A simple proof of the Isolation Lemma

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Abstract

We give a new simple proof for the Isolation Lemma, with slightly better parameters, that also gives non-trivial results even when the weight domain $m$ is smaller than the number of variables $n$.

1 The lemma and its proof

Let $n, m$ be two natural numbers, $\mathcal{F} \subseteq \mathcal{P}([n])$, where $[n] = \{1, \ldots, n\}$ and $\mathcal{P}([n])$ is the set of all subsets of $[n]$. A weight function is a function $w : \{1, \ldots, n\} \rightarrow \{1, \ldots, m\}$, given a weight function $w$, we extend it to sets by defining the weight of a set to be the sum of the weights of its elements, i.e., $w(S) = \sum_{x \in S} w(x)$.

Let $\min_w(\mathcal{F})$ be the family of sets with the smallest weight amongst $\mathcal{F}$ with respect to $w$, i.e.,

$$\min_w(\mathcal{F}) = \{ A \in \mathcal{F} | w(A) \text{ is minimal} \}.$$ 

When $|\min_w(\mathcal{F})| = 1$ the minimum weight under $w$ is attained uniquely.

Mulmuley, Vazirani and Vazirani [MVV87] proved:

**Theorem 1.** [MVV87] (The isolation lemma - original version) For every $\mathcal{F} \subseteq \mathcal{P}([n])$,

$$\Pr_{w : \{1, \ldots, n\} \rightarrow \{1, \ldots, m\}} (|\min_w(\mathcal{F})| = 1) \geq 1 - \frac{n}{m},$$

where the probability is over $w$ that is uniformly distributed over all functions from $[n]$ to $[m]$.

We give a new proof for the isolation lemma with slightly better parameters. Unlike Theorem 1 our proof gives non-trivial results even when $m \leq n$:

**Theorem 2.** For every $\mathcal{F} \subseteq \mathcal{P}([n])$,

$$\Pr_{w : \{1, \ldots, n\} \rightarrow \{1, \ldots, m\}} (|\min_w(\mathcal{F})| = 1) \geq \left( 1 - \frac{1}{m} \right)^n.$$ 

**Proof.** Fix a family $\mathcal{F} \subseteq \mathcal{P}([n])$. W.l.o.g. no set $S \in \mathcal{F}$ is a superset of another set $T \in \mathcal{F}$, as the weight of $S$ is always strictly bigger than the weight of $T$ and thus $S$ never affects whether there is a unique minimum or not and therefore we can drop it.

Let $W = \{ w : \{1, \ldots, n\} \rightarrow \{1, \ldots, m\} \}$ denote the set of all weight functions and $W_{\geq 1} = \{ w : \{1, \ldots, n\} \rightarrow \{2, \ldots, m\} \}$ denote the set of all weight functions that assign each element a weight that is strictly larger than 1. We

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define a mapping $\phi : W_{>1} \rightarrow W$ as follows: given a weight function $w \in W_{>1}$, fix an arbitrary set $S_0 \in \text{min}_w(F)$ and define the weight function $w' = \phi(w)$ to be:

$$w'(i) = \begin{cases} w(i) - 1 & \text{if } i \in S_0, \\ w(i) & \text{otherwise.} \end{cases}$$

We claim:

**Claim 3.**

1. If $w \in W_{>1}$ then $|\text{min}_{\phi(w)}(F)| = 1$

2. $\phi$ is one-to-one on $W_{>1}$.

Together, this shows that:

$$\Pr_{w:[n] \rightarrow [m]}(|\text{min}_w(F)| = 1) \geq \frac{\phi(W_{>1})}{|W|} = \frac{|W_{>1}|}{|W|} = \left(1 - \frac{1}{m}\right)^n.$$

We are left with proving the claim. To see the first item in the claim, notice that for all $S \in F$, $w'(S) = w(S) - |S \cap S_0|$. Thus, for all $S_0 \neq S \in F$,

$$w'(S_0) = w(S_0) - |S_0| \leq w(S) - |S_0| < w'(S),$$

where the first inequality is because $S_0$ gives minimal weight under $w$, and the second inequality is because the set $S_0$ is not contained in any other set in $F$ and therefore $|S \cap S_0| < |S_0|$.

The second item in the claim follows from the first one. If $w \in W_{>1}$ then there is a unique set $S_0 \in F$ achieving minimum value under $w' = \phi(w)$. If we take $w'$ and increment the weight it gives $S_0$ we recover $w$. Thus, $w'$ determines $w$ and $\phi$ is one-to-one on $W_{>1}$.

**References**