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Lower Bounds for Clique vs. Independent Set

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Abstract

We prove an $\omega(\log n)$ lower bound on the conondeterministic communication complexity of the Clique vs. Independent Set problem introduced by Yannakakis (STOC 1988, JCSS 1991). As a corollary, this implies superpolynomial lower bounds for the Alon–Saks–Seymour conjecture in graph theory. Our approach is to first exhibit a query complexity separation for the decision tree analogue of the UP vs. coNP question—namely, unambiguous DNF width vs. CNF width—and then "lift" this separation over to communication complexity using a result from prior work.

1 Introduction

Yannakakis's [Yan91] *Clique vs. Independent Set* problem, associated with an undirected n-node graph $G = ([n], E)$, is the following: Alice holds a clique $x \subseteq [n]$ in G, Bob holds an independent set $y \subseteq [n]$ in G, and their goal is to decide whether x and y intersect. As the underlying graph enforces $|x \cap y| \in \{0, 1\}$, we may define a boolean function by $\mathrm{CIS}_G(x, y) := |x \cap y|$.

Upper bounds. For every G there is an $\lceil \log n \rceil$-bit nondeterministic communication protocol for CIS_G that guesses the name of the unique node in the intersection $x \cap y$. Recall (e.g., [KN97, Juk12]) that, combinatorially, this means that the 1-entries of the communication matrix of CIS_G can be covered with n rectangles. We write this fact as $\mathsf{NP}^{\mathsf{cc}}(\mathrm{CIS}_G) \leq \lceil \log n \rceil$. Yannakakis further proved that $\mathsf{P}^{\mathsf{cc}}(\mathrm{CIS}_G) \leq O(\log^2 n)$, where P^{cc} stands for deterministic communication complexity. In particular, we have the same upper bound on the conondeterministic complexity:

$$\mathsf{coNP}^{\mathsf{cc}}(\mathrm{CIS}_G) \;\leq\; O(\log^2 n).$$

Lower bounds. It has been a relatively long-standing open problem to prove (for some choice of G) superlogarithmic lower bounds on $\mathsf{P}^{\mathsf{cc}}(\mathrm{CIS}_G)$, let alone on $\mathsf{coNP}^{\mathsf{cc}}(\mathrm{CIS}_G)$. See, for instance, the textbooks by Kushilevitz and Nisan [KN97, Exercise 1.8] and Jukna [Juk12, Research Problem 4.15]. See also Table 1 for a summary of previous bounds, and the works of Kushilevitz and Weinreb [KW09a, KW09b] for indirect attacks on the problem.

Our main result is to obtain such superlogarithmic lower bounds.

Theorem 1. *There is a family of graphs G such that*

$$\mathsf{coNP}^{\mathsf{cc}}(\mathrm{CIS}_G) \;\geq\; \Omega(\log^{1.128} n).$$

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Measure	Lower bound	Reference
P^{cc}	$2 \cdot \log n$	Kushilevitz, Linial, and Ostrovsky [KLO99]
$\mathsf{coNP}^{\mathsf{cc}}$	$6/5 \cdot \log n$	Huang and Sudakov [HS12]
$\mathsf{coNP}^{\mathsf{cc}}$	$3/2 \cdot \log n$	Amano [Ama14]
$\mathsf{coNP}^{\mathsf{cc}}$	$2 \cdot \log n$	Shigeta and Amano [SA14]
$\mathsf{coNP}^{\mathsf{cc}}$	$\omega(\log n)$	This work

Table 1: Lower bounds on the deterministic (P^{cc}) and conondeterministic ($\mathsf{coNP}^{\mathsf{cc}}$) communication complexities of the Clique vs. Independent Set problem.

Implications. The CIS problem admits several equivalent formulations, as explored in-depth by Bousquet, Lagoutte, and Thomassé [BLT14]. In particular, Theorem 1 refutes a certain "polynomial" version of the Alon–Saks–Seymour conjecture [BLT14, Conjecture 16]. The original conjecture stated that $\chi(G) \leq \mathrm{bp}(G) + 1$, that is, that the chromatic number of G can be bounded in terms of the *biclique packing number* of G (minimum number of complete bipartite graphs needed to partition the edges of G). Huang and Sudakov [HS12] disproved the original conjecture by showing that $\chi(G)$ can be polynomially larger than $\mathrm{bp}(G)$. Theorem 1 implies that the gap can be superpolynomial. See [BLT14] for more details about this and other connections.

1.1 Our approach

Unambiguity. The canonical $\lceil \log n \rceil$-cost nondeterministic protocol for CIS_G outlined above has the additional property of being *unambiguous*: on each input the protocol can accept at most one nondeterministic guess. That is, combinatorially, the rectangles covering the 1-entries do not overlap. Thus $\lceil \log n \rceil$ is an upper bound on the unambiguous communication complexity of CIS_G, which we write as $\mathsf{UP}^{\mathsf{cc}}(\mathrm{CIS}_G) \leq \lceil \log n \rceil$. In fact, it is known that the CIS_G family of problems is *complete* for unambiguous communication: for every two-party function $F \colon \mathcal{X} \times \mathcal{Y} \to \{0,1\}$ there is a graph G on $n = 2^{\mathsf{UP}^{\mathsf{cc}}(F)}$ nodes such that F appears as a subproblem of CIS_G. See Figure 1 for an illustration. In particular, we have $\mathsf{UP}^{\mathsf{cc}}(F) = \mathsf{UP}^{\mathsf{cc}}(\mathrm{CIS}_G) = \log n$. Given this structural perspective, our goal becomes to exhibit a total[1] function with a large $\mathsf{UP}^{\mathsf{cc}}$ vs. $\mathsf{coNP}^{\mathsf{cc}}$ gap.

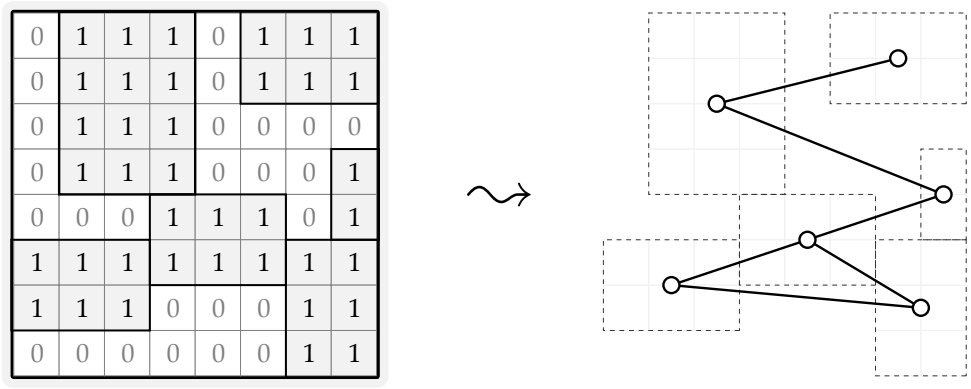
The following is a rephrasing of Theorem 1.

Theorem 2. *There is an F such that $\mathsf{coNP}^{\mathsf{cc}}(F) \geq \mathsf{UP}^{\mathsf{cc}}(F)^{\beta}$ where $\beta > 1.128$ is a constant.*

Query complexity. Instead of attacking Theorem 2 head-on, we first show an analogous separation in the simpler-to-understand world of decision tree complexity [BdW02]. Here one deals with plain boolean functions $f \colon \{0,1\}^n \to \{0,1\}$ (different n than above) without any two-party structure. We use the superscript "dt" for query complexity measures. For example, $\mathsf{NP}^{\mathsf{dt}}(f)$ denotes the nondeterministic decision tree complexity of f, or equivalently, the minimum k such that f can be written as a k-DNF. We also set $\mathsf{coNP}^{\mathsf{dt}}(f) \coloneqq \mathsf{NP}^{\mathsf{dt}}(\neg f)$. A DNF is *unambiguous* if on every input at most one of its terms (a.k.a. certificates) evaluates to true. We define $\mathsf{UP}^{\mathsf{dt}}(f)$ as the minimum k such that f can be written as an unambiguous k-DNF. (Unambiguous decision trees have been studied at least in [Sav02].)

The following is the query analogue of Theorem 2.

[1] It is easy to give a partial function (promise problem) with an exponential $\mathsf{UP}^{\mathsf{cc}}$ vs. $\mathsf{coNP}^{\mathsf{cc}}$ gap: the *unique set-intersection* function UINTER satisfies $\mathsf{UP}^{\mathsf{cc}}(\mathrm{UINTER}) \leq O(\log n)$ but $\mathsf{coNP}^{\mathsf{cc}}(\mathrm{UINTER}) \geq \Omega(n)$ [Raz92, KW14].



Partition of $F^{-1}(1)$ Graph G

Figure 1: Reduction $F \leq \mathrm{CIS}_G$ [Yan91, Lemma 1]: Fix an optimal partition of the 1-inputs of F using $2^{\mathsf{UP}^{\mathsf{cc}}(F)}$ rectangles. The nodes of the graph G are the rectangles, and two nodes are connected by an edge if the corresponding rectangles share a row. We have $F \leq \mathrm{CIS}_G$ via the following map: Alice maps her input x to the set of all rectangles intersecting row x, and Bob maps his input y to the set of all rectangles intersecting column y.

Theorem 3. *There is an f such that $\mathsf{coNP}^{\mathsf{dt}}(f) \geq \mathsf{UP}^{\mathsf{dt}}(f)^{\alpha}$ where $\alpha > 1.128$ is a constant.*

From query to communication. Given an f as above, we then apply a result from [GLM+14] that allows us to convert f into a communication problem while preserving its conondeterministic complexity. Specifically, we consider a composed function of the form $F := f \circ g^n$ where $g\colon \{0,1\}^b \times \{0,1\}^b \to \{0,1\}$ is a two-party "gadget" on a small number $b = \Theta(\log n)$ of inputs. The inputs to F are pairs $(x, y) \in \{0,1\}^{bn} \times \{0,1\}^{bn}$, which we view as being partitioned into n disjoint blocks $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ of b bits each. The task is to compute

$$F(x, y) \;:=\; f(g(x_1, y_1), \ldots, g(x_n, y_n)).$$

We use the following special case of a result proved in [GLM+14]—the full result there holds also for many randomised models. (We include a proof of this special case in Appendix A for completeness.)

Theorem 4 ([GLM+14])**.** *There is a gadget g on $b = \Theta(\log n)$ bits so that for all $f\colon \{0,1\}^n \to \{0,1\}$,*

$$\mathsf{coNP}^{\mathsf{cc}}(f \circ g^n) \;\geq\; \Omega(\mathsf{coNP}^{\mathsf{dt}}(f) \cdot b). \tag{1}$$

This establishes the lower bound on $\mathsf{coNP}^{\mathsf{cc}}(F)$. For the upper bound on $\mathsf{UP}^{\mathsf{cc}}(F)$, we use the simple fact that a protocol can always simulate a corresponding type of decision tree:

$$\mathsf{UP}^{\mathsf{cc}}(f \circ g^n) \;\leq\; O(\mathsf{UP}^{\mathsf{dt}}(f) \cdot b). \tag{2}$$

Indeed, when a nondeterministic decision tree queries the i-th input of f, we simulate this query in a nondeterministic protocol by a proof string of length $\Theta(b) = \Theta(\log n)$ that contains an encoding of i ($\lceil \log n \rceil$ bits), Alice's i-th input x_i (b bits), and Bob's i-th input y_i (b bits). Given this information, both players can learn the output $g(x_i, y_i)$ of the i-th gadget. Finally, it is straightforward to check that the resulting protocol is unambiguous iff the decision tree is unambiguous.

To conclude, Theorem 3 together with (1) and (2) imply a gap of $\mathsf{coNP}^{\mathsf{cc}}(F) \geq \Omega(\mathsf{UP}^{\mathsf{cc}}(F)^{\alpha} \cdot \log^{1-\alpha} n)$. Our F is actually going to satisfy $\mathsf{UP}^{\mathsf{cc}}(F) \geq n^{\Omega(1)}$ so the previous bound can be written as $\mathsf{coNP}^{\mathsf{cc}}(F) \geq \mathsf{UP}^{\mathsf{cc}}(F)^{\alpha - o(1)}$. This proves Theorem 2 for any $\beta < \alpha$.

The rest of this paper is devoted to proving Theorem 3.

2 Definitions

A size-k *certificate* for "$P(x)$"—where $P(\cdot)$ is a predicate and $x \in \{0,1\}^n$ is an input—is a partial assignment C to inputs, setting at most k variables, such that x agrees with C (we also say that C accepts x) and for every y agreeing with C it holds that $P(y)$. The *certificate complexity* of "$P(x)$" is the least size of a certificate for "$P(x)$". A nondeterministic ($\mathsf{NP}^{\mathsf{dt}}$) decision tree $\mathcal{T}$ of cost k for "$P(\cdot)$" is a collection of size-k certificates such that for every x, $P(x)$ holds iff there exists an $C \in \mathcal{T}$ that accepts x. We say that $\mathcal{T}$ is *unambiguous* ($\mathsf{UP}^{\mathsf{dt}}$), if on every input x there is at most one certificate $C \in \mathcal{T}$ that accepts x. We define $\mathsf{NP}^{\mathsf{dt}}(f)$ (resp. $\mathsf{UP}^{\mathsf{dt}}(f)$) as the least cost of a $\mathsf{NP}^{\mathsf{dt}}$ (resp. $\mathsf{UP}^{\mathsf{dt}}$) decision tree for "$f(\cdot) = 1$". Also let $\mathsf{coNP}^{\mathsf{dt}}(f) \coloneqq \mathsf{NP}^{\mathsf{dt}}(\neg f)$. Finally, $\mathsf{P}^{\mathsf{dt}}(f)$ denotes the deterministic decision tree complexity of f.

3 Warm-up discussion

The purpose of this section is to develop a "feel" for the $\mathsf{UP}^{\mathsf{dt}}$ vs. $\mathsf{coNP}^{\mathsf{dt}}$ question, and motivate some of the choices that are made in the upcoming sections.

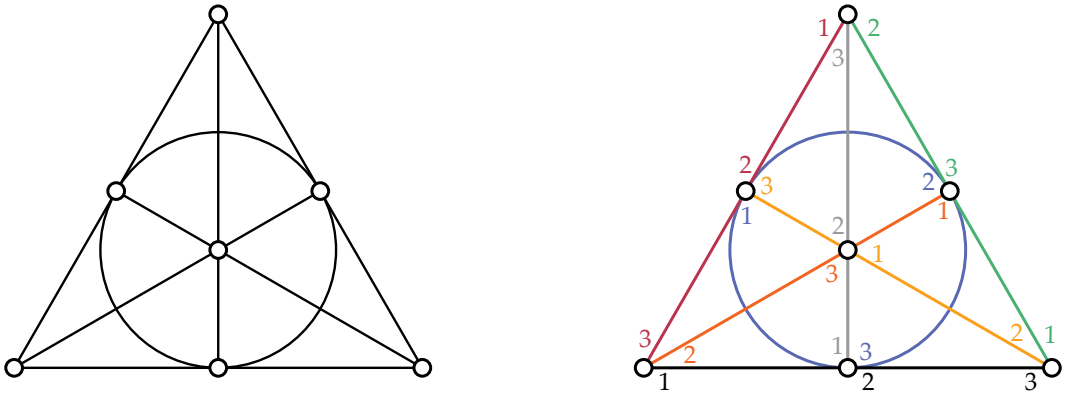
Upper bound on gap A well-known result for decision trees states that $\mathsf{P}^{\mathsf{dt}}(f) \leq \mathsf{NP}^{\mathsf{dt}}(f) \cdot \mathsf{coNP}^{\mathsf{dt}}(f)$; see [Juk12, §14.2]. A similar argument shows that $\mathsf{P}^{\mathsf{dt}}(f) \leq \mathsf{UP}^{\mathsf{dt}}(f)^2$. In particular, $\mathsf{coNP}^{\mathsf{dt}}(f) \leq \mathsf{UP}^{\mathsf{dt}}(f)^2$, and hence the $\mathsf{UP}^{\mathsf{dt}}$ vs. $\mathsf{coNP}^{\mathsf{dt}}$ gap cannot be too large (as in communication complexity). This argument is illuminating, so we present it here.

Proposition 5. $\mathsf{P}^{\mathsf{dt}}(f) \leq \mathsf{UP}^{\mathsf{dt}}(f)^2$.

Proof. Let $\mathcal{T}$ be a $\mathsf{UP}^{\mathsf{dt}}$ decision tree for f. A key observation is that any two certificates of $\mathcal{T}$ must intersect in variables—otherwise we could construct an input that satisfied two distinct certificates of $\mathcal{T}$, which contradicts unambiguity. The P^{dt} decision tree is this: Choose any certificate $C \in \mathcal{T}$ and query all its variables (at most $\mathsf{UP}^{\mathsf{dt}}(f)$ many). Let ρ be a partial assignment recording the values revealed, and let f_ρ be the restriction of f by ρ. If f_ρ is constant, we can output the corresponding value. Otherwise recursively run a decision tree for f_ρ. Here $\mathsf{UP}^{\mathsf{dt}}(f_\rho) \leq \mathsf{UP}^{\mathsf{dt}}(f) - 1$ as every remaining certificate $C' \neq C$ of $\mathcal{T}$ must intersect C in variables, so ρ already fixes at least one variable from each C'. □

Recursive composition It is easy to exhibit a constant-size function with *some* $\mathsf{UP}^{\mathsf{dt}}$ vs. $\mathsf{coNP}^{\mathsf{dt}}$ gap. For example, define a function on 3 bits by $f(x) = 1$ iff x has Hamming weight 1 or 2. We have $\mathsf{UP}^{\mathsf{dt}}(f) = 2$ since $\{x_1\bar{x}_2,\ x_2\bar{x}_3,\ x_3\bar{x}_1\}$ is an unambiguous collection of certificates for f. We also have $\mathsf{coNP}^{\mathsf{dt}}(f) = 3$ since the function is sensitive to flipping any of the bits on the all-0 input $\vec{0}$.

A standard trick in boolean function complexity [NS94, NW95, Sav02] is to take a constant-size function, such as f above, and to recursively compose it with itself: $f^{i+1}(\cdot) \coloneqq f(f^i(\cdot), f^i(\cdot), f^i(\cdot)))$. The hope is that at every level of recursion the gap between the two complexity measures of interest increases by some constant factor (e.g., 3/2 in the case of f). However, this approach works straightforwardly only when the complexity measures are two-sided, that is, they do not distinguish between f and $\neg f$. In our case of $\mathsf{UP}^{\mathsf{dt}}$ vs. $\mathsf{coNP}^{\mathsf{dt}}$ we are searching for a function whose 1-inputs are easy, but 0-inputs are hard. An obstacle associated with a recursively defined function f^i is that the only way to certify "$f^i(\cdot) = 1$"—which should be easy—often involves recursively certifying that "$f^{i-1}(\cdot) = 0$"—which should be hard!



(a) (b)

Figure 2: (a) Fano plane with 7 nodes and 7 hyperedges, and (b) a symmetric incidence ordering.

Large alphabets Our construction will employ recursion. In order to avoid the obstacle described above, we are going to enlarge the input/output alphabets of our functions. With large alphabets we assume a decision tree model where queries are still atomic, and reading a symbol from any input costs one query. We will study functions of the form $f\colon (\{0\} \cup \Sigma)^n \to \{0\} \cup \Sigma$, where the symbols in Σ are intuitively thought of as "easy to certify". The symbol 0 will be "hard to certify", and consequently we will always study the $\mathsf{coNP}^{\mathsf{dt}}$ complexity relative to the all-0 input $\vec{0}$. The idea is that certifying "$f^i(\,\cdot\,) = \sigma$" for $\sigma \in \Sigma$ is easy as we only need to recursively certify the same type of statements: "$f^{i-1}(\,\cdot\,) = \sigma'$" for $\sigma' \in \Sigma$. In particular, we make sure that an $\mathsf{UP}^{\mathsf{dt}}$ decision tree will never need to certify statements of the form "$f^i(\,\cdot\,) = 0$". Such $\mathsf{UP}^{\mathsf{dt}}$ decision trees will be called 0-*avoiding* later.

Note that if we have a function $f\colon \Sigma^n \to \{0,1\}$ with a large input alphabet, we may always convert it to a boolean function by composing it with n copies of some surjection $g\colon \{0,1\}^{\lceil \log |\Sigma| \rceil} \to \Sigma$. The following naïve bounds will suffice for our purposes:

$$\mathcal{C}(f) \;\le\; \mathcal{C}(f \circ g^n) \;\le\; \mathcal{C}(f) \cdot \lceil \log |\Sigma| \rceil \qquad \text{for all } \; \mathcal{C} \in \{\mathsf{UP}^{\mathsf{dt}}, \mathsf{coNP}^{\mathsf{dt}}\}. \tag{3}$$

4 Projective plane example

In this section we describe an example witnessing a constant-factor $\mathsf{UP}^{\mathsf{dt}}$ vs. $\mathsf{coNP}^{\mathsf{dt}}$ gap. The example is based on projective planes, and it will be suitable for recursive composition later.

Finite projective planes. For our purposes, a finite projective plane H is a k-uniform hypergraph with $n = k^2 - k + 1$ nodes and n hyperedges. Each node is incident to k edges, and each edge is incident to k nodes. Moreover, the edges are pairwise intersecting. See Figure 2a for an example of the case $k = 3$. It is well known that finite projective planes exist whenever $k - 1$ is a prime power.

Symmetric incidence ordering. Given an H as above, we fix for each node $v \in V(H)$ an ordering of its incident edges, and for each edge $e \in E(H)$ an ordering of its incident nodes. We make sure that the two orderings are *symmetric*: e is the i-th incident edge of $v \in e$ iff v is the i-th incident node of e; see Figure 2b. Such symmetric orderings are guaranteed to exist by Hall's

matching theorem [Die10, §2.1]. (Consider the k-regular bipartite incidence graph of H that has the original nodes $V(H)$ on the left and the original hyperedges $E(H)$ on the right and there is an edge $\{v, e\}$ iff $v \in e$. Then by Hall's theorem the edges of this graph can be partitioned into k disjoint perfect matchings—this encodes the desired orderings.)

Unweighted function. To turn H into a query problem f, we make the nodes of H correspond to input variables that take on values from $\{0\} \cup [k]$. These values are interpreted as *pointers*: we say that a node with input $i \in [k]$ *points* to its i-th incident edge. A node with input 0 does not point anywhere (0 is a *null pointer*). Moreover, we say that an assignment $x\colon V(H) \to \{0\} \cup [k]$ *satisfies* an edge $e \in E(H)$ if all the incident nodes $v \in e$ point to e according to x. Note that each x can satisfy at most a single edge, since every two edges share a node, and this node can only point to one of the two edges.

The function $f\colon (\{0\} \cup [k])^n \to \{0, 1\}$ is now defined so that $f(x) = 1$ iff x satisfies an edge. There is an obvious cost-k $\mathsf{UP}^{\mathsf{dt}}$ decision tree for f whose certificates are in one-to-one correspondence with the edges of H. Unfortunately, the certificate complexity of "$f(\vec{0}) = 0$" is not large: it is also k. We will next fix this by assigning *weights* to the inputs.

Input weights. We are going to modify the function f defined above so that querying the inputs x_j to f becomes harder: to decide whether "$x_j = i$" one needs to spend $w(i)$ queries (instead of 1) where $w\colon [k] \to \mathbb{N}$ is a *weighting* function of our choosing. We allow only positive weights, so we agree that $0 \notin \mathbb{N}$. Concretely, this is achieved by considering a composed function $f \circ g_w^n$ where g_w is a gadget that implements the weights w. More specifically, g_w is of the form $(\{0\} \cup [k])^m \to \{0\} \cup [k]$ where $m \coloneqq \max w([k])$ is the maximum weight and $g_w(x)$ is defined as follows: If $x = \vec{0}$, output 0. Otherwise let $x_j = i$ be the first non-0 coordinate of x. If $j \leq w(i)$, output i; otherwise output 0.

By construction, for deterministic decision trees and for every $i \in [k]$, it is both necessary and sufficient to make $w(i)$ queries in order to decide whether "$g_w(\,\cdot\,) = i$". Furthermore, we record for later use the following two properties that also follow straightforwardly from the construction.

(P1) The certificate complexity of "$g_w(\vec{0}) \neq i$" is $w(i)$.
(P2) Suppose $\hat{w}(i) \coloneqq \ell \cdot w(i)$ for all i. Then $\mathsf{coNP}^{\mathsf{dt}}(f_{\hat{w}}) = \ell \cdot \mathsf{coNP}^{\mathsf{dt}}(f_w)$ for every f.

Weighted function. Define w by $w(i) \coloneqq i$ and consider $f_w \coloneqq f \circ g_w^n$. We claim that

$$\begin{aligned} \mathsf{UP}^{\mathsf{dt}}(f_w) &\leq k(k+1)/2, \\ \mathsf{coNP}^{\mathsf{dt}}(f_w) &\geq k^2 - k + 1. \end{aligned}$$

That is, we have asymptotically a factor 2 separation.

Upper bound. We can devise a $\mathsf{UP}^{\mathsf{dt}}$ decision tree for f_w by simply composing a $\mathsf{UP}^{\mathsf{dt}}$ decision tree for f and optimal deterministic decision trees for the g_w's. A certificate corresponding to an edge $e \in V(H)$ in the $\mathsf{UP}^{\mathsf{dt}}$ decision tree for "$f(\,\cdot\,) = 1$" checks for each $i \in [k]$ that the i-th incident node of e is outputting i, which recursively involves checking that "$g_w(\,\cdot\,) = i$" for the corresponding gadget. For each i this amounts to $w(i) = i$ queries, which is $\sum_{i \in [k]} i = k(k+1)/2$ in total.

Lower bound. Our goal is to lower bound $\mathsf{coNP}^{\mathsf{dt}}(f_w)$ by analysing the 0-input $\vec{0}$. We start by highlighting a special property of the gadget that is afforded by our choice of w. The property states that to certify—on input $\vec{0}$—that none of a set of ℓ pointers appear as the output of g_w, one needs to make at least ℓ queries to the input variables.

Claim 6. *Let $S \subseteq [k]$. The certificate complexity of "$g_w(\vec{0}) \notin S$" is at least $|S|$.*

Proof. To certify "$g_w(\vec{0}) \notin S$" it is necessary to certify "$g_w(\vec{0}) \neq i$" where i is the largest number in S. But this latter task needs certificates of size $w(i) = i \geq |S|$ by *(P1)*. □

Our special property actually holds more generally across all the n gadgets. This generalised property states that to certify—on input $\vec{0}$—that none of a set of ℓ pointers (possibly associated with different nodes of H) appear in the output of the n gadgets g_w^n, one needs to make at least ℓ queries to the input variables of g_w^n. To state this more precisely, write $G \coloneqq g_w^n$ and let $G_v(\,\cdot\,)$ denote the output of the gadget corresponding to node $v \in V(H)$.

Claim 7. *For each v let $S_v \subseteq [k]$. The certificate complexity of "$\forall v : G_v(\vec{0}) \notin S_v$" is at least $\sum_v |S_v|$.*

Proof. Claim 6 proves this for each fixed v, but since the gadgets are defined on disjoint sets of variables, the individual certificate complexities sum up. □

Finally, the lower bound follows from the fact that any certificate for "$f_w(\vec{0}) = 0$" must certify that each edge $e \in E(H)$ lacks at least one pointer. This is $n = k^2 - k + 1$ pointers (and hence queries) in total.

5 Recursive composition

In this section we prove Theorem 3, restated below for convenience. In short, our idea is to recursively compose the example of Section 4.

Theorem 3. *There is an f such that $\mathsf{coNP}^{\mathsf{dt}}(f) \geq \mathsf{UP}^{\mathsf{dt}}(f)^{\alpha}$ where $\alpha > 1.128$ is a constant.*

Conventions. In what follows, we will consider pairs (f, w) where $f\colon (\{0\} \cup \Sigma)^n \to \{0,1\}$ is a function and $w\colon \Sigma \to \mathbb{N}$ assigns weights to the inputs of f. We also define $f_w \coloneqq f \circ g_w^n$ as before. A 0-*avoiding* $\mathsf{UP}^{\mathsf{dt}}$ decision tree for f is one whose certificates contain only values from Σ (i.e., not 0). More generally, a 0-*avoiding* $\mathsf{UP}^{\mathsf{dt}}$ decision tree for a weighted function f_w is a composition of a 0-avoiding $\mathsf{UP}^{\mathsf{dt}}$ decision tree for f and a deterministic decision tree for the g_w's; here a decision tree for g_w can—and indeed sometimes must—read 0's from the inputs. We let $\mathsf{UP}^{\mathsf{dt}}_\star(f_w)$ stand for the minimum cost of a 0-avoiding $\mathsf{UP}^{\mathsf{dt}}$ decision tree for f_w. We prove our $\mathsf{coNP}^{\mathsf{dt}}$ lower bound by considering the complexity of certifying "$f_w(\vec{0}) = 0$", that is, we only focus on the 0-input $\vec{0}$. Hence we introduce the notation $\mathsf{coNP}^{\mathsf{dt}}_\star(f_w)$ to stand for the certificate complexity of "$f_w(\vec{0}) = 0$".

Overview. Given a pair (f, w), we construct a new pair (f', w') with the following properties. (Here k is again a parameter, chosen later, such that $k-1$ is a prime power.)

(A1) Unambiguous complexity: $\mathsf{UP}^{\mathsf{dt}}_\star(f'_{w'}) \leq k(k+1)/2 \cdot \mathsf{UP}^{\mathsf{dt}}_\star(f_w)$.
(A2) Conondeterministic complexity: $\mathsf{coNP}^{\mathsf{dt}}_\star(f'_{w'}) \geq (k^2 - k + 1) \cdot \mathsf{coNP}^{\mathsf{dt}}_\star(f_w)$.

We proceed in two steps. In the first step (Section 5.1), we transform (f, w) into a pair $(\hat{f}, \hat{w})$ where $\hat{f}$ is a multi-valued function outputting values in $\{0\} \cup [k]$. The key property is the following:

> **If** f_w exhibits a gap between unambiguously certifying "$f_w(\,\cdot\,) = 1$" vs. nondeterministically certifying "$f_w(\vec{0}) = 0$", **then** for all $i \in [k]$, $\hat{f}_{\hat{w}}$ exhibits the same gap between unambiguously certifying "$\hat{f}_{\hat{w}}(\,\cdot\,) = i$" vs. nondeterministically certifying "$\hat{f}_{\hat{w}}(\vec{0}) \neq i$".

In the second step (Section 5.2), we begin thinking of $\hat{f}_{\hat{w}}$ as a gadget outputting pointer values: we plug several copies of $\hat{f}_{\hat{w}}$ as inputs to the projective plane example of Section 4. This will amplify the $\mathsf{UP}^{\mathsf{dt}}$ vs. $\mathsf{coNP}^{\mathsf{dt}}$ gap further and give us (f', w').

From this construction it is easy to derive Theorem 3 (Section 5.3).

5.1 First step: Multi-valued outputs

From $f\colon (\{0\} \cup \Sigma)^N \to \{0,1\}$ we will construct $\hat{f}\colon (\{0\} \cup \Sigma \times [k])^N \to \{0\} \cup [k]$. For notation, let $\pi_1\colon \Sigma \times [k] \to \Sigma$ and $\pi_2\colon \Sigma \times [k] \to [k]$ denote the natural projection maps, and extend $\pi_1(0) = \pi_2(0) = 0$ for convenience. For $x \in (\{0\} \cup \Sigma \times [k])^N$ we write $\pi_1(x)$ and $\pi_2(x)$ for the coordinate-wise application of π_1 and π_2 to x.

Definition of $(\hat{f}, \hat{w})$. Fix some optimal 0-avoiding $\mathsf{UP}^{\mathsf{dt}}$ decision tree $\mathcal{T}$ for f. We define $\hat{f}$ on input $x \in (\{0\} \cup \Sigma \times [k])^N$ as follows: If $f(\pi_1(x)) = 0$, output 0. Otherwise consider the unique certificate of $\mathcal{T}$ that accepts $\pi_1(x)$. This certificate reads some subset $S \subseteq [N]$ of inputs, where $x_j \neq 0$ for all $j \in S$, since $\mathcal{T}$ is 0-avoiding. If there is an $i \in [k]$ such that $\pi_2(x_j) = i$ for all $j \in S$, then output i; otherwise output 0.

The input weights $\hat{w}\colon \Sigma \times [k] \to \mathbb{N}$ are defined by $\hat{w}(\sigma, i) \coloneqq w(\sigma) \cdot i$.

Analysis. We claim that $(\hat{f}, \hat{w})$ satisfies the following.

(B1) There is a 0-avoiding $\mathsf{UP}^{\mathsf{dt}}$ decision tree for "$\hat{f}_{\hat{w}}(\,\cdot\,) = i$" with cost $i \cdot \mathsf{UP}^{\mathsf{dt}}_{\star}(f_w)$.

(B2) The certificate complexity of "$\hat{f}_{\hat{w}}(\vec{0}) \neq i$" is at least $i \cdot \mathsf{coNP}^{\mathsf{dt}}_{\star}(f_w)$.

(B1) holds: We can simply modify $\mathcal{T}$ slightly: to certify "$\hat{f}_{\hat{w}}(\,\cdot\,) = i$" the modified decision tree works as before but now additionally checks the new condition on $\pi_2(x)$ where x is the input to $\hat{f}$ (as encoded by $g_{\hat{w}}$). The cost of the modified tree is i times that of $\mathcal{T}$ as the relevant inputs are now i times heavier in the new $\hat{w}$ than in the old w.

(B2) holds: We prove this by a reduction argument. Fix $i \in [k]$. Consider deleting all symbols from the input alphabet of $\hat{f}$ except $\{0\} \cup \Sigma \times \{i\}$ and call the resulting subfunction $\hat{\hat{f}}$. Clearly the certificate complexity of "$\hat{f}_{\hat{w}}(\vec{0}) \neq i$" (which we are interested in) is at least that of "$\hat{\hat{f}}_{\hat{w}}(\vec{0}) \neq i$". Note that the range of $\hat{\hat{f}}$ is just $\{0, i\}$. Moreover, $\hat{\hat{f}}$ is isomorphic to f: identify the input alphabets via the map π_1, and the output alphabets via the map $0 \mapsto 0$, $i \mapsto 1$. Therefore the certificate complexities of "$\hat{\hat{f}}_{\hat{w}}(\vec{0}) \neq i$" and "$f_{\hat{w}}(\vec{0}) = 0$" are the same. (Here we abused notation by identifying the alphabets of f and $\hat{w}$ according to the isomorphism.) But $\hat{w}$ is nothing but w scaled by a factor of i (on the relevant alphabet $\Sigma \times \{i\}$), so by *(P2)* the certificate complexity of "$f_{\hat{w}}(\vec{0}) = 0$" is i times that of "$f_w(\vec{0}) = 0$", namely $i \cdot \mathsf{coNP}^{\mathsf{dt}}_{\star}(f_w)$.

5.2 Second step: Composition with a projective plane

Take a k-uniform projective plane hypergraph H as in Section 4 and let its associated unweighted function be $h\colon (\{0\} \cup [k])^n \to \{0,1\}$ where $n = k^2 - k + 1$. We define $f' \coloneqq h \circ \hat{f}^n$ together with the old weights $w' \coloneqq \hat{w}$ (so $f'_{w'} = h \circ \hat{f}^n_{\hat{w}}$). We claim that (f', w') satisfies *(A1)* and *(A2)*.

(A1) holds: We do the natural thing: The 0-avoiding $\mathsf{UP}^{\mathsf{dt}}$ decision tree for $f'_{w'}$ has a certificate for every edge $e \in E(H)$ that, for every $i \in [k]$, recursively checks that the i-th node incident to e points to e. The i-th recursive check involves certifying "$\hat{f}_{\hat{w}}(\,\cdot\,) = i$", which we can do in a 0-avoiding manner using *(B1)*. This has total cost $\sum_{i \in [k]} i \cdot \mathsf{UP}^{\mathsf{dt}}_{\star}(f_w) = k(k+1)/2 \cdot \mathsf{UP}^{\mathsf{dt}}_{\star}(f_w)$, as desired.

(A2) holds: We now begin thinking of the $\hat{f}_{\hat{w}}$'s as "gadgets". Under this nomenclature, we can simply repeat the argument from Section 4 verbatim. For example, our special property in the case of a single gadget states the following: to certify—on input $\vec{0}$—that none of a set of ℓ pointers appear as the output of $\hat{f}_{\hat{w}}$, one needs to make at least $\ell \cdot \mathsf{coNP}^{\mathsf{dt}}_{\star}(f_w)$ queries to the inputs of $\hat{f}_{\hat{w}}$.

Claim 8. *Let $S \subseteq [k]$. The certificate complexity of "$\hat{f}_{\hat{w}}(\vec{0}) \notin S$" is at least $|S| \cdot \mathsf{coNP}^{\mathsf{dt}}_{\star}(f_w)$.*

Proof. To certify "$\hat{f}_{\hat{w}}(\vec{0}) \notin S$" it is necessary to certify "$\hat{f}_{\hat{w}}(\vec{0}) \neq i$" where i is the largest number in S. But this latter task needs certificates of size $i \cdot \mathsf{coNP}^{\mathsf{dt}}_{\star}(f_w) \geq |S| \cdot \mathsf{coNP}^{\mathsf{dt}}_{\star}(f_w)$ by *(B2)*. □

As before, because the n gadgets $\hat{f}^n_{\hat{w}}$ are fed disjoint sets of variables, a more general property holds: to certify—on input $\vec{0}$—that none of a set of ℓ pointers (possibly associated with different nodes of H) appear in the output of the n gadgets $\hat{f}^n_{\hat{w}}$, one needs to make at least $\ell \cdot \mathsf{coNP}^{\mathsf{dt}}_{\star}(f_w)$ queries to the input variables of $\hat{f}^n_{\hat{w}}$.

Finally, the lower bound follows from the fact that any certificate for "$f'_{w'}(\vec{0}) = 0$" must certify that each edge $e \in E(H)$ lacks at least one pointer. This is $n = k^2 - k + 1$ pointers that require $(k^2 - k + 1) \cdot \mathsf{coNP}^{\mathsf{dt}}_{\star}(f_w)$ queries in total.

5.3 Putting everything together

Define a sequence of pairs (f^i, w^i) as follows: Initially, $f^0 \colon \{0,1\} \to \{0,1\}$ is the identity function and $w^0(1) \coloneqq 1$. Clearly the $\mathsf{UP}^{\mathsf{dt}}_{\star}$ and $\mathsf{coNP}^{\mathsf{dt}}_{\star}$ complexities for $f^0_{w^0}$ are both 1. Recursively define (f^{i+1}, w^{i+1}) as the "primed" versions of (f^i, w^i).

By inspecting the construction, we have the following properties.

- Number of inputs to f^i is $(k^2 - k + 1)^i$.
- Maximum weight of w^i is k^i.
- Therefore, the number of inputs to $f^i_{w^i}$ is $n \coloneqq (k^3 - k^2 + k)^i$.
- Input alphabet $\{0\} \cup \Sigma$ of $f^i_{w^i}$ has size $|\{0\} \cup \Sigma| = 1 + k^i$.
- *(A1)* implies: $\mathsf{UP}^{\mathsf{dt}}(f^i_{w^i}) \leq (k(k+1)/2)^i$.
- *(A2)* implies: $\mathsf{coNP}^{\mathsf{dt}}(f^i_{w^i}) \geq (k^2 - k + 1)^i$.

The last two items say that $\mathsf{coNP}^{\mathsf{dt}}(f^i_{w^i}) \geq \mathsf{UP}^{\mathsf{dt}}(f^i_{w^i})^\beta$ for $\beta \coloneqq \log(k^2 - k + 1)/\log(k(k+1)/2)$. This is maximised at $k = 8$ (among k such that $k - 1$ is a prime power), which yields $\beta = \log_{36} 57 > 1.128$.

As a final step, we need to replace the input alphabet of $f^i_{w^i}$ with a binary encoding. The above parameters reveal that $\log |\{0\} \cup \Sigma| \leq O(\log n)$ while our estimates for both $\mathsf{coNP}^{\mathsf{dt}}(f^i_{w^i})$ and $\mathsf{UP}^{\mathsf{dt}}(f^i_{w^i})$ are $n^{\Theta(1)}$. Hence the loose bounds (3) give us the gap in Theorem 3 for any $\alpha < \beta$.

A Appendix: Proof of Theorem 4

In this appendix—in order to make this work self-contained—we reproduce the proof of Theorem 4 originally proved in [GLM+14] (as part of a more general result).

Theorem 4 ([GLM+14])**.** *There is a gadget g on $b = \Theta(\log n)$ bits so that for all $f \colon \{0,1\}^n \to \{0,1\}$,*

$$\mathsf{coNP}^{\mathsf{cc}}(f \circ g^n) \;\geq\; \Omega(\mathsf{coNP}^{\mathsf{dt}}(f) \cdot b). \tag{1}$$

We choose g to be the inner-product gadget on $b := 100 \log n$ bits per party:

$$g(x, y) := \langle x, y \rangle \bmod 2, \qquad \text{where } x, y \in \{0,1\}^b.$$

Write $G := g^n$ for short. It is useful to think of a rectangle $R = X \times Y$ (where $X, Y \subseteq \{0,1\}^{bn}$) via a pair of random variables $\boldsymbol{xy}$ where $\boldsymbol{x}$ and $\boldsymbol{y}$ are uniformly (or otherwise) distributed on X and Y. A large rectangle corresponds to a pair $\boldsymbol{xy}$ where $\boldsymbol{x}$ and $\boldsymbol{y}$ have have high *min-entropy*, defined by $\mathbf{H}_\infty(\boldsymbol{x}) := \min_x \log(1/\mathbf{Pr}[\boldsymbol{x} = x])$ (see [Vad12] for more on min-entropy). We will use the fact that, if R is large, the output $\boldsymbol{z} := G(\boldsymbol{x}, \boldsymbol{y})$ has large support, that is, R accepts many different inputs to the outer function f. This is formalised in Lemma 9 below.

A.1 Density lemma

For $I \subseteq [n]$ and an input $x \in \{0,1\}^{bn}$, we write $x_I \in \{0,1\}^{b|I|}$ for the projection of x to the *blocks* indexed by I. We say that a pair $\boldsymbol{xy}$ of random variables is δ-*dense* if for all $I \subseteq [n]$ the blocks $\boldsymbol{x}_I\boldsymbol{y}_I$ have min-entropy rate at least δ, that is,

$$\mathbf{H}_\infty(\boldsymbol{x}_I\boldsymbol{y}_I) \geq \delta \cdot 2b|I|.$$

Lemma 9. *If $\boldsymbol{x}$ and $\boldsymbol{y}$ are independent and $\boldsymbol{xy}$ is 0.6-dense, then $\boldsymbol{z} := G(\boldsymbol{x}, \boldsymbol{y})$ has full support, namely $\{0,1\}^n$.*

Proof. [GLM+14, Lemma 13]. We actually prove a stronger claim: for $\epsilon = o(1)$, $\boldsymbol{z}$ is ϵ-*uniform*, meaning that $\mathbf{Pr}[\boldsymbol{z} = z] = (1 \pm \epsilon) \cdot 2^n$ for every $z \in \{0,1\}^n$. Clearly $\boldsymbol{z}$ has full support in this case.

First observe that for any $I \subseteq [n]$ the parity of the output bits $\boldsymbol{z}_I$ is simply $\langle \boldsymbol{x}_I, \boldsymbol{y}_I \rangle \bmod 2$. We use the fact that inner-product is a good two-source extractor to argue that this parity is close to an unbiased random bit. Indeed, by independence and 0.6-density we have $\mathbf{H}_\infty(\boldsymbol{x}_I) + \mathbf{H}_\infty(\boldsymbol{y}_I) = \mathbf{H}_\infty(\boldsymbol{x}_I\boldsymbol{y}_I) \geq 1.2 \cdot b|I|$ and this implies by a basic theorem of Chor and Goldreich [CG88, Theorem 9] that for $I \neq \emptyset$,

$$\left| \mathbf{Pr}[\langle \boldsymbol{x}_I, \boldsymbol{y}_I \rangle \bmod 2 = 0] - 1/2 \right| \leq 2^{-0.1 \cdot b|I|+1}. \tag{4}$$

This bound is enough to yield ϵ-uniformity for $\epsilon := 2^{-b/20}$, as we next verify using standard Fourier analysis. Let $\mathcal{D}$ be the distribution of $\boldsymbol{z}$. We think of $\mathcal{D}$ as a function $\{0,1\}^n \to [0,1]$ and write it in the Fourier basis as

$$\mathcal{D}(z) = \sum_{I \subseteq [n]} \widehat{\mathcal{D}}(I) \chi_I(z)$$

where $\chi_I(z) := (-1)^{\sum_{i \in I} z_i}$ and $\widehat{\mathcal{D}}(I) := 2^{-n} \sum_z \mathcal{D}(z)\chi_I(z) = 2^{-n} \cdot \mathbf{E}_{\boldsymbol{z} \sim \mathcal{D}}[\chi_I(\boldsymbol{z})]$. Note that $\widehat{\mathcal{D}}(\emptyset) = 2^{-n}$ because $\mathcal{D}$ is a distribution. In this language, property (4) says that, for all $I \neq \emptyset$, $2^n \cdot |\widehat{\mathcal{D}}(I)| = |\mathbf{E}[(-1)^{\langle \boldsymbol{x}_I, \boldsymbol{y}_I \rangle}]| \leq 2^{-0.1 \cdot b|I|+2}$, which is at most $\epsilon 2^{-2|I| \log n}$ by our definition of b and ϵ. Hence,

$$2^n \sum_{I \neq \emptyset} |\widehat{\mathcal{D}}(I)| \leq \epsilon \sum_{I \neq \emptyset} 2^{-2|I| \log n} = \epsilon \sum_{k=1}^{n} \binom{n}{k} 2^{-2k \log n} \leq \epsilon \sum_{k=1}^{n} 2^{-k \log n} \leq \epsilon.$$

We use this to show that $|\mathcal{D}(z) - 2^{-n}| \leq \epsilon 2^{-n}$ for all $z \in \{0,1\}^n$, which proves the lemma. To this end, let $\mathcal{U}$ denote the uniform distribution (note that $\widehat{\mathcal{U}}(I) = 0$ for all $I \neq \emptyset$) and let $\mathbb{1}_z$ denote the indicator for z defined by $\mathbb{1}_z(z) = 1$ and $\mathbb{1}_z(z') = 0$ for $z' \neq z$ (note that $|\widehat{\mathbb{1}}_z(I)| = 2^{-n}$ for all I). We can now calculate

$$\begin{aligned} |\mathcal{D}(z) - 2^{-n}| &= |\langle \mathbb{1}_z, \mathcal{D} \rangle - \langle \mathbb{1}_z, \mathcal{U} \rangle| = |\langle \mathbb{1}_z, \mathcal{D} - \mathcal{U} \rangle| = 2^n \cdot |\langle \widehat{\mathbb{1}}_z, \widehat{\mathcal{D}} - \widehat{\mathcal{U}} \rangle| \\ &\leq 2^n \cdot \textstyle\sum_{I \neq \emptyset} |\widehat{\mathbb{1}}_z(I)| \cdot |\widehat{\mathcal{D}}(I)| = \sum_{I \neq \emptyset} |\widehat{\mathcal{D}}(I)| \leq \epsilon 2^{-n}. \end{aligned}$$

□

A.2 Proof of Theorem 4

Overview. Let Π be a covering of the 0-inputs of $f \circ G$ using 2^c rectangles. Our goal will be to argue that for every $z \in f^{-1}(0)$ there is a small certificate for "$f(z) = 0$". More precisely, we are looking for a certificate C such that *(i)* C has size $O(c/b)$, *(ii)* C accepts z, and *(iii)* C only accepts 0-inputs of f. To find such a certificate, we find a subrectangle $R' \subseteq R$ of some $R \in \Pi$ such that $G(R')$ equals the set of inputs accepted by some C satisfying the properties prescribed above. More precisely, *(i)* $G(R')$ is a subcube of codimension $O(c/b)$, that is, $G(R')$ is fixed on $O(c/b)$ many coordinates and has full support elsewhere, *(ii)* $z \in G(R')$, and *(iii)* $G(R') \subseteq f^{-1}(0)$. Note that *any* $R' \subseteq R \in \Pi$ contains only 0-inputs of $f \circ G$ so that $G(R')$ contains only 0-inputs of f, that is, property *(iii)* will hold automatically.

Finding R'. Fix $z \in f^{-1}(0)$ and let $\boldsymbol{xy}$ be uniformly distributed on $G^{-1}(z)$. Using the fact that Π covers the whole support of $\boldsymbol{xy}$, we can find some $R \in \Pi$ such that $\mathbf{Pr}[\,\boldsymbol{xy} \in R\,] \geq 2^{-c}$. Denote by $(\boldsymbol{xy} \mid \boldsymbol{xy} \in R)$ the variables $\boldsymbol{xy}$ conditioned on the event "$\boldsymbol{xy} \in R$".

Let $I \subseteq [n]$ be a maximum-size subset for which 0.8-density is violated for $(\boldsymbol{xy} \mid \boldsymbol{xy} \in R)$, and let α be an outcome witnessing this: $\mathbf{Pr}[\,\boldsymbol{x}_I\boldsymbol{y}_I = \alpha \mid \boldsymbol{xy} \in R\,] > 2^{-1.6b|I|}$. We claim that conditioning further on the event "$\boldsymbol{x}_I\boldsymbol{y}_I = \alpha$", the remaining blocks indexed by $\bar{I} \coloneqq [n] \setminus I$ are 0.8-dense. Write $\boldsymbol{x}'\boldsymbol{y}' \coloneqq (\boldsymbol{xy} \mid \boldsymbol{xy} \in R,\ \boldsymbol{x}_I\boldsymbol{y}_I = \alpha)$ for short.

Claim 10. *$\boldsymbol{x}'_{\bar{I}}\boldsymbol{y}'_{\bar{I}}$ is 0.8-dense.*

Proof. Suppose not: there is some nonempty set $J \subseteq \bar{I}$ and a string β such that $\mathbf{Pr}[\,\boldsymbol{x}_J\boldsymbol{y}_J = \beta \mid \boldsymbol{xy} \in R,\ \boldsymbol{x}_I\boldsymbol{y}_I = \alpha\,] > 2^{-1.6b|J|}$. Now $\mathbf{Pr}[\,\boldsymbol{x}_I\boldsymbol{y}_I = \alpha \text{ and } \boldsymbol{x}_J\boldsymbol{y}_J = \beta \mid \boldsymbol{xy} \in R\,] = \mathbf{Pr}[\,\boldsymbol{x}_I\boldsymbol{y}_I = \alpha \mid \boldsymbol{xy} \in R\,] \cdot \mathbf{Pr}[\,\boldsymbol{x}_J\boldsymbol{y}_J = \beta \mid \boldsymbol{xy} \in R,\ \boldsymbol{x}_I\boldsymbol{y}_I = \alpha\,] > 2^{-1.6b|I|} \cdot 2^{-1.6b|J|} = 2^{-1.6b|I \cup J|}$. But this means that $(\boldsymbol{xy} \mid \boldsymbol{xy} \in R)$ violates 0.8-density on $I \cup J$, which contradicts the maximality of I. □

Claim 11. $|I| \leq O(c/b)$.

Proof. Our gadget is almost balanced: $g^{-1}(1), g^{-1}(0) \geq 2^{2b}/4 = 2^{2(b-1)}$. It follows that $\mathbf{H}_\infty(\boldsymbol{x}_I\boldsymbol{y}_I) \geq 2(b-1)|I|$ for all $I \subseteq [n]$. On the one hand, $\mathbf{H}_\infty(\boldsymbol{x}_I\boldsymbol{y}_I \mid \boldsymbol{xy} \in R) \geq \mathbf{H}_\infty(\boldsymbol{x}_I\boldsymbol{y}_I) - \log(1/\mathbf{Pr}[\,\boldsymbol{xy} \in R\,]) \geq 2(b-1)|I| - c$, where we used the fact that conditioning on an event with probability p lowers the min-entropy by at most $\log(1/p)$. On the other hand, $\mathbf{H}_\infty(\boldsymbol{x}_I\boldsymbol{y}_I \mid \boldsymbol{xy} \in R) < 1.6b|I|$ as $(\boldsymbol{xy} \mid \boldsymbol{xy} \in R)$ violates 0.8-density on I. These two bounds imply the claim. □

In summary, $\boldsymbol{x}'\boldsymbol{y}'$ is fixed on $O(c/b)$ many blocks I, and 0.8-dense on the remaining blocks $\bar{I}$. To apply Lemma 9 we need a pair of independent random variables, and currently $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are highly correlated (e.g., $G(\boldsymbol{x}', \boldsymbol{y}') = z$). Let $\boldsymbol{x}''$ and $\boldsymbol{y}''$ be independent copies of $\boldsymbol{x}'$ and $\boldsymbol{y}'$.

Claim 12. *$\boldsymbol{x}''_{\bar{I}}\boldsymbol{y}''_{\bar{I}}$ is 0.6 dense.*

Proof. Let $J \subseteq \bar{I}$. We want to show $\mathbf{H}_\infty(\boldsymbol{x}''_J\boldsymbol{y}''_J) \geq 1.2b|J|$. We calculate $\mathbf{H}_\infty(\boldsymbol{x}'_J) \geq \mathbf{H}_\infty(\boldsymbol{x}'_J\boldsymbol{y}'_J) - b|J| \geq 1.6b|J| - b|J| = 0.6b|J|$, where the first inequality follows since $b|J|$ is an upper bound on the support size of $\boldsymbol{y}'_J$ (measured in bits) and the second inequality uses the 0.8-density of $\boldsymbol{x}'_{\bar{I}}\boldsymbol{y}'_{\bar{I}}$. The same bound holds for $\mathbf{H}_\infty(\boldsymbol{y}'_J)$. Hence $\mathbf{H}_\infty(\boldsymbol{x}''_J\boldsymbol{y}''_J) = \mathbf{H}_\infty(\boldsymbol{x}''_J) + \mathbf{H}_\infty(\boldsymbol{y}''_J) = \mathbf{H}_\infty(\boldsymbol{x}'_J) + \mathbf{H}_\infty(\boldsymbol{y}'_J) \geq 0.6b|J| + 0.6b|J| = 1.2b|J|$. □

The subrectangle $R' \subseteq R$ we are looking for is now defined as the support of $\boldsymbol{x}''\boldsymbol{y}''$. We can apply Lemma 9 to $\boldsymbol{x}''_{\bar{I}}\boldsymbol{y}''_{\bar{I}}$ to deduce that $G(\boldsymbol{x}'', \boldsymbol{y}'') \in G(R')$ has full support on $\bar{I}$. Moreover, by construction, $G(R')$ is fixed on I and $z \in G(R')$. This concludes the proof of Theorem 4.

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