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Fooling Pairs in Randomized Communication Complexity

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Abstract

Fooling pairs are one of the standard methods for proving lower bounds for deterministic two-player communication complexity. We study fooling pairs in the context of randomized communication complexity. We show that every fooling pair induces far away distributions on transcripts of private-coin protocols. We then conclude that the private-coin randomized ε-error communication complexity of a function f with a fooling set $\mathcal{S}$ is at least order $\log \frac{\log|\mathcal{S}|}{\varepsilon}$. This is tight, for example, for the equality and greater-than functions.

1 Introduction

Communication complexity provides a mathematical framework for studying communication between two or more parties. It was introduced by Yao [Yao79] and has found numerous applications since. We focus on the two-player case, and provide a brief introduction to it. For more details see the textbook by Kushilevitz and Nisan [KN97].

In this model, there are two players called Alice and Bob. The players wish to compute a function $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$, where Alice knows $x \in \mathcal{X}$ and Bob knows $y \in \mathcal{Y}$. To achieve this goal, they need to communicate. The *communication complexity* of f measures the minimum number of bits the players must exchange in order to compute f. The communication is done according to a pre-determined protocol. Protocol may be deterministic or use private/public randomness. See Section 1.1 for definitions.

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A fundamental problem in this context is proving lower bounds on the communication complexity of a given function f. Lower bounds methods for deterministic communication complexity are based on the fact that any protocol for f defines a partition of $\mathcal{X} \times \mathcal{Y}$ to f-monochromatic rectangles. Thus, a lower bound on the size of a minimal partition of this kind readily translates to a lower bound on the communication complexity of f. Three basic bounds of this type are based on rectangle size, fooling sets, and matrix rank (see [KN97]). Both matrix rank and rectangle size lower bounds have natural and well-known analogues in the randomized setting: the approximate rank lower bound [LS09, Kra96] and the discrepancy lower bound [KN97] respectively. In this note we show that fooling sets also have natural counterparts in the randomized setting. A weaker variant of the structure we present is implicit in [BYJKS02], where it is used as part of a lower bound proof for the randomized communication complexity of the disjointness function.

1.1 Communication complexity

A *private-coin communication protocol* for computing a function $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ is a binary tree with the following generic structure. Each node in the protocol is owned either by Alice or by Bob. For every $x \in \mathcal{X}$, each internal node v owned by Alice is associated with a distribution $P_{v,x}$ on the children of v. Similarly, for every $y \in \mathcal{Y}$, each internal node v owned by Bob is associated with a distribution $P_{v,y}$ on the children of v. The leaves of the protocol are labeled by $\mathcal{Z}$.

On input x, y, a protocol π is executed as follows.

1. Set v to be the root node of the protocol-tree defined above.
2. If v is a leaf, then the protocol outputs the label of the leaf. Otherwise, if Alice owns the node v, she samples a child according to the distribution $P_{v,x}$ and sends a bit to Bob indicating which child was sampled. The case when Bob owns the node is analogous.
3. Set v to be the sampled node and return to the previous step.

A protocol is *deterministic* if for every internal node v, the distribution $P_{v,x}$ or $P_{v,y}$ has support of size one. A *public-coin* protocol is a distribution over private-coin protocols defined as follows: Alice and Bob first sample a shared random r to choose a protocol π_r, and they execute a private protocol π_r as above.

For an input (x, y), we denote by $\pi(x, y)$ the sequence of messages exchanged between the parties. We call $\pi(x, y)$ the *transcript* of the protocol π on input (x, y). Another way to think of $\pi(x,y)$ is as a leaf in the protocol-tree. We denote by $L(\pi(x,y))$ the label of the leaf $\pi(x,y)$ in the tree. The *communication complexity* of a protocol π, denoted by $\mathsf{CC}(\pi)$ is the depth of the protocol-tree of π. For a private-coin protocol π, we denote by $\Pi(x,y)$ the distribution of the transcript of $\pi(x,y)$.

For a function f, the *deterministic* communication complexity of f, denoted by $D(f)$, is the minimum of $\mathsf{CC}(\pi)$ over all deterministic protocols π such that $L(\pi(x,y)) = f(x,y)$ for every x, y. For $\varepsilon > 0$, we denote by $R_\varepsilon(f)$ the minimum of $\mathsf{CC}(\pi)$ over all public-coin protocols π such that for every (x,y), it holds that $\mathbb{P}[L(\pi(x,y)) \neq f(x,y)] \leq \varepsilon$ where the probability is taken over all coin flips in the protocol π. We call $R_\varepsilon(f)$ the *ε-error public-coin randomized* communication complexity of f. Analogously we define $R_\varepsilon^{\mathsf{pri}}(f)$ as the *ε-error private-coin* randomized communication complexity.

Although public-coin protocols are more general than private-coin ones, Newman [New91] proved that for boolean functions every public-coin protocol can be efficiently simulated by a private-coin protocol: If $f : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$ then for every $0 < \varepsilon < 1/2$,

$$R_{2\varepsilon}(f) \leq R_{2\varepsilon}^{\mathsf{pri}}(f) = O\left(R_\varepsilon(f) + \log \frac{\log(|\mathcal{X}||\mathcal{Y}|)}{\varepsilon}\right).$$

The additive logarithmic factor on the right-hand-side is often too small to matter, but it does make a difference in the bounds we prove below.

1.2 Fooling pairs and sets

Fooling sets are a well-known tool for proving lower bounds for $D(f)$. A pair $(x,y), (x',y') \in \mathcal{X} \times \mathcal{Y}$ is called *a fooling pair* for $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ if

- $f(x,y) = f(x',y')$, and
- either $f(x',y) \neq f(x,y)$ or $f(x,y') \neq f(x,y)$.

Observe that if (x,y) and (x',y') are a fooling pair then $x \neq x'$ and $y \neq y'$.

It is easy to see that if (x,y) and (x',y') form a fooling pair then there is no f-monochromatic rectangle that contains both of them. An immediate conclusion is the following:

Lemma 1.1 ([KN97]). *Let $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ be a function, let (x,y) and (x',y') be a fooling pair for f and let π be a deterministic protocol for f. Then*

$$\pi(x,y) \neq \pi(x',y').$$

A subset $\mathcal{S} \subseteq \mathcal{X} \times \mathcal{Y}$ is a *fooling set* if every $p \neq p'$ in $\mathcal{S}$ form a fooling pair. Lemma 1.1 implies the following basic lower bound for deterministic communication complexity.

Theorem 1.2 ([KN97]). *Let $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ be a function and let $\mathcal{S}$ be a fooling set for f. Then*

$$D(f) \geq \log_2(|\mathcal{S}|).$$

The same properties do not hold for randomized protocol, but the following variants are true. Let π be an ε-error private-coin protocol for f, and let $(x, y), (x', y')$ be a fooling pair for f.

Here we prove that the probabilistic analogue of $\pi(x, y) \neq \pi(x', y')$ holds: we have that $|\Pi(x, y) - \Pi(x', y')|$ is large, where $|\Pi(x, y) - \Pi(x', y')|$ is the statistical distance between the two distributions on transcripts.

Lemma 1.3 (Analogue of Lemma 1.1). *Let $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ be a function, let (x, y) and (x', y') be a fooling pair for f, and let π be an ε-error private-coin protocol for f. Then*

$$|\Pi(x, y) - \Pi(x', y')| \geq 1 - 2\sqrt{\varepsilon}.$$

Lemma 1.3 is not only an analogue of Lemma 1.1 but is actually a generalization of it. Indeed, plugging $\varepsilon = 0$ in Lemma 1.3 implies Lemma 1.1. Moreover, it implies that the bound from Theorem 1.2 holds also in the 0-error private-coin randomized case.

An analogue of Theorem 1.2 holds as well:

Theorem 1.4 (Analogue of Theorem 1.2). *Let $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ be a function and let $\mathcal{S}$ be a fooling set for f. Let $1/|\mathcal{S}| \leq \varepsilon < 1/3$. Then,*

$$R_{\varepsilon}^{\mathsf{pri}}(f) = \Omega\left(\log \frac{\log |\mathcal{S}|}{\varepsilon}\right).$$

The lower bound provided by the theorem above seems exponentially weaker than the one in Theorem 1.2, but it is tight. The equality function EQ over n-bit strings has a large fooling set of size 2^n, but it is well-known (see [KN97]) that

$$R_{\varepsilon}^{\mathsf{pri}}(\mathsf{EQ}) = O\left(\log \frac{n}{\varepsilon}\right).$$

Theorem 1.4 therefore provide a tight lower bound on $R_{\varepsilon}^{\mathsf{pri}}(\mathsf{EQ})$ in terms of both n and ε. It also provides a tight lower bound for the greater-than function. Moreover, Theorem 1.4 is a generalization of Theorem 1.2 and basically implies it by choosing $\varepsilon = 1/|\mathcal{S}|$.

The proof of the lower bound uses a general lower bound on the rank of perturbed identity matrices by Alon [Alo09]. Interestingly, although not every fooling set comes from an identity matrix (e.g. in the greater-than function), there is always some perturbed identity matrix in the background (the one used in the proof of Theorem 1.4).

We remark that for any constant $0 < \varepsilon < 1/3$, a version of Theorem 1.4 has been known for a long time. In particular, Håstad and Wigderson [HW07] give a proof of the following result[1] which appears in [Yao79] without proof: for every function f with a fooling set $\mathcal{S}$ and for every $0 < \varepsilon < 1/3$,

$$R_\varepsilon^{\mathsf{pri}}(f) = \Omega\left(\log\log|\mathcal{S}|\right). \tag{1.1}$$

The right-hand side above does not depend on ε. The same lower bound as in (1.1) also directly follows from Theorem 1.2 and from the following general result [KN97]: for every function f and for every $0 \leq \varepsilon < 1/2$,

$$R_\varepsilon^{\mathsf{pri}}(f) = \Omega(\log D(f)).$$

1.3 Two types of fooling pairs

Let $(x, y), (x', y')$ be a fooling pair for a boolean function f. For simplicity, consider the case $(x, y) = (0, 0)$ and $(x', y') = (1, 1)$. There are essentially two types of fooling pairs:

- The AND type for which $f(1, 0) \neq f(0, 1)$.
- The XOR type for which $f(1, 0) = f(0, 1)$.

A partial proof of Lemma 1.3 is implicit in [BYJKS02]. The case considered in [BYJKS02] corresponds to a fooling pair of the AND type. Let π be a private-coin ε-error protocol for f that is the AND of two bits. In this case, by definition it must hold that $\Pi(0, 0)$ is far away from $\Pi(1, 1)$. The cut-and-paste property (see Corollary 2.2) implies that the same holds for $\Pi(0, 1)$ and $\Pi(1, 0)$.

The case of a fooling pair of the XOR type was not analyzed before. If π is a private-coin ε-error protocol for XOR of two bits, then it does not immediately follow that $\Pi(0, 0)$ is far away from $\Pi(1, 1)$, nor that $\Pi(0, 1)$ is far away from $\Pi(1, 0)$. Lemma 1.3 implies that in fact both are true, but the argument can not use the cut-and-paste property. Our argument actually gives a better quantitative result for the XOR function as compared to the AND function.

The importance of the special case of Lemma 1.3 from [BYJKS02] is related to proving a lower bound on the randomized communication complexity of the disjointness

[1] In fact, the theorem in [Yao79, HW07] is more general than the one stated here. We state the theorem in this form since it fits well the focus of this text.

function DISJ defined over $\{0,1\}^n \times \{0,1\}^n$: $\mathsf{DISJ}(x,y) = 1$ if for all $i \in [n]$ it holds that $x_i \wedge y_i = 0$. They reproved that $R_{1/3}(\mathsf{DISJ}) \geq \Omega(n)$. This lower bound is extremely important and useful in many contexts and was first proved in [KS92].

On a high level, the proof of [BYJKS02] can be summarized in today's language as follows: Let π be a private-coin protocol with (1/3)-error for DISJ. We want to show that $\mathsf{CC}(\pi) = \Omega(n)$. The argument has two different parts: The first part of the argument essentially relates the *internal information cost* (as was later defined in [BBCR13]) of computing one copy of the AND function with the communication of the protocol π for DISJ. This is a direct-sum-esque result. More concretely, if μ is a distribution on $\{0,1\}^2$ such that $\mu(1,1) = 0$ then

$$\mathsf{IC}_\mu(\mathsf{AND}) \leq \frac{\mathsf{CC}(\pi)}{n},$$

where $\mathsf{IC}_\mu(\mathsf{AND})$ is the infimum over all (1/3)-error private-coin protocols τ for AND of the internal information cost of τ. The second part of the argument shows that if μ is uniform on the set $\{(0,0),(0,1),(1,0)\}$ then $\mathsf{IC}_\mu(\mathsf{AND}) > 0$. The challenge in proving the second part stems from the fact that μ is supported on the zeros of AND, so it is trivial to compute AND on inputs from μ. However, the protocols τ in the definition of $\mathsf{IC}_\mu(\mathsf{AND})$ are guaranteed to succeed for every x, y and not only on the support of μ. The authors of [BYJKS02] use the cut-and-paste property (see Corollary 2.2 below) to argue that indeed $\mathsf{IC}_\mu(\mathsf{AND}) > 0$.

The argument as described above is very specific to the AND function. Here we show that it follows from a more general fooling-set based method.

We conclude this discussion with another observation concerning the difference between the two types of fooling pairs. For any positive integer k, let $\mathsf{EQ}_k : [k] \times [k] \to \{0,1\}$ be the equality function on elements of the set $[k]$. Consider the following two seemingly similar functions on a pair of n-tuples of elements:

$$f_2(x,y) = \bigvee_{i=1}^{n} \mathsf{EQ}_2(x_i, y_i) \quad \text{and} \quad f_3(x,y) = \bigvee_{i=1}^{n} \mathsf{EQ}_3(x_i, y_i).$$

Since all the fooling pairs of EQ_2 are of the XOR type, the (1/3)-error private-coin communication complexity and internal information cost of f_2 are $O(\log n)$ (f_2 is basically equality on n-bit strings). On the other hand, since EQ_3 contains a fooling pair of the AND type, the (1/3)-error private-coin complexity and internal information cost of f_3 are $\Omega(n)$.

2 Fooling pairs and sets

2.1 Preliminaries

Communication. In the case of deterministic protocols, the set of inputs reaching a particular leaf forms a rectangle (a product set inside $\mathcal{X} \times \mathcal{Y}$). In the case of private-coin randomized protocols, the following holds (see for example Lemma 6.7 in [BYJKS02]).

Lemma 2.1 (Rectangle property for private-coin protocols)**.** *Let π be a private-coin protocol over inputs from $\mathcal{X} \times \mathcal{Y}$, and let $\mathcal{L}$ denote the set of leaves of π. There exist functions $\alpha : \mathcal{L} \times \mathcal{X} \to [0,1]$, $\beta : \mathcal{L} \times \mathcal{Y} \to [0,1]$ such that for every $(x,y) \in \mathcal{X} \times \mathcal{Y}$ and every $\ell \in \mathcal{L}$,*

$$\mathbb{P}[\pi(x,y) \text{ reaches } \ell] = \alpha(\ell,x) \cdot \beta(\ell,y).$$

Here too the lemma is in fact a generalization of what happens in the deterministic case where α, β take values in $\{0,1\}$ rather than in $[0,1]$.

The above implies the following property of private-coin protocols that is more commonly known as the cut-and-paste property [PS86, CK91]. The *Hellinger* distance between two distributions p, q over a finite set $\mathcal{U}$ is defined as

$$h(p,q) = \sqrt{1 - \sum_{u \in \mathcal{U}} \sqrt{p(u)q(u)}}.$$

Corollary 2.2 (Cut-and-paste property)**.** *Let (x,y) and (x',y') be inputs to a randomized private-coin protocol π. Then*

$$h(\Pi(x,y), \Pi(x',y')) = h(\Pi(x',y), \Pi(x,y')).$$

The next proposition immediately follows from the definitions.

Proposition 2.3. *Let $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ be a function and let (x,y) and (x',y') be such that $f(x,y) \neq f(x',y')$. Then, for any ε-error private-coin protocol π for f,*

$$|\Pi(x,y) - \Pi(x',y')| \geq 1 - 2\varepsilon.$$

Distances. We use the following relationship between Statistical and Hellinger Distances.

Lemma 2.4 (Statistical and Hellinger Distances)**.** *Let p and q be distributions such that the statistical distance $|p - q| \geq 1 - \varepsilon$ for $0 \leq \varepsilon \leq 1$. Then, $h^2(p,q) \geq 1 - \sqrt{2\varepsilon}$.*

Proof. In general, $h^2(p,q) \leq |p-q| \leq \sqrt{h^2(p,q)(2-h^2(p,q))}$. □

A geometric claim. We use the following technical claim that has a geometric flavor. For two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$, we denote by $\langle \mathbf{a}, \mathbf{b} \rangle$ the standard inner product between $\mathbf{a}, \mathbf{b}$. Denote by $\mathbb{R}_+$ the set of non-negative real numbers.

Claim 2.5. *Let $\varepsilon_1, \varepsilon_2, \delta_1, \delta_2 > 0$ and let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}_+^m$ be vectors such that*

$$\langle \mathbf{a}, \mathbf{b} \rangle \geq 1 - \varepsilon_1, \qquad \langle \mathbf{c}, \mathbf{d} \rangle \geq 1 - \varepsilon_2,$$
$$\langle \mathbf{a}, \mathbf{c} \rangle \leq \delta_1, \qquad \langle \mathbf{b}, \mathbf{d} \rangle \leq \delta_2.$$

Then,

$$\sum_{i \in [m]} |\mathbf{a}(i)\mathbf{b}(i) - \mathbf{c}(i)\mathbf{d}(i)| \geq 2 - (\varepsilon_1 + \varepsilon_2 + \delta_1 + \delta_2).$$

Proof.

$$\begin{aligned}
&\sum_{i \in [m]} |\mathbf{a}(i)\mathbf{b}(i) - \mathbf{c}(i)\mathbf{d}(i)| \\
&\quad \geq \sum_{i \in [m]} \left(\sqrt{\mathbf{a}(i)\mathbf{b}(i)} - \sqrt{\mathbf{c}(i)\mathbf{d}(i)}\right)^2 && \left(\forall t, s \geq 0 \ \ |t-s| \geq \left(\sqrt{t} - \sqrt{s}\right)^2\right) \\
&\quad = \langle \mathbf{a}, \mathbf{b} \rangle + \langle \mathbf{c}, \mathbf{d} \rangle - \sum_{i \in [m]} 2\sqrt{\mathbf{a}(i)\mathbf{b}(i)\mathbf{c}(i)\mathbf{d}(i)} \\
&\quad = \langle \mathbf{a}, \mathbf{b} \rangle + \langle \mathbf{c}, \mathbf{d} \rangle - \sum_{i \in [m]} 2\sqrt{\mathbf{a}(i)\mathbf{c}(i) \cdot \mathbf{b}(i)\mathbf{d}(i)} \\
&\quad \geq \langle \mathbf{a}, \mathbf{b} \rangle + \langle \mathbf{c}, \mathbf{d} \rangle - \sum_{i \in [m]} (\mathbf{a}(i)\mathbf{c}(i) + \mathbf{b}(i)\mathbf{d}(i)) && \text{(AM-GM inequality)} \\
&\quad = \langle \mathbf{a}, \mathbf{b} \rangle + \langle \mathbf{c}, \mathbf{d} \rangle - (\langle \mathbf{a}, \mathbf{c} \rangle + \langle \mathbf{b}, \mathbf{d} \rangle) \\
&\quad \geq 2 - (\varepsilon_1 + \varepsilon_2 + \delta_1 + \delta_2).
\end{aligned}$$

□

2.2 Fooling pairs induce far away distributions

Proof of Lemma 1.3. Let the fooling pair be (x, y) and (x', y') and assume without loss of generality that $f(x, y) = f(x', y') = 1$. We distinguish between the following two cases.

(a) $f(x', y) \neq f(x, y')$.

(b) $f(x', y) = f(x, y') = z$ where $z \neq 1$.

In the first case, Proposition 2.3 implies that $|\Pi(x', y) - \Pi(x, y')| \geq 1 - 2\varepsilon$. Proposition 2.2 implies that $h(\Pi(x, y), \Pi(x', y')) = h(\Pi(x', y), \Pi(x, y'))$. Lemma 2.4 thus implies that $|\Pi(x, y) - \Pi(x', y')| \geq 1 - 2\sqrt{\varepsilon}$.

Let us now consider the second case. Let $\mathcal{L}$ be the set of all leaves of π and let $\mathcal{L}_1$ denote those leaves which are labeled by 1. For $x \in \mathcal{X}$, $y \in \mathcal{Y}$, define the vectors $\mathbf{a}_x \in \mathbb{R}_+^{\mathcal{L}_1}$ as $\mathbf{a}_x(\ell) = \alpha(\ell, x)$, and the vectors $\mathbf{b}_y \in \mathbb{R}_+^{\mathcal{L}_1}$ as $\mathbf{b}_y(\ell) = \beta(\ell, y)$ where α and β are the functions from Lemma 2.1. Since $f(x, y) = 1$ and π is an ε-error protocol for f,

$$\langle \mathbf{a}_x, \mathbf{b}_y \rangle = \sum_{\ell \in \mathcal{L}_1} \alpha(\ell, x) \cdot \beta(\ell, y) = \mathbb{P}[L(\pi(x, y)) = 1] \geq 1 - \epsilon.$$

Similarly, we have $\langle \mathbf{a}_{x'}, \mathbf{b}_{y'} \rangle \geq 1 - \varepsilon$, $\langle \mathbf{a}_x, \mathbf{b}_{y'} \rangle \leq \varepsilon$ and $\langle \mathbf{a}_{x'}, \mathbf{b}_y \rangle \leq \varepsilon$. Observe

$$2|\Pi(x, y) - \Pi(x', y')| \geq \sum_{\ell \in \mathcal{L}_1} |\mathbf{a}_x(\ell)\mathbf{b}_y(\ell) - \mathbf{a}_{x'}(\ell)\mathbf{b}_{y'}(\ell)|.$$

Applying Proposition 2.5 with the vectors $\mathbf{a}_x, \mathbf{b}_y, \mathbf{a}_{x'}, \mathbf{b}_{y'}$ yields that $|\Pi(x, y) - \Pi(x', y')| \geq 1 - 2\varepsilon$. □

2.3 A lower bound based on fooling sets

The following result of Alon [Alo09] on the rank of perturbed identity matrices is a key ingredient.

Lemma 2.6. *Let $\frac{1}{2\sqrt{m}} \leq \varepsilon < \frac{1}{4}$. Let M be an $m \times m$ matrix such that $|M(i, j)| \leq \varepsilon$ for all $i \neq j$ in $[m]$ and $|M(i, i)| \geq \frac{1}{2}$ for all $i \in [m]$. Then,*

$$\mathsf{rank}(M) = \Omega\left(\frac{\log m}{\varepsilon^2 \log(\frac{1}{\varepsilon})}\right).$$

Proof of Theorem 1.4. Let $\mathcal{L}$ denote the set of leaves of π. Let $A \in \mathbb{R}^{\mathcal{S} \times \mathcal{L}}$ be the matrix defined by

$$A_{(x,y),\ell} = \sqrt{\mathbb{P}[\pi(x, y) = \ell]}.$$

Let

$$M = AA^T$$

where A^T is A transposed. First,

$$M_{(x,y),(x,y)} = 1.$$

Second, if $(x, y) \neq (x', y')$ in $\mathcal{S}$ then by Lemma 1.3 we know $|\Pi(x, y) - \Pi(x', y')| \geq 1 - 2\sqrt{\varepsilon}$

so by Lemma 2.4

$$h^2(\Pi(x,y),\Pi(x',y')) \geq 1 - 2\varepsilon^{1/4}$$

which implies

$$M_{(x,y),(x',y')} = 1 - h^2(\Pi(x,y),\Pi(x',y')) \leq 2\varepsilon^{1/4}.$$

Lemma 2.6 implies that[2] the rank of M is at least $\Omega\left(\frac{\log|\mathcal{S}|}{\sqrt{\varepsilon}\log\left(\frac{1}{\varepsilon^{1/4}}\right)}\right) = \Omega\left(\left(\frac{\log|\mathcal{S}|}{\varepsilon}\right)^{1/4}\right)$. On the other hand,

$$2^{CC(\pi)} \geq |\mathcal{L}| \geq \mathsf{rank}(M).$$

□

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[2] We may assume that say $\varepsilon < 2^{-12}$ by repeating the given randomized protocol a constant number of times.

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