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Relative Discrepancy does not separate Information and Communication Complexity

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Abstract. Does the information complexity of a function equal its communication complexity? We examine whether any currently known techniques might be used to show a separation between the two notions. Recently, Ganor *et al.* provided such a separation in the distributional setting for a specific input distribution μ. We show that in the non-distributional setting, the relative discrepancy bound they defined is, in fact, smaller than the information complexity, and hence it cannot be used to separate information and communication complexity. In addition, in the distributional case, we provide an equivalent linear program formulation for relative discrepancy and relate it to variants of the partition bound, resolving also an open question regarding the relation of the partition bound and information complexity. Last, we prove the equivalence between the adaptive relative discrepancy and the public-coin partition bound, which implies that the logarithm of the adaptive relative discrepancy bound is quadratically tight with respect to communication.

1 Introduction

The question of whether information complexity equals communication complexity is one of the most important outstanding questions in communication complexity. Communication complexity measures the amount of bits Alice and Bob need to communicate to each other in order to compute a function whose input is shared between them. On the other hand, information complexity measures the amount of information Alice and Bob must reveal about their inputs in order to compute the function. Equality between information and communication complexity is equivalent to a compression theorem in the interactive setting. It is well-known that a single message can be compressed to its information content [Sha48,Fan49] and here the question is whether such a compression is possible for an interactive conversation.

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In addition to being a very useful technique for proving lower bounds in communication complexity, one of the most important applications of information complexity is to prove direct sum theorems for communication complexity, namely to show that computing k instances of a function costs k times the communication of computing a single instance. This has been shown to be true in the simultaneous and one-way models [CWYS01,JRS08], for bounded-round two-way protocols under product distributions [JRS03,HJMR07] or non-product distributions [BR14], and also for specific functions like Disjointness [BYJKS04]; some non-trivial direct sum theorems have also been shown for general two-way randomized communication complexity [BBCR10]. Since the information complexity of a function is equal to its amortized communication complexity [BR14], the question of whether information and communication complexity are equal is also equivalent to whether communication complexity has a direct sum property [BR14,Bra12]. Note that in the case of deterministic, zero-error protocols, a separation between information and communication complexity is known for the equality function [Bra12].

Since information complexity deals with the information Alice and Bob transmit about their inputs, it is necessary to define a distribution on these inputs. For each fixed distribution μ, we define the distributional information complexity of a function f (also known as the information cost) as the information Alice and Bob transmit about their inputs in any protocol that solves f with small error according to μ [CWYS01,BR14]. The (non-distributional) information complexity of the function f is defined as its distributional information complexity for the worst distribution μ [Bra12]. In this paper we consider the internal information complexity.

Similarly, for the study of communication complexity, one may also consider a model with a distribution μ over the inputs, and the error probability of the protocol is taken over this distribution. This is called a distributional model, and Yao's minmax principle [Yao83] states that the randomized communication complexity of f is equal to its distributional communication complexity for the worst distribution μ, where the randomized communication complexity of a function f is defined as the minimum number of bits exchanged, in the worst case over the inputs, for a randomized protocol to compute the function with small error [Yao79].

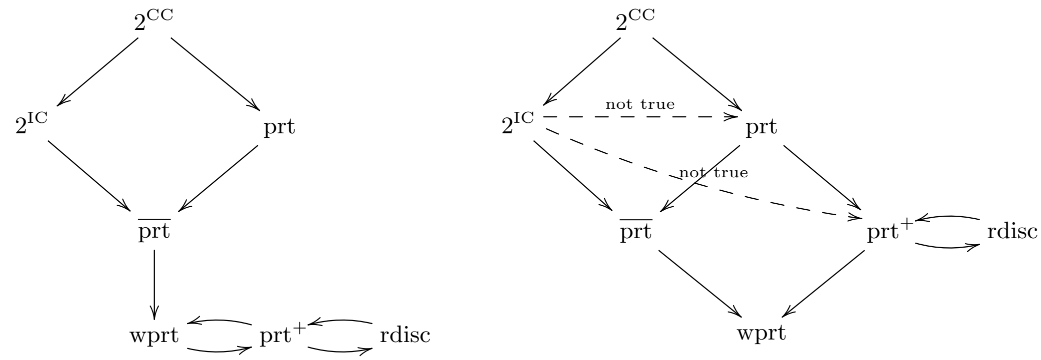
One can therefore ask whether the following stronger relation holds: is the distributional communication complexity equal to the distributional information complexity for all input distributions μ? A positive answer to this question would also imply a positive answer to the initial question, proving the equality of information and communication complexity.

In a recent breakthrough, Ganor *et al.* [GKR14a,GKR14b] gave an example of a function f and a distribution μ, for which there is an exponential separation between the distributional information and communication complexity. Does this settle the question of communication versus information? First, let us note that the gap, although exponential, is very small compared to the input size: a $\log\log(n)$ communication lower bound and a $\log\log\log(n)$ information upper bound, for inputs of size n. More importantly, Ganor *et al.*'s results prove that the *distributional* information and communication complexities are not equal for all distributions μ.

What will be needed to settle the question in the non-distributional setting? To prove a separation it is necessary to show that the communication complexity of a specific function is large, while its information complexity is small. In other words, we need some lower bound technique which provides a lower bound for communication but *not* for information. If we want to prove they are polynomially related, then it is useful to have formulations that are polynomially related to communication complexity.

In previous work, Kerenidis *et al.* [KLL$^+$12] showed that almost all known lower bound techniques for communication also provide lower bounds for information. More precisely, they studied the relaxed partition bound and proved that it subsumes all known lower bound techniques, including the rectangle-based, the norm-based, and the discrepancy methods, with the notable exception of the partition bound [JK10]. In addition, they proved that for any distribution μ, the distributional information complexity can be lower bounded by the relaxed partition bound. Since their result holds with respect to any distribution, it also holds in the non-distributional setting. An open question was whether the partition bound remained a candidate for separating information and communication complexity, or whether distributional information complexity could always be lower bounded by the partition bound, for any distribution.

The main question we ask in this paper is whether the techniques developed in the paper of Ganor *et al.* can help in proving, or disproving, the equality of information and communication complexity of a function f in the non-distributional setting. For their separation, Ganor *et al.* introduced a new communication lower bound technique called relative discrepancy. They showed that for a specific function f and a specific distribution μ, the relative discrepancy is high, while the distributional information complexity is low. In this paper, we study how large this new bound is compared to the other known lower bound techniques, and whether it can be used to separate information and communication complexity in the non-distributional setting. Our main results are:



(a) The non-distributional case. The fact that rdisc is upper bounded by the $\overline{\text{prt}}$ is given in Theorem 3.

(b) The distributional case. The equivalence between prt^+ and rdisc is given in Theorem 5. The separation given by Ganor *et al.* is between IC and rdisc.

Fig. 1: Relation among bounds. CC : comunication complexity; IC : information complexity; prt : the partition bound; $\overline{\text{prt}}$: the relaxed partition bound; prt^+: the LP formulation of the relative discrepancy bound rdisc; wprt : the weak partition bound. An arrow from one bound to another indicates that the former is at least as large as the latter.

Result 1: In the non-distributional case, we show that relative discrepancy is bounded above by the relaxed partition bound (**Theorem 3**). By the results of [KLL$^+$12], this means that relative discrepancy cannot be used to separate information and communication complexity.

Result 2: In the distributional case, we provide a clear relation between relative discrepancy, relaxed partition and partition bound. We give an equivalent linear program formulation for relative discrepancy (**Theorem 5**) and show how relative discrepancy and relaxed partition can be derived from the partition bound by imposing some simple extra constraints. This also answers negatively to the open question in [KLL$^+$12] regarding whether the partition bound is a lower bound on information.

Recently, another type of communication lower bound techniques has been proposed which use partitions instead of considering just rectangles. Jain, Lee and Vishnoi defined the public coin partition bound, and showed that its logarithm is quadratically related to the communication complexity [JLV14]. In addition, Ganor *et al.* introduced the adaptive relative discrepancy [GKR14b]. We study the relation between public coin partition and adaptive relative discrepancy and show the following:

Result 3: For any distribution μ, adaptive relative discrepancy and public-coin partition bound are equivalent (**Theorem 8**). This implies that the logarithm of the adaptive relative discrepancy bound is also quadratically tight with respect to communication.

In addition to providing a linear program formulation for relative discrepancy and adaptive relative discrepancy, the different variants of the partition bound have several other advantages. They can be defined for a wider range of problems, including non-boolean functions and relations; they have natural interpretations in terms of zero-communication protocols, a fact which has been succesfully used for relating information complexity to these bounds [KLL$^+$12] and for recent advances in the log rank conjecture [GL14]; and they have a natural interpretation in terms of efficiency and Bell inequalities, which makes them amenable to formulations that lower bound quantum communication complexity [LLR12].

In Section 2 we provide the necessary background and definitions. In Section 3 we prove that relative discrepancy is less than relaxed partition (in the non-distributional setting). In Section 4 we consider the setting with a fixed μ, and compare the partition bound and its variants to the relative discrepancy bound. In Section 5, we consider the adaptive relative discrepancy and compare it to the public coin partition bound.

2 Preliminaries

We will use the following notation throughout the paper. The sets $\mathbf{X}, \mathbf{Y}$ denote the set of inputs to the two players, and $\mathbf{Z}$ denotes the set of possible outputs. Since the discrepancy-based bounds studied in this paper apply naturally only to boolean functions, f will usually denote a (possibly partial) boolean function over $\mathbf{X} \times \mathbf{Y}$ taking values in $\mathbf{Z} = \{0, 1\}$, while μ denotes a probability distribution over $\mathbf{X} \times \mathbf{Y}$. We point out that the partition-based definitions apply naturally to non-boolean functions, relations, and bipartite distributions as well, but we do not give the full definitions in this paper for those settings.

2.1 Information and communication complexity

For any (possibly partial) function f over inputs $\mathbf{X} \times \mathbf{Y}$, and any $\epsilon \in (0, 1/2)$, the communication cost of a protocol that computes f with error probability at most ϵ is the number of bits sent for the worst case input.

Definition 1. *The communication complexity of f, denoted $R_\epsilon(f)$, is the best communication cost for any protocol that computes f with error at* *most ϵ for any input (x, y). For any distribution μ over the inputs, the distributional communication complexity of f, denoted $R_\epsilon(f, \mu)$, is the cost of the best protocol that computes f with error at most ϵ, where the error probability is taken over the input distribution.*

For information complexity, we are interested not in the number of bits exchanged, but the amount of information revealed about the inputs. We consider the internal information complexity in this paper. Here $I(X; Y)$ denotes the mutual information between random variables X and Y, and $I(X; Y|Z)$ is the mutual information conditioned on Z.

Definition 2 (Information complexity). *Fix f, μ, ε. Let (X, Y, Π) be the tuple distributed according to (X, Y) sampled from μ and then Π being the transcript of the protocol π applied to X, Y. Then define:*
1. $\mathrm{IC}_\mu(\pi) = I(X; \Pi \mid Y) + I(Y; \Pi \mid X)$
2. $\mathrm{IC}_\mu(f, \varepsilon) = \inf_\pi \mathrm{IC}_\mu(\pi)$, *where π computes f with error at most ϵ*
3. $\mathrm{IC}(f, \varepsilon) = \max_\mu \mathrm{IC}_\mu(f, \varepsilon)$

2.2 Lower bound techniques

For any family of variables $\{\beta_{x,y}\}_{(x,y)\in\mathbf{X}\times\mathbf{Y}}$ and any subset $E \subseteq \mathbf{X} \times \mathbf{Y}$, we will denote $\beta(E) = \sum_{(x,y)\in E} \beta_{x,y}$, and $\beta = \beta(\mathbf{X}\times\mathbf{Y})$. Unless otherwise specified "$\forall x, y$" means "$\forall x, y \in \mathbf{X} \times \mathbf{Y}$", "$\forall z$" means "$\forall z \in \mathbf{Z}$", "$\forall R$" means "for all rectangles R in $\mathbf{X}\times\mathbf{Y}$", and "$\forall P$" means "for all partitions P of $\mathbf{X} \times \mathbf{Y}$ into labeled rectangles (R, z)". We also denote by $|P|$ the size of the partition, that is, the number of rectangles (R, z) it contains.

Following Ganor *et al.* (making some small changes that do not affect the value of the bound) we define the relative discrepancy bound $\mathrm{rdisc}_\varepsilon(f, \mu)$, as follows.

Definition 3 (Relative discrepancy bound [GKR14b]). *Let μ be a distribution over $\mathbf{X} \times \mathbf{Y}$ and let $f : \mathrm{supp}(\mu) \to \{0, 1\}$ be a function.*

$$\begin{aligned}
\mathrm{rdisc}_\varepsilon(f, \mu) = \sup_{\kappa,\delta,\rho_{xy}} & \frac{1}{\delta}(\tfrac{1}{2} - \kappa - \varepsilon) \\
\textit{subject to} \quad & \left(\tfrac{1}{2} - \kappa\right) \cdot \rho(R) \le \mu(R \cap f^{-1}(z)) \qquad \forall R, z \ \textit{s.t.}\ \rho(R) \ge \delta \\
& \sum_{xy} \rho_{xy} = 1 \\
& 0 \le \kappa < \frac{1}{2}, \quad 0 < \delta < 1, \quad \rho_{xy} \ge 0 \quad \forall (x, y).
\end{aligned}$$

For the non-distributional case, we define $\mathrm{rdisc}_\varepsilon(f) = \max_\mu \mathrm{rdisc}_\varepsilon(f, \mu)$, where the maximum is over distributions μ over $\mathbf{X} \times \mathbf{Y}$, which implicitly adds nonnegativity and normalisation constraints on μ.

Notice that neither the constraints nor the objective function are linear in the variables. Throughout our proofs, we will assume that $\text{supp}(\mu) = \text{supp}(f)$ since this does not change the optimal value.

Using this formulation, Ganor *et al.* show:

Theorem 1. *[GKR14b] Let* $f : \text{supp}(\mu) \to \{0,1\}$ *be a (possibly partial) function. Then* $\log(\text{rdisc}_\varepsilon(f,\mu)) \le R_\varepsilon(f,\mu)$.

The relaxed partition bound was introduced by Kerenidis *et al.* [KLL+12] who proved that for any function, it is bounded above by its information complexity. Their results holds also relative to any fixed distribution on the inputs.[5]

Definition 4 (Relaxed partition bound [KLL+12]). *Let* μ *be a distribution over* $\mathbf{X} \times \mathbf{Y}$ *and let* $f : \text{supp}(\mu) \to \{0,1\}$ *be a function.*

$$\begin{aligned} \overline{\text{prt}}_\epsilon(f,\mu) = \max_{\alpha,\beta_{xy}} \quad & \beta - \alpha\epsilon \\ \textit{subject to :} \quad & \beta(R) - \alpha\mu(R \cap f^{-1}(z)) \le 1 \qquad \forall R, z \\ & \alpha \ge 0, \quad \alpha\mu_{xy} - \beta_{xy} \ge 0 \qquad \forall (x,y), \end{aligned}$$

where R *ranges over all rectangles,* $(x,y) \in \mathbf{X} \times \mathbf{Y}$ *and* $z \in \{0,1\}$. *The non-distributional relaxed partition bound is* $\overline{\text{prt}}_\epsilon(f) = \max_\mu \overline{\text{prt}}_\epsilon(f,\mu)$. *For the non-distributional case, we use* $\alpha_{x,y}$ *instead of* $\alpha\mu_{x,y}$ *(which is not linear if* μ *is no longer fixed), with* $\alpha_{x,y}$ *positive but non normalized.*

Theorem 2 ([KLL+12]). *For all* μ*, boolean functions* f *over the support of* μ *and all* $\varepsilon \in (0, \frac{1}{4}]$, $\Omega\left(\varepsilon^2 \log \overline{\text{prt}}_{2\varepsilon}(f,\mu)\right) = \text{IC}_\mu(f,\varepsilon) \le R_\varepsilon(f,\mu)$.

3 Relative discrepancy is bounded by relaxed partition

We show that the non-distributional relative discrepancy is bounded above by the relaxed partition. This implies that Ganor *et al.*'s separation strongly depends on the input distribution. To separate information from communication without fixing a distribution on the inputs, a technique stronger than relative discrepancy will be necessary. (See Figure 1a).

Theorem 3. *For any function* $f : \mathbf{X} \times \mathbf{Y} \to \{0,1\}$*, and any* $\epsilon \in (0, 2/3)$*,*

$$\text{rdisc}_{\frac{3}{2}\epsilon}(f) \le \overline{\text{prt}}_\epsilon(f).$$

[5] Compared with the original formulation [KLL+12], there is an implicit change of variables: we use $\beta_{x,y}$ here to denote what was $\alpha_{x,y} - \beta_{x,y}$ in the original notation.

Proof. It suffices to show that for any feasible solution of rdisc, there exists a feasible solution for $\overline{\text{prt}}$ whose objective value is at least as large. Let $(\kappa, \delta, \{\rho_{x,y}\}_{x,y}, \{\mu_{x,y}\}_{x,y})$ be a feasible solution of relative discrepancy for f. Define for any $(x,y) \in \mathbf{X} \times \mathbf{Y}$

$$\alpha_{x,y} = \frac{1}{\delta}(\tfrac{1}{2} - \kappa)\rho_{x,y} + \frac{1}{\delta}\mu_{x,y} \qquad , \qquad \beta_{x,y} = \frac{1}{\delta}(\tfrac{1}{2} - \kappa)\rho_{x,y}$$

We show that the relaxed partition constraints are satisfied. First, the sign constraints on $\alpha_{x,y}$ and $\alpha_{x,y} - \beta_{x,y}$ are satisfied. Moreover, for any rectangle R and output z,

$$\begin{aligned}
&\beta(R) - \alpha(R \cap f^{-1}(z)) \\
&= \frac{1}{\delta}(\tfrac{1}{2} - \kappa)\rho(R) - \frac{1}{\delta}\mu(R \cap f^{-1}(z)) - \frac{1}{\delta}(\tfrac{1}{2} - \kappa)\rho(R \cap f^{-1}(z)) \\
&\hspace{20em} \text{(by definition of } \alpha \text{ and } \beta) \\
&\leq \frac{1}{\delta}(\tfrac{1}{2} - \kappa)\rho(R) - \frac{1}{\delta}\mu(R \cap f^{-1}(z)) \qquad \text{(since } \rho_{xy} \geq 0 \text{ for any } (x,y))
\end{aligned}$$

There are two cases:

$\rho(R) \geq \delta$ **:** then $\frac{1}{\delta}(\frac{1}{2} - \kappa)\rho(R) - \frac{1}{\delta}\mu(R \cap f^{-1}(z)) \leq 0 \leq 1$ by the relative discrepancy constraint;

$\rho(R) < \delta$ **:** then $\frac{1}{\delta}(\frac{1}{2} - \kappa)\rho(R) - \frac{1}{\delta}\mu(R \cap f^{-1}(z)) < (\frac{1}{2} - \kappa) - \frac{1}{\delta}\mu(R \cap f^{-1}(z)) \leq \frac{1}{2} \leq 1$.

Finally we compare the objective values. Using the fact that ρ and μ are distributions, we get $\alpha = \frac{1}{\delta}\left(\frac{3}{2} - \kappa\right)$ and $\beta = \frac{1}{\delta}\left(\frac{1}{2} - \kappa\right)$, so

$$\beta - \epsilon\alpha = \frac{1}{\delta}\left[\tfrac{1}{2} - \kappa - (\tfrac{3}{2} - \kappa)\epsilon\right] \geq \frac{1}{\delta}(\tfrac{1}{2} - \kappa - \tfrac{3}{2}\epsilon)$$

which is the value of the objective function of rdisc for error $3\epsilon/2$.

Remark 1. Notice that our change of variables satisfies an additional constraint, that is,

$$\beta_{x,y} \geq 0 \text{ for any } (x,y) \in \mathbf{X} \times \mathbf{Y}. \tag{1}$$

since $\rho_{x,y} \geq 0$. We will examine the role of this constraint in Section 4. It turns out to be a key point in understanding how relative discrepancy relates to the partition bound and its variants. Also notice that $\alpha_{x,y}$ is not proportional to $\mu_{x,y}$, so this change of variable does not carry over to the distributional case, since $\alpha_{x,y}$ cannot be written as $\alpha\mu_{xy}$.

Combining Theorem 2 and Theorem 3 gives us that relative discrepancy is a lower bound on information complexity.

Corollary 1. *For all functions* $f : \mathbf{X} \times \mathbf{Y} \to \{0,1\}$ *and all* $\varepsilon \in (0, \frac{1}{6}]$, $\Omega\left(\varepsilon^2 \log(\text{rdisc}_{3\varepsilon}(f))\right) = \text{IC}(f, \varepsilon) \leq R_\varepsilon(f)$.

4 The distributional case

We have established in the previous section that relative discrepancy is bounded above by the relaxed partition bound, which we know provides a lower bound to information complexity [KLL+12]. Because of the counterexample of Ganor *et al.* we know that this cannot hold with respect to a fixed distribution μ [GKR14a,GKR14b]. In this section we study how the various bounds relate, relative to a fixed distribution μ, and uncover an elegant relationship between the bounds by adding simple positivity constraints to the partition bound.

We start with a fixed-distribution version of the partition bound [JK10], which we define below. It follows easily from the original proof that this is a lower bound on distributional communication complexity and that it equals the partition bound in the worst case distribution.

Definition 5 (Partition bound).

$$\begin{aligned}\mathrm{prt}_\epsilon(f,\mu) = \max_{\alpha,\beta_{xy}} \quad & \beta - \epsilon\alpha \\ \textit{subject to :} \quad & \beta(R) - \alpha\mu(R \cap f^{-1}(z)) \leq 1 \qquad \forall R, z \\ & \alpha \geq 0.\end{aligned}$$

The distributional bound is $\mathrm{prt}_\epsilon(f) = \max_\mu \mathrm{prt}_\epsilon(f,\mu)$. *Going from the non-distributional setting to a fixed distribution* μ, $\alpha_{x,y}$ *is replaced by* $\alpha \cdot \mu_{x,y}$, *that is,* $\{\alpha_{x,y}\}$ *is* $\{\mu_{x,y}\}$ *scaled by a factor* α.

Theorem 4. *([JK10]) Let* $f : \mathrm{supp}(\mu) \to \{0,1\}$ *be a (possibly partial) function. Then* $\log(\mathrm{prt}_\varepsilon(f,\mu)) \leq R_\varepsilon(f,\mu)$.

Note that the relaxed partition bound (Definition 4) is obtained from the partition bound by adding the constraint $\alpha\mu_{x,y} - \beta_{xy} \geq 0$ for all (x,y).

As suggested in the proof of Theorem 3, we now consider the constraint $\beta_{x,y} \geq 0$ for all x,y. Adding this constraint to the partition bound results in a new bound which we call the positive partition bound.

Definition 6 (Positive partition bound).

$$\begin{aligned}\mathrm{prt}^+_\epsilon(f,\mu) = \max_{\alpha,\beta_{xy}} \quad & \beta - \epsilon\alpha \\ \textit{subject to :} \quad & \beta(R) - \alpha\mu(R \cap f^{-1}(z)) \leq 1 \qquad \forall R, z \\ & \alpha \geq 0, \quad \beta_{xy} \geq 0 \qquad \forall (x,y).\end{aligned}$$

We also define $\mathrm{prt}^+_\epsilon(f) = \max_\mu \mathrm{prt}^+_\epsilon(f,\mu)$, *and use* $\alpha_{x,y}$ *instead of* $\alpha\mu_{x,y}$.

The weak partition bound is obtained by adding both constraints.

Definition 7 (Weak partition bound).

$$\begin{aligned}
\mathrm{wprt}_\epsilon(f,\mu) = \max_{\alpha,\beta_{xy}} \quad & \beta - \epsilon\alpha \\
\textit{subject to:} \quad & \beta(R) - \alpha\mu(R\cap f^{-1}(z)) \le 1 && \forall R, z, \\
& \alpha \ge 0, \quad \beta_{xy} \ge 0, \quad \alpha\mu_{xy} - \beta_{xy} \ge 0 && \forall (x,y).
\end{aligned}$$

We also define $\mathrm{wprt}_\epsilon(f) = \max_\mu \mathrm{wprt}_\epsilon(f,\mu)$.

Because we have added a constraint to a maximisation problem, it is easy to see that the following holds (see Figure 1b).

Proposition 1. *For all* f, μ, ϵ,

$\mathrm{wprt}_\epsilon(f,\mu) \le \mathrm{prt}^+_\epsilon(f,\mu) \le \mathrm{prt}_\epsilon(f,\mu)$ *and* $\mathrm{wprt}_\epsilon(f,\mu) \le \overline{\mathrm{prt}}_\epsilon(f,\mu) \le \mathrm{prt}_\epsilon(f,\mu)$.

We shall now see that the relative discrepancy bound is equivalent to the positive partition bound, up to constant factors.

Theorem 5. *Let* μ *be a distribution on* $\mathbf{X}\times\mathbf{Y}$ *and* f *be a boolean function on the support of* μ *such that either* $\mathrm{rdisc}_\epsilon(f,\mu) \ge 1$ *or* $\mathrm{prt}^+_{4\epsilon}(f,\mu) > 2$. *Then for any* $\epsilon \in (0, 1/4)$,

$$\frac{\epsilon}{2}\mathrm{prt}^+_{4\epsilon}(f,\mu) \le \mathrm{rdisc}_\epsilon(f,\mu) \le \mathrm{prt}^+_\epsilon(f,\mu).$$

The proof follows from Lemma 1 and Lemma 2.

Lemma 1. *Let* $0 < C < 1$, $\frac{1}{1-C}\epsilon < \epsilon' < 1$, μ *be a distribution on* $\mathbf{X}\times\mathbf{Y}$ *and* f *be a boolean function on the support of* μ *such that either* $\mathrm{rdisc}_\epsilon(f,\mu) \ge 1$ *or* $\mathrm{prt}^+_{\epsilon'}(f,\mu) > 1/C$. *Then, we have*

$$C \cdot (\epsilon'(1-C) - \epsilon) \cdot \mathrm{prt}^+_{\epsilon'}(f,\mu) \le \mathrm{rdisc}_\epsilon(f,\mu).$$

Proof. For the theorem, we just set $C = \frac{1}{2}$ and $\epsilon' = 4\epsilon$. Let us first assume that $\mathrm{prt}^+_{\epsilon'}(f,\mu) > 1/C$. It suffices to show that for any feasible solution of prt^+, there exists a feasible solution for rdisc whose objective value is at least as large. Let α, β_{xy} be a feasible solution of $\mathrm{prt}^+_{\epsilon'}(f,\mu)$ achieving value $\mathrm{prt}^+_{\epsilon'}(f,\mu) = \beta - \epsilon'\alpha > 1/C$. Note that this implies in particular that $\beta > 0$, and that we also necessarily have $\alpha > 0$ otherwise for $\alpha = 0$ the constraints of prt^+would imply $\beta \le 1$, in contradiction with $\mathrm{prt}^+_{\epsilon'}(f,\mu) = \beta > 1/C$. Let

$$\delta = \frac{1}{C\beta}, \qquad \kappa = \frac{1}{2} - \frac{\beta}{\alpha}(1-C), \qquad \rho_{xy} = \frac{\beta_{xy}}{\beta} \qquad \forall x, y.$$

We show that the relative discrepancy constraints are satisfied. First, we obtain from the prt^+constraint that for all z and for any rectangle R such that $\rho(R) \geq \delta$,

$$\mu(R \cap f^{-1}(z)) \geq \tfrac{1}{\alpha}(\beta(R) - 1) = \tfrac{1}{\alpha}(\beta\rho(R) - 1) \geq \tfrac{1}{\alpha}(\beta - \tfrac{1}{\delta})\rho(R) = (\tfrac{1}{2} - \kappa)\rho(R).$$

It remains to show that κ and δ satisfy the necessary constraints. For δ, we have $\delta > 0$ by definition and $\delta = \frac{1}{C\beta} \leq \frac{1}{C\mathrm{prt}^+_{\epsilon'}(f,\mu)} < 1$.

For κ, we have by definition $\kappa < \frac{1}{2}$ since $\beta > 1/C > 0$ and $C < 1$. Let us also recall that we have proved above that

$$\mu(R \cap f^{-1}(z)) \geq (\tfrac{1}{2} - \kappa)\rho(R)$$

for any z and any rectangle such that $\rho(R) \geq \delta$. Using the full rectangle (where $\rho(X \times Y) = 1 > \delta$) we have that

$$\mu(f^{-1}(z)) \geq \tfrac{1}{2} - \kappa$$

for all z. Summing over both values of z, we get $1 \geq 1 - 2\kappa$, that is $\kappa \geq 0$.

Finally we compare the objective values:

$$\begin{aligned}\frac{\frac{1}{2} - \kappa - \epsilon}{\delta} &= \left(\tfrac{\beta}{\alpha}(1 - C) - \epsilon\right) C\beta \\ &\geq C\left(\epsilon'(1 - C) - \epsilon\right)(\beta - \epsilon'\alpha),\end{aligned}$$

where the last inequality holds since $\beta - \epsilon'\alpha > 0$ implies $\beta/\alpha > \epsilon'$.

Note that the argument so far did not require $\mathrm{rdisc}_\epsilon(f,\mu) \geq 1$. Therefore, the only case that remains to be considered is when $\mathrm{rdisc}_\epsilon(f,\mu) \geq 1$ and $\mathrm{prt}^+_{\epsilon'}(f,\mu) \leq 1/C$. In that case, we have

$$\mathrm{rdisc}_\epsilon(f,\mu) \geq 1 \geq C\mathrm{prt}^+_{\epsilon'}(f,\mu) \geq C\left(\epsilon'(1 - C) - \epsilon\right)\mathrm{prt}^+_{\epsilon'}(f,\mu),$$

hence the lemma also holds.

Lemma 2. *Let μ be a distribution on $\mathbf{X} \times \mathbf{Y}$ and f be a boolean function on the support of μ. Then*

$$\mathrm{rdisc}_\epsilon(f,\mu) \leq \mathrm{prt}^+_\epsilon(f,\mu).$$

Proof. It suffices to show that for any such feasible solution of rdisc, there exists a feasible solution for prt^+ whose objective value is at least as large. Let κ, δ and $\{\rho_{xy}\}_{xy}$ be a feasible solution for the relative discrepancy bound. Let

$$\alpha = \frac{1}{\delta}, \qquad \beta_{xy} = \frac{\frac{1}{2} - \kappa}{\delta}\rho_{xy} \quad \forall x, y.$$

We first show that this yields a feasible point. By definition, we have $\alpha \geq 0$ and $\beta_{xy} \geq 0$ for all x, y. Moreover, for any rectangle R such that $\rho(R) < \delta$, we immediately have

$$\beta(R) - \alpha\mu(R \cap f^{-1}(z)) \leq \beta(R) = \frac{(\frac{1}{2} - \kappa)}{\delta}\rho(R) < 1,$$

so the constraint is satisfied. For any rectangle R such that $\rho(R) \geq \delta$, the first constraint in the relative discrepancy bound implies that

$$\left(\tfrac{1}{2} - \kappa\right) \cdot \rho(R) \leq \mu(R \cap f^{-1}(z)) \qquad \forall z,$$

Therefore, we have for such rectangles

$$\beta(R) - \alpha\mu(R \cap f^{-1}(z)) \leq \frac{\frac{1}{2} - \kappa}{\delta}\rho(R) - \frac{1}{\delta}(\tfrac{1}{2} - \kappa)\rho(R) = 0,$$

so the constraint is also satisfied.

It remains to compare the objective values. We have

$$\beta - \epsilon\alpha = \frac{\frac{1}{2} - \kappa}{\delta} - \frac{\epsilon}{\delta} = \frac{\frac{1}{2} - \kappa - \epsilon}{\delta}.$$

Revisiting the non-distributional case Once we have defined all these variants of the partition bound, we can revisit the non-distributional case. For the change of variables in the proof of Theorem 3, we have noted that the constraint $\beta_{xy} \geq 0$ holds $\forall (x, y)$ (see Inequality 1). This shows that, in the non-distributional case, relative discrepancy is, in fact, no larger than the weak partition bound, i.e. $\text{rdisc}_\epsilon(f) \leq \text{wprt}_{\frac{2}{3}\epsilon}(f)$.

One can prove Theorem 3 in a different way: First, by Lemma 2, for any distribution μ, $\text{rdisc}_\epsilon(f, \mu) \leq \text{prt}^+_\epsilon(f, \mu)$. Then, we can use the fact that in the non-distributional case, the positive partition bound is no larger than the weak partition bound (the reverse is true by definition).

Lemma 3. *For any function $f : \mathbf{X} \times \mathbf{Y} \to \{0, 1\}$, and any $\epsilon \in (0, 1/2)$,*

$$\text{prt}^+_\epsilon(f) \leq \text{wprt}_{\frac{\epsilon}{2}}(f) + \frac{\epsilon}{2}.$$

Proof. Let $\alpha_{x,y}, \beta_{x,y}$ be a feasible solution for prt^+, and consider the following assignment for wprt: $\alpha'_{x,y} = \alpha_{x,y} + \beta_{x,y}, \quad \beta'_{x,y} = \beta_{x,y}$. The constraint on rectangles is still satisfied, and the added positivity constraint $\alpha'_{x,y} - \beta'_{x,y} = \alpha_{x,y} \geq 0$ is also satisfied. Finally, the objective function for wprt with error $\frac{\epsilon}{2}$ is $\beta' - \frac{\epsilon}{2}\alpha' = \beta - \frac{\epsilon}{2}\beta - \frac{\epsilon}{2}\alpha \geq \beta - \epsilon\alpha - \frac{\epsilon}{2}$ (where we have used the constraint on $R = \mathbf{X} \times \mathbf{Y}$), as claimed.

Note that the change of variables in the proof of Theorem 3 is just the composition of the two changes of variables in Lemma 2 and Lemma 3. It is also now clearer how the distributional and the non-distributional settings differ. We know that it cannot be the case that $\mathrm{prt}^{+}_{\epsilon}(f,\mu) \leq \mathrm{wprt}_{\epsilon}(f,\mu)$ for fixed distribution, since Ganor *et al.* provide a counterexample. We can also see that for this specific change of variable, by setting $\alpha'_{x,y} = \alpha_{x,y} + \beta_{x,y}$, $\alpha'_{x,y}$ cannot be written as $\alpha_{x,y} = \alpha\mu_{x,y}$, as we would need in the distributional case, since it is a combination of α and β.

Negative relative discrepancy We have shown that for any μ, the relative discrepancy is equivalent to the positive partition bound by setting ρ proportional to β. One might ask what happens to the relative discrepancy bound if the positivity constraint on $\rho_{x,y}$ is relaxed? Recall that in the partition bound, the positivity constraint on $\beta_{x,y}$ is removed, so using essentially the same proof, we can show that this "negative relative discrepancy" bound is equivalent to the partition bound.

5 Adaptive relative discrepancy is equivalent to the public coin partition

Recently, two new lower bound techniques for communication have been introduced that involve partitions instead of just rectangles. First, the public coin partition bound, whose logarithm is polynomially related to randomized communication complexity ([JLV14]).

We give below a distributional version of the public-coin partition bound. Note that this is a simplified definition with respect to the original one by means of removing redundant variables and constraints in the primal formulation, taking the dual of the resulting expression, and replacing $\alpha_{x,y}$ by $\alpha\mu_{x,y}$, where the distribution μ is fixed:

Definition 8 (Public coin partition bound [JLV14]).

$$\begin{aligned}
\mathrm{pprt}_{\epsilon}(f,\mu) = \max_{\alpha,\beta} \quad & \beta - \epsilon\alpha \\
\textit{subject to :} \quad & \beta - \sum_{(R,z)\in P} \alpha\mu(R\cap f^{-1}(z)) \leq |P| \qquad \forall P \\
& \alpha \geq 0, \beta \geq 0
\end{aligned}$$

The authors proved the following result (that we restate only for boolean functions in the context of this paper), that we extend to the distributional setting

Theorem 6. *([JLV14]) Let f be a (possibly partial) boolean function over $\mathbf{X} \times \mathbf{Y}$, μ any distribution and $\epsilon \in (0, 1/2)$. Then,*

$$\log(\mathrm{pprt}_\epsilon(f,\mu)) \le R_\epsilon(f,\mu) \le \left(\log \mathrm{pprt}_{\epsilon/2}(f,\mu) + \log\frac{1}{\epsilon} + 2\right)^2$$

The second bound is the adaptive relative discrepancy bound of Ganor *et al.*, which can be expressed (with some minor changes that do not affect the optimal value) as follows:

Definition 9 (Adaptive relative discrepancy [GKR14b]).

$$\begin{aligned}
\mathrm{ardisc}_\varepsilon(f,\mu) = \sup_{\kappa,\delta,\rho^P_{x,y}} \quad & \frac{1}{\delta}\left(\frac{1}{2} - \kappa - \varepsilon\right) \\
\textit{subject to :} \quad & \left(\frac{1}{2} - \kappa\right)\rho^P(R) \le \mu(R \cap f^{-1}(z)) \quad \forall P, \forall (z,R) \in P s.t.\ \rho^P(R) \ge \delta \\
& \rho^P = 1 \quad \forall P \\
& 0 \le \kappa < \tfrac{1}{2}, \quad 0 < \delta < 1, \quad \rho^P_{x,y} \ge 0 \quad \forall P, \forall (x,y).
\end{aligned}$$

Then $\mathrm{ardisc}_\varepsilon(f) = \max_\mu \mathrm{ardisc}_\varepsilon(f,\mu)$.

Notice again that this is not a linear program. Also, the total weight ρ^P does not depend on P. It is clear that a solution for the relative discrepancy bound provides a solution for the adaptive relative discrepancy bound. Moreover, Ganor *et al.* have proven the following result.

Theorem 7. *([GKR14b]) Let $f : \mathrm{supp}(\mu) \to \{0,1\}$ be a (possibly partial) function. Then $\log(\mathrm{ardisc}_\varepsilon(f,\mu)) \le R_\varepsilon(f,\mu)$.*

We show that the two bounds are equivalent up to constant factors.

Theorem 8. *For any distribution μ, any function $f : \mathrm{supp}(\mu) \to \{0,1\}$ and $\epsilon \in (0, \frac{1}{4})$ such that either $\mathrm{ardisc}_\epsilon(f,\mu) \ge 1$ or $\mathrm{pprt}_{4\epsilon}(f,\mu) > 2$,*

$$\frac{\epsilon}{2}\mathrm{pprt}_{4\epsilon}(f,\mu) \le \mathrm{ardisc}_\epsilon(f,\mu) \le \mathrm{pprt}_\epsilon(f,\mu)$$

Once we have expressed the public coin partition bound as in Definition 8, the equivalence proof is similar to the proof of Theorem 5 and follows from the following two lemmata (one for each inequality).

Lemma 4. *For any $0 \le C < 1$ and any $\frac{1}{1-C}\epsilon < \epsilon' < 1$ such that either $\mathrm{ardisc}_\epsilon(f,\mu) \ge 1$ or $\mathrm{pprt}_{\epsilon'}(f,\mu) > 1/C$, we have*

$$C(\epsilon'(1-C) - \epsilon)\mathrm{pprt}_{\epsilon'}(f,\mu) \le \mathrm{ardisc}_\epsilon(f,\mu).$$

Proof. For the theorem, it is enough to take $C = \frac{1}{2}$ and $\epsilon' = 4\epsilon$. Let us first assume that $\mathrm{pprt}_{\epsilon'}(f,\mu) > 1/C$. It suffices to show that for any feasible solution of pprt, there exists a feasible solution for ardisc whose objective value is at least as large. Consider $\alpha \geq 0, \beta \geq 0$ a feasible solution of $\mathrm{pprt}_{\epsilon'}(f,\mu)$ such that $\beta - \epsilon'\alpha > 0$. Note that this implies in particular that $\beta > 0$, and that we also necessarily have $\alpha > 0$ otherwise for $\alpha = 0$ the constraints of pprt would imply $\beta \leq 1$, in contradiction with $\mathrm{pprt}_{\epsilon'}(f,\mu) = \beta > 1/C$. Let

$$v_{z,R} = 1 + \alpha\mu(R \cap f^{-1}(z)), \qquad \rho^P(R) = \frac{v_{z,R}}{v(P)}$$
$$\delta = \frac{1}{C\beta}, \qquad \kappa = \frac{1}{2} - \frac{\beta}{\alpha}(1-C)$$

where z is the only output label such that $(R,z) \in P$ and $v(P) = \sum_{(R,z)\in P} v_{z,R}$. Observe that for any P, ρ^P is a distribution since $v_{z,R} \geq 0$ and it is normalized.

Note that by the first pprt constraint, we have for all P

$$v(P) = \sum_{(z,R)\in P} v_{z,R} = |P| + \sum_{(z,R)\in P} \alpha\mu(R \cap f^{-1}(z)) \geq \beta. \tag{2}$$

First, we check that the ardisc constraint is satisfied. We have successively:

$$\begin{aligned}
\mu(R \cap f^{-1}(z)) &= \frac{1}{\alpha}(v_{z,R} - 1) && \text{(by definition of } v_{z,R}\text{)}\\
&= \frac{1}{\alpha}\left(\rho^P(R)v(P) - 1\right) && \text{(by definition of } \rho^P\text{)}\\
&\geq \frac{1}{\alpha}\left(\rho^P(R)\beta - 1\right) && \text{(since } v(P) \geq \beta\text{, see (2))}
\end{aligned}$$

Hence if $\rho^P(R) \geq \delta$ (otherwise, there is no constraint to satisfy in ardisc), then $-1 \geq -\frac{\rho^P(R)}{\delta}$ and we have

$$\begin{aligned}
\mu(R \cap f^{-1}(z)) &\geq \frac{1}{\alpha}\left(\beta - \frac{1}{\delta}\right)\rho^P(R)\\
&\geq \left(\frac{1}{2} - \kappa\right)\rho^P(R) && \text{(by definitions of } \kappa \text{ and } \delta\text{)}
\end{aligned}$$

We check now the constraints on δ and κ:

- for δ: $\delta > 0$ by definition and $\delta = \frac{1}{C\beta} \leq \frac{1}{C\mathrm{pprt}_{\epsilon'}(f,\mu)} < 1$ by assumption.

– for κ: $\kappa < \frac{1}{2}$ by definition and $\kappa \geq 0$ by summing ardisc's main constraint for $R = \mathbf{X} \times \mathbf{Y}$ over both values of z.

Finally for the objective values, we can easily verify that :

$$\frac{\frac{1}{2} - \kappa - \varepsilon}{\delta} = \left(\frac{\beta}{\alpha}(1-C) - \varepsilon\right) C\beta \geq C\left(\epsilon'(1-C) - \epsilon\right)(\beta - \epsilon'\alpha),$$

where the last inequality holds since $\beta - \varepsilon'\alpha > 0$ implies $\beta/\alpha > \varepsilon'$.

It remains to consider the case where $\text{ardisc}_\epsilon(f,\mu) \geq 1$ and $\text{pprt}_{\epsilon'}(f,\mu) \leq 1/C$. In that case, we have

$$\text{ardisc}_\epsilon(f,\mu) \geq 1 \geq C\text{pprt}_{\epsilon'}(f,\mu) \geq C\left(\epsilon'(1-C) - \epsilon\right)\text{pprt}_{\epsilon'}(f,\mu),$$

hence the claim also holds.

Lemma 5. $\text{ardisc}_\epsilon(f,\mu) \leq \text{pprt}_\epsilon(f,\mu)$

Proof. It suffices to show that for any such feasible solution of ardisc, there exists a feasible solution for pprt whose objective value is at least as large. We define

$$\alpha = \frac{1}{\delta}, \qquad \beta = \frac{\frac{1}{2} - \kappa}{\delta}.$$

We show that this yields a feasible point. The ardisc constraint implies for all $(R,z) \in P$:

$$\left(\frac{1}{2} - \kappa\right)\rho^P(R) \leq \mu(R \cap f^{-1}(z)) + \delta\left(\frac{1}{2} - \kappa\right).$$

Summing over the $(R,z) \in P$ gives

$$\left(\frac{1}{2} - \kappa\right) - \sum_{(R,z)\in P} \mu(R \cap f^{-1}(z)) - \delta\left(\frac{1}{2} - \kappa\right) \cdot |P| \leq 0.$$

Dividing by δ and using the definitions of α and β, we obtain

$$\beta - \sum_{(R,z)\in P} \alpha\mu(R \cap f^{-1}(z)) \leq (\tfrac{1}{2} - \kappa)|P| \leq |P|,$$

so the constraint is satisfied. It remains to compare the objective values. We have

$$\beta - \epsilon\alpha = \frac{\frac{1}{2} - \kappa}{\delta} - \frac{\epsilon}{\delta} = \frac{\frac{1}{2} - \kappa - \epsilon}{\delta}.$$

Combining Theorem 6 and Theorem 8 we have

Corollary 2. *For any* μ, $f : \text{supp}(\mu) \to \{0,1\}$ *and* $\epsilon \in (0, \frac{1}{8})$,

$$\log(\text{ardisc}_\epsilon(f,\mu)) \leq R_\epsilon(f,\mu) \leq \left(\log \text{ardisc}_{\epsilon/8}(f,\mu) + 2\log\frac{1}{\epsilon} + 6\right)^2$$

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