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Lines Missing Every Random Point

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Abstract. This paper proves that there is, in every direction in Euclidean space, a line that misses every computably random point. Our proof of this fact shows that a famous set constructed by Besicovitch in 1964 has computable measure 0.

Keywords: randomness, effective geometric measure theory, computable analysis

1 Introduction

One objective of the theory of computing is to investigate the fine-scale geometry of algorithmic information in Euclidean space. Recent work along these lines has included algorithmic classifications of points lying on computable curves and arcs [6, 13, 18, 21, 23] and in more exotic sets [8, 16, 17].

This paper concerns a simple, fundamental question: Can the direction of a line in Euclidean space force the line to meet at least one random point? That is, can the set of Martin-Löf random points, which is everywhere dense and contains almost every point in Euclidean space, be avoided by lines in every direction? For example, it is reasonable to conjecture that every line of random slope in $\mathbb{R}^2$ contains a random point. We show here that this conjecture is false, and in fact that—regardless of slope—every line can be translated so that it contains no Martin-Löf random point. Moreover, the line can miss the larger class of all computably random points.

Our solution of this problem builds on a very old—and ongoing—line of research in geometric measure theory. In 1917 Fujiwara and Kakeya [12,14] posed the question of the minimum area of a plane set in which a unit segment can be continuously reversed without leaving the set, a *Kakeya needle set*. This question was resolved in 1928 by Besicovitch and Perron [2, 20]: such a set can have arbitrarily small measure. The work made use of a construction by Besicovitch from 1919 [1] (but not widely circulated until its republication in 1928 [3]) of a plane set of area 0 containing a unit line segment in every direction, a *Kakeya*

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set. This set was constructed using a clever iterated process of partitioning and translating the pieces of an equilateral triangle.

In 1964, Besicovitch gave a simpler construction of a plane set with area 0 that contains a line in every direction, a *Besicovitch set* [5]. This line set B is the point-line dual of a simply defined "fractal dust," and for slopes $m \in [0, 1]$ the y-intercept for the line of slope m is given by applying a permutation on $\mathbb{Z}^4$ digit-by-digit to the base-4 expansion of m. The set B is described in detail in section 4, and our main result is achieved by showing that B has computable measure 0, as does its Cartesian product with $\mathbb{R}^n$, for every $n \in \mathbb{N}$.

More recent work on the "sizes" of Besicovitch sets and Kakeya sets has focused on their dimensions. Davies showed that every Kakeya set in $\mathbb{R}^2$ has Hausdorff dimension 2 [7], and the famous Kakeya conjecture states that Kakeya sets in $\mathbb{R}^n$ have Hausdorff dimension n for all $n \geq 2$. For more on this history, consult [10, 15].

The remainder of the paper is organized as follows. Section 2 contains preliminary information regarding computable measure and randomness in $\mathbb{R}^n$. In section 3, we present a class of martingales for betting on open sets. In section 4, we describe the set from Besicovitch's 1964 construction and prove the main theorem in $\mathbb{R}^2$. Section 5 extends the main theorem to $\mathbb{R}^n$. Section 6 mentions open problems.

2 Computable Randomness in $\mathbb{R}^n$

We now discuss the elements of computable measure and randomness in $\mathbb{R}^n$. For each $r \in \mathbb{N}$ and each $\mathbf{u} = (u_1, ..., u_n) \in \mathbb{Z}^n$, let

$$Q_r(\mathbf{u}) = [u_1 \cdot 2^{-r}, (u_1 + 1) \cdot 2^{-r}) \times ... \times [u_n \cdot 2^{-r}, (u_n + 1) \cdot 2^{-r})$$

be the r-*dyadic cube* at $\mathbf{u}$. Note that each $Q_r(\mathbf{u})$ is "half-open, half-closed" in such a way that, for each $r \in \mathbb{N}$, the family

$$\mathcal{Q}_r = \left\{ Q_r(\mathbf{u}) \;\middle|\; \mathbf{u} \in \{0, ..., 2^r - 1\}^n \right\}$$

is a partition of the unit cube $Q_0(\mathbf{0}) = [0, 1)^n$. The family

$$\mathcal{Q} = \bigcup_{r=0}^{\infty} \mathcal{Q}_r$$

is the set of all *dyadic cubes* in $[0, 1)^n$.

A *martingale* on $[0, 1)^n$ is a function $d : \mathcal{Q} \to [0, \infty)$ satisfying

$$d(Q_r(\mathbf{u})) = 2^{-n} \sum_{\mathbf{a} \in \{0,1\}^n} d(Q_{r+1}(2\mathbf{u} + \mathbf{a})) \tag{1}$$

for all $Q_r(\mathbf{u}) \in \mathcal{Q}$. Intuitively, a martingale d is a strategy for placing successive bets on the location of a point $\mathbf{x} \in [0, 1)^n$. After r bets have been placed, the bettor's capital is

$$d^{(r)}(\mathbf{x}) = d(Q_r(\mathbf{u})),$$

where $\mathbf{u}$ us the unique element of $\{0, ..., 2^r - 1\}^n$ such that $\mathbf{x} \in Q_r(\mathbf{u})$. The bettor's next bet is on which of the 2^n immediate subcubes $Q_{r+1}(2\mathbf{u} + \mathbf{a})$ of $Q_r(\mathbf{u})$ has $\mathbf{x}$ as an element. The condition (1) says that the bettor's expected capital after this bet is exactly the bettor's capital before the bet, i.e., the payoffs are fair. A martingale d *succeeds* at a point $\mathbf{x} \in [0, 1)^n$ if

$$\limsup_{r \to \infty} d^{(r)}(\mathbf{x}) = \infty.$$

A well known theorem of Ville [22], restated in the present setting, says that a set $E \subseteq [0, 1)^n$ has Lebesgue measure 0 if and only if there is a martingale d that succeeds at every point $\mathbf{x} \in E$. It follows easily by the countable additivity and translation invariance of Lebesgue measure that a set $E \subseteq \mathbb{R}^n$ has Lebesgue measure 0 if and only if there is a martingale d that succeeds at every point $\mathbf{x} \in E^{\#}$, where

$$E^{\#} = [0, 1)^n \cap \bigcup_{\mathbf{t} \in \mathbb{Z}^n} (E + \mathbf{t}). \tag{2}$$

Let

$$J = \left\{(r, \mathbf{u}) \in \mathbb{N} \times \mathbb{Z}^n \;\middle|\; \mathbf{u} \in \{0, ..., 2^r - 1\}^n\right\}.$$

Then a martingale $d : \mathcal{Q} \to [0, \infty)$ is *computable* if there is a computable function $\widehat{d} : \mathbb{N} \times J \to \mathbb{Q} \cap [0, \infty)$ such that, for all $(s, r, \mathbf{u}) \in \mathbb{N} \times J$,

$$\left| \widehat{d}(s, r, \mathbf{u}) - d(Q_r(\mathbf{u})) \right| \leq 2^{-s}.$$

A set $E \subseteq \mathbb{R}^n$ is defined to have *computable measure* 0 if there is a computable martingale d that succeeds at every point $\mathbf{x} \in E^{\#}$, where $E^{\#}$ is defined as in (2). A point $\mathbf{x} \in \mathbb{R}^n$ is *computably random* if it is not an element of any set of computable measure 0, i.e., if there is no computable martingale that succeeds at $\mathbf{x}$. It is well known [9, 19] that every random point in $\mathbb{R}^n$ (i.e., every Martin-Löf random point in $\mathbb{R}^n$) is computably random and that the converse does not hold. In particular, then, almost every point in $\mathbb{R}^n$ is computably random.

3 Betting on Open Sets

In this section we describe a class of martingales that are used in the proof of the main theorem in section 4. These martingales are also likely to be useful in future investigations.

For any set $G \subseteq [0, 1)^n$ with $m(G) > 0$, define a martingale $d_G : \mathcal{Q} \to [0, \infty)$ recursively as follows.

(i) $d_G(Q_0(\mathbf{0})) = 1$.
(ii) For all $r \geq 0$, $\mathbf{u} \in \{0, ..., 2^r - 1\}^n$, and $\mathbf{a} \in \{0, 1\}^n$,

$$d_G(Q_{r+1}(2\mathbf{u} + \mathbf{a})) = \begin{cases} 0 & \text{if } d_G(Q_r((\mathbf{u})) = 0 \\ 2^n d_G(Q_r(\mathbf{u})) \frac{m(G \cap Q_{r+1}(2\mathbf{u}+\mathbf{a}))}{m(G \cap Q_r(\mathbf{u}))} & \text{otherwise.} \end{cases}$$

That is, for each cube $Q \in \mathcal{Q}_r$, the values of the martingale on the immediate subcubes of Q are proportional to the measures of the subcubes' intersections with E.

Theorem 1 *For every nonempty set G that is open as a subset of the subspace $[0,1)^n$ of $\mathbb{R}^n$ and every $\mathbf{x} \in G$, $d_G^{(r)}(\mathbf{x}) = 1/m(G)$ for all sufficiently large r.*

Proof. Let G be a nonempty open set in the subspace $[0,1)^n$ of $\mathbb{R}^n$. Then $m(G) > 0$, and by a routine induction argument, for any $r \in \mathbb{N}$ and $\mathbf{u} \in \{0, ..., 2^r - 1\}^n$,

$$d_G(Q_r(\mathbf{u})) = 2^{nr} \frac{m(G \cap Q_r(\mathbf{u}))}{m(G)}. \tag{3}$$

For any $\mathbf{x} \in G$ there exists $\varepsilon > 0$ such that $\mathcal{B}_\varepsilon(\mathbf{x}) \subseteq G \cap [0,1)^n$. Let $r > -log(\varepsilon)$ and $Q \in \mathcal{Q}_r$ such that $\mathbf{x} \in Q$. Then $d_G^{(r)}(\mathbf{x}) = d_G(Q)$, and $2^{-r} < \varepsilon$, so $Q \subseteq \mathcal{B}_\varepsilon(\mathbf{x}) \subseteq E$. Applying (3),

$$d_G^{(r)} = 2^{nr} \frac{m(G \cap Q)}{m(G)} = 2^{nr} \frac{m(Q)}{m(G)} = \frac{1}{m(G)}.$$

□

4 Betting on Besicovitch

In 1964 Besicovitch [5] (see also [10, 11]) gave an elegant construction of a Lebesgue measure 0 set $B \subseteq \mathbb{R}^2$ (our notation, not his) that contains a line in every direction. This section reviews this construction and proves that the set B in fact has computable measure 0. Hence B contains a line in every direction in $\mathbb{R}^2$, and each of these lines misses every computably random point in $\mathbb{R}^2$.

For each $m, b \in \mathbb{R}$, let $\mathcal{L}_{m,b} \subseteq \mathbb{R}^2$ be the line with slope m and y-intercept b. Besicovitch defined the *line set operator*

$$\mathcal{L} : \mathcal{P}(\mathbb{R}^2) \to \mathcal{P}(\mathbb{R}^2)$$

by

$$\mathcal{L}(F) = \bigcup \{\mathcal{L}_{m,b} \mid (m, b) \in F\}$$

for all $F \subseteq \mathbb{R}^2$. (We call $\mathcal{L}(F)$ the *line set* of F.) It is easy to verify that the operator $\mathcal{L}$ is monotone, maps open sets to open sets, and maps closed sets to closed sets.

We are interested in the line set of a particular self-similar fractal F, which we now define. Consider the alphabet $\Sigma = \{0, 1, 2, 3\}$. For each $i \in \Sigma$ define the contraction $S_i : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$S_i(x, y) = \frac{1}{4}((x, y) + (i, a_i)),$$

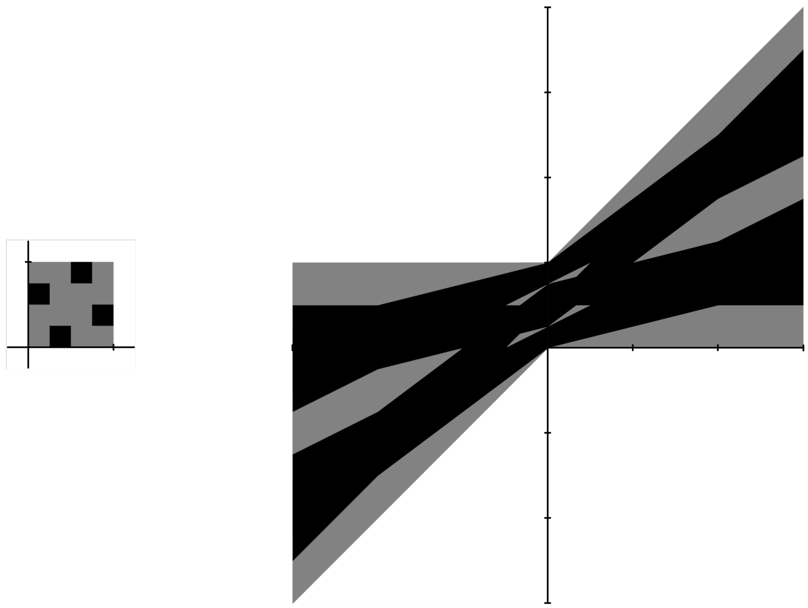

Fig. 1. F_0 and F_1, along with their line sets. F_0 and $\mathcal{L}(F_0)$ are shaded gray; F_1 and $\mathcal{L}(F_1)$ are black.

where $a_0 = 2$, $a_1 = 0$, $a_2 = 3$, and $a_3 = 1$. For each $w \in \Sigma^*$ define the set $F(w) \subseteq \mathbb{R}^2$ by the recursion

$$F(\lambda) = [0,1]^2;$$
$$F(iw) = S_i(F(w))$$

for all $i \in \Sigma$ and $w \in \Sigma^*$. For each $k \in \mathbb{N}$ let

$$F_k = \bigcup \{F(w) \mid w \in \Sigma^k\}.$$

The sets F_0 and F_1, along with their line sets, are depicted in Figure 1. We are interested in the set

$$F = \bigcap_{k=0}^{\infty} F_k.$$

This set F is an uncountable, totally disconnected set, informal called a "fractal dust." More formally it is the *attractor* of the *iterated function system* (S_0, S_1, S_2, S_3), i.e., it is a *self-similar fractal.*

Let $\mathrm{Ref}_Y : \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathrm{Rot}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection across the y-axis and rotation about the origin by the angle θ, respectively. The set

$$B = \mathcal{L}(F) \cup \mathrm{Rot}_{\frac{\pi}{2}}(\mathcal{L}(F)) \cup \mathrm{Ref}_Y\big(\mathcal{L}(F) \cup \mathrm{Rot}_{\frac{\pi}{2}}(\mathcal{L}(F))\big) \tag{4}$$

is the Besicovitch set mentioned in the first paragraph of this section.

Observation 2 *The set B contains a line in every direction in $\mathbb{R}^2$.*

Proof. Let $m \in [0,1]$. By (4) it suffices to show that $\mathcal{L}(F)$ contains a line of slope m. But this is clear, since each F_k, and hence F, contains a point of the form (m,b). □

Using an ingenious duality principle and some nontrivial fractal geometry, Besicovitch also proved the following.

Lemma 3 (Besicovitch [5]) *The set B has Lebesgue measure 0.*

It is not obvious whether or how Besicovitch's proof of Lemma 3 can be effectivized. Nevertheless we prove the following.

Theorem 4 (main theorem, in $\mathbb{R}^2$) *The set B has computable measure 0. Hence there is, in every direction in $\mathbb{R}^2$, a line that misses every computably random point.*

The remainder of this section is devoted to proving Theorem 4. We begin with a property of the line set operator.

Lemma 5 *Let I and J be closed intervals of finite length, so that $R = I \times J$ is a solid rectangle. Let $R' = (I \times J^\circ) \cup (I^\circ \times J)$ be R with its four corners removed. Then*

$$\mathcal{L}(R') \subseteq \mathcal{L}(R)^\circ \cup Y,$$

where $Y = \{(0,y) \mid y \in \mathbb{R}\}$ is the y-axis of $\mathbb{R}^2$.

Proof. See appendix.

Lemma 5 has the following consequence for the stages F_k in the construction of F.

Corollary 6 *For every $n \in \mathbb{N}$*

$$\mathcal{L}(F_{k+1}) \subseteq \mathcal{L}(F_k)^\circ \cup Y.$$

Proof. It suffices to note that F_{k+1} dos not contain any of the corners of the squares comprising F_k. □

Proof of Theorem 4. By (4) it suffices to prove that $\mathcal{L}(F)$ has computable measure 0. We do this by presenting a computable martingale d that succeeds at every point $\mathbf{x} \in \mathcal{L}(F)^\#$, where

$$\mathcal{L}(F)^\# = [0,1)^2 \cap \bigcup_{\mathbf{t} \in \mathbb{Z}^2} (\mathcal{L}(F) + \mathbf{t})$$

is defined as in (2). See appendix for details. □

5 Higher Dimensions

For every $n \in \mathbb{N}$, the set $B \times \mathbb{R}^n$ contains a line in every direction in $\mathbb{R}^{n+2}$, and Fubini's theorem implies that this set has Lebesgue measure 0 [11]. In this section we show that $B \times \mathbb{R}^n$ also has computable measure 0.

For any set $E \subseteq \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$, for $1 \leq m < n$, define

$$E_{\mathbf{y}} = \{(x_1, ..., x_{n-m}) \in \mathbb{R}^{n-m} \mid (x_1, ..., x_{n-m}, y_1, ..., y_m) \in E\}.$$

The following computable Fubini theorem may be known, but we do not know a reference at the time of this writing.

Theorem 7 *Let $E \in \mathbb{R}^n$. If there is a computable martingale d on $[0,1)^{n-m}$ such that the set*

$$N_E(d) = \{\mathbf{y} \in [0,1)^m \mid \exists\, \mathbf{x} \in E^{\#}_{\mathbf{y}} \textit{ such that } d \textit{ does not succeed at } \mathbf{x}\}$$

has computable measure 0, then E has computable measure 0.

Proof. Let d_1 be such a martingale for E and let d_2 be a computable martingale on $[0,1)^m$ that succeeds at every $\mathbf{y} \in N_E(d_1)$. Define two martingales on $[0,1)^n$, d_1' and d_2', by

$$d_1'(Q_r(u_1, ..., u_n)) = d_1(Q_r(u_1, ..., u_{n-m}));$$
$$d_2'(Q_r(u_1, ..., u_n)) = d_2(Q_r(u_{n-m+1}, ..., u_n)).$$

Note that both are computable.

Now let $\mathbf{x} = (x_1, ..., x_n) \in E^{\#}$. If $(x_{n-m+1}, ..., x_n) \in N_E(d_1)$, then d_2' succeeds at $\mathbf{x}$; otherwise, d_1' succeeds at $\mathbf{x}$. We conclude that the computable martingale $d = d_1' + d_2'$ succeeds at every $\mathbf{x} \in E^{\#}$, hence E has computable measure 0. □

Corollary 8 *For every computable measure 0 set E and $n \in \mathbb{N}$, the set $E \times \mathbb{R}^n$ has computable measure 0.*

Theorem 9 (main theorem, in $\mathbb{R}^n$) *For every $n \geq 2$ there is, in every direction in $\mathbb{R}^n$, a line that misses every computably random point.*

Proof. By Theorem 4, B has computable measure 0. Thus by Corollary 8, $B \times \mathbb{R}^{n-2}$ has computable measure 0 for every $n \geq 3$. □

6 Open Problems

It would be interesting to know whether there exist lines in every direction missing larger classes of random points. In particular, is there a line in every direction missing every polynomial time random point in Euclidean space?

Besicovitch's duality idea for constructing the set B came soon after, and was perhaps prompted by, the Mathematical Association of America's production of a film in which he explained his 1919 solution of the Kakeya needle problem. (The article [4] is based on this film.) Does a copy of this film still exist?

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Appendix

Proof of Theorem 4. Trivial martingale transformations show that the sets of computable measure 0 in $\mathbb{R}^2$ are closed under 90° rotations, reflections about the coordinate axes, and finite unions. Hence by (4) it suffices to prove that $\mathcal{L}(F)$ has computable measure 0. We do this by presenting a computable martingale d that succeeds at every point $\mathbf{x} \in \mathcal{L}(F)^{\#}$, where

$$\mathcal{L}(F)^{\#} = [0,1)^2 \cap \bigcup_{\mathbf{t}\in\mathbb{Z}^2} (\mathcal{L}(F) + \mathbf{t})$$

is defined as in (2).

For each $\mathbf{t} \in \mathbb{Z}^2$ and $k \in \mathbb{N}$ let

$$H_{\mathbf{t},k} = [0,1)^2 \cap (\mathcal{L}(F_k)^\circ + \mathbf{t}),$$

noting that $H_{\mathbf{t},k}$ is an open set in the subspace $[0,1)^2$ of $\mathbb{R}^2$. The sets F_k are so simply defined that the function $h : \mathbb{Z}^2 \times \mathbb{N} \to \mathbb{Q}$ defined by

$$h(\mathbf{t},k) = m(H_{\mathbf{t},k})$$

is computable. For each $\mathbf{t} \in \mathbb{Z}^2$ Lemma 3 tells us that

$$\begin{aligned}
0 &= m([0,1)^2 \cap (\mathcal{L}(F) + \mathbf{t})) \\
&= m(\bigcap_{k=0}^{\infty}([0,1)^2 \cap (\mathcal{L}(F_k) + \mathbf{t}))) \\
&= \lim_{k\to\infty} m([0,1)^2 \cap (\mathcal{L}(F_k) + \mathbf{t})) \\
&= \lim_{k\to\infty} h(\mathbf{t},k).
\end{aligned}$$

Hence the function $k : \mathbb{Z}^2 \times \mathbb{N} \to \mathbb{N}$ defined by

$$k(\mathbf{t},j) = \text{the least } k \text{ such that } g(\mathbf{t},k) \leq 2^{-j}$$

is also computable.

For each $\mathbf{t} = (t_1,t_2) \in \mathbb{Z}^2$ and $j \in \mathbb{N}$, define the set $G_{\mathbf{t},j}$ and the coefficient $c_{\mathbf{t},j}$ as follows.

(i) If $H_{\mathbf{t},k(\mathbf{t},|t_1|+|t_2|+j)} \neq \emptyset$, then

$$G_{\mathbf{t},j} = H_{\mathbf{t},k(\mathbf{t},|t_1|+|t_2|+j)}$$

and

$$c_{\mathbf{t},j} = m(G_{\mathbf{t},j}).$$

(ii) Otherwise,

$$G_{\mathbf{t},j} = [0,1)^2$$

and

$$c_{\mathbf{t},j} = 2^{-(|t_1|+|t_2|+j)}.$$

Define the special-purpose martingale d_Y by

$$d_Y(Q_r(\mathbf{u})) = \begin{cases} 2^r & \text{if } u_1 = 0 \\ 0 & \text{if } u_1 > 0 \end{cases}$$

for all $r \in \mathbb{N}$ and $\mathbf{u} = (u_1, u_2) \in \{0, ..., 2^r - 1\}^2$. Finally, let

$$d = d_Y + \sum_{\mathbf{t} \in \mathbb{Z}^2} \sum_{j=0}^{\infty} c_{\mathbf{t},j} d_{G_{\mathbf{t},j}} \ ,$$

where each $d_{G_{\mathbf{t},j}}$ is defined from $G_{\mathbf{t},j}$ as in section 3. Then

$$\begin{aligned} d([0,1)^2) &\leq 1 + \sum_{\mathbf{t} \in \mathbb{Z}^2} \sum_{j=0}^{\infty} 2^{-(|t_1|+|t_2|+j)} \\ &= 19 < \infty, \end{aligned}$$

so d is a martingale.

To see that d is computable, define

$$\widehat{d} : \mathbb{N} \times J \to \mathbb{Q}$$

(where J is defined as in section 2) by

$$\widehat{d}(s, r, \mathbf{u}) = d_Y(Q_r(\mathbf{u})) + \sum_{t_1=-p}^{p} \sum_{t_2=-p}^{p} \sum_{j=0}^{p} c_{\mathbf{t},j} d_{G_{\mathbf{t},j}}(Q_r(\mathbf{u})),$$

where $p = s + 2r + 6$. Then $\widehat{d}$ is computable, and it is clear that $\widehat{d}(s, r, \mathbf{u}) \leq d(Q_r(\mathbf{u}))$ holds for all $(s, r, \mathbf{u}) \in \mathbb{N} \times J$. We now fix $(s, r, \mathbf{u}) \in \mathbb{N} \times J$, let $p = s + 2r + 6$, and estimate the difference $d(Q_r(\mathbf{u})) - \widehat{d}(s, r, \mathbf{u})$.

For each index set $I \subseteq \mathbb{Z}^2 \times \mathbb{N}$ define the sums

$$\sigma(I) = \sum_{(\mathbf{t},j) \in I} c_{\mathbf{t},j} d_{G_{\mathbf{t},j}}(Q_r(\mathbf{u}))$$

and

$$\tau(I) = \sum_{(\mathbf{t},j) \in I} 2^{-(|t_1|+|t_2|+j)}.$$

By the trivial bound $d_{G_{\mathbf{t},j}}(Q_r(\mathbf{u})) \leq 4^r$ and the fact that $c_{\mathbf{t},j} \leq 2^{-(|t_1|+|t_2|+j)}$ always holds, we have

$$\sigma(I) \leq 4^r \tau(I)$$

for every $I \subseteq \mathbb{Z}^2 \times \mathbb{N}$. Now

$$d(Q_r(\mathbf{u})) = d_Y(Q_r(\mathbf{u})) + \sigma(\mathbb{Z}^2 \times \mathbb{N}),$$

and

$$\widehat{d}(s, r, \mathbf{u}) = d_Y(Q_r(\mathbf{u})) + \sigma(I_0),$$

where

$$I_0 = \{(t_1, t_2, j) \mid -p \le t_1 \le p,\ -p \le t_2 \le p,\ j \le p\},$$

so

$$\begin{aligned} d(Q_r(\mathbf{u})) - \widehat{d}(s, r, \mathbf{u}) &= \sigma((\mathbb{Z}^2 \times \mathbb{N}) \smallsetminus I_0) \\ &\le 4^r \tau((\mathbb{Z}^2 \times \mathbb{N}) \smallsetminus I_0). \end{aligned}$$

If we let

$$I_a = \big\{(t_1, t_2, j) \bigm| |t_a| > p\big\}$$

for $a \in \{1, 2\}$ and

$$I^+ = \big\{(t_1, t_2, j) \bigm| j > p\big\},$$

then

$$(\mathbb{Z}^2 \times \mathbb{N}) \smallsetminus I_0 \subseteq I_1 \cup I_2 \cup I^+,$$

so

$$d(Q_r(\mathbf{u})) - \widehat{d}(s, r, \mathbf{u}) \le 4^r(\tau(I_1) + \tau(I_2) + \tau(I^+)).$$

Now

$$\begin{aligned} \tau(I_1) &= \tau(I_2) \\ &= 2 \sum_{t_1=p+1}^{\infty} 2^{-t_1} \sum_{t_2=-\infty}^{\infty} 2^{-|t_2|} \sum_{j=0}^{\infty} 2^{-j} \\ &= 12 \sum_{t_1=p+1}^{\infty} 2^{-t_1} \\ &= 12 \cdot 2^{-p}, \end{aligned}$$

and

$$\begin{aligned} \tau(I^+) &= \sum_{t_1=-\infty}^{\infty} 2^{-|t_1|} \sum_{t_2=-\infty}^{\infty} 2^{-|t_2|} \sum_{j=p+1}^{\infty} 2^{-j} \\ &= 9 \cdot 2^{-p}, \end{aligned}$$

so

$$\begin{aligned} d(Q_r(\mathbf{u})) - \widehat{d}(s, r, \mathbf{u}) &\le 4^r \cdot 33 \cdot 2^{-p} \\ &= 33 \cdot 2^{-(s+6)} \\ &< 2^{-s} \end{aligned}$$

Hence $\widehat{d}$ testifies that d is computable.

To see that d succeeds at every point in $\mathcal{L}(F)^{\#}$, let $\mathbf{x} \in [0, 1)^2 \cap (\mathcal{L}(F) + \mathbf{t})$. By Corollary 6 we have two cases.

Case 1. $\mathbf{x} \in Y$. Then

$$\begin{aligned}\limsup_{r\to\infty} d^{(r)}(\mathbf{x}) &\geq \limsup_{r\to\infty} d_Y^{(r)}(\mathbf{x}) \\ &= \limsup_{r\to\infty} 2^r \\ &= \infty,\end{aligned}$$

so d succeeds at $\mathbf{x}$.

Case 2. $\mathbf{x} \in \mathcal{L}(F_k)$ for every $k \in \mathbb{N}$. Then $\mathbf{x} \in H_{\mathbf{t},k}$ for every $k \in \mathbb{N}$, so clause (i) holds in the definitions of $G_{\mathbf{t},j}$ and $c_{\mathbf{t},j}$ for every $j \in \mathbb{N}$, with $\mathbf{x} \in G_{\mathbf{t},j}$. By Theorem 1, this implies that

$$\begin{aligned}\limsup_{r\to\infty} d^{(r)}(\mathbf{x}) &\geq \limsup_{r\to\infty} \sum_{j=0}^{\infty} c_{\mathbf{t},j} d_{G_{\mathbf{t},j}}^{(r)}(\mathbf{x}) \\ &= \limsup_{r\to\infty} \sum_{j=0}^{\infty} m(G_{\mathbf{t},j}) d_{G_{\mathbf{t},j}}^{(r)}(\mathbf{x}) \\ &= \infty,\end{aligned}$$

whence d succeeds at $\mathbf{x}$. □

Proof of Lemma 5. First let $(m, b) \in I \times J^\circ$. Then there exists $\varepsilon > 0$ such that $\{m\} \times (b - \varepsilon, b + \varepsilon) \subseteq I \times J$. Then $\mathcal{L}_{m,b'} \subseteq \mathcal{L}(I \times J) = \mathcal{L}(R)$ holds for all $b' \in (b - \varepsilon, b + \varepsilon)$, so $\mathcal{L}_{m,b} \subseteq \mathcal{L}(R)^\circ$. This shows that $\mathcal{L}(I \times J^\circ) \subseteq \mathcal{L}(R)^\circ$.

Now let $(m, b) \in I^\circ \times J$. Then there exists $\varepsilon > 0$ such that $(m-\varepsilon, m+\varepsilon) \times \{b\} \subseteq I \times J$. Then $\mathcal{L}_{m',b} \subseteq \mathcal{L}(I, J) = \mathcal{L}(R)$ holds for all $m' \in (m - \varepsilon, m + \varepsilon)$, so $\mathcal{L}_{m,b} \subseteq \mathcal{L}(R)^\circ \cup \{(0, b)\} \subseteq \mathcal{L}(R)^\circ \cup Y$. This shows that $\mathcal{L}(I^\circ \times J) \subseteq \mathcal{L}(R) \cup Y$. □

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