rfloor$. We have

$$r \cdot l = r \cdot \lfloor \frac{p-1}{r} \rfloor \leq p - 1$$

$$\begin{aligned} (r + 1) \cdot l &\geq (r + 1) \cdot \left(\frac{p-1}{r} - 1\right) \\ &= (p - 1) + \frac{p-1}{r} - r - 1 > (p - 1) \end{aligned}$$

The first claim now follows from Lemma 21 and Corollary 16.

If Cramer's conjecture is true then there is a constant c such that for every integer x there is a prime number in $[x, x + c \log^2(x)]$. Then there is a prime number p in the interval $[2cr \log^2 r, 2cr \log^2 r + c \log^2(2cr \log^2 r)]$ and we can choose $l = 2c \log^2 r$. Then the time complexity will be

$$\text{poly}(n) \cdot O\left(r^{\frac{k}{r} + o(\frac{k}{r})}\right).$$

□

The following table summarizes the result for $r \leq 11$. See Lemma 21.

r	Result	Field and l
1	2^k [18]	\mathbb{F}_2 , $l = 1$
2	1.73^k	\mathbb{F}_3 , $l = 1$
3	1.912^k	\mathbb{F}_7 , $l = 2$
4	1.495^k	\mathbb{F}_5 , $l = 1$
5	1.615^k	\mathbb{F}_{11} , $l = 2$
6	1.383^k	\mathbb{F}_7 , $l = 1$
7	1.533^k	\mathbb{F}_{23} , $l = 3$
8	1.424^k	\mathbb{F}_{17} , $l = 2$
9	1.387^k	\mathbb{F}_{19} , $l = 2$
10	1.27^k	\mathbb{F}_{11} , $l = 1$
11	1.329^k	\mathbb{F}_{23} , $l = 2$

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