A simple proof of the Isolation Lemma

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Abstract

We give a new simple proof for the Isolation Lemma, with slightly better parameters, that also gives non-trivial results even when the weight domain m is smaller than the number of variables n.

1 The lemma and its proof

Let n, m be two natural numbers, $\mathcal{F} \subseteq \mathrm{P}([n])$, where $[n] = \{1, \ldots, n\}$ and $\mathrm{P}([n])$ is the set of all subsets of $[n]$. A *weight function* is a function $w : [n] \to [m]$. Given a weight function w, we extend it to sets by defining the weight of a set to be the sum of the weights of its elements, i.e., $w(S) = \sum_{x \in S} w(x)$.

Let $min_w(\mathcal{F})$ be the family of sets with the smallest weight amongst $\mathcal{F}$ with respect to w, i.e.,

$$min_w(\mathcal{F}) = \{A \in \mathcal{F} \mid w(A) \text{ is minimal}\}.$$

When $|min_w(\mathcal{F})| = 1$ the minimum weight under w is attained uniquely.
Mulmuley, Vazirani and Vazirani [MVV87] proved:

Theorem 1. *[MVV87] (The isolation lemma - original version) For every $\mathcal{F} \subseteq \mathrm{P}([n])$,*

$$\Pr_{w:[n]\to[m]}(|min_w(\mathcal{F})| = 1) \geq 1 - \frac{n}{m},$$

where the probability is over w that is uniformly distributed over all functions from $[n]$ to $[m]$.

We give a new proof for the isolation lemma with slightly better parameters. Unlike Theorem 1 our proof gives non-trivial results even when $m \leq n$:

Theorem 2. *For every $\mathcal{F} \subseteq \mathrm{P}([n])$,*

$$\Pr_{w:[n]\to[m]}(|min_w(\mathcal{F})| = 1) \geq \left(1 - \frac{1}{m}\right)^n.$$

Proof. Fix a family $\mathcal{F} \subseteq \mathrm{P}([n])$. W.l.o.g. no set $S \in \mathcal{F}$ is a superset of another set $T \in \mathcal{F}$, as the weight of S is always strictly bigger than the weight of T and thus S never affects whether there is a unique minimum or not and therefor we can drop it.

Let $W = \{w : [n] \to [m]\}$ denote the set of all weight functions and $W_{>1} = \{w : [n] \to \{2, \ldots, m\}\}$ denote the set of all weight functions that assign each element a weight that is strictly larger than 1. We

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define a mapping $\phi : W_{>1} \to W$ as follows: given a weight function $w \in W_{>1}$, fix an arbitrary set $S_0 \in min_w(\mathcal{F})$ and define the weight function $w' = \phi(w)$ to be:

$$w'(i) \quad = \quad \begin{cases} w(i) - 1 & \text{if } i \in S_0, \\ w(i) & \text{otherwise.} \end{cases}$$

We claim:

Claim 3. *1. If $w \in W_{>1}$ then $|min_{\phi(w)}(\mathcal{F})| = 1$*

2. ϕ is one-to-one on $W_{>1}$.

Together, this shows that:

$$\Pr_{w:[n]\to[m]}(|min_w(\mathcal{F})| = 1) \;\geq\; \frac{|\phi(W_{>1})|}{|W|} \;=\; \frac{|W_{>1}|}{|W|} \;=\; \left(1 - \frac{1}{m}\right)^n.$$

We are left with proving the claim. To see the first item in the claim, notice that for all $S \in \mathcal{F}$, $w'(S) = w(S) - |S \cap S_0|$. Thus, for all $S_0 \neq S \in \mathcal{F}$,

$$w'(S_0) \quad = \quad w(S_0) - |S_0| \leq w(S) - |S_0| < w'(S),$$

where the first inequality is because S_0 gives minimal weight under w, and the second inequality is because the set S_0 is not contained in any other set in $\mathcal{F}$ and therefore $|S \cap S_0| < |S_0|$.

The second item in the claim follows from the first one. If $w \in W_{>1}$ then there is a unique set $S_0 \in \mathcal{F}$ achieving minimum value under $w' = \phi(w)$. If we take w' and increment the weight it gives S_0 we recover w. Thus, w' determines w and ϕ is one-to-one on $W_{>1}$. □

References

[MVV87] Ketan Mulmuley, Umesh V Vazirani, and Vijay V Vazirani. Matching is as easy as matrix inversion. In *Proceedings of the nineteenth annual ACM symposium on Theory of computing*, pages 345–354. ACM, 1987.