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Clique, Permanent and Monotone projections

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1 The Clique polynomial

In this note we are interested in the following two polynomial families,

$$\mathsf{Perm}_n := \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{i,\sigma(i)}, \quad \text{and} \quad \mathsf{Clique}_n := \sum_{\substack{S \subseteq [n] \\ |S| = \sqrt{n}}} \prod_{\substack{i,j \in S \\ i<j}} x_{i,j}.$$

It is motivated by the question, *whether* Clique_n *is a monotone* p*-projection of* Perm_n*?* This question was raised by Jukna [Juk14]. In fact, Jukna asked whether the Hamiltonian cycle polynomial is a monotone p-projection of the permanent. The first progress was made by Grochow [Gro15]. He proved that the Hamiltonian cycle polynomial is *not* a monotone sub-exponential-size projection of Perm_n, but left open the possibility that Clique_n itself is a monotone p-projection of Perm_n. We rule this out as well, using the same approach. Thus this possibility of transferring monotone circuit lower bounds for clique to permanent cannot work. It should be noted that Grochow's connection between extended formulation and monotone projection (Lemma 1.2) easily allows one to obtain lower bounds against monotone projection (for example, see [Gro15, MS16]).

For any polynomial p in n variables, let $\mathsf{Newt}(p)$ denote the polytope in $\mathbb{R}^n$ that is convex hull of the vectors of exponents of monomials of p. The *correlation polytope* $\mathsf{COR}(n)$ is defined as the convex hull of $n \times n$ binary symmetric matrices of rank 1. That is, $\mathsf{COR}(n) := \text{convex hull}\{vv^{\mathrm{t}} \mid v \in \{0,1\}^n\}$.

For a polytope P, let $\mathsf{c}(P)$ denote the minimal number of linear inequalities needed to define P. A polytope $Q \subseteq \mathbb{R}^m$ is an *extension* of $P \subseteq \mathbb{R}^n$ if there is a linear map $\pi \colon \mathbb{R}^m \to \mathbb{R}^n$ such that $\pi(Q) = P$. The *extension complexity* of P, denoted $\mathsf{xc}(P)$, is the minimum size $c(Q)$ of any extension Q (of any dimension) of P.

We use the following recent results.

Fact 1.1. 1. *[Gro15]* $\mathsf{c}(\mathsf{Newt}(\mathsf{Perm}_n)) \leqslant 2n$.

2. *[FMP+15] If polytope Q is an extension of polytope P, then $xc(P) \leqslant \mathsf{xc}(Q)$.*

Lemma 1.2 ([Gro15])**.** *Let $f(x_1, \dots, x_n)$ and $g(y_1, \dots, y_m)$ be polynomials over a totally ordered semi-ring R, with non-negative coefficients. If f is a monotone projection of g, then the intersection of* $\mathsf{Newt}(g)$ *with some linear subspace is an extension of* $\mathsf{Newt}(f)$*. In particular,* $\mathsf{xc}(\mathsf{Newt}(f)) \leqslant m + \mathsf{c}(\mathsf{Newt}(g))$.

Theorem 1.1 ([FMP+15])**.** *There exists some constant $C > 0$ such that for all n,* $\mathsf{xc}(\mathsf{COR}(n)) \geqslant 2^{Cn}$.

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We now show that Clique_n is **not** a monotone p-projection of Perm_n. To establish this we will consider a different polynomial $\mathsf{Clique}^* = (\mathsf{Clique}^*{}_n)$ that counts all cliques in a graph. More formally,

$$\mathsf{Clique}^*{}_n := \sum_{S \subseteq [n]} \prod_{\substack{i,j \in S \\ i<j}} x_{i,j}.$$

We first claim that proving monotone projection lower bound against Clique^* suffices to establish lower bound against Clique. The proof is basically the VNP-completeness proof of Clique_n (see [Hru15]).

Lemma 1.3. *The family* Clique^* *is a monotone* p*-projection of the family* Clique. *In particular,* $\mathsf{Clique}^*{}_n$ *is a monotone projection of* Clique_{n^2}.

Theorem 1.2. *Over the reals (or any totally ordered semi-ring), the family* Clique^* *is not monotone* p*-projections of the* Perm *family. In fact, if* $\mathsf{Clique}^*{}_n$ *is a monotone projection of* $\mathsf{Perm}_{t(n)}$*, then* $t(n) \geqslant 2^{\Omega(n)}$.

Proof. Let Q be the Newton polytope of $\mathsf{Clique}^*{}_n$. It resides in N dimensions, where $N = \binom{n}{2}$, and is the convex hull of vectors of the form $\langle \tilde{a} \rangle$ where $\tilde{a} \in \{0,1\}^N$ is a characteristic vector of set of edges of a clique in the complete undirected graph K_n. Let $\{v_1, \ldots, v_n\}$ be the vertex set of K_n.

Define the polytope R, also in N dimensions, to be the intersection of Q with the constraint $\sum_{\text{e is incident on } v_n} a_e \geqslant 1$. That is, R is the convex hull of all cliques that contain the vertex v_n. Also, define a linear map $\ell\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{(n-1)\times(n-1)}$, as follows, $\ell(A) = B$ where $B_{i,j} = A_{i,j}$ if $i \neq j$, and $B_{i,i} = A_{n,i}$. It easily follows that $\ell(R) = \mathsf{COR}(n-1)$. Thus R is an extension of $\mathsf{COR}(n-1)$, so by Fact 1.1 (2), $\mathsf{xc}(\mathsf{COR}(n-1)) \leqslant \mathsf{xc}(R)$. Further, we can obtain an extension of R from any extension of Q by adding 1 inequality; hence $\mathsf{xc}(R) \leqslant 1 + \mathsf{xc}(Q)$.

Suppose $\mathsf{Clique}^*{}_n$ is a monotone projection of $\mathsf{Perm}_{t(n)}$. By Fact 1.1 (1) and Lemma 1.2, $\mathsf{xc}(\mathsf{Newt}(\mathsf{Clique}^*{}_n)) = \mathsf{xc}(Q) \leqslant t(n)^2 + c(\mathsf{Perm}_{t(n)}) \leqslant O(t(n)^2)$. From the preceding discussion and By Theorem 1.1, we get $2^{\Omega(n)} \leqslant \mathsf{xc}(\mathsf{COR}(n-1)) \leqslant \mathsf{xc}(R) \leqslant 1 + \mathsf{xc}(Q) \leqslant 1 + O(t(n)^2)$. It follows that $t(n)$ is at least $2^{\Omega(n)}$. ∎

Theorem 1.3. *Over the reals (or any totally ordered semi-ring), the family* Clique *is not monotone* p*-projections of the* Perm *family. In fact, if* Clique_n *is a monotone projection of* $\mathsf{Perm}_{t(n)}$*, then* $t(n) \geqslant 2^{\Omega(\sqrt{n})}$.

Proof. Suppose Clique_n is a monotone projection of $\mathsf{Perm}_{t(n)}$. From Lemma 1.3, it follows that $\mathsf{Clique}^*{}_n$ is a monotone projection of $\mathsf{Perm}_{t(n^2)}$. Hence, from Theorem 1.2 we get $t(n^2) \geqslant 2^{\Omega(n)}$. Thus, $t(n) \geqslant 2^{\Omega(\sqrt{n})}$. ∎

Remark 1.1. *It is easily seen that if a polynomial* f *over* n*-variables is an affine projection of* Perm_m*, then* f *is a (simple) projection of* $\mathsf{Perm}_{m(n+1)}$*. Hence, Theorem 1.2 and Theorem 1.3 holds even when we consider monotone affine projections of the permanent.*

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