Electronic Colloquium on Computational Complexity, Report No. 79 (2014)

Counting the Number of Perfect Matchings in K_5-free Graphs

Simon Straub, Ulm University
Thomas Thierauf, Aalen University
Fabian Wagner, Ulm University

June 11, 2014

Abstract

Counting the number of perfect matchings in arbitrary graphs is a #P-complete problem. However, for some restricted classes of graphs the problem can be solved efficiently. In the case of planar graphs, and even for $K_{3,3}$-free graphs, Vazirani showed that it is in NC^2. The technique there is to compute a *Pfaffian orientation* of a graph.

In the case of K_5-free graphs, this technique will not work because some K_5-free graphs do not have a Pfaffian orientation. We circumvent this problem and show that the number of perfect matchings in K_5-free graphs can be computed in polynomial time. We also parallelize the sequential algorithm and show that the problem is in TC^2.

1 Introduction

Counting the number of perfect matchings in bipartite graphs is #P-complete [Val79]. Nonetheless for some classes of restricted graphs the problem can be solved efficiently. Kasteleyn [Kas67] showed that it is in P for planar graphs. The technique is to compute a Pfaffian orientation. Such an orientation assures that each term in the Pfaffian of the Tutte matrix of a graph has the same sign. Since every term in the Pfaffian corresponds to a perfect matching, this yields an efficient algorithm.

The Pfaffian calculation over skew-symmetric integer matrices is GapL-complete [MSV04]. The result is based on a preliminary work on determinants of Mahajan and Vinay [MV97]. Hence computing the number of perfect matchings in a planar graph is in GapL. Counting the number of perfect matchings is necessary to prove the best known upper bound SPL on the construction problem for bipartite planar graphs [DKR10]. Despite counting is possible in planar graphs it remains an intriguing open question if a planar perfect matching can be constructed in parallel. On arbitrary graphs perfect matchings are constructible in P although counting is #P-hard [Edm65, MV80].

Little [Lit74] showed that any $K_{3,3}$-free graph has a Pfaffian orientation which can be computed in P. Vazirani [Vaz89] improved the bound to NC^2. They utilize the fact that any biconnected graph can be decomposed into triconnected components and each component is planar or the K_5.

A challenging open problem is to compute the number of perfect matchings in K_5-free graphs efficiently. In this paper, we solve this problem. The Pfaffian orientation technique is not applicable here because some K_5-free graphs have no such orientation. The $K_{3,3}$ is an example. We solve this problem by decomposing a given K_5-free graph G into its triconnected components. It is known that non-planar triconnected components of G are either the Möbius ladder M_8, or their decomposition into 4-connected components yields only

ISSN 1433-8092

planar components. Such a decomposition can be computed in logspace [DNTW09, TW09]. The number of perfect matchings in the planar components can be computed efficiently. The major task to be accomplished here is to calculate the overall number of perfect matchings from them. The difficulty thereby is, that the graph is decomposed along separating pairs or triples which have a copy in every component they split off. However, when counting perfect matchings, the nodes of the separating pairs and triples should be matched only in one of the components. We use gadgets to replace components such that every node is matched only in one component. We adapt some of Valiants matchgates as gadgets to prove our result. Valiant [Val08] introduced the notion of *holographic algorithms* and showed various counting problems to be in P. Thereby matchgates are crucial, which constitute matchgrids.

After some preliminaries, the decomposition of a K_5-free graph and its properties are explained. Subsequently it is shown how to obtain the number of perfect matchings of the input graph based on this decomposition.

2 Definitions and Notations

A graph $G = (V, E)$ consists of a finite set of vertices $V(G) = V$ and edges $E(G) = E \subseteq V \times V$. Graph G is called *weighted* if there is a weight function $w : E \to \mathbb{R}$. For $U \subseteq V$ let $G - U$ be the *induced subgraph* of G on $V - U$.

A graph G is called *undirected* if E is symmetric. An undirected graph G is *connected* if there is a path between any two vertices in G.

A *tree* T is an undirected, connected, acyclic graph. The *root* of a tree is a designated vertex r in the tree. The *parent* of a vertex v in T is the neighbor of v along a simple path from v to the root of T. The other neighbors of v are *children* of v. Let $T(v)$ be the subtree of T with root v. The number of nodes in $T(v)$ is denoted by $|T(v)|$. A child u of v is called a *large child* if $|T(u)| > |T(v)|/2$. A *large child path* in T is a path of maximal length such that every node on the path, except the first node, is a large child of its parent in T.

Let G be an undirected graph and $S \subseteq V$ with $|S| = k$. We call S a *k-separating set*, if $G - S$ is not connected. For $u, v \in V$ we say that *S separates u from v in G*, if $u \in S$, $v \in S$, or u and v are in different components of $G - S$. For sets of vertices $V_1, V_2 \subseteq V$ we say that *S separates V_1 from V_2 in G*, if S separates every $v_1 \in V_1$ from every $v_2 \in V_2$.

A k-separating set is called *articulation point* (or *cut vertex*) for $k = 1$, *separating pair* for $k = 2$, and *separating triple* for $k = 3$.

A graph G is *k-connected* if it contains no $(k-1)$-separating set. Hence a 1-connected graph is simply a connected graph. A 2-connected graph is also called *biconnected*, a 3-connected graph is one type of *triconnected* graphs.

Let S be a k-separating set in a k-connected graph G. Let G' be a connected component in $G - S$. A *split graph* or a *split component of* S in G is the induced subgraph of G on vertices $V(G') \cup S$, where we add *virtual edges* between all pairs of vertices in S. Note that the vertices of a separating set S occur in several split graphs of G.

A *$K_{3,3}$-free graph* is an undirected graph which does not contain a $K_{3,3}$ as a minor. A *K_5-free graph* is an undirected graph which does not contain a K_5 as a minor. In particular, planar graphs are $K_{3,3}$-free and K_5-free [Wag37].

Let $G = (V, E)$ be an undirected graph, $|V| = n$. A *perfect matching* in G is a set $M \subseteq E$ such that every vertex of G occurs in exactly one edge of M. A consequence is that $|M| = n/2$.

The number of perfect matchings in G is denoted by $\#\mathrm{pm}(G)$. We extend the notion to weighted graphs. Let w be a weight funtion. The *weighted number of perfect matchings* in G is defined as

$$\#\mathrm{pm}(G) = \sum_{M} \prod_{(u,v)\in M} w(u,v)\,,$$

where the sum is over all perfect matchings M in G. Weight function $w(u,v) = 1$ for all $(u,v) \in E$ is equivalent to the unweighted case.

For the definition of complexity classes we refer the reader to any standard text book, for example [Vol99]. In short, circuit classes NC^k, AC^k, and TC^k consist of polynomial size circuits of depth $O(\log^k n)$, where n is the length of the input. NC-circuits have fan-in two and-or-gates, whereas AC-circuits have unbounded fan-in and-or-gates. TC-circuits have unbounded fan-in gates as AC, and additionally threshold gates. It is known that for all $k \geq 0$

$$\mathrm{NC}^k \subseteq \mathrm{AC}^k \subseteq \mathrm{TC}^k \subseteq \mathrm{NC}^{k+1}\,.$$

The ciruit classes are interleaved by logspace complexity classes. Class L stands for problems recognized by logspace bounded Turing machines, NL is its nondeterministic counterpart. The function class #L counts the number of accepting computations of a nondeterministic logspace bounded Turing machine. An extension of #L is GapL which is the *difference* of the number of accepting and rejecting computations of a nondeterministic logspace bounded Turing machine. It is known that

$$\mathrm{NC}^1 \subseteq \mathrm{L} \subseteq \mathrm{NL} \subseteq \mathrm{AC}^1$$

and

$$\#\mathrm{L} \subseteq \mathrm{GapL} \subseteq \mathrm{TC}^1 \subseteq \mathrm{NC}^2\,.$$

The interest for GapL stems from the fact that it characterizes the complexity of computing the determinant and the Pfaffian of integer matrices. Also, the number of perfect matchings in planar graphs can be computed in GapL [MSV04]. Our parallel algorithm to count the number of perfect matchings in K_5-free graphs has up to $\log n$ levels where perfect matchings are counted in planar components. This will result in a TC^2-circuit.

3 Decomposition of Graphs

Wagner [Wag37] studied the decomposition of K_5-free graphs into 2-, 3- and 4-connected components. He showed that the components will be planar at some point, except for one type of component which has constant size. The number of perfect matchings in a planar graph can be computed in polynomial time [Kas67], in fact in NC [Vaz89]. Our goal is to use Wagners result to reduce the problem of computing the number of perfect matchings in a K_5-free graph to the one for planar graphs.

Let $G = (V,E)$ be a graph. If G is not connected, then the number of perfect matchings in G is the product of the number of perfect matchings in the connected components of G. Hence we may assume in the following that G is connected.

But actually we assume that G is biconnected. Otherwise there is an articulation point a in G. Let $G_1,\ldots,G_l$ be the connected components of $G - a$. For a perfect matching to exist, G must have an even number of vertices. Therefore, exactly one of $G_1,\ldots,G_l$ has an odd number of vertices, say G_1, and the others have even size. Hence the number of perfect matchings in G is the product of the number of perfect matchings in $G_1 \cup a, G_2,\ldots,G_l$. We can continue to split these components along the remaining articulation points until we end up with biconnected components only. Hence it suffices to determine the number of perfect matchings in biconnected graphs. We assume in the following that G is biconnected.

There is an extensive literature on graph decomposition, see for example [Tut66, Har69, HT73, BT89, BT96, MR92]. We follow the exposition of [DNTW09, TW14] which works for parallel computation, in fact in logspace.

3.1 Triconnected components

Definition 3.1. [DNTW09] *Let $G = (V, E)$ be a biconnected graph. A separating pair $\{a, b\}$ is called* 3-connected *if there are three vertex-disjoint paths between a and b in G.*

The triconnected components *of G are the split graphs we obtain from G by splitting G successively along all 3-connected separating pairs, in any order. If a separating pair $\{a, b\}$ is connected by an edge in G, then we also define a* 3-bond *for $\{a, b\}$ as a triconnected component, i.e., a multigraph with two vertices $\{a, b\}$ and three edges between them.*

The definition yields three types of triconnected components of a biconnected graph: 3-connected components, cycle components, and 3-bonds. The 3-bonds represent edges of the graph between separating pairs because they are replaced by virtual edges in the components. The cycle components are not 3-connected, but they are not decomposed further anyway.

Definition 3.2. *Let $G = (V, E)$ be a biconnected graph. The* triconnected component tree $\mathcal{T}$ of *G is the following graph. There is a node for each triconnected component and for each 3-connected separating pair of G. There is an edge in $\mathcal{T}$ between the node for triconnected component C and the node for a separating pair $\{a, b\}$, if a, b belong to C.*

Lemma 3.3. [DNTW09, TW14] The triconnected component tree can be computed and traversed in logspace.

Fix one component node in $\mathcal{T}$ as the root of $\mathcal{T}$. Hence we can talk of a parent and the children of a node. For a component node C in $\mathcal{T}$ let $\mathcal{T}(C)$ be the subtree of $\mathcal{T}$ with root C. When we reverse the decomposition process just on $\mathcal{T}(C)$ we obtain the graph denoted by $G(\mathcal{T}(C))$, the graph *associated with* $\mathcal{T}(C)$. Analogously we define $\mathcal{T}(\pi)$ and $G(\mathcal{T}(\pi))$ for a separating pair π.

3.2 Four-connected components

There are some subtleties in the decomposition of a 3-connected graph into 4-connected components. We refer to the exposition in [TW14]. In a nutshell, it is an inductive process that splits off components along separating triples. However, the separating triples might overlap each other, and even worse, their split components might overlap. In this case separating triples are called *crossing* in [TW14]. It is shown that one can select one of the crossing triples and throw away the other to obtain a decomposition into 4-connected components.

The components we get are

- separating triples where the vertices are connected by virtual edges,
- 4-connected components that contain the separating triples where they are split off. Again there are virtual edges between the vertices of the separating triples,

- 3-bonds for every pair a, b of vertices that are part of a separating triple and there is an edge $(a, b) \in E$.

Let C be a 3-connected component of G. The 4-*connected component tree* $\mathcal{T}_C$ *of* C has a node for every component as described above that occurs in the decomposition process of C. There is an edge between a 4-connected component D and a separating triple τ in $\mathcal{T}_C$ if τ belongs to D. If there is a 3-bond for two vertices a, b which are in τ, then we also have an edge between the 3-bond and τ in $\mathcal{T}_C$.

Lemma 3.4. [TW14] The 4-connected component tree can be computed and traversed in logspace.

Fix one component node as the root of $\mathcal{T}_C$. Let D be a component node and τ be a separating triple. Similar as for the triconnected component tree we define $\mathcal{T}_C(D)$ and $\mathcal{T}_C(\tau)$ as the subtrees of $\mathcal{T}_C$ rooted at D and τ, respectively. The graphs associated with the subtrees are denoted by $G(\mathcal{T}_C(D))$ and $G(\mathcal{T}_C(\tau))$, respectively.

3.3 Properties of K_5-free graphs

The crucial theorem about K_5-free graphs is due to Wagner [Wag37].

Theorem 3.5. [Wag37] *A* 3*-connected non-planar component of a* K_5*-free biconnected graph is either the Möbius ladder* M_8 *or its* 4*-connected components are all planar.*

The Möbius ladder M_8 is shown in Figure 1. It is a 3-connected graph on 8 vertices which is non-planar because it contains a $K_{3,3}$.

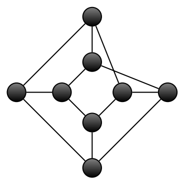


Figure 1: *The Möbius ladder* M_8.

The Möbius ladder M_8 has 5 perfect matchings. However, we will also have weights on the edges. In this case, we have to count the weighted perfect matchings of M_8. Since M_8 has constant size this is computationally a simple task.

Theorem 3.5 describes the route we follow: we decompose the given biconnected graph G into 3-connected components. When we are lucky, a 3-connected component C is already planar or M_8. In this case we directly compute the number of perfect matchings in C. Otherwise, we decompose C further into 4-connected components. These are all planar now and we can again compute the number of perfect matchings there. What makes things a bit tricky is, that we have to consider all possibilities of assigning the separating pairs and triples to the components they are part of. This will be the major challenge for the complexity bound on computing the number of perfect matchings.

4 Counting Perfect Matchings in K_5-free Graphs

In this section we prove the following theorem.

Theorem 4.1. *The number of perfect matchings in a K_5-free graph can be computed in polynomial time.*

Let $G = (V, E)$ be a biconnected K_5-free graph. We will decompose G into triconnected components, and, if necessary, into 4-connected components. Thereby we end up with components that are either planar, or the Möbius ladder M_8. The number of perfect matchings of these components can be computed in polynomial time. Note that the Möbius ladder M_8 has constant size. The critical part of our algorithm is to put these numbers together to obtain the number of perfect matchings of G.

Consider the triconnected component tree $\mathcal{T}$ of G. We will compute the number of perfect matchings of the components in a bottom-up fashion according to $\mathcal{T}$ by dynamic programming. If a component C is non-planar and $\neq M_8$, then we decompose C and consider its 4-connected component tree $\mathcal{T}_C$. Then we compute the number of perfect matchings of C by dynamic programming according to $\mathcal{T}_C$. Note that the separating pairs and triples occur in several components. However, when we consider perfect matchings in G, we should match the vertices of these pairs or triples only in one of the components, respectively. Hence we have to consider all possibilities to put the vertices of the separating pairs and triples into the split components.

Our algorithm will successively replace components by gadgets. The gadgets will have weighted edges. Hence we will compute the *weighted* number of perfect matchings. In the given graph G, edges have no weights. Equivalently we can say that all edges have weight one. In the decomposition of G into tri- and 4-connected components we introduce virtual edges between the vertices of the separating pairs and triples. The virtual edges that do not have an associated 3-bond are defined to have weight zero. These are the edges which are not present in G. With weight zero they do not contribute to the number of perfect matchings.

We start by considering the algorithm for the triconnected component tree $\mathcal{T}$ of G. Then we look at 4-connected component trees.

4.1 The triconnected component tree

Let $\mathcal{T}$ be the triconnected component tree of G. One component node of $\mathcal{T}$ is labeled as the root of $\mathcal{T}$. We describe an algorithm that computes the number of perfect matchings by dynamic programming. We start with the leaf nodes of $\mathcal{T}$. These are component nodes. Then we inductively work our way up to the root of $\mathcal{T}$.

Let C be a leaf in $\mathcal{T}$ and $\pi = \{a, b\}$ be the parent separating pair of C in $\mathcal{T}$. We compute the number of perfect matchings in C for every possibility of keeping a or b in C or not. If edge (a, b) is present in G, we should put it only into one of the split components of $\{a, b\}$ in order to get the correct number of perfect matchings. We will put edge (a, b) into its parent component, by giving it weight one there. Therefore, we define the weight of edge (a, b) to be zero in C.

- If C has odd size, we compute $p_a(C) = \#\mathrm{pm}(C - a)$ and $p_b(C) = \#\mathrm{pm}(C - b)$.
- If C has even size, we compute $p_\emptyset(C) = \#\mathrm{pm}(C)$ and $p_{ab}(C) = \#\mathrm{pm}(C - \pi)$.

This works directly only when C is a planar component or $C = M_8$. Otherwise C is non-planar and we decompose C into 4-connected components. We show in the next subsection how to compute the number of perfect matchings in this case.

In the inductive step, let $\pi = \{a, b\}$ be a separating pair node in $\mathcal{T}$. Let C_0 be the parent of $\{a, b\}$ in $\mathcal{T}$, and $C_1, C_2, \ldots, C_\ell$ be the children of π. Vertices a, b are contained in all these components. We should match a and b only in one of the components, respectively.

Define n_i to be the number of vertices of the subgraph $G(\mathcal{T}(C_i))$ of G. At most two of $n_1, \ldots, n_\ell$ can be odd, otherwise there is no perfect matching in G.

There are three cases:

- $n_1, \ldots, n_\ell$ are all even. Then a and b have both to be matched within one component C_i, for some $i \in \{1, \ldots, \ell\}$, or within C_0. Hence there are $\ell + 1$ possibilities to assign a, b.
- One of $n_1, \ldots, n_\ell$ is odd, say n_i. Then either a has to be matched within C_i and b within C_0, or vice versa. Hence there are two ways to assign a, b.
- In case where two of $n_1, \ldots, n_\ell$ are odd, we assign one of a, b to each of the two odd components. There are again two ways to assign a, b.

Any assignment of a and b other than the ones described above will result in zero perfect matchings.

We keep track of the assignments of a and b by a vector $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_\ell)$ and β_0, where $\beta_i \subseteq \pi$ are the vertices that should *not* be matched in C_i. Such a vector $\boldsymbol{\beta}$ is called *legal w.r.t.* β_0, if it corresponds to an assignment of a and b as explained above. That is, let $\overline{\beta}_i = \pi - \beta_i$. Then $\boldsymbol{\beta}$ is legal w.r.t. β_0 if the $\overline{\beta}_i$'s are pairwise disjoint and $\bigcup_{i \geq 0} \overline{\beta}_i = \pi$. Moreover, the assignment defined by the vector should respect the odd-even cases explained above. There are $\leq \ell$ legal vectors $\boldsymbol{\beta}$ for a fixed β_0.

Recall that $\mathcal{T}(C_i)$ is the subtree of $\mathcal{T}$ rooted at node C_i and $G(\mathcal{T}(C_i))$ is the graph associated with $\mathcal{T}(C_i)$. Inductively assume that we have already computed

$$p_\beta(C_i) = \#\mathrm{pm}(G(\mathcal{T}(C_i)) - \beta),$$

for every $\beta \subseteq \pi$ and $i = 1, 2, \ldots, \ell$. For a legal vector $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_\ell)$ w.r.t. β_0, define

$$\begin{aligned} p_{\boldsymbol{\beta}}(\pi) &= \prod_{i=1}^{\ell} p_{\beta_i}(C_i) \\ p_{\beta_0}(\pi) &= \sum_{\boldsymbol{\beta} \text{ legal w.r.t.} \beta_0} p_{\boldsymbol{\beta}}(\pi) \end{aligned}$$

Then we have

$$p_{\beta_0}(\pi) = \#\mathrm{pm}(G(\mathcal{T}(\pi)) - \beta_0).$$

There are only two possibilities for β_0. We compute $p_{\beta_0}(\pi)$ for both of these values.

The other case in the inductive step is to consider a component node C in $\mathcal{T}$. Let $\pi_0 = \{a_0, b_0\}$ be the parent separating pair of C in $\mathcal{T}$, and $\pi_1 = \{a_1, b_1\}, \cdots, \pi_\ell = \{a_\ell, b_\ell\}$ be the children of C. As already explained in the leaf-case above, edge (a_i, b_i) gets weight one if it is an edge in G, for $i = 1, 2, \ldots, \ell$. Edge (a_0, b_0) gets weight zero.

Inductively assume that we have already computed

$$p_\beta(\pi_i) = \#\mathrm{pm}(G(\mathcal{T}(\pi_i)) - \beta),$$

for every $\beta \subseteq \pi_i$ and $i = 1, 2, \ldots, \ell$. Our goal is to compute

$$p_\beta(C) = \#\mathrm{pm}(G(\mathcal{T}(C)) - \beta),$$

for every $\beta \subseteq \pi_0$. To do so, we replace the subgraphs $G(\mathcal{T}(\pi_i))$ of $G(\mathcal{T}(C))$ by appropriate weighted gadgets, for $i = 1, 2, \ldots, \ell$. That is, we take component C and add the gadgets at the separating pairs π_i.

- If $G(\mathcal{T}(\pi_i))$ has an odd number of vertices, we add one new vertex v_i to C and
  - an egde (v_i, a_i) of weight $p_{b_i}(\pi_i)$ and
  - an egde (v_i, b_i) of weight $p_{a_i}(\pi_i)$.
- If $G(\mathcal{T}(\pi_i))$ has an even number of vertices, we add two new vertices u_i, v_i to C and
  - an egde (u_i, a_i) of weight $p_\emptyset(\pi_i)$,
  - an egde (v_i, b_i) of weight 1, and
  - an egde (u_i, v_i) of weight $p_{a_i b_i}(\pi_i)$.

Figure 2 shows the gadgets. These gadgets were also used in [Val08]. Figure 3 shows an example of the construction.

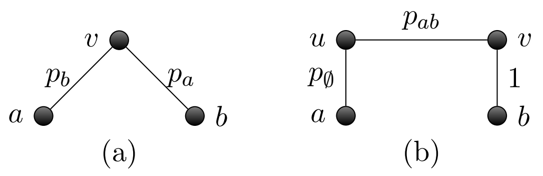


Figure 2: *The gadget for separating pair $\pi = \{a, b\}$ in case $G(\mathcal{T}(\pi))$ has (a) an odd number and (b) an even number of vertices. The gadgets replace $G(\mathcal{T}(\pi))$ in G.*

Let C' be the resulting component. The construction is such that the number of weighted perfect matchings of C' is the same as the number of perfect matchings of $G(\mathcal{T}(C))$. If C is planar, then also C' is planar and we can directly compute $\#\mathrm{pm}(G(\mathcal{T}(C)) - \beta)$, for every $\beta \subseteq \pi_0$. The same holds if $C = M_8$ because then also C' has constant size.

The third case is that C is non-planar and $\neq M_8$. This case we handle slightly different. Namely we do *not* place the gadgets right now in C. Instead, we first decompose C into 4-connected components. The reason is that C' is not 3-connected because of the gadgets. For every separating pair π_i we choose one 4-connected component where π_i occurs and put the gadget there.

The algorithm runs until C is the root of $\mathcal{T}$. In this case C has no parent separating pair. Then $\#\mathrm{pm}(G(\mathcal{T}(C)))$ is our result, the number of perfect matchings in G.

4.2 The 4-connected component tree

Let $C \neq M_8$ be a non-planar 3-connected component with weighted edges. Let a, b be two vertices in C that are a separating pair in G, and $\beta \subseteq \{a, b\}$. Recall that β contains the vertices that should not be matched within C, i.e., we want to compute the weighted number

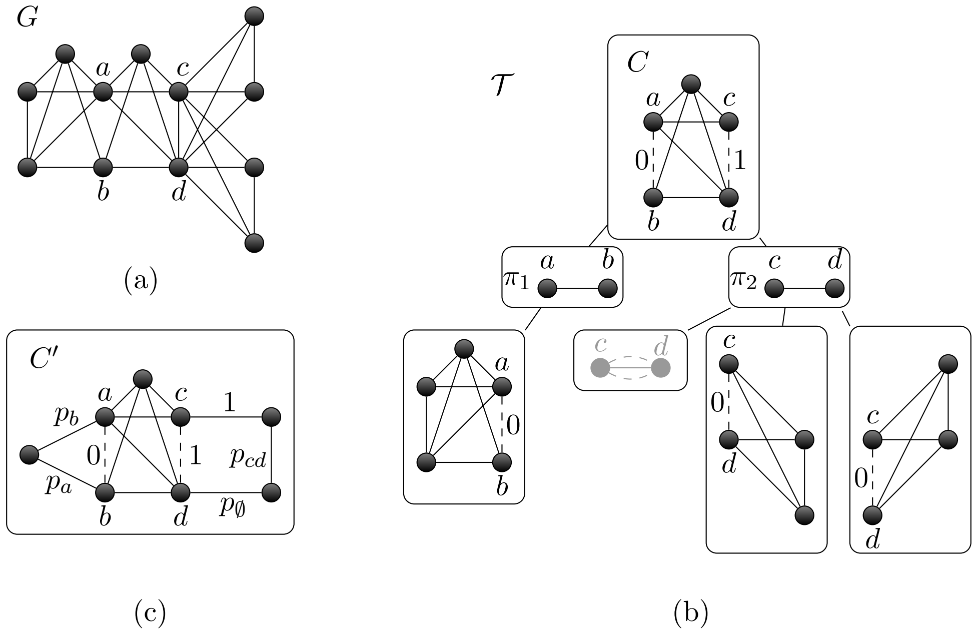


Figure 3: *(a) A biconnected graph G.*
(b) The triconnected component tree $\mathcal{T}$ of G. Component C is the root of $\mathcal{T}$. In the components, the virtual edges are indicated with dashed lines together with their weights.
(c) The component C' constructed from C. The subgraphs that correspond to the subtrees below C in $\mathcal{T}$ are replaced by weighted gadgets in C'.

of perfect matchings in $C - \beta$. In the exposition below we omit β for better readability. β has to be subtracted from every component we consider below.

Let $\mathcal{T}_C$ be the 4-connected component tree of C. Recall that we postponed the placement of the gadgets from the child separating pairs $\pi_1, \pi_2, \ldots, \pi_\ell$ of C in the triconnected component tree $\mathcal{T}$. Our first step now is to choose one component node in $\mathcal{T}_C$ for each π_i where π_i occurs and place the gadget there. We still call the component C in the following.

The algorithm to compute the number of perfect matchings in C is similar to the one for the triconnected component tree. One 4-connected component node is labeled as the root of $\mathcal{T}_C$. We start at the leafs of $\mathcal{T}_C$ and inductively proceed to the root of $\mathcal{T}_C$.

Let D be a leaf in $\mathcal{T}_C$ and $\tau = \{a, b, c\}$ be the parent separating triple of D in $\mathcal{T}_C$. The edges between vertices a, b, c which are present in G should be put only into one of the split components of τ in order to get the correct number of perfect matchings. We will put these edges into the parent component of τ, by giving them weight one there. Therefore, we define the weight of the three virtual edges within τ to be zero in D.

We compute the number of perfect matchings in D for every possibility of keeping a, b or c in D or not. That is, we compute $p_\gamma(D) = \#\text{pm}(D - \gamma)$, for all $\gamma \subseteq \tau$. If D has odd size, it suffices to take γ odd, and if D has even size, we can restrict γ to be even. Recall that the 4-connected components are all planar. Hence we can directly compute the number of perfect matchings.

In the inductive step, let $\tau = \{a, b, c\}$ be a separating triple node in $\mathcal{T}_C$. Let D_0 be the parent of τ in $\mathcal{T}_C$, and $D_1, D_2, \ldots, D_\ell$ be the children of τ. We consider the possible cases where to match a, b and c.

Define n_i to be the number of vertices of the subgraph $G(\mathcal{T}_C(D_i))$ of G. At most three of $n_1, \ldots, n_\ell$ can be odd, otherwise there is no perfect matching in G.

There are four cases:

- $n_1, \ldots, n_\ell$ are all even. Then either two vertices out of τ are matched within one of $D_1, \ldots, D_\ell$ and the remaining vertex within D_0, or all vertices of τ are matched within D_0. These are $3\ell + 1$ possibilities to assign a, b, c.

- One of $n_1, \ldots, n_\ell$ is odd, say n_i. Then either a, b, c are all matched within D_i, or just one of a, b, c is matched within D_i and the other two in D_j, for some $j \neq i$, or in D_0. Hence there are again $3\ell + 1$ possibilities to assign a, b, c.

- Two of $n_1, \ldots, n_\ell$ are odd, say n_i and n_j. Then one vertex of a, b, c is matched within D_i, one within D_j, and the remaining one in D_0. There are 6 ways to assign a, b, c.

- In case where three of $n_1, \ldots, n_\ell$ are odd, we assign one of a, b, c to each of the corresponding components. There are again 6 ways to assign a, b, c.

Any assignment of a, b and c other than the ones described above will result in zero perfect matchings.

We administrate the assignments of a, b and c again by a vector $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_\ell)$ and γ_0, where $\gamma_i \subseteq \tau$ are the vertices *not* matched in D_i. Similar as for the $\boldsymbol{\beta}$-vector in the triconnected component tree, we define $\boldsymbol{\gamma}$ to be *legal w.r.t.* γ_0 if it represents an assignment of a, b, c to the D_i's as explained above.

Inductively we have already computed

$$p_\gamma(D_i) = \#\mathrm{pm}(G(\mathcal{T}_C(D_i)) - \gamma),$$

for every $\gamma \subseteq \tau$ and $i = 1, 2, \ldots, \ell$. For a legal vector $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_\ell)$ w.r.t. γ_0, define

$$\begin{aligned} p_{\boldsymbol{\gamma}}(\tau) &= \prod_{i=1}^{\ell} p_{\gamma_i}(D_i) \\ p_{\gamma_0}(\tau) &= \sum_{\boldsymbol{\gamma} \text{ legal w.r.t.} \gamma_0} p_{\boldsymbol{\gamma}}(\tau) \end{aligned}$$

Then we have

$$p_{\gamma_0}(\tau) = \#\mathrm{pm}(G(\mathcal{T}_C(\tau)) - \gamma_0).$$

There are ≤ 4 possibilities for γ_0. We compute $p_{\gamma_0}(\tau)$ for all of these values.

The second case in the inductive step is to consider a component node D in $\mathcal{T}_C$. Let $\tau_0 = \{a_0, b_0, c_0\}$ be the parent separating triple of D in $\mathcal{T}_C$, and $\tau_1 = \{a_1, b_1, c_1\}, \ldots, \tau_\ell = \{a_\ell, b_\ell, c_\ell\}$ be the children of D. As already explained in the leaf-case above, the edges within τ_0 get weight zero, and the edges between the vertices in τ_i which are present in G get weight one in D, for $i = 1, 2, \ldots, \ell$. If $\tau_i \cap \tau_0 \neq \emptyset$ for some $i \geq 1$, there might be an edge e within both, τ_0 and τ_i. Then e gets weight zero in D.

Inductively assume that we have already computed

$$p_\gamma(\tau_i) = \#\mathrm{pm}(G(\mathcal{T}_C(\tau_i)) - \gamma),$$

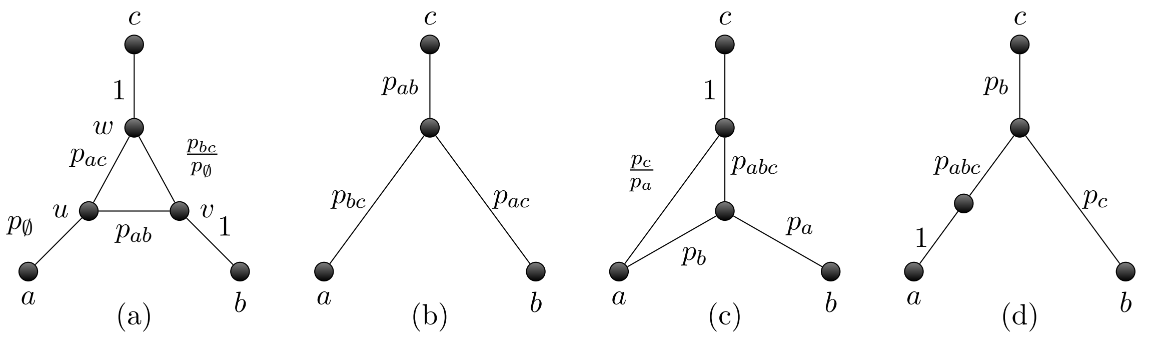


Figure 4: *The gadget for separating triple $\tau = \{a, b, c\}$ in case $G(\mathcal{T}_C(\tau))$ has (a) an even number and (c) an odd number of vertices. The gadgets replace $G(\mathcal{T}_C(\tau))$ in G. In (a), an odd number of a, b, c should be matched within the gadget, in (c) an even number.*
In (a), when $p_\emptyset = 0$, we use the gadget shown in (b) instead. Similarly, when $p_a = 0$ in (c) we use the gadget from (d).
For example in (a), if a, b, c should all be matched within the gadget, we use the edges (a, u), (b, v), (c, w). These edges contribute weight $p_\emptyset$ to a perfect matching. If only b should be matched within the gadget, we use edges (b, v) and (u, w). These two edges contribute weight p_{ac} to a perfect matching. Then a and c have to be matched in the rest of the graph. Similarly, when a should be matched within the gadget, we use edges (a, u) and (v, w). These two edges contribute weight $p_\emptyset \cdot \frac{p_{bc}}{p_\emptyset} = p_{bc}$ to a perfect matching.

for every $\gamma \subseteq \tau_i$ and $i = 1, 2, \ldots, \ell$. Our goal is to compute

$$p_\gamma(D) = \#\mathrm{pm}(G(\mathcal{T}_C(D)) - \gamma),$$

for every $\gamma \subseteq \tau_0$. We replace the subgraphs $G(\mathcal{T}_C(\tau_i))$ of $G(\mathcal{T}_C(D))$ again by appropriate weighted gadgets, for $i = 1, 2, \ldots, \ell$. That is, we add the gadgets at the separating triples τ_i of D. The gadgets incorporate all possibilities to match some vertices within $G(\mathcal{T}_C(\tau))$ and some vertices in the rest of G. We have again two gadgets. We distinguish the cases whether $G(\mathcal{T}_C(\tau_i))$ has an odd or even number of vertices. The gadgets are shown in Figure 4. They are similar to those given in [Val08].

Let D' be the resulting component. The construction guarantees that the number of weighted perfect matchings of D' equals the number of perfect matchings of $G(\mathcal{T}_C(D))$. Since D is planar, also D' is planar. Hence we can compute $\#\mathrm{pm}(D')$ in polynomial time. Figure 5 shows an example of the construction. This completes the proof of Theorem 4.1.

5 Counting Perfect Matchings in K_5-free Graphs in Parallel

We parallelize the algorithm from Section 4. We explain a circuit construction and argue that it has polylogarithmic depth. Our plan is to transform the sequential algorithm of Section 4 into a circuit. The decomposition of the given graph into 3- or 4-connected components is in logspace, and hence can be parallelized. However, since the resulting component trees may have depth $O(n)$, we cannot simply evaluate the tree bottom-up as in the sequential case.

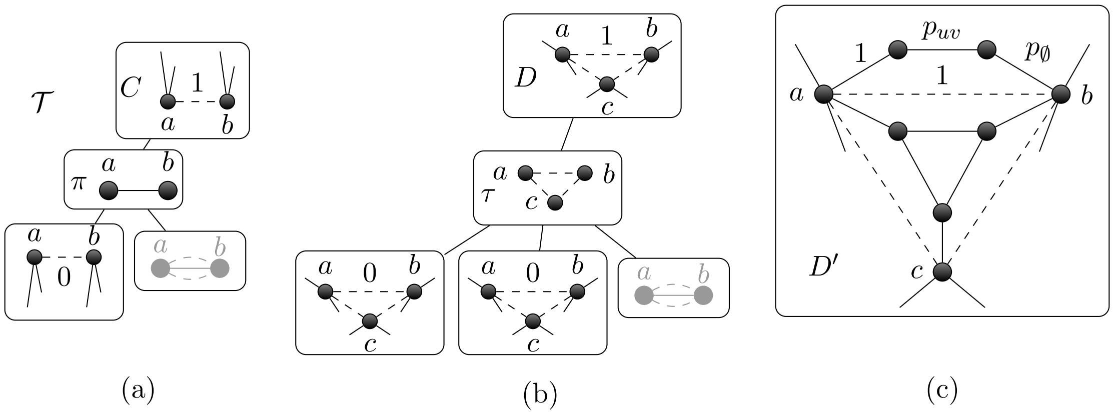


Figure 5: *In (a), a triconnected component tree $\mathcal{T}$ is shown with C as a triconnected component node and π as a separating pair node.*
In (b), the situation is shown where $C \neq M_8$ is not planar and hence, is decomposed into a 4-connected component tree $\mathcal{T}_C$. The more interesting case is, when $\pi \subseteq \tau$, then there are several occurrences of π as virtual edges $\{a, b\}$ in 4-connected component nodes. Only one of these virtual edges gets weight 1.
In (c), in D, the parent component of τ, the gadget corresponding to $\mathcal{T}(\pi)$ is embedded above the virtual edge $\{a, b\}$ which inherits weight 1. The gadget (without indicating weights) corresponding to $\mathcal{T}_C(\tau)$ is plugged inside the triangle of vertices a, b, c.

To get around this problem we identify *large child paths* in the component trees. These paths lead to a large depth. We show how to handle large child paths in parallel. This will suffice to obtain a small depth circuit.

Theorem 5.1. *Counting perfect matchings in K_5-free graphs is in* TC^2.

Recall that the number of perfect matchings in planar graphs can be computed in TC^1. Hence, for every component node C which is planar or M_8 and has parent π_0, we have a TC^1-circuit which computes $\#\mathrm{pm}(G(\mathcal{T}(C)) - \beta)$ for every $\beta \subseteq \pi_0$. We combine these circuits according to the edge relation in the component trees. I.e., we connect the output of a subcircuit for a node to the input of the subcircuit corresponding to its parent node. To obtain a circuit with polylogarithmic depth we deviate from the circuit construction at *large children* in the tree. Recall that u is a large child of v in a tree T if $|T(u)| > |T(v)|/2$. A *large child path* is a path $v_0, v_1, \ldots, v_k$ of maximal length in T such that v_i is a large child of v_{i-1}, for $i = 1, \ldots, k$.

We will compute the number of perfect matchings in the components on a large child path in parallel. We do this for all possible input values of a subcircuit. Recall that a subcircuit gets edge weights as input which are maybe large. Therefore we use the Chinese remainder representation to represent large numbers, i.e., all computations are done modulo p for enough small prime numbers p.

5.1 The computation graph

Recall that we have tri- *and* 4-connected component trees in the decomposition process of a K_5-free connected graph. We define a new tree, the *computation graph* $\mathcal{K}$ of a K_5-free biconnected graph G which combines the tri- and 4-connected component trees into one tree. Informally we start with the triconnected component tree $\mathcal{T}$ of G and replace every non-planar component node $C \neq M_8$ of the triconnected component tree $\mathcal{T}$ by the 4-connected component tree $\mathcal{T}_C$. More precisely, $\mathcal{K}$ is defined as follows.

Definition 5.2. *The* computation graph $\mathcal{K}$ *of a K_5-free biconnected graph G has the same nodes as the triconnected component tree $\mathcal{T}$ of G, but instead of the node for a non-planar component $C \neq M_8$, it has all the nodes of the 4-connected component tree $\mathcal{T}_C$.*

The edges between nodes within $\mathcal{T}$ or within a $\mathcal{T}_C$ are the same as in these trees, respectively. For a non-planar component $C \neq M_8$ in $\mathcal{T}$ let π_0 be the parent of C in $\mathcal{T}$ and $\pi_1, \ldots, \pi_\ell$ be its children. Let D be the root of $\mathcal{T}_C$. Then we define an edge between π_0 and D. Furthermore for every child π_i, $i \in \{1, \ldots, \ell\}$ there is a unique node in $\mathcal{T}_C$ connected to it. This unique node is the node where the gadget of $G(\mathcal{T}(\pi_i))$ of the algorithm of Section 4 is plugged in.

Note that when we plug in a tree $\mathcal{T}_C$ in $\mathcal{T}$ for some non-planar nodes $C \neq M_8$ in $\mathcal{T}$, then the children of C and its parent are connected to exactly one node in $\mathcal{T}_C$. Therefore $\mathcal{K}$ is again a tree. We can assume that $\mathcal{K}$ is a rooted tree.

Lemma 5.3. *The computation graph $\mathcal{K}$ of a biconnected graph G is a tree. It can be computed in logspace.*

Proof. The tri- and 4-connected component trees of G can be computed in logspace [TW14]. Hence it remains to compute the edges between the separating pair nodes in $\mathcal{T}$ and the 4-connected component nodes. With the notation from Definition 5.2, there is an edge between separating pair π_i and a 4-connected component D in $\mathcal{T}_C$, for some D that contains the vertices of π_i. We may simply choose the first such D that is computed by the logspace algorithm that computes $\mathcal{T}_C$. Therefore also these edges can be computed in logspace. $\square$

Let C be a component in $\mathcal{T}$ with parent separating pair π. We want to compute $\#\mathrm{pm}(G(\mathcal{T}(C)) - \beta)$, for any $\beta \subseteq \pi$. If $C \neq M_8$ is non-planar and D is a component in $\mathcal{T}_C$ with parent separating triple τ, then we also want to compute $\#\mathrm{pm}(G(\mathcal{T}_C(D)) - \gamma - \beta)$, for any $\gamma \subseteq \tau$. Hence component D does not only depend on its parent separating set node τ, but additionally on a separating pair π. We define a set $\mathcal{V}_K$ which covers all possibilities of β and γ.

Let R_0 be the root in $\mathcal{K}$ and let S be any separating pair or triple node in $\mathcal{K}$. Let P be a simple path from R_0 to S. Let $\widehat{S}$ be the separating pair node with shortest distance to S on P. I.e. $\widehat{S} = S$ in case S is a separating pair node. We define $\mathcal{V}_K(S) = \mathcal{P}(S) \cup \mathcal{P}(\widehat{S})$, where $\mathcal{P}$ denotes the power set. Let $\mu = |\mathcal{V}_K(S)| \leq 2^2 + 2^3$.

For a node N in $\mathcal{K}$, the subtree of $\mathcal{K}$ with root N is denoted by $\mathcal{K}(N)$. The subgraph of G corresponding to $\mathcal{K}(N)$ is denoted by $G(\mathcal{K}(N))$. That is, $G(\mathcal{K}(N))$ is the subgraph of G induced by the vertices that occur in some component node of $\mathcal{K}(N)$.

Because all component nodes in $\mathcal{K}$ are planar components or M_8, the number of perfect matchings in these components can be computed in TC^1.

Lemma 5.4. *Let S be a separating set node and $S_1, \dots, S_\ell$ be all the descendant separating set nodes at distance two in $\mathcal{K}$. The following function f is in* TC^1*: on input of* $\#\mathrm{pm}(G(\mathcal{K}(S_i)) - \kappa_i)$ *for* $i = 1, \dots, \ell$*,* $\kappa_i \in \mathcal{V}_K(S_i)$ *and* $\kappa \subseteq \mathcal{V}_K(S)$ *the output of* f *is* $\#\mathrm{pm}(G(\mathcal{K}(S)) - \kappa)$*.*

Proof. Let N be a component in the level between S and the S_i's. The first step is to remove the vertices from κ. Let N' be the resulting graph. Clearly, N' is still planar. Hence, the number of weighted perfect matchings in N' can be computed in TC^1 [MSV04]. Recall that the weights come from the input values $\#\mathrm{pm}(G(\mathcal{K}(S_i)) - \kappa_i)$ which are placed in the gadgets in N'.

This has to be done in all the component nodes between S and the S_i's. Then we have a sum and product of many numbers modulo some small prime number. Recall that modulo division as well as addition and multiplication of n numbers with n bits is in TC^0 [Vol99]. □

The lemma handles the case of component nodes and it remains to stick the results together at the nodes of separating pairs and triples in K. Therefore we reduce K by removing all the component nodes.

Definition 5.5. *Let $\mathcal{K}$ be the computation graph of G. The* reduced computation graph $\widehat{\mathcal{K}}$ *of $\mathcal{K}$ is defined by the following process: for every component node N in $\mathcal{K}$ with parent S and children $S_1, \dots, S_\ell$, remove N from $\mathcal{K}$ and instead draw edges between S and $S_1, \dots, S_\ell$.*

Similar as $\mathcal{K}$, the reduced computation graphs $\widehat{\mathcal{K}}$ is a tree and can be computed in logspace.

The tree $\widehat{\mathcal{K}}$ may have linear depth. To evaluate $\widehat{\mathcal{K}}$ efficiently in parallel, we have to do some depth reduction. The reason for the large depth are the large children: let T be a tree with root r. If v is a non-large child of r, then we have $|T(v)| \leq |T(r)|/2$. Hence, any simple path in T from r to a leaf along non-large children has length $\leq \log n$. However, paths that contain large children could be long. The next lemma states that there are only few large child paths on any path in T.

Lemma 5.6. *Let p be a path from the root to a leaf node in a tree T. Then we have*

(i) the number of large child paths on p is $\leq \log n$,

(ii) the number of nodes on p that are not large children is $\leq \log n$.

Proof. Consider two consecutive large child paths p_1, p_2 on p. Say, the first path p_1 goes from s_1 to t_1, and p_2 goes from s_2 to t_2. Because we defined large child paths to be of maximal length, t_1 has no large child. Hence we have

$$|T(s_2)| \leq |T(t_1)|/2 < |T(s_1)|/2\,.$$

Now the claim follows. □

5.2 Circuit construction

The computation tree $\mathcal{K}$ can be computed in logspace from the input graph G. Since $\mathrm{L} \subseteq \mathrm{TC}^1$, we may think of $\mathcal{K}$, respectively $\widehat{\mathcal{K}}$, being the output of a TC^1-circuit, in some appropriate coding. The output contains information about

- the vertices of G that are in one component node or separating set node in $\mathcal{K}$,
- the edges that are between the nodes of $\mathcal{K}$ and $\widehat{\mathcal{K}}$,

- the type of a node, i.e. whether it is the root or a leaf, or a large child,
- all the large child paths in $\widehat{\mathcal{K}}$.

Our goal is to evaluate $\mathcal{K}$, as we did in the sequential algorithm in Section 5. However, $\mathcal{K}$ depends on the input graph G and our circuit has to work for all graphs with the same number n of vertices. We construct the circuit in levels, where there is a subcircuit for *every* node of $\mathcal{K}$ in each level. Note that $\mathcal{K}$ has $O(n)$ nodes. Every subcircuit in one level is connected to every subcircuit of the next level. These connections represent the potential edge connections in $\mathcal{K}$. The actual edges in a given $\mathcal{K}$ are then activated by the results of the TC^1-circuit that computes $\mathcal{K}$.

Consider a node S be a node in $\widehat{\mathcal{K}}$ and let $S_1, \dots, S_\ell$ be its children in $\widehat{\mathcal{K}}$. We want to compute $\#\mathrm{pm}(G(\mathcal{K}(S)) - \kappa_S)$, for $\kappa_S \in \mathcal{V}_K(S)$. If S has no large child then there is a TC^1-circuit as described in Lemma 5.4, where the input values are obtained from lower circuit levels.

Because the depth of $\mathcal{K}$ can be as large as $O(n)$, we cannot afford such a level of subcircuits at any depth of $\mathcal{K}$. That is, we have to deviate from the sequential bottom-up evaluation of $\mathcal{K}$ and do some kind of depth-reduction. What causes the large depth are the large child paths. We will parallelize the computation along the large child paths with the balanced binary tree method, see [GR88]. By Lemma 5.6, the number of large child paths is bounded by $O(n \log n)$.

Consider a large child path $S = \widetilde{S}_0, \widetilde{S}_1, \dots, \widetilde{S}_t$ in $\widehat{\mathcal{K}}$. For each $\widetilde{S}_i$ we place many TC^1-circuits in parallel as shown in Figure 6, namely one circuit for each possible value of

- $\#\mathrm{pm}(G(\mathcal{K}(\widetilde{S}_{i+1})) - \kappa_{\widetilde{S}_{i+1}})$ modulo p,
- $\kappa_{\widetilde{S}_{i+1}} \in \mathcal{V}_K(\widetilde{S}_{i+1})$, and
- prime p.

Assume for the moment, that for each $\widetilde{S}_i$ of the large child path, the subtrees at the non-large children of $\widetilde{S}_i$ have already been evaluated. We use a flag to indicate when the assumption is fulfilled. We compose the functions computed by the circuits for each $\widetilde{S}_i$ in a binary tree like fashion. In the bottom layer, the composition of the circuits for $\widetilde{S}_{2i-1}$ and $\widetilde{S}_{2i}$ means: for each circuit C for $\widetilde{S}_{2i}$ we put an AC^0-circuit to select the circuit for $\widetilde{S}_{2i-1}$ which uses the output of C as input. Such a circuit exists, since we have a circuit for $\widetilde{S}_{2i-1}$ for every possible output of C. Clearly, we combine only circuits for the same prime p.

We continue to combine the resulting circuits in higher levels similarly. After $\log t$ levels, we have composed the circuits of the whole large child path. Then the correct values $\#\mathrm{pm}(G(\mathcal{K}(S)) - \kappa_S)$ for all $\kappa_S \in \mathcal{V}_K(S)$ are computed at the output gates of the constructed circuit. The whole composition circuit is in AC^1. A schematic view is shown in Figure 7.

We bound the depth of the resulting circuit. By Lemma 5.6 there are $\leq \log n$ nodes which are non-large children on every path. Therefore $\log n$ levels suffice to evaluate $\widehat{\mathcal{K}}$. Each level consists of TC^1-circuits to compute the number of perfect matchings in some planar component, followed by AC^1-circuits to evaluate large child paths. Therefore we obtain circuits in TC^2, for every prime p.

It remains to combine the results for the different primes. An obstacle thereby are the fractions that occur as weights in the gadgets in the 4-connected components. Let D be a 4-connected component. We do the following modification. For every edge $e = (u, v)$ of a

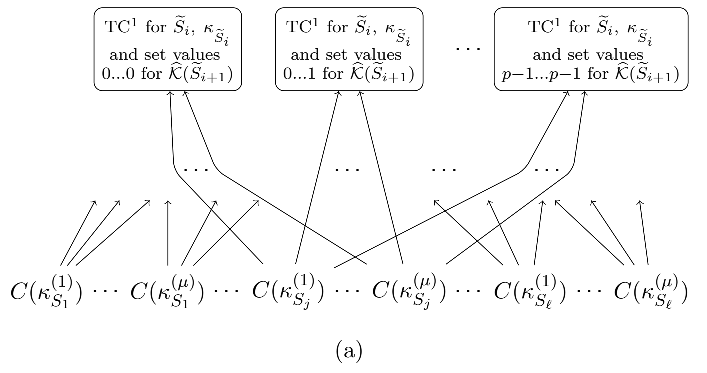
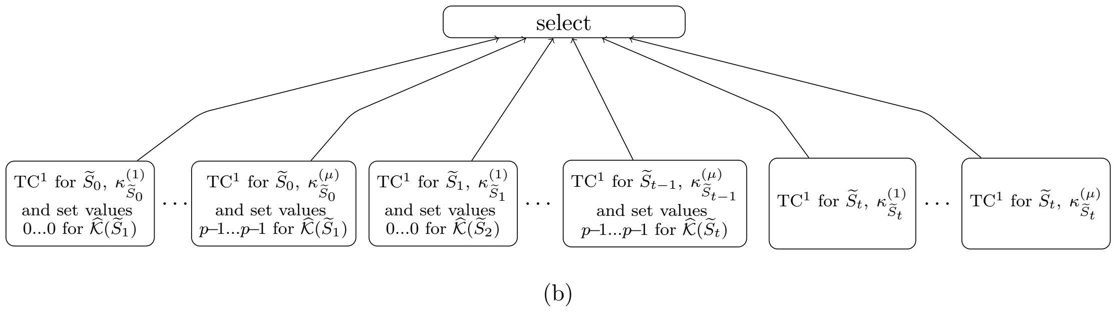

TC^1 for $\widetilde{S}_i$, $\kappa_{\widetilde{S}_i}$ and set values 0...0 for $\widehat{\mathcal{K}}(\widetilde{S}_{i+1})$
TC^1 for $\widetilde{S}_i$, $\kappa_{\widetilde{S}_i}$ and set values 0...1 for $\widehat{\mathcal{K}}(\widetilde{S}_{i+1})$
TC^1 for $\widetilde{S}_i$, $\kappa_{\widetilde{S}_i}$ and set values $p-1...p-1$ for $\widehat{\mathcal{K}}(\widetilde{S}_{i+1})$
$C(\kappa^{(1)}_{S_1})$ $C(\kappa^{(\mu)}_{S_1})$ $C(\kappa^{(1)}_{S_j})$ $C(\kappa^{(\mu)}_{S_j})$ $C(\kappa^{(1)}_{S_\ell})$ $C(\kappa^{(\mu)}_{S_\ell})$
(a)

select
TC^1 for $\widetilde{S}_0$, $\kappa^{(1)}_{\widetilde{S}_0}$ and set values 0...0 for $\widehat{\mathcal{K}}(\widetilde{S}_1)$
TC^1 for $\widetilde{S}_0$, $\kappa^{(\mu)}_{\widetilde{S}_0}$ and set values p–1...p–1 for $\widehat{\mathcal{K}}(\widetilde{S}_1)$
TC^1 for $\widetilde{S}_1$, $\kappa^{(1)}_{\widetilde{S}_1}$ and set values 0...0 for $\widehat{\mathcal{K}}(\widetilde{S}_2)$
TC^1 for $\widetilde{S}_{t-1}$, $\kappa^{(\mu)}_{\widetilde{S}_{t-1}}$ and set values p–1...p–1 for $\widehat{\mathcal{K}}(\widetilde{S}_t)$
TC^1 for $\widetilde{S}_t$, $\kappa^{(1)}_{\widetilde{S}_t}$
TC^1 for $\widetilde{S}_t$, $\kappa^{(\mu)}_{\widetilde{S}_t}$
(b)

Figure 6: *(a) We place many* TC^1*-circuits for each* $\widetilde{S}_i$ *in parallel, namely one for each possible weighting scheme for the gadget of* $\widehat{\mathcal{K}}(\widetilde{S}_{i+1})$. *The circuits for the non-large children* $S_1, ..., S_\ell$ *are indicated by* $C(\kappa^{(r)}_{S_j})$, *for* $\kappa^{(r)}_{S_j} \in \mathcal{V}_K(S_j)$ *and* $j \in \{1, \ldots, \ell\}$. *They are connected to all the* TC^1*-circuits for* $\widetilde{S}_i$.
(b) For all nodes $\widetilde{S}_0, \widetilde{S}_1, \ldots, \widetilde{S}_{t-1}$ *along a large child path, there are* μp^μ *many circuits in parallel, since for a node* $\widetilde{S}_i$, *there are* μ *different sets* $\kappa_{\widetilde{S}_i} \in \mathcal{V}_K(\widetilde{S}_i)$ *and* p^μ *many different possibilities for the values of* $\widehat{\mathcal{K}}(\widetilde{S}_{i+1})$.

gadget with rational weight $w(e) = a/b$ placed in D we do the following. Select one vertex of e, say v. For every edge incident to v, multiply the corresponding weights by the denominator b. Let w' be the resulting weight function. The weights of w' are integers.

Lemma 5.7. *Let* $b_1, \ldots, b_m$ *be all the denominators of rational weights in gadgets placed in a 4-connected component* D. *Then* $\#\mathrm{pm}_w(D) = \#\mathrm{pm}_{w'}(D)/(b_1 \cdots b_m)$.

Proof. If $w(e) = a/b$, then $w'(e) = a$. The other edges around v have weights multiplied with b. Every perfect matching matches exactly one edge around v. Hence $\#\mathrm{pm}_{w'}(D) = b \cdot \#\mathrm{pm}_w(D)$. Now the claim follows by an induction on the number of weighted edges. □

Hence, we move denominators step by step upwards in the tree. In two final steps we compute integers from the Chinese remainder representation and calculate the divisions. The quotient at the root is the number of perfect matchings in the input graph G. This finishes the proof of Theorem 5.1.

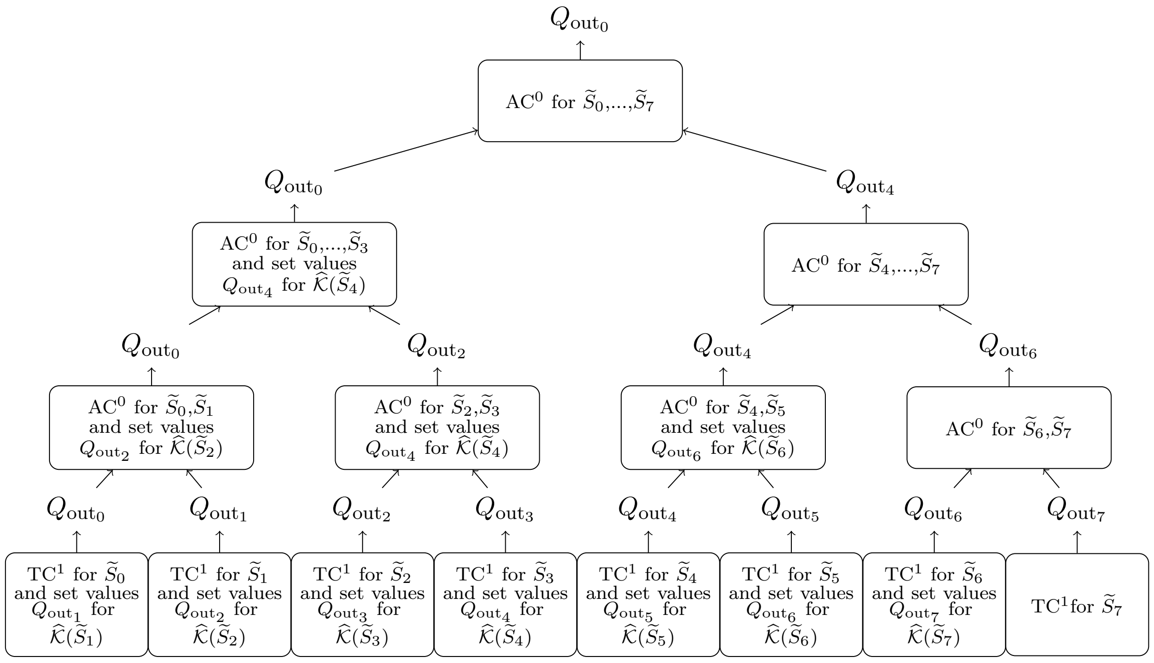

Q_{out_0}
AC^0 for $\widetilde{S}_0,\dots,\widetilde{S}_7$
Q_{out_0}
Q_{out_4}
AC^0 for $\widetilde{S}_0,\dots,\widetilde{S}_3$ and set values Q_{out_4} for $\widehat{\mathcal{K}}(\widetilde{S}_4)$
AC^0 for $\widetilde{S}_4,\dots,\widetilde{S}_7$
Q_{out_0}
Q_{out_2}
Q_{out_4}
Q_{out_6}
AC^0 for $\widetilde{S}_0,\widetilde{S}_1$ and set values Q_{out_2} for $\widehat{\mathcal{K}}(\widetilde{S}_2)$
AC^0 for $\widetilde{S}_2,\widetilde{S}_3$ and set values Q_{out_4} for $\widehat{\mathcal{K}}(\widetilde{S}_4)$
AC^0 for $\widetilde{S}_4,\widetilde{S}_5$ and set values Q_{out_6} for $\widehat{\mathcal{K}}(\widetilde{S}_6)$
AC^0 for $\widetilde{S}_6,\widetilde{S}_7$
Q_{out_0}
Q_{out_1}
Q_{out_2}
Q_{out_3}
Q_{out_4}
Q_{out_5}
Q_{out_6}
Q_{out_7}
TC^1 for $\widetilde{S}_0$ and set values Q_{out_1} for $\widehat{\mathcal{K}}(\widetilde{S}_1)$
TC^1 for $\widetilde{S}_1$ and set values Q_{out_2} for $\widehat{\mathcal{K}}(\widetilde{S}_2)$
TC^1 for $\widetilde{S}_2$ and set values Q_{out_3} for $\widehat{\mathcal{K}}(\widetilde{S}_3)$
TC^1 for $\widetilde{S}_3$ and set values Q_{out_4} for $\widehat{\mathcal{K}}(\widetilde{S}_4)$
TC^1 for $\widetilde{S}_4$ and set values Q_{out_5} for $\widehat{\mathcal{K}}(\widetilde{S}_5)$
TC^1 for $\widetilde{S}_5$ and set values Q_{out_6} for $\widehat{\mathcal{K}}(\widetilde{S}_6)$
TC^1 for $\widetilde{S}_6$ and set values Q_{out_7} for $\widehat{\mathcal{K}}(\widetilde{S}_7)$
TC^1 for $\widetilde{S}_7$

Figure 7: *A schematic view of the balanced binary tree method is shown for a large child path $\widetilde{S}_0,\dots,\widetilde{S}_7$. Except for the rightmost one, every box in the bottom row represents μp^μ many circuits, one for every possible result of $\widehat{\mathcal{K}}(\widetilde{S}_i)$ and every possible κ.*

Final Remarks

We presented efficient sequential and parallel algorithms to compute the number of perfect matchings in K_5-free graphs. This extends work of Kasteleyn for planar graphs and complements work of Little and Vazirani for $K_{3,3}$-free graphs.

Both, K_5 and $K_{3,3}$ can be drawn in the plane with one crossing, they have *crossing number* one. Robertson and Seymour showed a decomposition for H-free graphs, for any graph H with crossing number one, which is similar to that for K_5-free and $K_{3,3}$-free graphs. Based on this decomposition, and techniques similar to the ones presented in this paper, Radu Curticapean independently obtained the following result [personal communication]: The number of perfect matchings in H-free graphs can be computed in polynomial time, for any fixed graph H with crossing number at most one.

References

[BT89] G. Di Battista and R. Tamassia. Incremental planarity testing. In *IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 436–441, 1989.

[BT96] G. Di Battista and R. Tamassia. On-line maintenance of triconnected components with SPQR-trees. *Algorithmica*, 15(4):302–318, 1996.

[DKR10] S. Datta, R. Kulkarni, and S Roy. Deterministically isolating a perfect matching in bipartite planar graphs. *Theory of Computing Systems*, 47:737–757, 2010.

[DNTW09] S. Datta, P. Nimbhorkar, T. Thierauf, and F. Wagner. Isomorphism for $K_{3,3}$-free and K_5-free graphs is in log-space. In *Proceedings of the 29th annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS)*, pages 145–156, 2009.

[Edm65] J. Edmonds. Paths, trees, and flowers. *Canadian Journal of Mathematics*, 17:449–467, 1965.

[GR88] A. Gibbons and W. Rytter. *Efficient Parallel Algorithms.* Cambridge University Press, 1988.

[Har69] F. Harary. *Graph Theory.* Addison-Wesley, 1969.

[HT73] J. E. Hopcroft and R. E. Tarjan. A v log v algorithm for isomorphism of triconnected planar graphs. *Journal of Computer and System Sciences*, 7(3):323 – 331, 1973.

[Kas67] P. W. Kasteleyn. Graph theory and crystal physics. In F. Harary, editor, *Graph Theory and Theoretical Physics*, pages 43–110. Academic Press, 1967.

[Lit74] C. H. C. Little. An extension of Kasteleyn's method of enumerating the 1-factors of planar graphs. In D. A. Holton, editor, *Combinatorial Mathematics*, volume 403 of *Lecture Notes in Mathematics*, pages 63–72. Springer Berlin Heidelberg, 1974.

[MR92] G. L. Miller and V. Ramachandran. A new graph triconnectivity algorithm and its parallelization. *Combinatorica*, 12:53–76, 1992.

[MSV04] M. Mahajan, P.R. Subramanya, and V. Vinay. The combinatorial approach yields an NC algorithm for computing Pfaffians. *Discrete Applied Mathematics*, 143(13):1 – 16, 2004.

[MV80] S. Micali and V. Vazirani. An $O(V^{1/2}E)$ algorithm for finding maximum matching in general graphs. In *IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 17–27, 1980.

[MV97] M. Mahajan and V. Vinay. Determinant: Combinatorics, algorithms, and complexity. *Chicago Journal of Theoretical Computer Science*, 1997(5), December 1997.

[Tut66] W. T. Tutte. *Connectivity in graphs.* University of Toronto Press, 1966.

[TW09] T. Thierauf and F. Wagner. Reachability in $K_{3,3}$-free graphs and K_5-free graphs is in unambiguous log-space. In M. Kutylowski, W. Charatonik, and M. Gebala, editors, *Fundamentals of Computation Theory*, volume 5699 of *Lecture Notes in Computer Science*, pages 323–334. Springer Berlin Heidelberg, 2009.

[TW14] T. Thierauf and F. Wagner. Reachability in $K_{3,3}$-free graphs and K_5-free graphs is in unambiguous log-space. *Chicago Journal of Theoretical Computer Science*, 2014. To appear.

[Val79] L. Valiant. The complexity of computing the permanent. *Theoretical Computer Science*, 8:189–201, 1979.

[Val08] L. Valiant. Holographic algorithms. *SIAM Journal on Computing*, 37(5):1565–1594, 2008.

[Vaz89] V. Vazirani. NC algorithms for computing the number of perfect matchings in $K_{3,3}$-free graphs and related problems. *Information and computation*, 80(2):152–164, 1989.

[Vol99] H. Vollmer. *Introduction to Circuit Complexity*. Springer, 1999.

[Wag37] K. Wagner. Über eine Eigenschaft der ebenen Komplexe. *Mathematische Annalen*, 114(1):570–590, 1937.