

Sub-linear Upper Bounds on Fourier dimension of Boolean Functions in terms of Fourier sparsity

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Abstract

We prove that the Fourier dimension of any Boolean function with Fourier sparsity s is at most $O(s^{2/3})$. Our proof method yields an improved bound of $\tilde{O}(\sqrt{s})$ assuming a conjecture of Tsang *et. al.* [TWXZ13], that for every Boolean function of sparsity s there is an affine subspace of \mathbb{F}_2^n of co-dimension $O(\text{poly log } s)$ restricted to which the function is constant. This conjectured bound is tight upto poly-logarithmic factors as the Fourier dimension and sparsity of the address function are quadratically separated. We obtain these bounds by observing that the Fourier dimension of a Boolean function is equivalent to its non-adaptive parity decision tree complexity, and then bounding the latter.

1 Introduction

The study of Boolean functions involves studying various properties of Boolean functions and their inter-relationships. Two such properties, which we investigate in this article, are the Fourier dimension and the Fourier sparsity, which were first studied in the context of property testing by Gopalan *et. al.* [GOS⁺09]. Given a Boolean function $f : \mathbb{F}_2^n \rightarrow \{1, -1\}$ with Fourier expansion

$$f(x) = \sum_{\gamma \in \widehat{\mathbb{F}_2^n}} \widehat{f}(\gamma) \chi_\gamma(x),$$

Fourier dimension and Fourier sparsity are defined as follows.

Definition 1.1 (Fourier dimension and sparsity). *For a Boolean function $f : \mathbb{F}_2^n \rightarrow \{1, -1\}$ with Fourier expansion*

$$f(x) = \sum_{\gamma \in \widehat{\mathbb{F}_2^n}} \widehat{f}(\gamma) \chi_\gamma(x),$$

the Fourier support of f , denoted by $\text{supp}(f)$, is defined as

$$\text{supp}(\widehat{f}) := \{\gamma \in \widehat{\mathbb{F}_2^n} : \widehat{f}(\gamma) \neq 0\}.$$

The Fourier sparsity of f , denoted by $\text{sparsity}(f)$, is defined as the size of the support, i.e.,

$$\text{sparsity}(f) := |\text{supp}(\widehat{f})|,$$

while the Fourier dimension $\text{dim}(f)$ of f is defined as the dimension of span of $\text{supp}(\widehat{f})$.

The following inequalities easily follow from the definition of Fourier sparsity and dimension.

$$\log_2 \text{sparsity}(f) \leq \text{dim}(f) \leq \text{sparsity}(f). \tag{1.1}$$

There are functions (e.g., indicator functions of subspaces) for which the first inequality is tight. For the second inequality, the function known to us having the closest gap between dimension and sparsity is the *address function* $\text{Add}_s : \{0, 1\}^{\frac{1}{2} \log s + \sqrt{s}} \rightarrow \{0, 1\}$, defined as

$$\text{Add}_s(x, y_1, y_2, \dots, y_{\sqrt{s}}) := y_x, \quad x \in \{0, 1\}^{\frac{1}{2} \log s}, y_i \in \{0, 1\}.$$

In other words, at any input (x, y) , $\text{Add}_s(x, y)$ is the value of the addressee input bit y_x indexed by the addressing variables x . The address function¹ has sparsity s and dimension at least \sqrt{s} . It is believed that this is the tight upper bound for $\text{dim}(f)$ in terms of $\text{sparsity}(f)$. I. e., it is believed that the upper bound in (1.1) can be improved to $\text{dim}(f) \leq \sqrt{\text{sparsity}(f)}$ ².

Our main result is the following, which to our knowledge is the first improvement over the trivial $\text{dim}(f) \leq \text{sparsity}(f)$ bound.

Theorem 1.2. *Let f be a Boolean function with $\text{sparsity}(f) = s$. Then, $\text{dim}(f) = O(s^{2/3})$.*

This result is proved using a lemma of Tsang *et. al.* [TWXZ13] bounding the co-dimension of an affine subspace restricted to which the function reduces to a constant, in terms of Fourier sparsity of the function.

Lemma 1.3 (Corollary of [TWXZ13, Lemma 30]). *Let $f : \mathbb{F}_2^n \rightarrow \{1, -1\}$ be a Boolean function with Fourier sparsity s . Then there is an affine subspace V of \mathbb{F}_2^n of co-dimension $O(\sqrt{s})$ such that f is constant on V .*

Tsang *et. al.* [TWXZ13] proved a more general result in terms of Fourier l_1 -norm (see Section 2 for more details). Tsang *et. al.* proved this result while trying to investigate the *log rank conjecture* in communication complexity for *xor* functions. The log rank conjecture is a long standing and important conjecture in communication complexity. The statement of the conjecture is that the deterministic communication complexity of a Boolean function is asymptotically bounded above by some fixed poly-logarithm of the rank of its communication matrix. Tsang *et. al.* [TWXZ13] suggested a direction towards proving log-rank conjecture for an important class of functions called *xor* functions. A Boolean function $f(x, y)$ on two n bit inputs is a *xor* function if there exists a Boolean function F on n bits such that $f(x, y) = F(x \oplus y)$. In particular, they propose a protocol for such a f based on the parity decision tree of f and show that the communication complexity of this proposed protocol is polylogarithmic in rank of the communication matrix if the following related conjecture is true.

¹To be precise, we should consider the ± 1 version of the address function described here, where the 0 and 1 in the range are interpreted as $+1$ and -1 respectively.

²This is one of the conjectures proposed in the open problem session at the Simons workshop on Real Analysis in Testing, Learning and Inapproximability.

Conjecture 1.4 ([TWXZ13, Conjecture 27]). *There exists a constant $c > 0$ such that for every Boolean function f with Fourier sparsity s , there exists an affine subspace of co-dimension $O(\log^c s)$ on which f is constant.*

Tsang *et. al.* prove the above conjecture for certain classes of functions, which include functions with constant \mathbb{F}_2 degree and prove [Lemma 1.3](#) for general functions. Our next result shows that if we assume this conjecture instead of [Lemma 1.3](#), we can improve the bound in [Theorem 1.2](#) to the following (which is optimal upto poly-logarithmic factors).

Theorem 1.5. *Let f be a Boolean function with Fourier sparsity s . Assuming [Conjecture 1.4](#), $\dim(f) = \tilde{O}(\sqrt{s})$.*

Proof Idea: We begin by making a simple, but crucial observation that the Fourier dimension of a Boolean function is equivalent to its non-adaptive parity decision tree complexity (see [Proposition 2.7](#)). This offers us a potential approach towards upper bounding the Fourier dimension of a Boolean function: exhibiting a shallow non-adaptive parity decision tree of the function.

Towards this end, we first recall the construction of the (adaptive) parity decision tree of Tsang *et. al.* [[TWXZ13](#)], which in turn improves on an earlier construction due to Shpilka *et. al.* [[STIV14](#), [Theorem 1.1](#)]. The broad idea of their construction is as follows: At any point in time, a partial tree is maintained whose leaves are functions which are restrictions of the original function on different affine subspaces. Then a non-constant leaf is picked arbitrarily, and a small set of linear restrictions is obtained by invoking [Lemma 1.3](#), such that the restricted function at that leaf becomes constant. The next step is observing that if the function at the same leaf is restricted to all the affine subspaces obtained by setting the same set of parities in all possible ways, the sparsity of each of the corresponding restricted functions is at most half of that of the original function. This is because, in the former restriction, since the function becomes constant, the Fourier coefficients corresponding to non-constant characters must disappear in the restricted space. This can only happen if every non-constant parity gets identified with at least one other parity. This identification leads to halving of the support. Proceeding in this way, they obtain a parity decision tree of depth $O(\sqrt{s})$.

Note that the choice of parities depends on the leaf (function) chosen, and hence on the outcomes of the preceding queries. Thus the constructed tree is an adaptive one. In this article, we make this tree non-adaptive, at the cost of a small increase in depth. At each level, we choose an appropriate function, invoke [Lemma 1.3](#), and obtain the restrictions which make it constant. Then we query the same set of parities at every leaf. The next step is arguing that this leads to a significant reduction of sparsity in the next level. This is done using the Uncertainty Principle ([Theorem 2.4](#)). Continuing in this fashion, we show that in a small number of levels, the size of the union of the Fourier supports of all the leaves becomes so small that we can query all of them, thereby turning all the leaves into constants.

2 Preliminaries

let $f : \mathbb{F}_2^n \rightarrow \{+1, -1\}$ be a Boolean function. We think of the range $\{+1, -1\}$ as a subset of \mathbb{R} . The inputs to f are n variables x_1, \dots, x_n which take values in \mathbb{F}_2 . We identify the additive group in \mathbb{F}_2 with the group $\{+1, -1\}$ under real number multiplication, and think of the variables as taking $+1$ and -1 values, where 0 and 1 of \mathbb{F}_2 get mapped to $+1$ and -1 respectively. We denote this group

isomorphism by $(-1)^{(\cdot)}$, i.e., $(-1)^0$ is 1 and $(-1)^1$ is -1 . When the x_i 's are ± 1 , it is well known that every Boolean function $f(x)$ (where x stands for x_1, \dots, x_n) can be uniquely written as

$$f(x) = \sum_{S \subseteq [n]} \widehat{f}(S) \prod_{i \in S} x_i.$$

Thus, when the variables are ± 1 , f can be written as a multilinear real polynomial. For every $S \subseteq [n]$, the product $\prod_{i \in S} x_i$ is the logical xor of the bits in S , and $\widehat{f}(S)$ is a real number. These products are exactly the *characters* of \mathbb{F}_2^n , which are ± 1 versions of the linear forms belonging to the dual vector space $\widehat{\mathbb{F}_2^n}$ of \mathbb{F}_2^n . We adopt the following notation in this paper:

$$f(x) = \sum_{\gamma \in \widehat{\mathbb{F}_2^n}} \widehat{f}(\gamma) \chi_\gamma(x).$$

Here, each $\gamma \in \widehat{\mathbb{F}_2^n}$ is a linear function from \mathbb{F}_2^n to \mathbb{F}_2 , and χ_γ is $(-1)^\gamma$.

We recall some standard definitions and facts about the Fourier coefficients.

Definition 2.1. Let $f(x) = \sum_{\gamma \in \widehat{\mathbb{F}_2^n}} \widehat{f}(\gamma) \chi_\gamma(x)$ be a Boolean function. The p -th spectral norm $\|\widehat{f}\|_p$ of f is defined as:

$$\|\widehat{f}\|_p := \left[\sum_{\gamma \in \widehat{\mathbb{F}_2^n}} |\widehat{f}(\gamma)|^p \right]^{1/p}.$$

Lemma 2.2 (Parseval's identity). For a Boolean function f , $\|\widehat{f}\|_2 = 1$.

The 1st spectral norm of a Boolean function can be bounded via sparsity as follows.

Claim 2.3. For a Boolean function f , $\|\widehat{f}\|_1 \leq \sqrt{s}$.

Proof.

$$\|\widehat{f}\|_1 \leq \|\widehat{f}\|_2 \cdot \sqrt{s} = \sqrt{s}.$$

The first inequality follows due to Cauchy-Schwarz inequality while the second equality follows from Parseval's identity. \square

For proving our results, we shall use the following version of the Uncertainty Principle. The reader is referred to [O'D] for a proof.

Theorem 2.4 (Uncertainty Principle). Let $p : \mathbb{R}^n \rightarrow \mathbb{R}$ be a real multilinear n -variate polynomial with sparsity s (i.e., it has s monomials with non-zero coefficients). Let U_n denote the uniform distribution on $\{+1, -1\}^n$. Then

$$\Pr_{x \sim U_n} [p(x) \neq 0] \geq \frac{1}{s}.$$

As stated in the introduction, we need the following theorem due to Tsang *et. al.* [TWXZ13].

Theorem 2.5 ([TWXZ13, Lemma 30]). Let $f : \mathbb{F}_2^n \rightarrow \{1, -1\}$ be such that $\|\widehat{f}\|_1 = A$. Then there is an affine subspace V of \mathbb{F}_2^n of co-dimension $O(A)$ such that f is constant on V .

Lemma 1.3 is a simple corollary of this theorem via Claim 2.3.

We end this section by a simple proposition which is crucial to our proofs.

Definition 2.6 (non-adaptive parity decision tree complexity). *Let f be a Boolean function. The non-adaptive parity decision tree complexity of f , (denoted by $NADT_{\oplus}(f)$), is defined as the minimum integer t such that there exist t linear forms $\gamma_1, \dots, \gamma_t \in \widehat{\mathbb{F}}_2^n$ such that f is a junta of $\gamma_1, \dots, \gamma_t$. In other words, on every input, specifying the outputs of the γ_i 's specifies the output of f .*

Proposition 2.7. *For a Boolean function f , $NADT_{\oplus}(f) = \dim(f)$.*

Proof. If the outputs of a basis of span of $\text{supp}(\widehat{f})$ is specified, then that clearly specifies the outputs of all characters in $\text{supp}(\widehat{f})$, and hence it specifies the output of the function. Thus $NADT_{\oplus}(f) \leq \dim(f)$.

Now, Let $NADT_{\oplus}(f) = t$. Let the outputs of $\gamma_1, \dots, \gamma_t$ specify the output of f , and without loss of generality assume these linear forms to be linearly independent as vectors in $\widehat{\mathbb{F}}_2^n$. Arbitrarily extend $\gamma_1, \dots, \gamma_t$ to a basis $\gamma_1, \dots, \gamma_n$ of $\widehat{\mathbb{F}}_2^n$. For $x = (x_1, \dots, x_n) \in \mathbb{F}_2^n$, let $L(x) = (\gamma_1(x), \dots, \gamma_n(x))$. L is easily seen to be an invertible linear transformation from \mathbb{F}_2^n onto itself. Now, $\forall x \in \mathbb{F}_2^n, \forall i = 1, \dots, n, \gamma_i(x) = (L(x))_i$. Replacing x by $L^{-1}(x)$ we have $\gamma_i(L^{-1}(x)) = x_i$. Now consider the Boolean function $g(x) = f(L^{-1}(x)) = \sum_{\gamma \in \widehat{\mathbb{F}}_2^n} \widehat{f}(\gamma) (-1)^{\gamma(L^{-1}(x))}$. Clearly $\dim(g) = \dim(f)$. Also, g is completely specified by the outputs of $\gamma_i(L^{-1}(x))$'s for $i = 1, \dots, t$. Since $\gamma_i(L^{-1}(x)) = x_i$, we have that g is a junta of x_1, \dots, x_t . Thus all the monomials in $\text{supp}(\widehat{g})$ contain only the variables x_1, \dots, x_t . Thus $\dim(f) = \dim(g) \leq t = NADT_{\oplus}(f)$.

The proposition follows by combining the two inequalities. □

3 Upper Bounding Parity Decision Tree Complexity

In this section, we upper bound the non-adaptive parity decision tree complexity of a Boolean function f with Fourier sparsity at most s . Consider the following procedure, parametrized by a parameter $\tau \in \mathbb{N}$ (that we will set later) that constructs the non-adaptive parity decision tree.

NON-ADAPTIVE-PARITY-DECISION-TREE-PROCEDURE $_{\tau}(f)$

Input: Boolean function $f : \mathbb{F}_2^n \rightarrow \{1, -1\}$; Parameter: $\tau \in \mathbb{N}$

1. Set $\Gamma \leftarrow \emptyset$, $\mathcal{S} \leftarrow \text{supp}(\widehat{f})$ and $\mathcal{F} \leftarrow \{f\}$.
2. While $|\mathcal{S}| > \tau$, do
 - (a) Let g be a function in \mathcal{F} with the largest Fourier sparsity. Let $\gamma_1, \dots, \gamma_{n_g}$ be linear functions and $b_1, \dots, b_{n_g} \in \mathbb{F}_2$ be such that a largest affine subspace on which g is constant is $\{x \in \mathbb{F}_2^n : \gamma_1(x) = b_1, \dots, \gamma_{n_g}(x) = b_{n_g}\}$. Query $\gamma_1, \dots, \gamma_{n_g}$.
 - (b) Set $\Gamma \leftarrow \Gamma \cup \{\gamma_1, \dots, \gamma_{n_g}\}$.
 - (c) For each $b = (b_{\gamma})_{\gamma \in \Gamma} \in \mathbb{F}_2^{|\Gamma|}$, let V_b be the affine subspace $\{x \in \mathbb{F}_2^n : \forall \gamma \in \Gamma, \gamma(x) = b_{\gamma}\}$. Set $\mathcal{F} \leftarrow \bigcup_{b \in \mathbb{F}_2^{|\Gamma|}} \{f|_{V_b}\}$.
 - (d) $\mathcal{S} \leftarrow \bigcup_{h \in \mathcal{F}} \text{supp}(\widehat{h})$.

3. Query all the parities in \mathcal{S} .

Notation: After each iteration of the while loop in the procedure, Γ is the set of parities that have been queried so far, \mathcal{F} is the set of all restrictions of f to the affine subspaces obtained by different assignments to parities in Γ and \mathcal{S} the union of the Fourier supports of functions in \mathcal{F} . Let $\Gamma^{(i)}$, $\mathcal{F}^{(i)}$ and $\mathcal{S}^{(i)}$ denote Γ , \mathcal{F} and \mathcal{S} respectively at the end of the i -th iteration of the while loop.

For each i , let $b = (b_\gamma)_{\gamma \in \Gamma^{(i)}} \in \mathbb{F}_2^{|\Gamma^{(i)}|}$ and let V_b be the affine subspace defined by linear constraints $\{\gamma(x) = b_\gamma : \gamma \in \Gamma^{(i)}\}$. In V_b , more than one linear functions of the original space may get identified as same.³ More specifically, δ_1 and δ_2 get identified as same in V_b if and only if $\delta_1 + \delta_2 \in \text{span } \Gamma^{(i)}$. Thus, $\text{supp}(\widehat{f})$ gets partitioned into equivalence classes, such that for each class, for every $b \in \mathbb{F}_2^{|\Gamma^{(i)}|}$, the linear functions belonging to that class are identified as same in V_b .

Let $l^{(i)}$ denote the number of cosets of the subspace $\text{span } \Gamma^{(i)}$ with which $\text{supp}(\widehat{f})$ has non-empty intersection. For $j = 1, \dots, l^{(i)}$, let $\beta_j^{(i)}$ be some representative element in $\text{supp}(\widehat{f})$ of the j -th coset of $\text{span } \Gamma^{(i)}$ having non-empty intersection with $\text{supp}(\widehat{f})$. For each j , let $\beta_j^{(i)} + \alpha_{j,1}^{(i)}, \dots, \beta_j^{(i)} + \alpha_{j,k_j}^{(i)}$ be the $k_j^{(i)} (\geq 1)$ elements in $\text{supp}(\widehat{f})$ which are in the same coset of $\text{span } \Gamma^{(i)}$ as $\beta_j^{(i)}$. For each i, j , define the polynomials $P_j^{(i)}(x) := \sum_{l=1}^{k_j} \widehat{f}(\beta_j^{(i)} + \alpha_{j,l}^{(i)}) \chi_{\alpha_{j,l}^{(i)}}(x)$. Note that the polynomials $P_j^{(i)}$, $j = 1, \dots, l^{(i)}$, are non-zero.

Given this notation, we can then write the Fourier expansion of f in the following form:

$$f(x) = \sum_{j=1}^{l^{(i)}} P_j^{(i)}(x) \chi_{\beta_j^{(i)}}(x).$$

Observation 3.1. $\forall i, \sum_{j=1}^{l^{(i)}} k_j^{(i)} = s$.

Observation 3.2. $|\mathcal{S}^{(i)}| = l^{(i)}$.

We now argue that after every iteration of the while loop, there exists a function $h \in \mathcal{F}^{(i)}$ which has large support.

Lemma 3.3. *After i -th iteration, there exists a $h \in \mathcal{F}^{(i)}$ such that $|\text{supp}(\widehat{h})|$ is at least $(l^{(i)})^2 / s$.*

Proof. Consider any function $f|_{V_b} \in \mathcal{F}^{(i)}$. The Fourier decomposition of $f|_{V_b}$ is given by $f|_{V_b} = \sum_{j=1}^{l^{(i)}} P_j^{(i)}(b) \chi_{\beta_j^{(i)}}(x)$. Thus, $|\text{supp}(\widehat{f|_{V_b}})|$ is exactly the number of polynomials $P_j^{(i)}$, $j = 1, \dots, l^{(i)}$ such that $P_j^{(i)}(b)$ is non-zero. We analyze this quantity as follows. Pick a $b \in \mathbb{F}_2^{|\Gamma^{(i)}|}$ uniformly at random. For each j , $j = 1, \dots, l^{(i)}$, by [Theorem 2.4](#), $\Pr_b[P_j^{(i)}(b) \neq 0] \geq \frac{1}{k_j^{(i)}}$ (since each $P_j^{(i)}$ is a

³By ‘same’ we also include their being negations of each other as the smaller subspace is an affine space and not always a vector space.

non-zero polynomial). Thus,

$$\begin{aligned} \mathbb{E}_b \left[|\text{supp}(\widehat{f|_{V_b}})| \right] &\geq \sum_{j=1}^{l^{(i)}} \frac{1}{k_j^{(i)}} \geq l^{(i)} \cdot \frac{1}{\left(\sum_{j=1}^{l^{(i)}} k_j^{(i)} \right) / l^{(i)}} && \text{[By convexity of } 1/x \text{]} \\ &= \frac{(l^{(i)})^2}{s} && \text{[By Observation 3.1].} \end{aligned}$$

Hence, there exists a $h \in \mathcal{F}^{(i)}$ such that $|\text{supp}(\widehat{h})|$ is at least $(l^{(i)})^2 / s$. \square

Lemma 3.4. *Assume that NON-ADAPTIVE-PARITY-DECISION-TREE-PROCEDURE $_{\tau}(f)$ runs for t iterations. Then for all $i, i = 1, \dots, t - 1$,*

$$l^{(i+1)} \leq l^{(i)} - \frac{(l^{(i)})^2 / s - 1}{2}.$$

Proof. Let g be the chosen function at Step 2a in the $(i + 1)$ -th iteration of the procedure. Let $\gamma_1, \dots, \gamma_{n_g}$ be the parities queried at that step. Hence there is $b = (b_1, \dots, b_{n_g}) \in \mathbb{F}_2^{n_g}$ such that g is constant on the affine subspace V_b obtained by setting each γ_j to b_j for $j = 1, \dots, n_g$. Since g is constant on V_b , each non-zero parity in its Fourier support must disappear in V_b . Thus, for every $b' = (b')_j \in \mathbb{F}_2^{n_g}$, in the affine space $V_{b'}$ obtained by restricting each γ_j to b'_j , every non-zero parity in $\text{supp}(\widehat{g})$ is matched to some other parity in $\text{supp}(\widehat{g})$. Since $\text{supp}(\widehat{g}) \subseteq \mathcal{S}^{(i)}$, it follows that $|\mathcal{S}^{(i+1)}|$ is at least $\frac{|\text{supp}(\widehat{g})| - 1}{2}$ less than $|\mathcal{S}^{(i)}|$. The proof now follows from Lemma 3.3 and Observation 3.2. \square

Lemma 3.5. *Let NON-ADAPTIVE-PARITY-DECISION-TREE-PROCEDURE be run with parameter $\tau \geq \sqrt{2s}$. Assume that it runs for t iterations. Then for $i = 1, \dots, t$, $l^{(i)} \leq \frac{4s}{i}$.*

Proof. We will prove it by induction on i . Base case, $i = 1$, is trivial as $l^{(1)}$ can be at most s .

Now let us assume that the statement is true for all $i \leq m$. From Lemma 3.4, we have that $l^{(m+1)} \leq l^{(m)} - \left((l^{(m)})^2 / s - 1 \right) / 2$. Since $\gamma \geq \sqrt{2s}$, $((l^{(m)})^2 / s - 1) / 2$ can be lower bounded by $(l^{(m)})^2 / 4s$. We thus have

$$\begin{aligned} l^{(m+1)} &\leq l^{(m)} - \frac{(l^{(m)})^2}{4s} = s - \left(\sqrt{s} - \frac{l^{(m)}}{2\sqrt{s}} \right)^2 \\ &\leq s - \left(\sqrt{s} - \frac{2s}{m\sqrt{s}} \right)^2 && \text{[By inductive hypothesis]} \\ &= \frac{4s(m-1)}{m^2} = \left(\frac{4s}{m+1} \right) \left(\frac{m^2-1}{m^2} \right) \\ &\leq \frac{4s}{m+1}. \end{aligned}$$

This completes the proof of the lemma. \square

Theorems 1.2 and 1.5 follow easily from the above lemma using Lemma 1.3 and Conjecture 1.4 respectively as follows.

Proof of Theorem 1.2. Run the NON-ADAPTIVE-PARITY-DECISION-TREE-PROCEDURE with parameter $\tau = \Theta(s^{2/3})$. Since $l^{(i)} \leq \frac{4s}{i}$ (Lemma 3.5), the procedure terminates after $t = O(s^{1/3})$ iterations. From Lemma 1.3, in the i -th iteration, the number of parities set is at most $O(\sqrt{l^{(i)}})$. Thus the total number of queries made by the procedure is

$$\begin{aligned}
\sum_{i=1}^t O(\sqrt{l^{(i)}}) + \tau &= \sum_{i=1}^t O\left(\sqrt{\frac{s}{i}}\right) + \tau && \text{[By Lemma 3.5]} \\
&= \sqrt{s} \left(\sum_{i=1}^t O\left(\sqrt{\frac{1}{i}}\right) \right) + \tau && = \sqrt{s} \left(O\left(\int_1^t \frac{dx}{\sqrt{x}}\right) \right) + \tau = O(\sqrt{s} \cdot \sqrt{t}) + \tau \\
&= O\left(s^{1/2+1/6}\right) + \tau && \text{[Since } t = O(s^{1/3})\text{]} \\
&= O\left(s^{2/3}\right) && \text{[Since } \tau = \Theta(s^{2/3})\text{]}.
\end{aligned}$$

Thus, $\text{NADT}_{\oplus}(f) = O(s^{2/3})$. From Proposition 2.7, it follows that $\dim(f) = \text{NADT}_{\oplus}(f) = O(s^{2/3})$. \square

Proof of Theorem 1.5. Run the NON-ADAPTIVE-PARITY-DECISION-TREE-PROCEDURE with parameter $\tau = 2\sqrt{s}$. By Lemma 3.5, it runs for at most $\frac{4s}{2\sqrt{s}} = 2\sqrt{s}$ iterations. If Conjecture 1.4 is true, the total number of parities set is at most $O(\log^c s) 2\sqrt{s} + \tau$ which is $\tilde{O}(\sqrt{s})$. \square

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