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The Complexity of Theorem Proving in Circumscription and Minimal Entailment

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Abstract. We provide the first comprehensive proof-complexity analysis of different proof systems for propositional circumscription. In particular, we investigate two sequent-style calculi: *MLK* defined by Olivetti [28] and *CIRC* introduced by Bonatti and Olivetti [8], and the tableaux calculus *NTAB* suggested by Niemelä [26]. In our analysis we obtain exponential lower bounds for the proof size in *NTAB* and *CIRC* and show a polynomial simulation of *CIRC* by *MLK*. This yields a chain $NTAB \leq_p CIRC \leq_p MLK$ of proof systems for circumscription of strictly increasing strength with respect to lengths of proofs.

1 Introduction

Circumscription is one of the main formalisms for non-monotonic reasoning. It uses reasoning with minimal models, the key idea being that minimal models have as few exceptions as possible. Therefore circumscription embodies common sense reasoning. Indeed, circumscription is known to be equivalent to reasoning under the extended closed world assumption, one of the main formalisms for reasoning with incomplete information. Apart from its foundational relation to human reasoning, circumscription has wide-spread applications, e.g. in AI, description logics [7] and SAT solving [21]. Circumscription is used both in first-order as well as in propositional logic, and we concentrate in this paper on the propositional case.

The semantics and complexity of circumscription have been the subject of intense research (see e.g. the recent articles [7, 14, 29]). In particular, deciding circumscriptive inference is harder than for propositional logic as it is complete for Π_2^p, the second level of the polynomial hierarchy [11,16]. Likewise, from the proof-theoretic side there are a number of formal systems for circumscription ranging from sequent calculi [8,28] to tableau methods [25,26,28].

The contribution of the present paper is a comprehensive analysis of these formal systems from the perspective of proof complexity. The main objective in proof complexity is a precise understanding of lengths of proofs. The two main tools for this are *lower bound methods* for the size of proofs for specific proof systems as well as *simulations* between proof systems. While lower bounds provide exact information on proof size, simulations compare the relative strength of proof systems and determine whether proofs can be efficiently translated between different formalisms. In this paper our results will employ both of these paradigms. While the bulk of research in proof complexity has concentrated on propositional proofs the last decade has seen ever increasing interest in proof complexity of non-classical logics (cf. [4] for a survey). In particular, very impressive results have been obtained for modal and intuitionistic logics [20, 22].

Prior to this paper, very little was known about the proof complexity of propositional circumscription. Our analysis concentrates on three of the main formalisms for circumscription: the tableau system *NTAB* introduced by Niemelä [26], the analytic sequent calculus *CIRC*

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by Bonatti and Olivetti [8], and the sequent calculus *MLK* by Olivetti [28]. Our main results are exponential lower bounds for the proof size in the tableau system *NTAB* and the sequent calculus *CIRC* (Theorems 6 and 19) as well as an efficient simulation of *CIRC* by *MLK* (Theorem 13). Together with the simulation of *NTAB* by *CIRC* shown by Bonatti and Olivetti [8] this gives a hierarchy of proof systems $NTAB \leq_p CIRC \leq_p MLK$. Moreover, this hierarchy is strict as our results provide separations between the proof systems (Theorems 8 and 19). While the systems *NTAB* and *MLK* only work for minimal entailment — the most important special case of circumscription — we also extend the results on *MLK* to the calculus *DMLK* from [28] for general circumscription (Theorem 16).

In related research, Egly and Tompits [15] investigated the proof-theoretic strength of circumscription in a first-order version of Bonatti and Olivetti's sequent calculus. They showed that for some formulas, first-order *CIRC* has much shorter proofs than classical first-order *LK*. Also in [1,5] the authors investigated the proof complexity of propositional default logic and autoepistemic logic, two other main approaches to non-monotonic reasoning. Although there are several translations between the different non-monotonic logics, we stress that none of these previous results imply lower bounds or simulations for propositional circumscription.

This paper is organised as follows. In Sect. 2 we review background information and notation about circumscription and proof complexity. In particular, we discuss the antisequent calculus *AC*. Section 3 contains our first main result: the exponential lower bound for *CIRC*. In Sect. 4 we prove the simulation of *CIRC* by *MLK* for minimal entailment; and this is extended to full circumscription and the calculus *DMLK* in Sect. 5. Section 6 then contains the comparison to Niemelä's tableau calculus *NTAB*, obtaining a separation between this tableau and *CIRC*. We conclude in Sect. 7 with a discussion and some open problems.

2 Preliminaries

Our propositional language contains the logical symbols $\bot, \top, \neg, \rightarrow, \vee, \wedge$. The notation $A[x/y]$ indicates that in the formula A every occurrence of formula x is replaced by formula y. For a set of formulae Σ, $\mathrm{VAR}(\Sigma)$ is the set of all atoms that occur in Σ. For a set P of atoms we set $\neg P = \{\neg p \mid p \in P\}$.

Circumscription is a non-monotonic logic introduced by McCarthy [24]. It looks at finding the 'minimal' situations that can occur, given our assumptions (cf. McCarthy's famous example of the "missionaries and cannibals" problem [24]). For circumscription, the propositional atoms are partitioned into three sets: P is the set of all atoms that are *minimised*, R is the set of *fixed* atoms, and Z denotes all remaining atoms, which may vary from the minimisation but are not themselves minimised. We usually only display P and R in the notation.

A *model* is a subset of the propositional atoms Σ_{Prop}. We define a pre-order $\leq_{P;R}$ on models I, J as follows: $I \leq_{P;R} J \Leftrightarrow I \cap P \subseteq J \cap P$ and $I \cap R = J \cap R$. The relation $\leq_{P;R}$ is transitive and minimality can be defined for models. Let $I \models \Gamma$. We say that I is a $(P;R)$-*minimal model* of Γ (and denote it by $I \models_{P;R} \Gamma$) if and only if for any model J; if $J \models \Gamma$ then $(J \leq_{P;R} I) \Rightarrow (I \leq_{P;R} J)$.

If ϕ is a formula, then $\Gamma \vDash_{P;R} \phi$ means that ϕ holds in all $(P;R)$-minimal models of Γ. This is the notion of semantic entailment in circumscription. A few special cases can be noted. When $P = \emptyset$ then $\vDash_{P;R}$ coincides with $\vDash$, the classical entailment. When P is the set of all variables appearing in the formulae of the sequent then entailment is known as *minimal entailment*, and we denote it with the symbol $\vDash_M$.

Let us give an example. Consider $a \rightarrow b, a \vee c \vDash_{b,c;a} b \rightarrow a$. There are only two possible minimal models. If we set a to true then b must be true, but we minimise and so c is false. If we take a as false then c must be true and we minimise so b is false. Hence the two models are $\{c\}$ and $\{a, b\}$. This example also demonstrates the difference between circumscription and classical logic as the expression $a \rightarrow b, a \vee c \vDash_{b,c;a} b \rightarrow a$ is not true, since model $\{b, c\}$ falsifies it.

When taking the same example and adding the atom b to the premise, we now get a different set of minimal models; $\{b, c\}$ and $\{a, b\}$. The succedent is now no longer true. This demonstrates the *non-monotonicity* of circumscription.

Noteworthy of this example is that even though the right hand side is an implication, bizarrely taking b as a premise does not entail a, namely $b, a \rightarrow b, a \vee c \nvDash_{b,c;a} a$. This is because $\{b, c\}$ and $\{a, b\}$ are now the only minimal models, each with a different truth value of a. So the deduction theorem does not hold in circumscription.

Proof Complexity. A *proof system* (Cook, Reckhow [12]) for a language L over alphabet Γ is a polynomial-time computable partial function $f : \Gamma^\star \rightharpoondown \Gamma^\star$ with $rng(f) = L$. An *f-proof* of string y is a string x such that $f(x) = y$.

From this we can start defining proof size. For f a proof system for language L and string $x \in L$ we define $s_f(x) = \min(|w| : f(w) = x)$. Thus the partial function s_f tells us the minimum proof size of a theorem. We can overload the notation by setting $s_f(n) = \max(s_f(x) : |x| \leq n)$ where $n \in \mathbb{N}$. For a function $t : \mathbb{N} \rightarrow \mathbb{N}$, a proof system f is called *t-bounded* if $\forall n \in \mathbb{N}, s_f(n) \leq t(n)$.

Proof systems are compared by simulations. We say that a proof system f *simulates* g ($g \leq f$) if there exists a polynomial p such that for every g-proof π_g there is an f-proof π_f with $f(\pi_f) = g(\pi_g)$ and $|\pi_f| \leq p(|\pi_g|)$. If π_f can even be constructed from π_g in polynomial time, then we say that f *p-simulates* g ($g \leq_p f$). Two proof systems f and g are *(p-)equivalent* ($g \equiv_{(p)} f$) if they mutually (p-)simulate each other.

Gentzen's system LK is one of the historically first and best studied proof system [18]. It operates with sequents. Formally, a *sequent* is a pair (Γ,Δ) with Γ and Δ finite sets of formulae. A sequent is usually written in the form $\Gamma \vdash \Delta$. In classical logic $\Gamma \vdash \Delta$ is true if every model for $\bigwedge \Gamma$ is also a model of $\bigvee \Delta$, where the disjunction of the empty set is taken as $\bot$ and the conjunction as $\top$. The system can be used both for propositional and first-order logic; the propositional rules are displayed in Fig. 1. Notice that the rules here do not contain structural rules for contraction or exchange. These come for free as we chose to operate with sets of formulae rather than sequences. Note the soundness of rule ($\bullet \vdash$), which gives us monotonicity of classical propositional logic.

A useful ingredient for working towards a calculus for non-monotonic logics is the notion of *underivability.* We use $\Gamma \nvDash \phi$ to denote that "there is a model M that satisfies all formulae in Γ but for which $\neg\phi$ holds". An *antisequent* is a pair of sets Γ, Δ of formulae, denoted $\Gamma \nvdash \Delta$. Semantically, an antisequent $\Gamma \nvdash \Sigma$ is true if there is some model $M \models \Gamma$ so that for all ϕ in Σ $M \models \neg\phi$. This is equivalent to saying that we cannot derive $\Gamma \vdash \Sigma$.

Bonatti [6] devised an *antisequent calculus AC;* its rules are given in Fig. 2. Correctness and completeness of AC was proven by Bonatti.

Theorem 1. ***(Bonatti [6])*** *An antisequent is true if and only if it is derivable in the antisequent calculus AC.*

$$\frac{}{A \vdash A}\,(\vdash) \qquad \frac{}{\bot \vdash}\,(\bot \vdash) \qquad \frac{}{\vdash \top}\,(\vdash \top)$$

$$\frac{\Gamma \vdash \Sigma}{\Delta, \Gamma \vdash \Sigma}\,(\bullet \vdash) \qquad \frac{\Gamma \vdash \Sigma}{\Gamma \vdash \Sigma, \Delta}\,(\vdash \bullet) \qquad \frac{\Gamma \vdash \Sigma, A}{\neg A, \Gamma \vdash \Sigma}\,(\neg \vdash) \qquad \frac{A, \Gamma \vdash \Sigma}{\Gamma \vdash \Sigma, \neg A}\,(\vdash \neg)$$

$$\frac{A, \Gamma \vdash \Sigma}{B \wedge A, \Gamma \vdash \Sigma}\,(\bullet\wedge \vdash) \qquad \frac{A, \Gamma \vdash \Sigma}{A \wedge B, \Gamma \vdash \Sigma}\,(\wedge\bullet \vdash) \qquad \frac{\Gamma \vdash \Sigma, A \qquad \Gamma \vdash \Sigma, B}{\Gamma \vdash \Sigma, A \wedge B}\,(\vdash \wedge)$$

$$\frac{A, \Gamma \vdash \Sigma \qquad B, \Gamma \vdash \Sigma}{A \vee B, \Gamma \vdash \Sigma}\,(\vee \vdash) \qquad \frac{\Gamma \vdash \Sigma, A}{\Gamma \vdash \Sigma, B \vee A}\,(\vdash \bullet\vee) \qquad \frac{\Gamma \vdash \Sigma, A}{\Gamma \vdash \Sigma, A \vee B}\,(\vdash \vee\bullet)$$

$$\frac{A, \Gamma \vdash \Sigma, B}{\Gamma \vdash \Sigma, A \to B}\,(\vdash\to) \qquad \frac{\Gamma \vdash \Sigma, A \qquad B, \Delta \vdash \Lambda}{A \to B, \Gamma, \Delta \vdash \Sigma, \Lambda}\,(\to\vdash)$$

$$\frac{\Gamma, \vdash \Sigma, \Delta \qquad \Delta, \Gamma \vdash \Sigma}{\Gamma \vdash \Sigma}\,(\text{cut})$$

Fig. 1. Rules of the sequent calculus *LK* [18]

The truth of an antisequent tells us of the existence of a model that satisfies the left hand side but contradicts the right hand side. While this does not point immediately to the model itself, it can be constructed from an AC-proof:

Proposition 2. *Given an AC-proof of an antisequent $\Gamma \nvdash \Delta$ we can construct in polynomial-time a model M that satisfies Γ and falsifies Δ.*

Proof. Given a derivation of an antisequent in AC we only have unary rules (rules that require only one premise) and a single axiom (requires no premises) so any sequent results from unary rules applied to an axiom ($\nvdash$). The axiom ($\nvdash$) gives us a trivial model that satisfies its RHS but not LHS; we take the RHS atoms to be true and all other atoms to be false. Next we can observe that in every application of the unary rules, the model that makes the premise true also makes the conclusion true. We will only verify this for an example. If $\Gamma \nvdash \Sigma, \alpha, \beta$ is a true antisequent, then there is some model M that satisfies Γ, but not any of Σ, α or β. So it must not satisfy $\alpha \vee \beta$. Thus M witnesses the conclusion sequent of rule ($\nvdash \vee$); $\Gamma \nvdash \Sigma, \alpha \vee \beta$ to be true. □

We mention that Proposition 2 implies that AC is presumably not *automatizable*, *i.e.*, it is not possible to construct AC-proofs in polynomial-time (even though AC-proofs are always of quadratic size [5]). In fact, using Proposition 2 it can be shown that automatizability of AC is equivalent to a complexity assumption Q, studied in [17] and shown to be equivalent to the p-optimality of the standard proof system for SAT in [3].

3 A lower bound for the sequent calculus *CIRC*

Bonatti and Olivetti [8] devised sequent calculi for several non-monotonic logics, among them was circumscription in a sequent calculus referred to as *CIRC*. A new item known as a *constraint* has been added to the sequents; Σ which is a set of atoms disjoint from R, so the *circumscriptive sequents* are of form $\Sigma; \Gamma \vdash_{P;R} \Delta$ (which may be regarded as a 5-tuple).

$$\frac{}{\Gamma \nvdash \Sigma} (\nvdash) \quad \text{where } \Gamma \text{ and } \Sigma \text{ are disjoint sets of propositional variables}$$

$$\frac{\Gamma \nvdash \Sigma, \alpha}{\Gamma, \neg\alpha \nvdash \Sigma} (\neg \nvdash) \qquad \frac{\Gamma, \alpha \nvdash \Sigma}{\Gamma \nvdash \Sigma, \neg\alpha} (\nvdash \neg)$$

$$\frac{\Gamma, \alpha, \beta \nvdash \Sigma}{\Gamma, \alpha \wedge \beta \nvdash \Sigma} (\wedge \nvdash) \qquad \frac{\Gamma \nvdash \Sigma, \alpha}{\Gamma \nvdash \Sigma, \alpha \wedge \beta} (\nvdash \bullet\wedge) \qquad \frac{\Gamma \nvdash \Sigma, \beta}{\Gamma \nvdash \Sigma, \alpha \wedge \beta} (\nvdash \wedge\bullet)$$

$$\frac{\Gamma \nvdash \Sigma, \alpha, \beta}{\Gamma \nvdash \Sigma, \alpha \vee \beta} (\nvdash \vee) \qquad \frac{\Gamma, \alpha \nvdash \Sigma}{\Gamma, \alpha \vee \beta \nvdash \Sigma} (\bullet\vee \nvdash) \qquad \frac{\Gamma, \beta \nvdash \Sigma}{\Gamma, \alpha \vee \beta \nvdash \Sigma} (\vee\bullet \nvdash)$$

$$\frac{\Gamma, \alpha \nvdash \Sigma, \beta}{\Gamma \nvdash \Sigma, \alpha \to \beta} (\nvdash\to) \qquad \frac{\Gamma \nvdash \Sigma, \alpha}{\Gamma, \alpha \to \beta \nvdash \Sigma} (\bullet \to\nvdash) \qquad \frac{\Gamma, \beta \nvdash \Sigma}{\Gamma, \alpha \to \beta \nvdash \Sigma} (\to \bullet \nvdash)$$

Fig. 2. Inference rules of the antisequent calculus AC by Bonatti [6]

As defined by Bonatti and Olivetti [8], the sequent $\Sigma; \Gamma \vdash_{P;R} \Delta$ is true when: "In every $(P \cup \Sigma; R)$-minimal model of Γ that satisfies Σ there is a formula $\phi \in \Delta$ that holds."

When Σ is empty we omit it from the notation, and these are the circumscriptive sequents we are primarily interested in. The rules of the calculus *CIRC* comprise the rules given in Fig. 3 together with all rules from *LK* and *AC*. Bonatti and Olivetti proved the correctness and completeness of *CIRC*:

Theorem 3. ***(Bonatti, Olivetti [8])*** *A sequent $\Sigma; \Gamma \vdash_{P;R} \Delta$ is true if and only if it is derivable in CIRC.*

$$\frac{\Gamma, \neg P \nvdash q}{q, \Sigma; \Gamma \vdash_{P;\emptyset} \Delta} (C1) \qquad \frac{\Sigma, \Gamma \vdash \Delta}{\Sigma; \Gamma \vdash_{P;R} \Delta} (C2)$$

$$\frac{q, \Sigma; \Gamma \vdash_{P;R} \Delta \qquad \Sigma; \Gamma, \neg q \vdash_{P;R} \Delta}{\Sigma; \Gamma \vdash_{P,q;R} \Delta} (C3) \qquad \frac{\Sigma; \Gamma, q \vdash_{P;R} \Delta \qquad \Sigma; \Gamma, \neg q \vdash_{P;R} \Delta}{\Sigma; \Gamma \vdash_{P;R,q} \Delta} (C4)$$

In all rules q is atomic and does not occur in P or R, and $\neg P = \{\neg p : p \in P\}$.

Fig. 3. Inference rules of the circumscription calculus *CIRC* of Bonatti & Olivetti [8]

To start a proof-theoretic investigation of *CIRC* we need the following notion:

Definition 4. *Let π be a CIRC-proof of a circumscriptive sequent $\Gamma \vdash_{P;R} \Delta$; and let s be a sequent occurring in π. We call s* involved *in π if either s is $\Gamma \vdash_{P;R} \Delta$ or is used as premise for some rule whose conclusion is an involved sequent. We call s* intermediate *if s is involved in π and occurs in π as a conclusion of any of rules (C1)–(C4).*

Thus the intermediate sequents form the "essential *CIRC*-part" of the proof on which we will focus our analysis. The whole proof can be much larger due to *LK* and *AC*-derivations. The next lemma shows that intermediate sequences are always of a special form.

Lemma 5. *Let π be a proof of the minimal entailment formula $\Gamma \vdash_{\mathrm{VAR}(\Gamma\cup\Delta);\emptyset} \Delta$. Then every intermediate line in π (in the sense of Definition 4) is of the form $P^+;\Gamma,\neg P^- \vdash_{P^0;\emptyset} \Delta$, where* $\mathrm{VAR}(\Gamma\cup\Delta) = P^0 \sqcup P^+ \sqcup P^-$.

Proof. Let π be a *CIRC* proof of length n of $\Gamma \vdash_{\mathrm{VAR}(\Gamma\cup\Delta);\emptyset} \Delta$. We show the lemma by induction on the distance of involved sequents from the bottom of the proof.

Induction Hypothesis on the distance k from the end of the proof: For all $r \geq n-k$ if the rth line of π is intermediate then it is of the form $P^+;\Gamma,\neg P^- \vdash_{P^0;\emptyset} \Delta$, where $\mathrm{VAR}(\Gamma\cup\Delta) = P^0 \sqcup P^+ \sqcup P^-$.

Base Case: For the base case, $k=0$ and we need to show that $\Gamma \vdash_{\mathrm{VAR}(\Gamma\cup\Delta);\emptyset} \Delta$ is of the specified form. This is true when choosing P^+, P^- as empty and P^0 as $\mathrm{VAR}(\Gamma\cup\Delta)$.

Inductive Step: By induction hypothesis all intermediate lines with higher or same index than the $(n-k)$-th line are of the desired form and we need to show that if the $(n-1-k)$-th is intermediate it is of the desired form. If line $n-1-k$ is intermediate then it must be a circumscriptive sequent and it must be a premise of some rule that derives an involved line. However, the only rules that allow circumscriptive sequents as premises in *CIRC* are (C3) and (C4). By induction hypothesis, all intermediate lines have empty R. So the rule cannot be (C4), but must use (C3). Using the induction hypothesis again, the conclusion is of form $P^+;\Gamma,\neg P^- \vdash_{P^0,p;\emptyset} \Delta$ where $\mathrm{VAR}(\Gamma,\cup\Delta) = P^0 \sqcup P^+ \sqcup P^- \sqcup \{p\}$ and so either the $(n-1-k)$-th line is $P^+;\Gamma,\neg P^-,\neg p \vdash_{P^0;\emptyset} \Delta$ or it is $P^+,p;\Gamma,\neg P^-, \vdash_{P^0;\emptyset} \Delta$ for some variable atom p. Both sequents are of the desired form. □

Our first result shows an exponential lower bound to the proof size of *CIRC*.

Theorem 6. *CIRC needs exponential-size proofs, i.e.,* $s_{CIRC}(n) \in 2^{\Omega(n/\log n)}$.

Proof. The idea is to construct a class of formulae which are of size $O(n\log n)$, but whose proof size grows exponentially. We use propositional variables $P_n = \{p_i, q_i : 1 \leq i \leq n\}$ and define antecedant $\Gamma_n := \{p_i \vee q_i : 1 \leq i \leq n\}$ and succedent $\Delta_n := \bigwedge_{1\leq i\leq n}(p_i \wedge \neg q_i)\vee(q_i\wedge\neg p_i)$. We consider the class of sequents $\Gamma_n \vdash_{P_n;\emptyset} \Delta_n$.

Intuitively the sequents express $\bigwedge_{1\leq i\leq n} p_i \vee q_i \vdash_M \bigwedge_{1\leq i\leq n} p_i \oplus q_i$, which is not classically true. But they are true circumscriptive sequents, because every minimal model of Γ_n will include p_i or q_i but cannot include both as these models are not minimal. Notice that the size of the sequents is bounded by $O(n\log n)$ because to represent of each of the n variables we need $O(\log n)$ bits.

Let now π be a *CIRC*-proof of $\emptyset;\Gamma_n \vdash_{P_n;\emptyset} \Delta_n$. We now argue inductively.

Induction Hypothesis (on k for $k \leq n$): Let $P^+;\Gamma_n,\neg P^- \vdash_{P^0;\emptyset} \Delta_n$ be an intermediate sequent of π (we know it is this form by Lemma 5) with $k = n - |P^- \sqcup P^+|$. Then the sub-proof of $P^+;\Gamma_n,\neg P^- \vdash_{P^0;\emptyset} \Delta_n$ in π contains at least 2^k lines of the form $B;\Gamma_n,\neg A \vdash_{C;\emptyset} \Delta_n$, where A, B, C are sets of atoms, with $P^+ \subseteq B$, $P^- \subseteq A$, and with B, A disjoint in any line.

Base Case (when $k=0$): A single line is needed to state the end result $P^+;\Gamma_n,\neg P^- \vdash_{P^0;\emptyset} \Delta_n$, and it suffices to take $B = P^+$, $A = P^-$.

Inductive Step: Assume the induction hypothesis holds for $k-1$. Our aim is to show that if $1 \leq k \leq n$, then $P^+;\Gamma_n,\neg P^- \vdash_{P^0;\emptyset} \Delta_n$ can only be inferred in *CIRC* by using (C3) in the form of

$$\frac{s, P^+;\Gamma_n,\neg P^- \vdash_{P^0\setminus\{s\};\emptyset} \Delta_n \qquad P^+;\Gamma_n,\neg P^-,\neg s \vdash_{P^0\setminus\{s\};\emptyset} \Delta_n}{P^+;\Gamma_n,\neg P^- \vdash_{P^0;\emptyset} \Delta_n}$$

for some s in P^0. Lemma 5 tells us that $P^+ \sqcup P^- \sqcup P^0 = P_n$. As $k < n$ there is some i, $1 \leq i \leq n$, such that $p_i, q_i \notin P^+ \sqcup P^-$ and so $p_i, q_i \in P^0$.

Suppose that $P^+; \Gamma_n, \neg P^- \vdash_{P^0;\emptyset} \Delta_n$ is inferred via (C1). Then, for some $p \in P^+$, the sequent $\Gamma_n, \neg P^-, \neg P^0 \nvdash p$ must be obtainable in the antisequent calculus. But as $p_i, q_i \in P^0$ and $p_i \vee q_i \in \Gamma_n$ the set $\Gamma_n, \neg P^-, \neg P^0$ is inconsistent and has no models. Hence $\Gamma_n, \neg P^-, \neg P^0 \models p$ and $\Gamma_n, \neg P^-, \neg P^0 \nvdash p$ is not derivable in AC.

Suppose instead that it is inferred via (C2). Then $P^+, \Gamma_n, \neg P^- \models \Delta_n$ must be true. However, as $p_i, q_i \notin P^+ \sqcup P^-$ the model which takes p_i, q_i as both true is consistent with the antecedent but not the succedent; so (C2) cannot be used.

Rule (C4) cannot be used either as the resulting sequent always has an element in R. Hence, (C3) is used to infer $P^+; \Gamma_n, \neg P^- \vdash_{P^0;\emptyset} \Delta_n$.

The inductive case needs proofs of both $s, P^+; \Gamma_n, \neg P^- \vdash_{P^0\backslash\{s\};\emptyset} \Delta_n$ and $P^+; \Gamma_n, \neg P^-, \neg s \vdash_{P^0\backslash\{s\};\emptyset} \Delta_n$ to construct the full proof. By the induction hypothesis each takes at least 2^{n-k-1} many lines of our desired form. Atom s is either in B or in A but not both. Therefore the lines are all distinct and there are $2 \cdot 2^{n-k-1}$ many lines, hence at least 2^{n-k} lines for the inductive step.

Finally, when $k = n$ we get that the full proof π of $\emptyset; \Gamma_n \vdash_{P_n;\emptyset} \Delta_n$ contains at least 2^n applications of (C3). □

In fact the proof even shows an exponential lower bound to the number of lines, *i.e.*, the proof length, which is a stronger statement.

4 Separating the sequent calculi *CIRC* and *MLK*

We now focus our attention on minimal entailment. In particular we will discuss Olivetti's sequent calculus *MLK* from [28] and compare its proof complexity with *CIRC*. *MLK* operates with sequents $\Gamma \vdash_M \Delta$. Semantically, $\Gamma \vdash_M \Delta$ is true if $\bigvee \Delta$ holds in all $(\mathrm{VAR}(\Gamma \cup \Delta); \emptyset)$-minimal models of Γ.

To introduce derivability we use the property of a *positive* atom in a formula from [28], defined inductively as follows. Atom p is positive in formula p. Atom p is positive in formula ϕ if and only if it is negative in $\neg\phi$. If atom p is positive in formula ϕ or χ, it is positive in $\phi \wedge \chi$ and $\phi \vee \chi$. If atom p is negative formula in ϕ or positive in χ then it is positive in $\phi \to \chi$.

The *MLK* calculus comprises all rules detailed in Fig. 4 together with all rules from *LK*. Olivetti showed soundness and completeness of *MLK*.

Theorem 7. ***(Olivetti [28])*** *A sequent $\Gamma \vdash_M \Delta$ is true if and only if it is derivable in MLK.*

We first show that for minimal entailment, *CIRC* is not better than *MLK*.

Theorem 8. *CIRC does not p-simulate MLK for minimal entailment.*

Proof. We use the hard examples from Theorem 6 and show that they can be proved in *MLK* in polynomial size. Using the same notation as in the proof of Theorem 6 we define Γ^i as $\Gamma_n \backslash \{p_i \vee q_i\}$. Consider the following *MLK* derivation.

$$\frac{}{\Gamma \vdash_M \neg p}\;(\vdash_M) \qquad\qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash_M \Delta}\;(\vdash\vdash_M)$$

for p atomic and not positive in any formula in Γ

$$\frac{\Gamma, \vdash_M \Sigma, \Delta \qquad \Delta, \Gamma \vdash_M \Lambda}{\Gamma \vdash \Sigma, \Lambda}\;(\text{M-cut}) \qquad\qquad \frac{\Gamma \vdash_M \Sigma \qquad \Gamma \vdash_M \Delta}{\Gamma, \Sigma \vdash_M \Delta}\;(\bullet \vdash_M)$$

$$\frac{\Gamma \vdash_M \Sigma, A \qquad \Gamma \vdash_M \Sigma, B}{\Gamma \vdash_M \Sigma, A \wedge B}\;(\vdash_M \wedge) \qquad\qquad \frac{A, \Gamma \vdash_M \Sigma \qquad B, \Gamma \vdash_M \Sigma}{A \vee B, \Gamma \vdash_M \Sigma}\;(\vee \vdash_M)$$

$$\frac{\Gamma \vdash_M \Sigma, A}{\Gamma \vdash_M \Sigma, B \vee A}\;(\vdash_M \bullet\vee) \qquad\qquad \frac{\Gamma \vdash_M \Sigma, A}{\Gamma \vdash_M \Sigma, A \vee B}\;(\vdash_M \vee\bullet)$$

$$\frac{A, \Gamma \vdash_M \Sigma}{\Gamma \vdash_M \Sigma, \neg A}\;(\vdash_M \neg) \qquad\qquad \frac{A, \Gamma \vdash_M \Sigma, B}{\Gamma \vdash_M \Sigma, A \to B}\;(\vdash_M \to)$$

Fig. 4. Rules of the sequent calculus *MLK* for minimal entailment (Olivetti [28])

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p_i \vdash p_i}\,(\vdash)}{\Gamma^i, p_i \vdash p_i}\,(\bullet\vdash)}{\Gamma^i, p_i \vdash_M p_i}\,(\vdash\vdash_M) \quad \dfrac{}{\Gamma^i, p_i \vdash_M \neg q_i}\,(\vdash_M)}{\Gamma^i, p_i \vdash_M p_i \wedge \neg q_i}\,(\vdash_M \wedge)}{\Gamma^i, p_i \vdash_M (p_i \wedge \neg q_i) \vee (q_i \wedge \neg p_i)}\,(\vdash_M \vee\bullet) \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q_i \vdash q_i}\,(\vdash)}{\Gamma^i, q_i \vdash q_i}\,(\bullet\vdash)}{\Gamma^i, q_i \vdash_M q_i}\,(\vdash\vdash_M) \quad \dfrac{}{\Gamma^i, q_i \vdash_M \neg p_i}\,(\vdash_M)}{\Gamma^i, q_i \vdash_M q_i \wedge \neg p_i}\,(\vdash_M \wedge)}{\Gamma^i, q_i \vdash_M (p_i \wedge \neg q_i) \vee (q_i \wedge \neg p_i)}\,(\vdash_M \bullet\vee)}{\Gamma_n \vdash_M (p_i \wedge \neg q_i) \vee (q_i \wedge \neg p_i)}\,(\vee \vdash_M)$$

This proof tree shows that $\Gamma_n \vdash_M (p_i \wedge \neg q_i) \vee (q_i \wedge \neg p_i)$ can be proved in linear length. By repeated use (at most a linear number of times) of rule $(\vdash_M \wedge)$ we build the big conjunction and obtain $\Gamma_n \vdash_M \Delta_n$ in polynomial size. □

The next lemma provides a translation of intermediate *CIRC*-sequent to *MLK*-sequents, which is easy to verify model-theoretically.

Lemma 9. *Let* $\mathrm{VAR}(\Gamma, \Delta) = P^0 \sqcup P^+ \sqcup P^-$. *Then* $P^+; \Gamma, \neg P^- \vdash_{P^0;\emptyset} \Delta$ *is true if and only if* $\Gamma, \neg P^- \vdash_M \Delta, \neg P^+$ *is true.*

Proof. Assume that $P^+; \Gamma, \neg P^- \vdash_{P^0;\emptyset} \Delta$ holds and let N be a $(P^+ \sqcup P^0; \emptyset)$-minimal model of $\Gamma, \neg P^-$. N is also $(\mathrm{VAR}(\Gamma \cup \Delta); \emptyset)$-minimal for $\Gamma, \neg P^-$ as it cannot satisfy any $p \in P^-$. Either N satisfies P^+, in which case it must satisfy Δ, or it must satisfy the disjunction of $\neg P^+$ hence $\Gamma, \neg P^- \vdash_M \Delta, \neg P^+$.

Conversely, if $\Gamma, \neg P^- \vdash_M \Delta, \neg P^+$, then let N be a $(P^+ \sqcup P^0; \emptyset)$-minimal model of $\Gamma, \neg P^-$. N must not satisfy any $p \in P^-$, hence it is $(\mathrm{VAR}(\Gamma \cup \Delta); \emptyset)$-minimal, which means if it satisfies P^+ it must satisfy Δ. Hence $P^+; \Gamma, \neg P^- \vdash_{P^0} \Delta$. □

Given a minimal entailment sequent $\Lambda \vdash_{\mathrm{VAR}(\Lambda,\Delta);\emptyset} \Delta$ and its proof $(t_i)_{0 \leq i \leq n}$ in *CIRC* we define a map τ that acts on intermediate sequents of the form $\Sigma; \Gamma \vdash_{P;\emptyset} \Delta$ and maps them to the *MLK*-sequent $\Gamma \vdash_M \Delta, \neg\Sigma$. This map is well defined as Lemma 5 guarantees that all intermediate sequents are exactly of the form that allow the translation in Lemma 9.

To compare *MLK* with *CIRC* we need a few facts on *LK* (cf. the appendix).

Lemma 10. *1. For sets of formulae Γ, Δ and disjoints sets of atoms Σ^+, Σ^- with* $\mathrm{VAR}(\Gamma \cup \Delta) = \Sigma^+ \sqcup \Sigma^-$ *we have* $s_{LK}(\Sigma^+, \neg\Sigma^-, \Gamma \vdash \Delta) \in O(|\Sigma^+, \neg\Sigma^-, \Gamma \vdash \Delta|^2)$*.*

2. For formulae ϕ, χ we have $s_{LK}(\chi \vdash \phi[\chi/\bot]) \in O(|\chi| + |\phi|)$.

Lemma 11. *Let Σ, Γ Δ be sets of formulae. From a sequent $\Sigma, \bigwedge\Gamma \vdash_M \Delta$ of size n we can derive $\Sigma, \Gamma \vdash_M \Delta$ in an $O(n^3)$ size MLK proof.*

Proof. Informally, the idea is that writing a conjunction or a list of formulae is semantically the same thing, but must be treated as different objects in a proof. The lemma demonstrates the ability of *MLK* to prove one direction of the equivalence in polynomial size. The strategy used is to inductively prove $\Sigma, \bigwedge\Gamma, \Gamma' \vdash_M \Delta$ for $\Gamma' \subseteq \Gamma$. We use proof by induction on the number of elements r of Γ'.

Induction hypothesis (on r): We can derive $\Sigma, \bigwedge\Gamma, \Gamma' \vdash_M \Delta$ in a size $O(rn^2)$ proof from $\Sigma, \bigwedge\Gamma \vdash_M \Delta$.

Base case ($r = 0$): Γ' is empty and so the sequent we need to prove is exactly the one we have started with.

Inductive step: Below is the extension we get the proof for the iterative step on the number of elements of Γ', we use an additional assumption that we can prove $\Sigma^+, \bigwedge\Gamma, \Gamma'\backslash\{\phi\} \vdash \phi$ in $O(n^2)$ size for $\phi \in \Gamma'$. This works by using axiom $\phi \vdash \phi$ and building the big conjunction using $(\bullet\wedge \vdash)$ and $(\wedge\bullet \vdash)$ rules. Then weakening with $(\bullet \vdash)$ to get the sequent. This derived sequent takes proof size $O(n^2)$.

$$\dfrac{\Sigma, \bigwedge\Gamma, \Gamma'\backslash\{\phi\} \vdash_M \Delta \qquad \dfrac{\Sigma, \bigwedge\Gamma, \Gamma'\backslash\{\phi\} \vdash \phi}{\Sigma, \bigwedge\Gamma, \Gamma'\backslash\{\phi\} \vdash_M \phi}\,(\vdash\vdash_M)}{\Sigma, \bigwedge\Gamma, \Gamma' \vdash_M \Delta}\,(\bullet \vdash_M)$$

Using our induction hypothesis we have total size as $O((r-1)n^2) + O(n^2)$ which is size $O(rn^2)$ as required.

Now after the induction, we have the $r = |\Gamma|$ case, $\Sigma, \bigwedge\Gamma, \Gamma \vdash_M \Delta$ and now we can cut on $\bigwedge\Gamma$. We prove $\Sigma, \Gamma \vdash \bigwedge\Gamma$ and in quadratic size using $(\vdash)$,$(\bullet \vdash)$ and $(\vdash \wedge)$.

$$\dfrac{\dfrac{\Sigma, \Gamma \vdash \bigwedge\Gamma}{\Sigma, \Gamma \vdash_M \bigwedge\Gamma}\,(\vdash\vdash_M) \qquad \Sigma, \bigwedge\Gamma, \Gamma \vdash_M \Delta}{\Sigma, \Gamma \vdash_M \Delta}\,(\text{M-cut})$$

This retains the $O(n^3)$ bound. □

Remark 12. As can be seen the M-cut rule is very powerful and allows us to manipulate the minimal entailment sequents, by using classical sequents. In fact, even when omitting all rules $(\vdash_M \wedge), (\vee \vdash_M), (\vdash_M \bullet\vee), (\vdash_M \vee\bullet), (\vdash_M \vee\bullet), (\vdash_M \neg), (\vdash_M \rightarrow)$ from *MLK* we still obtain a calculus that is complete for minimal entailment and p-simulates the original *MLK*. An example illustrating this for $(\vdash_M \neg)$ is given below.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\vdash A, \neg A}\,(\vdash \neg)}{\Gamma \vdash A, \neg A}\,(\text{repeated use of } \bullet \vdash)}{\Gamma \vdash_M A, \neg A}\,(\vdash\vdash_M) \qquad \Gamma, A \vdash_M \Delta}{\Gamma \vdash_M \Delta, \neg A}\,(\text{M-cut})$$

The next theorem is the main result in this section. Together with Theorem 8 it shows that *MLK* is strictly stronger than *CIRC* for minimal entailment.

Theorem 13. *MLK p-simulates CIRC for minimal entailment.*

Proof. Let π be a *CIRC* proof of minimal entailment sequent $\Lambda \vdash_{\mathrm{VAR}(\Lambda,\Delta)} \Delta$. We will show that there exists constants a and b (independent of π and the sequent) such that there is a proof $\pi^\star$ of $\Lambda \vdash_M \Delta$ in *MLK* with $|\pi^\star| \leq a|\pi|^4 + b$.

The induction argument forms from taking each line of the proof in *CIRC* and translating it using τ defined after Lemma 9 and showing it can be derived in quartic size in *MLK*.

Induction hypothesis (on the number r of applications of (C3) and (C4)): Let $\Lambda \vdash_{\mathrm{VAR}(\Lambda,\Delta)} \Delta$ be a minimal entailment sequent with *CIRC* proof π of size n. Let $\Sigma; \Gamma \vdash_{P;\emptyset} \Delta$ be an intermediate sequent of π (as in Definition 4), which is preceded by r applications of rules (C3) and (C4) in π, and the sub-proof up to that line is of size k. Then $\tau(\Sigma; \Gamma \vdash_{P;\emptyset} \Delta)$ can be derived in an $(ak^3 + b)$-size *MLK* proof.

Base Case ($r = 0$): For the base cases we only have to consider conclusions of rules (C1) and (C2).

C1: The problem with rule (C1) is that it uses the antisequent calculus, which is not incorporated in *MLK*. When using (C1) in *CIRC* proof π we would start with premise $\Gamma, \neg P \nvdash q$ and end with conclusion $q, \Sigma; \Gamma \vdash_{P;\emptyset} \Delta$, so we have to find an *MLK* proof starting with the axioms of the *MLK* calculus that is quartic in size and reaches conclusion $\tau(q, \Sigma; \Gamma \vdash_{P;\emptyset} \Delta) = \Gamma \vdash_M \Delta, \neg q, \neg\Sigma$.

Suppose that the intermediate sequent $q, \Sigma; \Gamma \vdash_{P;\emptyset} \Delta$ is inferred via (C1) in the *CIRC* proof π. Then $\Gamma, \neg P \nvdash q$ holds; so there is some model N in which $\Gamma, \neg P$ and $\neg q$ hold. Moreover, since we have the *AC*-proof we can efficiently construct this N by Proposition 2, which is needed to get a p-simulation.

Consider the sets of atoms $\Sigma^+ = \mathrm{VAR}(\Gamma) \cap N$ and $\Sigma^- = \mathrm{VAR}(\Gamma) \setminus (N \cup \{q\} \cup P)$. We claim that $\Sigma^+ \subseteq \Sigma \subseteq \Sigma^+ \sqcup \Sigma^-$. The first inclusion holds because $q, \Sigma; \Gamma \vdash_{P;\emptyset} \Delta$ is intermediate. Atoms in Σ^+ each must fall in one of the three sets of atoms stated in Lemma 5; each of these atoms is positive in N and hence of the three disjoint sets it can be in it must be in the one that is made up of q, Σ, as the others contain only negative atoms. Since q is negative in N, it must be in Σ. For the second inclusion, we use the fact that all the atoms in our sequent occur in Σ^+, Σ^-, P or q. Since atoms in Σ do not occur in P or q they occur in $\Sigma^+ \sqcup \Sigma^-$.

Therefore we can find $\Sigma^\star \subseteq \Sigma^-$ such that $\Sigma = \Sigma^+ \sqcup \Sigma^\star$.

For set of atoms $A = \{a_1, \ldots, a_l\}$ let us define $\hat{\Gamma}(A) = \bigwedge \Gamma[a_1/\bot, \ldots, a_l/\bot]$. This notation allows us to replace the variables with their assigned value, and treat the antecedent as a single formula. Let $m = |\Lambda \vdash_{\mathrm{VAR}(\Lambda,\Delta)} \Delta|$.

Let $U \sqcup Q = \Sigma^- \cup P$. Then $\Sigma^+ \vdash_M \hat{\Gamma}(U)$ is true. This is because all atoms in Q and U are minimised to not true, and the remaining positive atoms of N are all true, hence the minimal model is N and so Γ is satisfied. We incorporate these sequents in a proof by induction where we replace $\bot$ with atoms in $\hat{\Gamma}$ one by one.

Induction hypothesis (on the number $|Q|$ of reintroduced variables): We claim that $\Sigma^+ \vdash_M \hat{\Gamma}(U)$ has an *MLK*-proof of size $O(m^2)$ where $U \sqcup Q = \Sigma^- \cup P$.

Base case ($|Q| = 0$): We need to derive $\Sigma^+ \vdash_M \hat{\Gamma}(\Sigma^- \cup P)$. It follows from Lemma 10 that $\neg q, \Sigma^+ \vdash \hat{\Gamma}(\Sigma^- \cup P)$ in a $O(m^2)$-size proof. We augment this with the derivation:

$$\dfrac{\dfrac{\neg q, \Sigma^+ \vdash \hat{\Gamma}(\Sigma^- \cup P)}{\neg q, \Sigma^+ \vdash_M \hat{\Gamma}(\Sigma^- \cup P)}(\vdash\vdash_M) \quad \dfrac{}{\Sigma^+ \vdash_M \neg q}(\vdash_M)}{\Sigma^+ \vdash_M \hat{\Gamma}(\Sigma^- \cup P)}\text{(M-cut)}$$

Inductive step: Using Lemma 10 we derive $\Sigma^+, \hat{\Gamma}(U \cup \{p\}), \neg p \vdash \hat{\Gamma}(U)$ in an MLK-proof of linear size. We augment this by

$$\dfrac{\Sigma^+ \vdash_M \hat{\Gamma}(U \cup \{p\}) \quad \dfrac{\dfrac{\Sigma^+, \hat{\Gamma}(U \cup \{p\}), \neg p \vdash \hat{\Gamma}(U)}{\Sigma^+, \hat{\Gamma}(U \cup \{p\}), \neg p \vdash_M \hat{\Gamma}(U)}(\vdash\vdash_M) \quad \dfrac{}{\Sigma^+, \hat{\Gamma}(U \cup \{p\}) \vdash_M \neg p}(\vdash_M)}{\Sigma^+, \hat{\Gamma}(U \cup \{p\}) \vdash_M \hat{\Gamma}(U)}\text{(M-cut)}}{\Sigma^+ \vdash_M \hat{\Gamma}(U)}\text{(M-cut)}$$

Since we only add a linear size extension on each inductive step we retain a quadratic bound.

Now we have the inductive and base case, and all can be proved in quadratic size because we need to supply a linear number of iterations of a linear size proof each. Therefore, for $Q = \Sigma^- \cup P$ we obtain an MLK-proof of $\Sigma^+ \vdash_M \bigwedge \Gamma$ of size $O(m^2)$. We proceed extending the proof.

$$\dfrac{\Sigma^+ \vdash_M \bigwedge \Gamma \quad \dfrac{}{\Sigma^+ \vdash_M \neg q}(\vdash_M)}{\Sigma^+, \bigwedge \Gamma \vdash_M \neg q}(\bullet \vdash_M)$$

Using Lemma 11 we can add a cubic size proof to get $\Sigma^+, \Gamma \vdash_M \neg q$. Now we wish to weaken the right hand side. To do this we start with the axiom $\neg q \vdash \neg q$. Then use the weakening rules of LK to get $\Sigma^+, \Gamma, \neg q \vdash \neg q, \neg \Sigma^\star, \Delta$. We then continue with

$$\dfrac{\Sigma^+, \Gamma \vdash_M \neg q \quad \dfrac{\Sigma^+, \Gamma, \neg q \vdash \neg q, \neg \Sigma^\star, \Delta}{\Sigma^+, \Gamma, \neg q \vdash_M \neg q, \neg \Sigma^\star, \Delta}(\vdash\vdash_M)}{\Sigma^+, \Gamma \vdash_M \neg q, \neg \Sigma^\star, \Delta}\text{(M-cut)}$$

Repeated use of rule $(\vdash_M \neg)$ on sequents derives $\Gamma \vdash_M \Delta, \neg q, \neg \Sigma$, which is equivalent to the conclusion in (C1) under translation τ.

C2: We start with the classical sequent $\Sigma, \Gamma \vdash \Delta$ and then continue with

$$\dfrac{\dfrac{\Sigma, \Gamma \vdash \Delta}{\Sigma, \Gamma \vdash_M \Delta}(\vdash\vdash_M)}{\Gamma \vdash_M \Delta, \neg \Sigma}\text{repeated use of } (\vdash_M \neg)$$

to obtain $\Gamma \vdash_M \Delta, \neg \Sigma = \tau(\Sigma; \Gamma \vdash_{P;\emptyset} \Delta)$.

Inductive step: In our overall induction we still need to consider the cases of applications of rules (C3) and (C4).

C3: For (C3) our premises translated under τ must be $\Lambda, \neg P^- \vdash_M \Delta, \neg P^+, \neg p$ and $\Lambda, \neg P^-, \neg p \vdash_M \Delta, \neg P^+$, yielding

$$\dfrac{\Lambda, \neg P^- \vdash_M \Delta, \neg P^+, \neg p \qquad \Lambda, \neg P^-, \neg p \vdash_M \Delta, \neg P^+}{\Lambda, \neg P^- \vdash_M \Delta, \neg P^+}\text{(M-cut)}$$

C4: Since we have no fixed elements (C4) can be ignored.

Finally, using the induction on the entire proof gives us a cubic size proof of the sequent $\tau(\Lambda \vdash_{\text{VAR}(\Lambda,\Delta);\emptyset} \Delta)$, and this is $\Lambda \vdash_M \Delta$ as required. Since our proof is constructive we even obtain a p-simulation. □

5 Extending the simulation to full circumscription

While *MLK* only works for minimal entailment Olivetti [28] also augmented this calculus to obtain a sequent calculus for full circumscription. The rules of this calculus *DMLK* are shown in Figure 5. To distinguish between the different sequent calculi we use the notation $\Gamma \rhd_{P;R} \Delta$ for derivability in *DMLK*.

$$\frac{}{\Gamma \rhd_{P;R} \neg p}\ (P\text{-int}) \qquad \frac{\Gamma, N(U) \rhd_{P;R} \Delta}{\Gamma, N(z), U \to z \rhd_{P;R} \Delta}\ (Z\text{-int}) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \rhd_{P;R} \Delta}\ (\vdash \rhd)$$

for $p \in P$ and not positive in any formula in Γ
for $z \in Z$ and $z \notin \Gamma, \Delta, U$ and formula U occurring negatively in $N(U)$

$$\frac{\Gamma, \rhd_{P;R} \Sigma, \Delta \qquad \Delta, \Gamma \rhd_{P;R} \Lambda}{\Gamma \vdash \Sigma, \Lambda}\ (\rhd\text{-cut}) \qquad \frac{\Gamma \rhd_{P;R} \Sigma \qquad \Gamma \rhd_{P;R} \Delta}{\Gamma, \Sigma \rhd_{P;R} \Delta}\ (\bullet\rhd)$$

$$\frac{\Gamma \rhd_{P;R} \Sigma, A \qquad \Gamma \rhd_{P;R} \Sigma, B}{\Gamma \rhd_{P;R} \Sigma, A \wedge B}\ (\rhd\wedge) \qquad \frac{A, \Gamma \rhd_{P;R} \Sigma \qquad B, \Gamma \rhd_{P;R} \Sigma}{A \vee B, \Gamma \rhd_{P;Z} \Sigma}\ (\vee\rhd)$$

$$\frac{\Gamma \rhd_{P;R} \Sigma, A}{\Gamma \rhd_{P;R} \Sigma, B \vee A}\ (\rhd \bullet \vee) \qquad \frac{\Gamma \rhd_{P;R} \Sigma, A}{\Gamma \rhd_{P;R} \Sigma, A \vee B}\ (\rhd \vee \bullet)$$

$$\frac{A, \Gamma \rhd_{P;R} \Sigma}{\Gamma \rhd_{P;R} \Sigma, \neg A}\ (\rhd\neg) \qquad \frac{A, \Gamma \rhd_{P;R} \Sigma, B}{\Gamma \rhd_{P;R} \Sigma, A \to B}\ (\rhd \to)$$

Fig. 5. Rules of the sequent calculus *DMLK* for circumscription (Olivetti [28])

Theorem 14. ***(Olivetti [28])*** *DMLK is sound and complete for circumscription.*

If we want to prove a p-simulation of *DMLK* by *CIRC* it is necessary to make use of the (Z-int) rule. This seems problematic as the (Z-int) rule is syntactically quite restrictive and specialised for Olivetti's proof of Theorem 14. We therefore alternatively suggest to incorporate the antisequent calculus, adding rules of *AC* and the following new rule

$$\frac{\Gamma, R^+, \neg R^-, \neg P^-, \neg P^0 \nvdash p}{\Gamma, R^+, \neg R^-, \neg P^- \rhd_{P;R} \neg P^+}\ (\nvdash \rhd)$$

for $p \in P^+$, $P^- \sqcup P^0 \sqcup P^+ = P$, and $R^+ \sqcup R^- = R$. This still yields a sequent calculus $DMLK + (\nvdash \rhd)$ which is sound and complete for circumscription.

Similarly to Lemmas 5 and 9, the next lemma provides a translation of circumscriptive sequents to $\rhd$-sequents.

Lemma 15. *Let $\Gamma \vdash_{P;R} \Delta$ be a circumscriptive sequent with a CIRC-proof π.*

1. *Every intermediate sequent of π is of form $P^+; \Gamma, \neg P^-, R^+, \neg R^- \vdash_{P^0;R^0} \Delta$, where P is partitioned into sets P^+, P^-, P^0; R is partitioned analogously.*
2. *Let σ be the function that takes intermediate sequents of π of the form $P^+; \Gamma, \neg P^-, R^+, \neg R^- \vdash_{P^0;R^0} \Delta$ to sequents $\Gamma, \neg P^-, R^+, \neg R^- \rhd_{P;R} \Delta, \neg P^+$. Let A be an intermediate sequent of π, then $\sigma(A)$ is a true sequent.*

Proof. For item 1 we use proof by induction similar to Lemma 5.

Induction Hypothesis (on the distance k from the end of the proof): for all $r \geq n-k$, if the rth line is intermediate then it is of the form $P^+; \Gamma, \neg P^-, R^+, \neg R^- \vdash_{P^0;R^0} \Delta$, where P is partitioned into sets P^+, P^-, P^0, and R is partitioned similarly.

Base Case ($k = 0$): We need to show $\Gamma \vdash_{P;R} \Delta$ is of the required form. If we take P^+, P^-, R^+, R^- as empty, $P^0 = P$ and $R^0 = R$ then this holds.

Inductive Step: The induction hypothesis tells us that all intermediate lines with higher index than the $(n-k)$th line are of the desired form and we only need to show that if the $(n-1-k)$th is intermediate it is of the desired form. If line $n-1-k$ is intermediate then it must be a circumscription sequent and it must be a premise of some rule that concludes with an involved line. However, the only rules that allow circumscriptive sequents as premises in *CIRC* are (C3) and (C4), which means in this instance the conclusion is an intermediate sequent and by the induction hypothesis we can write that conclusion as $P^+; \Gamma, \neg P^-, R^+, \neg R^- \vdash_{P^0;R^0} \Delta$, with corresponding partitions.

If the rule used is (C3) then the $(n-1-k)$th line is either of the form $P^+, p; \Gamma, \neg P^-, R^+, \neg R^- \vdash_{P^0\setminus\{p\};R^0} \Delta$ or $P^+; \Gamma, \neg P^-, \neg p, R^+, \neg R^- \rhd_{P^0\setminus\{p\};R^0} \Delta$. If the rule used is (C4) then the $(n-1-k)$th line is either $P^+; \Gamma, \neg P^-, R^+, r, \neg R^- \vdash_{P^0;R^0\setminus\{r\}} \Delta$ or $P^+; \Gamma, \neg P^-, R^+, \neg R^-, \neg r \vdash_{P^0;R^0\setminus\{r\}} \Delta$. In every case we prove the inductive step.

The proof of item 2 is exactly the same as the proof of Lemma 9. ☐

We can now state the simulation.

Theorem 16. *DMLK* $+ (\nvdash \rhd)$ *p-simulates CIRC.*

Proof. Let π be a *CIRC* proof of sequent $\Lambda \vdash_{P;R} \Delta$. We will show that a proof $\pi^\star$ of polynomially similar size can be constructed in *DMLK* $+ (\nvdash \rhd)$.

The induction argument forms from taking each intermediate line of the proof in *CIRC* and translating it using σ defined in Lemma 15 and showing it can be inferred in polynomial size (for fixed polynomial w) in the calculus *DMLK* $+ (\nvdash \rhd)$.

Induction Hypothesis on the number r of applications of (C3) and (C4): Let $\Gamma \vdash_{P;R} \Delta$ be a circumscriptive sequent with *CIRC*-proof π of size n. Let $\Sigma; \Psi \vdash_{P^0;R^0} \Delta$ be an intermediate sequent of π (as in Definition 4) of π, which is preceded by r applications of rules (C3) and (C4) in π, and the sub-proof up to that line is of size k. Then $\sigma(\Sigma; \Psi \vdash_{P;\emptyset} \Delta)$ can be derived in an $w(r)$-size *DMLK* $+ (\nvdash \rhd)$-proof $\pi^\star$.

Base Case $r = 0$: For the base cases we are only concerned about the conclusions of the rules (C1) and (C2).

C1: In π we would start with a sequence in AC which can be copied exactly into $\pi^\star$. Using Lemma 15, the conclusion of (C1) in π is $P^+; \Gamma, \neg P^-, R^+, \neg R^- \vdash_{P^0;\emptyset} \Delta$ (for $P = P^- \sqcup P^0 \sqcup P^+$, $R = R^- \sqcup R^+$), and hence the premise must be $\Gamma, \neg P^-, \neg P^0, R^+, \neg R^- \nvdash p$ for $p \in P^+$. If we apply σ to the conclusion we can repeat the derivation using rule $(\nvdash \rhd)$:

$$\frac{\Gamma, \neg P^-, \neg P^0, R^+, \neg R^- \nvdash p}{\Gamma, \neg P^-, R^+, \neg R^- \rhd_{P^0;\emptyset} \neg P^+}$$

For each formula $\delta \in \Delta$ we can add it to the $\rhd -$ *sequent* by utilising the ($\rhd$-cut) rule.

C2: In π we would start with a proof in *LK* which can be copied for $\pi^\star$. This proof will end with $\Sigma, \Psi \vdash \Delta$. In π it immediately uses (C2) to derive $\Sigma; \Psi \vdash_{P;R} \Delta$, but in $\pi^\star$ we repeatedly use $(\vdash \neg)$ until we have $\Psi \vdash \Delta, \neg\Sigma$; and then we use $(\vdash \rhd)$ to get $\Psi \rhd_{P;R} \Delta, \neg\Sigma$ as required.

C3: For (C3) our premises translated under σ must be $\Gamma, \neg P^-, R^+, \neg R^- \rhd_{P;R} \Delta, \neg P^+, \neg p$ and $\Gamma, \neg P^-, \neg p, R^+, \neg R^- \rhd_{P;R} \Delta, \neg P^+$. Then we get

$$\frac{\Gamma, \neg P^-, R^+, \neg R^- \rhd_{P;R} \Delta, \neg P^+, \neg p \qquad \Gamma, \neg P^-, \neg p, R^+, \neg R^- \rhd_{P;R} \Delta, \neg P^+}{\Gamma_n, \neg P^-, R^+, \neg R^- \rhd_{P;R} \Delta, \neg P^+} \ (\rhd\text{-cut})$$

C4: For (C4) our premises translated under σ must be $\Gamma, \neg P^-, R^+, \neg R^-, r \rhd_{P;R} \Delta, \neg P^+$ and $\Gamma, \neg P^-, R^+, \neg R^-, \neg r \rhd_{P;R} \Delta, \neg P^+$. We derive

$$\frac{\dfrac{\Gamma, \neg P^-, R^+, \neg R^-, r \rhd_{P;R} \Delta, \neg P^+}{\Gamma, \neg P^-, R^+, \neg R^- \rhd_{P;R} \Delta, \neg P^+, \neg r} \qquad \Gamma, \neg P^-, R^+, \neg R^-, \neg r \rhd_{P;R} \Delta, \neg P^+}{\Gamma_n, \neg P^-, R^+, \neg R^- \rhd_{P;R} \Delta, \neg P^+}$$

Since this proof is constructive, we obtain a p-simulation. □

6 Comparison to Niemelä's tableau calculus

We now discuss the relations of these sequent calculi to a tableau calculus for minimal entailment. This tableau works for clausal theories and was introduced by Niemelä [26]. In this paper we will refer to this tableau calculus as *NTAB*.

For clausal theory Γ and formula ϕ, a Niemelä-tableau is defined as follows. We start the construction of the tableau T with a single branch $(C_i)_{0 \leq i \leq k}$ containing all the clauses of $\Gamma \cup \Delta$, where Δ is $\neg\phi$ expressed in CNF (conjunctive normal form). There are two rules for extending a branch, where the premises must occur earlier in the branch. Figure 6 gives these two rules where those clauses above the line indicate the premises needed to use the rule, and the clauses below indicate the extensions.

$$\frac{\{a_1, a_2, \ldots, a_m, \neg b_1, \neg b_2, \ldots, \neg b_n\}, \{b_1\}, \ldots, \{b_n\}, \{\neg a_1\}, \ldots, \{\neg a_{j-1}\}, \{\neg a_{j+1}\}, \ldots, \{\neg a_m\}}{\{a_j\}} \ (N1)$$

$$\frac{\{a_1, a_2, \ldots, a_m, \neg b_1, \neg b_2, \ldots, \neg b_n\}, \{b_1\}, \ldots, \{b_n\}}{\{a_j\} \mid \{\neg a_j\}} \ (N2)$$

Fig. 6. Rules of Niemelä's tableau *NTAB* [26]. The notation $\{a_j\} \mid \{\neg a_j\}$ indicates that the branch splits.

Niemelä's tableau *NTAB* uses the following conditions to close branches.

1. A branch B is *(classically) closed* when for some atoms $b_1, \ldots, b_n$ the clauses $\{\neg b_1, \ldots, \neg b_n\}$, $\{b_1\}, \ldots, \{b_n\}$ occur in the same branch.
2. Let $N_\Gamma(B) = \{\neg c \mid c \text{ is an atom}, \{c\} \text{ does not occur in } B, \text{ and } \exists C \in \Gamma \text{ s.t. } c \in C\}$. A branch B is *ungrounded* when B contains a unit clause $\{a\}$, for which $N_\Gamma(B) \cup \Gamma \not\models a$.
3. A branch is *MM-closed* if it is either closed or ungrounded.

The correctness and completeness of *NTAB* was shown by Niemelä:

Theorem 17. ***(Niemelä [26])*** *For clausal Γ and arbitrary ϕ there is an NTAB proof for Γ, ϕ with all its branches MM-closed if and only if $\Gamma \models_M \phi$.*

In the same work [8], where Bonatti and Olivetti introduce *CIRC*, they also compare it to *NTAB*, showing that tableaux in *NTAB* can be efficiently translated into *CIRC*-proofs.

Theorem 18. ***(Bonatti, Olivetti [8])*** *CIRC p-simulates NTAB.*

We will now show that the converse simulation does not hold, *i.e.*, we will prove a separation between *NTAB* and *CIRC*. This separation uses the well-known *pigeonhole principle* PHP_n^{n+1}. This an elementary, but famous principle for which a wealth of lower bounds is known in proof complexity (cf. [2,19]). PHP_n^{n+1} uses variables $x_{i,j}$ with $i \in [n+1]$ and $j \in [n]$, indicating that pigeon i goes into hole j. PHP_n^{n+1} consists of the clauses $\bigvee_{j\in[n]} x_{i,j}$ for all pigeons $i \in [n+1]$ and $\neg x_{i_1,j} \vee \neg x_{i_2,j}$ for all choices of distinct pigeons $i_1, i_2 \in [n+1]$ and holes $j \in [n]$. We use these formulas to obtain an exponential separation between *NTAB* and *CIRC*.

Theorem 19. *NTAB does not simulate CIRC for minimal entailment.*

Proof. We first show that $s_{NTAB}(PHP_n^{n+1} \vdash \bot) \geq \exp(cn^k)$. The crucial observation is that any tableau in *NTAB* for the pigeonhole principle, is in fact a refutation using the DPLL algorithm [13]. This can be seen as follows. The formula $\neg\bot$ in conjunctive normal form is just the empty set. So each tableau has as starting nodes just the clauses of PHP_n^{n+1}. In any MM-closed tableau for this sequent, every branch must be closed. This holds as PHP_n^{n+1} is inconsistent; so the antisequent $N_\Gamma(B), \Gamma \nvdash a$ is untrue and the ungrounded condition never holds for any branch.

The only clauses that can be derived by (N1) and (N2) are unit clauses. The unit clauses being derived by rule (N2) can be interpreted as the branching labels in the DPLL algorithm. Using (N1) is a restricted form of unit propagation; this step can be done at any point in the DPLL algorithm, and normally it is done automatically between each branching step. Using (N2) is equivalent to branching on a variable. When a branch is (classically) closed this means that the empty clause can be inferred by unit propagation in a constant number of steps. Therefore each proof of $PHP_n^{n+1} \vdash \bot$ in *NTAB* can be efficiently turned into a DPLL execution.

It is well known that runs of the DPLL algorithm can be efficiently translated into resolution refutations. Therefore the exponential lower bound for PHP_n^{n+1} of Haken [19] applies and each *NTAB*-proof of $PHP_n^{n+1} \vdash \bot$ must be of exponential size. On the other hand, Buss [10] showed that the pigeonhole formulas admit polynomial-size Frege proofs; and Frege systems are known to be p-equivalent to *LK* (cf. [23]). As *LK* is part of *CIRC* we obtain polynomial-size *CIRC*-proofs of $PHP_n^{n+1} \vdash_M \bot$. □

7 Conclusion

Combining results from this paper together with earlier results from [8] we obtain the p-simulations $NTAB \leq_p CIRC \leq_p MLK$ of proof systems for propositional circumscription. Moreover, all these systems are exponentially separated. While this tells us that *MLK* is the best proof systems with respect to size of proofs, this might be different when it comes to proof search. In fact, *NTAB* and *CIRC* are both analytic which enables efficient proof search strategies (cf. [8]), whereas for *MLK* the restricted cut rule is very powerful, making the system highly non-analytic. This is in line with the experience from classical proof complexity and

SAT solving where strong proof systems are known to be *not automatizable* under suitable assumptions (cf. [9]); and modern SAT solvers all build on rather weak proof systems [27].

In terms of proof complexity, the main question left open by this paper is to show lower bounds for *MLK*. Clearly, as circumscription is complete for the second level Π_2^p of the polynomial hierarchy [11,16], there exist at least super-polynomial lower bounds for *MLK* assuming $\mathsf{NP} \neq \Pi_2^p$. However, it might be very hard to show such bounds unconditionally. We note that for default logic and autoepistemic logic it is even known that showing lower bounds for the sequent calculi of these logics from [8] is as hard as showing lower bounds for classical *LK* [1,5], which is the main open problem in propositional proof complexity. We leave open whether a similar connection as in [1,5] can also be shown between *LK* and *MLK*.

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Appendix

This appendix contains proofs of some facts for LK, which have been omitted in the main part of the paper.

Lemma 10. *1. For sets of formulae Γ, Δ and disjoints sets of atoms Σ^+, Σ^- with* $\mathrm{VAR}(\Gamma \cup \Delta) = \Sigma^+ \sqcup \Sigma^-$ *we have $s_{LK}(\Sigma^+, \neg\Sigma^-, \Gamma \vdash \Delta) \in O(|\Sigma^+, \neg\Sigma^-, \Gamma \vdash \Delta|^2)$.*
2. For formulae ϕ, χ we have $s_{LK}(\chi \vdash \phi[\chi/\bot]) \in O(|\chi| + |\phi|)$.

Proof of Lemma 10 part 1

Proof. First let us consider individual formulas, here ϕ will be a well formed formula in $\Gamma \cup \Delta$. Because we have only one model either $\Sigma \vDash \phi$ or $\Sigma \vDash (\neg\phi)$.

Induction hypothesis (on the number of connectives of ϕ): The proof length (now denoted as $S(\phi)$) of either $\Sigma^+, \neg\Sigma^- \vdash \phi$ or $\Sigma^+, \neg\Sigma^- \vdash \neg\phi$ is bounded above by fixed polynomial $p(w) \in O(w^2)$ with $w = |\Sigma| + |\phi|$.

Firstly let $\Sigma = \Sigma^+ \cup \neg\Sigma^-$.

Base Case ($|\phi| = 1$): If $\phi = \top$ then $\Sigma^+ \vDash \phi$. We use the proof

$$\dfrac{\dfrac{}{\vdash \top}\,(\vdash \top)}{\Sigma \vdash \top}\text{ repeated (linearly bounded) use of } (\bullet \vdash)$$

Since we repeatedly use the weakening rule we can place a quadratic bound on the proof size.

If $\phi = \bot$ it can be proved false.

$$\dfrac{\dfrac{\dfrac{}{(\bot \vdash \emptyset)}\,(\bot \vdash)}{(\vdash \neg\bot)}\,(\vdash \neg)}{(\Sigma \vdash \neg\bot)}\text{ repeated (linearly bounded) use of } (\bullet \vdash)$$

Since we use repeated use of the weakening rule we can place a quadratic bound on the proof size.

If ϕ is atomic and true then $\phi \in \Sigma^+$, $\phi \vdash \phi$ is obtained via axiom $(\vdash)$, the remaining LHS is obtained via $(\bullet \vdash)$.

Since we use repeated use of the weakening rule we can place a quadratic bound on the proof size.

If ϕ is atomic and false then $\phi \in \Sigma^-$,

$$\dfrac{\dfrac{\dfrac{}{\phi \vdash \phi}\,(\vdash)}{\emptyset \vdash \phi, \neg\phi}\,(\vdash \neg)}{\neg\phi \vdash \neg\phi}\,(\neg \vdash)$$

and then by weakening with $(\bullet \vdash)$ the right hand side.

Since we use repeated use of the weakening rule we can place a quadratic bound on the proof size.

Hence our inductive hypothesis is true for $|\phi| = 1$.

Inductive Step:

- If $\phi = \neg\chi$ then if ϕ is true in the model then χ is false so it is already done by the proof in quadratic size of the negation of χ by the induction hypothesis.
- If $\phi = \neg\chi$ and if ϕ is false in the model then χ is true so

$$\cfrac{\cfrac{\Sigma \vdash \chi}{\Sigma, \neg\chi \vdash \emptyset}\ (\neg\vdash)}{\Sigma \vdash \neg\phi}\ (\vdash\neg)$$

  And we can simply add this to the end of the proof and retain the quadratic size from the induction hypothesis.
- If $\phi = \chi \vee \psi$ and if ϕ is true in the model then either $\Sigma \vdash \chi$ or $\Sigma \vdash \psi$. In the case that $\Sigma \vdash \chi$ we can use our inductive hypothesis. Then proceed as follows;

$$\frac{\Sigma \vdash \chi}{\Sigma \vdash \phi}\ (\vdash \vee\bullet)$$

  In the case that $\Sigma \vdash \psi$ we can use our inductive hypothesis. Then proceed as follows;

$$\frac{\Sigma \vdash \psi}{\Sigma \vdash \phi}\ (\vdash \bullet\vee)$$

  The proof length is bounded above by $S(\chi) + S(\psi) + O(w)$, which allows us to retain a quadratic bound.
- If $\phi = \chi \vee \psi$ and if ϕ is false in the model, then $\Sigma \vdash (\neg\chi)$ and $\Sigma \vdash (\neg\psi)$. We use the proof below to obtain sequents $\Sigma, \chi \vdash$ and $\Sigma, \psi \vdash$ similarly.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\chi \vdash \chi}\ (\vdash)}{\chi, \neg\chi \vdash}\ \neg\vdash}{\chi \vdash \neg\neg\chi}\ (\vdash\neg)}{\Sigma, \chi \vdash \neg\neg\chi}\ \text{repeated (linearly bounded) use of } (\bullet\vdash) \qquad \cfrac{\Sigma \vdash \neg\chi}{(\Sigma, \neg\neg\chi \vdash)}\ (\neg\vdash)}{\Sigma, \chi \vdash}\ (\text{cut})$$

  We then do similarly for ψ and use both to complete the proof.

$$\cfrac{\cfrac{\Sigma, \chi \vdash \qquad \Sigma, \psi \vdash}{\Sigma, \chi \vee \psi \vdash}\ (\vee\vdash)}{\Sigma \vdash \neg\phi}\ (\vdash\neg)$$

  By the repeated use of weakening, the proof length is bounded above by $S(\chi) + S(\psi) + O(w)$, which allows us to retain the quadratic bound.
- If $\phi = \chi \wedge \psi$ and if ϕ is true in the model, then $\Sigma \vdash \chi$ and $\Sigma \vdash \psi$ as with a proof length bounded by our inductive hypothesis, the proof for ψ follows as;

$$\frac{\Sigma \vdash \chi \qquad \Sigma \vdash \psi}{\Sigma \vdash \psi}\ (\vdash\wedge)$$

  The proof length is bounded above by $S(\chi) + S(\phi) + O(w)$, which allows us to retain the quadratic bound.
- If $\phi = \chi \wedge \psi$ and if ϕ is false in the model, then either $\Sigma \vdash \neg\chi$ or $\Sigma \vdash \neg\psi$. Without loss of generality if $\Sigma \vdash \neg\chi$, then we use its bounded proof via the induction hypothesis.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\chi \vdash \chi}\,(\vdash)}{\chi, \neg\chi \vdash}\,(\neg \vdash)}{\chi \vdash \neg\neg\chi}\,(\vdash \neg)}{\Sigma, \chi \vdash \neg\neg\chi}\,\text{repeated (linearly bounded) use of } (\bullet \vdash) \qquad \dfrac{\Sigma \vdash \neg\chi}{\Sigma, \neg\neg\chi \vdash}\,(\neg \vdash)}{\Sigma, \chi \vdash}\,(\text{cut})}{\Sigma, \phi \vdash}\,(\wedge\bullet \vdash)
$$
$$
\dfrac{\Sigma, \phi \vdash}{\Sigma \vdash \neg\phi}\,(\vdash \neg)
$$

By the repeated use of weakening, the proof length is bounded above by $S(\chi)+S(\phi)+O(w)$, which allows us to retain the quadratic bound.

- If $\phi = \chi \to \psi$ and ϕ is true in the model, then either $\Sigma \vdash \neg\chi$ or $\Sigma \vdash \psi$. If $\Sigma \vdash (\neg\chi)$ we use the short proof from the induction hypothesis.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\chi \vdash \chi}\,(\vdash)}{\chi, \neg\chi \vdash}\,(\neg \vdash)}{\chi \vdash \neg\neg\chi}\,(\vdash \neg)}{\Sigma, \chi \vdash \neg\neg\chi}\,\text{repeated (linearly bounded) use of } (\bullet \vdash) \qquad \dfrac{\Sigma \vdash \neg\chi}{\Sigma, \neg\neg\chi \vdash}\,(\neg \vdash)}{\Sigma, \chi \vdash}\,(\text{cut})}{\Sigma, \chi \vdash \psi}\,(\vdash \bullet)
$$
$$
\dfrac{\Sigma, \chi \vdash \psi}{\Sigma \vdash \phi}\,(\vdash\to)
$$

If instead $\Sigma \vdash \psi$ we use the short proof of it from the inductive hypothesis and proceed as follows;

$$
\dfrac{\dfrac{\Sigma \vdash \psi}{\Sigma, \chi \vdash \psi}\,(\bullet \vdash)}{\Sigma \vdash \phi}\,(\vdash\to)
$$

By the repeated use of weakening, in either case it is bounded above by $S(\chi) + O(w)$, which allows us to retain the quadratic bound.

- If $\phi = \chi \to \psi$ and ϕ is false in the model, then $\Sigma \vdash \chi$ and $\Sigma \vdash \neg\psi$. We use the short proof of $\phi \vdash \phi$ again in the same way;

$$
\dfrac{\dfrac{\dfrac{\dfrac{\psi \vdash \psi}{\psi, \neg\psi \vdash}\,(\neg \vdash)}{\psi \vdash \neg\neg\psi}\,(\vdash \neg)}{\Sigma, \psi \vdash \neg\neg\psi}\,\text{repeated (linearly bounded) use of } (\bullet \vdash) \qquad \dfrac{\Sigma \vdash \neg\psi}{\Sigma, \neg\neg\psi \vdash}\,(\neg \vdash)}{\Sigma, \psi \vdash}\,(\text{cut})
$$

And finish the proof as follows;

$$
\dfrac{\dfrac{\Sigma, \psi \vdash \qquad \Sigma \vdash \chi}{\Sigma, \phi \vdash}\,(\to\vdash)}{\Sigma \vdash \neg\phi}\,(\vdash \neg)
$$

By the repeated use of weakening, the proof length is bounded above by $S(\chi)+S(\phi)+O(w)$, which allows us to retain the quadratic bound.

In order to complete the proof for $\Sigma, \Gamma \vdash \Delta$ we can use the rules $(\vdash \bullet)$, $(\bullet \vdash)$ for weakening. In the case that the left hand side is inconsistent because $\Sigma \vdash \neg\phi$ for some $\phi \in \Gamma$ it can be done quickly by using the $(\neg \vdash)$ rule to bring over $\neg\neg\phi$ and change this into ϕ using the same trick that we have done in many of the proofs above. The remainder follows from weakening. □

Proof of Lemma 10 part 2

Proof. **Induction hypothesis:**(on the logical depth of ϕ) $s_{LK}(\phi(\bot), \neg a \vdash \phi(a)) \leq p(n) \geq s_{LK}(\phi(a), \neg a \vdash \phi(\bot))$

We can split this into two hypotheses

i)$s_{LK}(\phi(\bot), \neg a \vdash \phi(a)) \leq p(n)$

ii)$s_{LK}(\phi(a), \neg a \vdash \phi(\bot)) \leq p(n)$

Base Case (when $\phi(a) = \phi(\bot)$ or $\phi(a) = a$):

When we have $\phi(a) = \phi(\bot)$

$$\dfrac{\dfrac{}{\phi(a) \vdash \phi(\bot)}\ (\vdash)}{\phi(a), \neg a \vdash \phi(\bot)}\ (\bullet \vdash)$$

Here case i) is identical to case ii)

Another part of the base case is when $\phi(a) = a$. We can prove case i) easily as below

$$\dfrac{\dfrac{\dfrac{}{\bot \vdash}\ (\bot \vdash)}{\bot, \neg a \vdash}\ (\bullet \vdash)}{\bot, \neg a \vdash a}\ (\vdash \bullet)$$

For case ii)

$$\dfrac{\dfrac{\dfrac{}{a \vdash a}\ (\vdash)}{a, \neg a \vdash}\ (\neg \vdash)}{a, \neg a \vdash \bot}\ (\vdash \bullet)$$

Inductive Step:

Suppose $\phi(a) = \neg\chi(a)$ by the induction hypothesis we have $\chi(a), \neg a \vdash \chi(\bot)$, this can be used to derive case i).

$$\dfrac{\dfrac{\chi(a), \neg a \vdash \chi(\bot)}{\neg a \vdash \chi(\bot), \neg\chi(a)}\ (\vdash \neg)}{\neg\chi(\bot), \neg a \vdash, \neg\chi(a)}\ (\neg \vdash)$$

Also by the induction hypothesis we have $\chi(\bot), \neg a \vdash \chi(a)$, this can be used to derive case ii).

$$\dfrac{\dfrac{\chi(\bot), \neg a \vdash \chi(a)}{\neg a \vdash \chi(a), \neg\chi(\bot)}\ (\vdash \neg)}{\neg\chi(a), \neg a \vdash, \neg\chi(\bot)}\ (\neg \vdash)$$

Suppose $\phi(a) = \chi(a) \vee \psi(a)$ by the induction hypothesis we have $\chi(\bot), \neg a \vdash \chi(a)$ and $\psi(\bot), \neg a \vdash \psi(a)$. We use this to derive case i).

$$\dfrac{\dfrac{\chi(\bot), \neg a \vdash \chi(a)}{\chi(\bot), \neg a \vdash \chi(a) \vee \psi(a)}\ (\vdash \vee\bullet) \qquad \dfrac{\psi(\bot), \neg a \vdash \psi(a)}{\psi(\bot), \neg a \vdash \chi(a) \vee \psi(a)}\ (\vdash \bullet\vee)}{\chi(\bot) \vee \psi(\bot), \neg a \vdash \chi(a) \vee \psi(a)}\ (\vee \vdash)$$

By the induction hypothesis we have $\chi(a), \neg a \vdash \chi(\bot)$ and $\psi(a), \neg a \vdash \psi(\bot)$. We use this to derive case ii).

$$\dfrac{\dfrac{\chi(a), \neg a \vdash \chi(\bot)}{\chi(a), \neg a \vdash \chi(\bot) \vee \psi(\bot)}\,(\vdash \vee\bullet) \qquad \dfrac{\psi(a), \neg a \vdash \psi(\bot)}{\psi(a), \neg a \vdash \chi(\bot) \vee \psi(\bot)}\,(\vdash \bullet\vee)}{\chi(a) \vee \psi(a), \neg a \vdash \chi(\bot) \vee \psi(\bot)}\,(\vee \vdash)$$

Suppose $\phi(a) = \chi(a) \wedge \psi(a)$ by the induction hypothesis we have $\chi(\bot), \neg a \vdash \chi(a)$ and $\psi(\bot), \neg a \vdash \psi(a)$. We use this to derive case i).

$$\dfrac{\dfrac{\chi(\bot), \neg a \vdash \chi(a)}{\chi(\bot) \wedge \psi(\bot), \neg a \vdash \chi(a)}\,(\wedge\bullet \vdash) \qquad \dfrac{\psi(\bot), \neg a \vdash \psi(a)}{\chi(\bot) \wedge \psi(\bot), \neg a \vdash \psi(a)}\,(\bullet\wedge \vdash)}{\chi(\bot) \wedge \psi(\bot), \neg a \vdash \psi(a)}\,(\vdash \wedge)$$

By the induction hypothesis we have $\chi(a), \neg a \vdash \chi(\bot)$ and $\psi(a), \neg a \vdash \psi(\bot)$. We use this to derive case ii).

$$\dfrac{\dfrac{\chi(a), \neg a \vdash \chi(\bot)}{\chi(a) \wedge \psi(a), \neg a \vdash \chi(\bot)}\,(\wedge\bullet \vdash) \qquad \dfrac{\psi(a), \neg a \vdash \psi(\bot)}{\chi(a) \wedge \psi(a), \neg a \vdash \psi(\bot)}\,(\bullet\wedge \vdash)}{\chi(a) \wedge \psi(a), \neg a \vdash \psi(\bot)}\,(\vdash \wedge)$$

Suppose $\phi(a) = \chi(a) \rightarrow \psi(a)$ by the induction hypothesis we have $\chi(a), \neg a \vdash \chi(\bot)$ and $\psi(\bot), \neg a \vdash \psi(a)$. From this we can prove case i).

$$\dfrac{\dfrac{\dfrac{\chi(a), \neg a \vdash \chi(\bot)}{\chi(a), \neg a \vdash \chi(\bot), \psi(a)}\,(\vdash \bullet)}{\neg a \vdash \chi(\bot), \chi(a) \rightarrow \psi(a)}\,(\vdash\rightarrow) \qquad \dfrac{\dfrac{\psi(\bot), \neg a \vdash \psi(a)}{\chi(a), \psi(\bot) \neg a \vdash, \psi(a)}\,(\bullet \vdash)}{\psi(\bot), \neg a \vdash \chi(\bot), \chi(a) \rightarrow \psi(a)}\,(\vdash\rightarrow)}{\chi(\bot) \rightarrow \psi(\bot), \neg a \vdash \chi(a) \rightarrow \psi(a)}\,(\rightarrow\vdash)$$

By the induction hypothesis we have $\chi(\bot), \neg a \vdash \chi(a)$ and $\psi(a), \neg a \vdash \psi(\bot)$. From this we can prove case ii).

$$\dfrac{\dfrac{\dfrac{\chi(\bot), \neg a \vdash \chi(a)}{\chi(\bot), \neg a \vdash \chi(a), \psi(\bot)}\,(\vdash \bullet)}{\neg a \vdash \chi(a), \chi(\bot) \rightarrow \psi(\bot)}\,(\vdash\rightarrow) \qquad \dfrac{\dfrac{\psi(a), \neg a \vdash \psi(\bot)}{\chi(\bot), \psi(a) \neg a \vdash, \psi(\bot)}\,(\bullet \vdash)}{\psi(a), \neg a \vdash \chi(\bot), \chi(\bot) \rightarrow \psi(\bot)}\,(\vdash\rightarrow)}{\chi(a) \rightarrow \psi(a), \neg a \vdash \chi(\bot) \rightarrow \psi(\bot)}\,(\rightarrow\vdash)$$

Our steps that have the highest order polynomial size are linear factors in the base case and hence we can find a linear proof. $\square$

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