

Sculpting Quantum Speedups

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Abstract

Given a problem which is intractable for both quantum and classical algorithms, can we find a sub-problem for which quantum algorithms provide an exponential advantage? We refer to this problem as the “sculpting problem.” In this work, we give a full characterization of sculptable functions in the query complexity setting. We show that a total function f can be restricted to a promise P such that $Q(f|_P) = O(\text{polylog } N)$ and $R(f|_P) = N^{\Omega(1)}$, if and only if f has a large number of inputs with large certificate complexity. The proof uses some interesting techniques: for one direction, we introduce new relationships between randomized and quantum query complexity in various settings, and for the other direction, we use a recent result from communication complexity due to Klartag and Regev. We also characterize sculpting for other query complexity measures, such as $R(f)$ vs. $R_0(f)$ and $R_0(f)$ vs. $D(f)$.

Along the way, we prove some new relationships for quantum query complexity: for example, a nearly quadratic relationship between $Q(f)$ and $D(f)$ whenever the promise of f is small. This contrasts with the recent super-quadratic query complexity separations, showing that the maximum gap between classical and quantum query complexities is indeed quadratic in various settings – just not for total functions!

Lastly, we investigate sculpting in the Turing machine model. We show that if there is any BPP-bi-immune language in BQP, then *every* language outside BPP can be restricted to a promise which places it in PromiseBQP but not in PromiseBPP. Under a weaker assumption, that some problem in BQP is hard on average for P/poly, we show that every *paddable* language outside BPP is sculptable in this way.

1 Introduction

When are quantum algorithms useful? In general, quantum algorithms are believed to provide exponential speedups for certain structured problems, such as factoring [Sho97], but not for unstructured problems like NP-complete problems.

In this work, we ask the question in a new way. Given a problem for which quantum algorithms are not useful, can we nevertheless find a sub-problem on which they provide an exponential advantage over classical algorithms? We call this the “sculpting” question: our goal is to sculpt the original intractable problem into a sub-problem that’s still classically intractable, but for which there exists a fast quantum algorithm. The sculpting question arises, for example, in adiabatic quantum computation: while it is not believed that adiabatic quantum computing can solve NP-complete problems in polynomial time, a widely discussed question is whether there is a sub-problem of SAT on which adiabatic computing provides an exponential advantage over classical algorithms.

We study the sculpting question primarily in the query complexity model. The utility of the model comes from its relative tractability: for example, in query complexity, Shor’s period finding algorithm provides a provable exponential speedup over any classical algorithm [Cle04].

In query complexity, we’re given a (possibly partial) function $f : \{0, 1\}^N \rightarrow \{0, 1\}$ and an oracle access to a string $x \in \{0, 1\}^N$. The goal is to evaluate $f(x)$ using as few oracle calls to the entries of x as possible. The minimum number of queries required by an algorithm for computing $f(x)$ (over the worst-case choice of x) is the query complexity of f . If the algorithm in question is deterministic, we denote this by $D(f)$; if it is (bounded error) randomized, we denote this by $R(f)$; and if it is (bounded error) quantum, we denote it by $Q(f)$.

In this query complexity setting, the sculpting question can be phrased as follows: given a total function $f : \{0, 1\}^N \rightarrow \{0, 1\}$ for which $R(f)$ and $Q(f)$ are both large (say, $N^{\Omega(1)}$), is there a promise $P \subseteq \{0, 1\}^N$ such that $f|_P$, the restriction of f to P , has $Q(f|_P) = O(\text{polylog } N)$ and $R(f|_P) = N^{\Omega(1)}$?

For example, if f is the OR function, such sculpting is not possible, as follows from [Aar06]. As another example, if f is defined to be 1 when Simon’s condition is satisfied and 0 otherwise, then sculpting is possible: the promise will simply restrict to inputs that either satisfy Simon’s condition or are far from satisfying it; this promise suffices for an exponential quantum speedup [BFNR08].

We fully characterize the functions f for which such a promise exists. In particular, we show that sufficiently “rich” functions, such as PARITY or MAJORITY, are sculptable. Our sculpting construction uses communication complexity in a novel way. In the other direction, to prove non-sculptability, we prove new query complexity relationships. As a corollary, we get nearly quadratic relationships between classical and quantum query complexities for a wider class of functions than previously known.

Results

H-indices

We introduce a new query complexity measure, $H(C_f)$, defined as the maximum number h for which there are 2^h inputs to f with certificate complexity at least h . We call this the H-index of certificate complexity (motivated by the citation H-index sometimes used to measure research productivity [Hir05]). This quantity measures the number of inputs there are to a function f that have large certificate complexity. We prove various properties of $H(C_f)$; most notably, we show

that for total functions, it is nearly quadratically related to $H(\text{bs}_f)$, the H-index of block sensitivity. This is analogous to the quadratic relationship between C and bs .

Sculpting in Query Complexity

Our main result is the following theorem, which neatly characterizes sculptability in the query complexity model in terms of the H-index of certificate complexity.

Theorem 1. *Let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be a total function. Then there is a promise $P \subseteq \{0, 1\}^N$ such that $R(f|_P) = N^{\Omega(1)}$ and $Q(f|_P) = N^{o(1)}$, if and only if $H(C_f) = N^{\Omega(1)}$. Furthermore, in this case we also have $Q(f|_P) = O(\log^2 N)$.*

This theorem follows as an immediate corollary of the following more general characterization theorem.

Theorem 2. *For all total functions $f : \{0, 1\}^N \rightarrow \{0, 1\}$ and all promises $P \subseteq \{0, 1\}^N$, we have*

$$R(f|_P) = O(Q(f|_P)^2 H(C_f)^2).$$

Conversely, for all total functions $f : \{0, 1\}^N \rightarrow \{0, 1\}$, there is a promise $P \subseteq \{0, 1\}^N$ such that

$$R(f|_P) = \Omega\left(\frac{H(C_f)^{1/6}}{\log^{13/6} N}\right) \quad \text{and} \quad Q(f|_P) = O(\log^2 H(C_f)).$$

We also prove an analogous theorem for D vs. R_0 , showing that the same $H(C_f) = N^{\Omega(1)}$ condition also characterizes sculpting $D(f)$ vs. $R_0(f)$. On the other hand, we show that sculpting $R_0(f)$ vs. $R(f)$ is *always* possible: for every total function f with $R_0(f) = N^{\Omega(1)}$, there is a promise P such that $R_0(f|_P) = N^{\Omega(1)}$ and $R(f|_P) = O(1)$.

Query Complexity on Small Promises

On the way to proving [Theorem 2](#), we prove the following theorem, providing a quadratic relationship between $Q(f)$ and $D(f)$ when the domain of f is small. This provides a ironic twist to the query complexity story: for a long time, it was believed that $D(f)$ and $Q(f)$ are quadratically related when the domain of f is very large (in particular, for total functions). This conjecture was recently disproven by [\[ABB⁺15\]](#) (who showed a $D(f) \sim Q(f)^4$ separation) and by [\[ABK15\]](#) (who showed an f such that $R(f) \sim Q(f)^{2.5}$). Instead, we now show that the quadratic relationship holds when the domain of f is very *small*.

Theorem 3. *Let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be a partial function, and let $\text{Dom}(f)$ denote the domain of f . Then*

$$Q(f) = \Omega\left(\frac{\sqrt{D(f)}}{\log |\text{Dom}(f)|}\right).$$

Query Complexity for Unbalanced Functions

We show two relationships similar to [Theorem 3](#) that hold for functions whose domain is large, but which are unbalanced: they contain very few 0-inputs compared to 1-inputs, or vice versa.

Theorem 4. *Let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be a partial function. Define the measure $\text{Bal}(f) \in [0, N]$ as $\text{Bal}(f) := 1 + \min\{\log |f^{-1}(0)|, \log |f^{-1}(1)|\}$ (or 0 if f is constant). Then*

$$R(f) = O(Q(f)^2 \text{Bal}(f))$$

$$D(f) = O(R_0(f) \text{Bal}(f)).$$

A similar polynomial relationship between $R_0(f)$ and $R(f)$ does not hold in general.

New Relationship for Total Functions

We prove the following new query complexity relationship for total functions, generalizing the known relationship $D(f) = O(Q(f)^2 C(f))$.

Theorem 5. *Let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be a total function. Then*

$$D(f) = O(Q(f)^2 H(\sqrt{C_f})^2).$$

Here $H(\sqrt{C_f})$ denotes the H-index of the square root of certificate complexity; this is the maximum number h such that there are at least 2^h inputs to f for which $\sqrt{C(f)}$ is at least h . We note that $H(\sqrt{C_f})^2 \leq C(f)$ for all total functions, so this is an improvement over the relationship $D(f) = O(Q(f)^2 C(f))$. Moreover, when $f = \text{OR}$, we have $H(\sqrt{C_f})^2 = 1$ and $C(f) = N$, so this improvement is strict.

We remark that this result could let us improve the relationship $D(f) = O(Q(f)^6)$ if we could show $H(\sqrt{C_f})^2 = o(Q(f)^4)$. [Theorem 5](#) therefore provides a new approach for this long-standing open problem.

Sculpting in the Turing Machine Model

In [Section 8](#), we examine sculpting in the Turing machine model. We say that a language L is *sculptable* if there is a promise set P such that the promise problem of deciding if an input from P is in L is in PromiseBQP but not in PromiseBPP. We prove two sculptability theorems, both of them providing evidence that most or all languages outside of BPP are sculptable.

Theorem 6. *Assume PromiseBQP is hard on average for P/poly. Then every paddable language outside of BPP is sculptable.*

Theorem 7. *Assume there exists a BPP-bi-immune language in BQP. Then every language outside of BPP is sculptable.*

For the definitions of paddability and bi-immunity, see [Section 8](#). These theorems assume very little about BQP and BPP, and analogous statements hold for other pairs of complexity classes.

2 Preliminaries

For a (possibly partial) function $f : \{0, 1\}^N \rightarrow \{0, 1\}$, we use $D(f)$, $R_0(f)$, $R(f)$, and $Q(f)$ to denote the deterministic query complexity, zero-error randomized query complexity, bounded-error randomized query complexity, and bounded-error quantum query complexity of f , respectively. For the definitions of these measures, see [BdW02].

A partial assignment is a string p in $\{0, 1, *\}^N$ that represents partial knowledge of a string in $\{0, 1\}^N$. For $x \in \{0, 1\}^N$, we say that p is a partial assignment of x if x extends p ; that is, if x and p agree on all the non- $*$ entries of p . A partial assignment p for x is called a *certificate* for x if all strings that extend p have that same value under f as x ; that is, if for all $y \in \{0, 1\}^N$ that extend p , we have $f(y) = f(x)$. The certificate complexity of f on input x , denoted by $C_f(x)$, is the minimum number of bits in any certificate of x with respect to f . The certificate complexity of f , denoted by $C(f)$, is defined as the maximum of $C_f(x)$ over all strings x in the domain of f .

The certificate complexity $C_f(x)$ can be thought of as the deterministic query complexity of f given the promise that the input is either x or else an input y such that $f(x) \neq f(y)$. Motivated by this observation, Aaronson [Aar06] defined the *randomized certificate complexity* of x , denoted by $RC_f(x)$, to be the (bounded-error) randomized query complexity of f on this promise. He defined the quantum certificate complexity $QC_f(x)$ analogously. As with C , we use $RC(f)$ to denote the maximum of $RC_f(x)$ over all x in the domain of f , and define $QC(f)$ similarly.

For any string $x \in \{0, 1\}^N$ and set of bits B , denote by x^B the string x with the bits in B flipped. For any $f : \{0, 1\}^N \rightarrow \{0, 1\}$, if $f(x) \neq f(x^B)$, we say that B is a *sensitive block* for x with respect to f . The *block sensitivity* of x , denoted by $bs_f(x)$, is the maximum number of disjoint sensitive blocks for x . Note that the block sensitivity is the packing number of the collection of sensitive blocks of x . It can be seen that $C_f(x)$ can be interpreted as the hitting number of that collection (the minimum number of bits required to hit all the blocks). Moreover, $RC_f(x)$ is simply the fractional packing number (which equals the fractional hitting number). In other words, $RC_f(x)$ can be interpreted as the fractional block sensitivity (or fractional certificate complexity). These observations are implicit in [Aar06], and were made explicit in [KT13].

Actually, the fractional block sensitivity differs by a constant factor from Aaronson's original definition of RC . In this work we will use RC to denote the fractional block sensitivity. Another property of RC that we will need is that $1/RC_f(x)$ is equal to the minimum infinity-norm distance between x and the convex hull of the set of inputs y such that $f(y) \neq f(x)$. That is, if $f(x) = 0$ and $S = f^{-1}(1)$, we have

$$\frac{1}{RC_f(x)} = \min_{\mu \in \Delta_S} \max_i \Pr_{y \sim \mu} [x_i \neq y_i],$$

where Δ_S is the set of all probability distributions over S (equivalently, the convex hull of S). In particular, this minimum is attained, so there is a probability distribution μ over $f^{-1}(1)$ such that for all $i = 1, 2, \dots, n$, if we sample $y \sim \mu$ we get $\Pr[y_i \neq x_i] \leq 1/RC_f(x)$.

Clearly, for all $f : \{0, 1\}^N \rightarrow \{0, 1\}$ we have

$$\{QC(f), bs(f)\} \leq RC(f) \leq \{C(f), R(f)\} \leq R_0(f) \leq D(f),$$

with $Q(f)$ lying between $QC(f)$ and $R(f)$. Aaronson [Aar06] showed that $RC_f(x) = \Theta(QC_f(x)^2)$ for all f and x , so in particular, $RC(f) = \Theta(QC(f)^2)$. In addition, when f is total, we can relate all these measures to each other: we have

$$D(f) = O(bs(f)^3) = O(RC(f)^3) = O(QC(f)^6) = O(Q(f)^6).$$

Balance and H Indices

We will use $\text{Dom}(f)$ to denote the domain of a partial function f . We define $\text{Bal}(f)$ to be 0 if f is constant, and otherwise, to be the minimum of $1 + \log |f^{-1}(0)|$ and $1 + \log |f^{-1}(1)|$ (we use \log to denote logarithm base 2). Note that since $|f^{-1}(0)| + |f^{-1}(1)| = |\text{Dom}(f)| \leq 2^N$, we have $\text{Bal}(f) \leq N$. Thus $\text{Bal}(f) \in [0, N]$.

We will use a new set of query complexity measures called H-indices (the name is motivated by the H-index measure of citations, a common metric for research output). For a given function $g : \{0, 1\}^N \rightarrow [0, \infty)$, we will define the H-index of g , denoted by $\text{H}(g)$, as the maximum number h such that there are at least 2^h inputs with $g(x) \geq h$. Alternatively, the H-index of g can be defined as the minimum number h such that there are at most 2^h inputs with $g(x) > h$. It is not obvious that these definitions are equivalent (or even that the minimum and maximum are attained); we prove this in [Appendix A](#).

Note that $\text{H}(g) \in [0, N]$, and $\text{H}(g) \leq \max_x g(x)$. Also, if $g(x) \geq g'(x)$ for all $x \in \{0, 1\}^N$, we have $\text{H}(g) \geq \text{H}(g')$.

We'll primarily be interested in measures like $\text{H}(C_f)$, $\text{H}(\text{RC}_f)$, and $\text{H}(\text{bs}_f)$. We have $\text{H}(C_f) \leq C(f)$, $\text{H}(\text{RC}_f) \leq \text{RC}(f)$, and $\text{H}(\text{bs}_f) \leq \text{bs}(f)$. We also have

$$\text{H}(\text{bs}_f) \leq \text{H}(\text{RC}_f) \leq \text{H}(C_f).$$

The H-index of certificate complexity can be much smaller than the certificate complexity itself. For example, the OR function has only one certificate of size greater than 1, so $\text{H}(C_{\text{OR}}) = 1$, even though $C(\text{OR}) = n$.

In [Appendix A](#) we show that if $\alpha : [0, \infty) \rightarrow [0, \infty)$ is an increasing function, then

$$\text{H}(\alpha \circ g) \leq \max\{\text{H}(g), \alpha(\text{H}(g))\}.$$

In particular, this will imply $\text{H}(C_f^2) \leq \text{H}(C_f)^2$.

Shattering and the Sauer-Shelah Lemma

For a set of indices $A \subseteq \{1, 2, \dots, N\}$, let $S|_A \subseteq \{0, 1\}^{|A|}$ be the set of restrictions of each string in S to the indices in A . We say A is shattered by S if $S|_A = \{0, 1\}^{|A|}$. In other words, A is shattered by S if S has all possible behaviors on A . The Sauer-Shelah lemma [[Sau72](#), [She72](#)] is a classic result that upper-bounds the size of S in terms of the size of A . We will use the following corollary of it.

Lemma 8. *Let $S \subseteq \{0, 1\}^N$ be a collection of strings. Then there is a shattered set of indices of size at least*

$$\frac{\log |S|}{\log(N + 1)}.$$

[Lemma 8](#) follows straightforwardly from the Sauer-Shelah lemma, as we prove in [Appendix B](#). We will often use the weaker bound $\frac{\log |S|}{2 \log N}$ instead, which holds for $N \geq 2$. This will sometimes lead to simpler formulas.

3 Non-Sculptability Theorems

In this section, we prove the non-sculptability direction of [Theorem 2](#). The proof has two parts: in [Section 3.1](#), we prove a relationship between randomized and quantum query complexities for

“unbalanced” functions, and in [Section 3.2](#), we use this to prove a sculpting lower bound in terms of the H-index of certificate complexity.

3.1 Query Complexity for Unbalanced Functions

We wish to show a nearly-quadratic relationship between randomized and quantum query complexities for functions f for which $\text{Bal}(f)$ is small. Note that this is a generalization of the relation $\text{RC}_f(x) = O(Q_f(x)^2)$ from [\[Aar06\]](#). That is, [\[Aar06\]](#) showed that for the task of distinguishing one input from a (possibly large) set of alternatives, randomized and quantum algorithms are quadratically related. We want a similar relationship for the task of distinguishing a *small set* of inputs from a (possibly large) set of alternatives.

We start with the following lemma.

Lemma 9. *Let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be a partial function. For $a \notin f^{-1}(0)$, let $f_{a,0}$ be the problem of distinguishing a from $f^{-1}(0)$. That is, $f_{a,0}$ is the function $f_{a,0} : \{a\} \cup f^{-1}(0) \rightarrow \{0, 1\}$ with $f(x) = 1$ iff $x = a$. For $a \notin f^{-1}(1)$, define $f_{a,1}$ analogously. Then for all $a \in \{0, 1\}^N$, we have either $R(f_{a,0}) = O(Q(f)^2)$ or $R(f_{a,1}) = O(Q(f)^2)$.*

Note that this holds even when a is not in the promise of f . The constant in the big- O notation is a universal constant independent of a , f , and N .

Proof. Let Q be the quantum algorithm that achieves $Q(f)$ quantum query complexity in determining the value of f on a given input. When run on any $a \in f^{-1}(0)$, Q will output 0 with probability at least $2/3$, and when run on $a \in f^{-1}(1)$, it will output 1 with probability at least $2/3$.

Consider running Q on an input $a \notin \text{Dom}(f)$. Then Q will output 0 with some probability p and output 1 with probability $1 - p$. If $p \geq 1/2$, then Q distinguishes a from $f^{-1}(1)$ with constant probability. If $p \leq 1/2$, then Q distinguishes a from $f^{-1}(0)$ with constant probability. Thus for all $a \in \{0, 1\}^N$, we have either $Q(f_{a,0}) = O(Q(f))$ or $Q(f_{a,1}) = O(Q(f))$. From [\[Aar06\]](#), we have $\text{RC}(g) = O(\text{QC}(g)^2) = O(Q(g)^2)$ for all functions g , so we conclude that either $\text{RC}(f_{a,0}) = O(Q(f)^2)$ or $\text{RC}(f_{a,1}) = O(Q(f)^2)$.

Finally, note that for a problem of distinguishing one input from the rest, randomized query complexity equals randomized certificate complexity. Thus we get that for all $a \in \{0, 1\}^N$, either $R(f_{a,0}) = O(Q(f)^2)$ and or $R(f_{a,1}) = O(Q(f)^2)$. \square

We’re now ready to prove the desired relationship between R and Q .

Theorem 10. *Let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be a partial function. Then*

$$R(f) = O(Q(f)^2 \text{Bal}(f)).$$

Proof. Without loss of generality, assume $|f^{-1}(0)| \leq |f^{-1}(1)|$. We use [Lemma 9](#) to construct a randomized algorithm for determining $f(x)$ given oracle access to x , assuming that $f^{-1}(0)$ is small. The idea is to keep track of the subset $Z \subseteq f^{-1}(0)$ of strings that the input x might feasibly be (consistent with the queries seen so far). We then construct a string a from a majority vote of the elements of Z ; that is, for each index $i \in [n]$, a_i will be the majority of y_i over all $y \in Z$ (with ties broken arbitrarily).

This string a need not be in $\text{Dom}(f)$. The important property of it is that if we query an index i of the input x and discover that $x_i \neq a_i$, we can eliminate at least half of the strings from Z , since they are no longer feasible possibilities for x .

We then get the following randomized algorithm for evaluating $f(x)$:

- Initialize $Z = f^{-1}(0)$.
- While $Z \neq \emptyset$:
 1. Calculate a from the entry-wise majority vote of Z .
 2. Determine $b \in \{0, 1\}$ such that $R(f_{a,b}) = O(Q(f)^2)$ (this exists by [Lemma 9](#)).
 3. Run the randomized algorithm evaluating $f_{a,b}$ on x with some amplification (to be specified later).
 4. If its output is 1 (i.e. the algorithm thinks $x = a$ rather than $x \in f^{-1}(b)$), output $1 - b$ and halt.
 5. If its output is 0, a bit i was queried to reveal $x_i \neq a_i$, so update Z (removing at least half its elements).
- If $Z = \emptyset$, output 1.

We note a few things about this algorithm. First, in step 3, notice that x need not be in the domain of $f_{a,b}$. However, we may still run the randomized algorithm that evaluates $f_{a,b}$, and use the fact that if x does happen to be in the domain (in particular, if $x \in f^{-1}(b)$), then the algorithm will work correctly. This is exactly what we use in step 4: if the algorithm that distinguishes a from $f^{-1}(b)$ says that x is equal to a , it need not mean that x is in fact equal to a , but it does mean that $x \notin f^{-1}(b)$.

Secondly, step 5 assumes that the randomized algorithm for evaluating $f_{a,b}$ will only conclude that an input x is not equal to a if it finds a disagreement with a . This is a safe assumption, as argued in Lemma 5 of [\[Aar06\]](#).

Finally, we determine the number of queries this algorithm uses. The outer loop happens at most $\lfloor \log |f^{-1}(0)| \rfloor + 1 \leq \text{Bal}(f)$ times. Step 3 in the loop is the only one which queries the input string. Since the loop repeats at most $\text{Bal}(f)$ times, we can safely amplify the algorithm in step 3 $O(\log \text{Bal}(f))$ times. This gives a query complexity of $O(Q(f)^2 \log \text{Bal}(f))$ for step 3, so the overall number of queries is $O(Q(f)^2 \text{Bal}(f) \log \text{Bal}(f))$.

We can get rid of the log factor by being more careful with the amplification. Note that if we ever find a disagreement with a when running the algorithm, we may immediately stop amplifying and proceed to step 5. We keep a count c_0 of how many times we had to amplify in step 3 for functions of the form $f_{a,0}$, and a count c_1 for functions of the form $f_{a,1}$.

If c_0 ever reaches $2 \text{Bal}(f)$, we output 1 and halt. Similarly, if c_1 ever reaches $2 \text{Bal}(f)$, we output 0 and halt. This ensures the total amplification is $O(\text{Bal}(f))$, so the total query complexity of the algorithm is $O(Q(f)^2 \text{Bal}(f))$.

Note that if $f(x) = 0$ and the output of the algorithm was 1, it means that we ran the algorithm evaluating $f_{a,0}$ (for varying values of a) $2 \text{Bal}(f)$ times, and at most $\text{Bal}(f)$ of those times the algorithm said that $x \in f^{-1}(0)$. For each individual run, the probability is at least $2/3$ that the algorithm would say that $x \in f^{-1}(0)$. An application of the Chernoff bound shows that the probability of this happening is exponentially small. Similarly, the probability of the algorithm giving 0 when in actuality $f(x) = 1$ is also exponentially small.

We conclude that $R(f) = O(Q(f)^2 \text{Bal}(f))$, as desired. □

3.2 Application to Non-Sculptability

Theorem 10 immediately gives the following non-sculptability result, which says that unbalanced functions cannot be sculpted.

Corollary 11. *Let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be a total function. For any promise $P \subseteq \{0, 1\}^N$, we have*

$$R(f|_P) = O(Q(f|_P)^2 \text{Bal}(f)).$$

Proof. Note that $\text{Bal}(f|_P) \leq \text{Bal}(f)$ for any f and P . Then, by **Theorem 10**, we have

$$R(f|_P) = O(Q(f|_P)^2 \text{Bal}(f|_P)) = O(Q(f|_P)^2 \text{Bal}(f)).$$

□

We extend this result by showing that any function with a small number of large certificates also cannot be sculpted. This gives us a non-sculptability result in terms of the H-index of certificate complexity.

Theorem 12. *Let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be a total function. For any promise $P \subseteq \{0, 1\}^N$, we have*

$$R(f|_P) = O(Q(f|_P)^2 H(C_f^2)).$$

Proof. We design a deterministic algorithm that reduces the set of possibilities for the input to an unbalanced set. Specifically, the algorithm will reduce the possibilities for the input to a set $S \subseteq \{0, 1\}^N$ such that $\text{Bal}(f|_S) \leq H(C_f^2) + 1$. We then use **Theorem 10** to get the desired non-sculptability result.

Note that every 1-certificate of f must conflict with every 0-certificate of f in at least one bit. Therefore, by querying all non-* entries of a 0-certificate, we reveal at least one entry of each 1-certificate.

We design a deterministic algorithm for computing f on an input from P . The algorithm proceeds as follows: it repeatedly picks a 0-certificate p for f of size at most $\sqrt{H(C_f^2)}$ that is consistent with all the entries of the input that were revealed so far. It then queries all the non-* entries of p . This is repeated $\sqrt{H(C_f^2)}$ times, or until there are no 0-certificates of size at most $\sqrt{H(C_f^2)}$ (whichever happens first). Finally, the algorithm returns the set S of strings that are consistent with the revealed entries of the input.

This algorithm uses at most $H(C_f^2)$ queries. We check its correctness by examining the set S . Clearly, the input is in S . Furthermore, if any certificate of f was revealed, then f is constant on S , so S contains either no 0-inputs or no 1-inputs.

There are at most $2^{H(C_f^2)}$ inputs with certificate complexity larger than $\sqrt{H(C_f^2)}$.

If the algorithm terminated because there were no consistent 0-certificates, then the only 0-inputs in S have certificates of size larger than $\sqrt{H(C_f^2)}$. There are at most $2^{H(C_f^2)}$ of them, so S has at most $2^{H(C_f^2)}$ 0-inputs to f . Conversely, if the algorithm went through $\sqrt{H(C_f^2)}$ iterations of querying consistent 0-certificates, then it must have revealed $\sqrt{H(C_f^2)}$ entries of each 1-certificate to f . If no 1-certificate was discovered, it means the revealed entries contradicted all 1-certificates

of size at most $\sqrt{H(C_f^2)}$. Thus the only 1-inputs in S have certificate size greater than $\sqrt{H(C_f^2)}$, from which it follows that there are less than $2^{H(C_f^2)}$ of them.

We conclude that S contains either at most $2^{H(C_f^2)}$ 0-inputs to f or at most $2^{H(C_f^2)}$ 1-inputs to f . This gives $\text{Bal}(f|_S) \leq H(C_f^2) + 1$.

We design a randomized algorithm for $f|_P$ as follows. First, we run the above deterministic algorithm to reduce the problem of computing $f|_P$ to the problem of computing $f|_{S \cap P}$. This costs $H(C_f^2)$ queries. By [Theorem 10](#), there is a randomized algorithm that uses

$$O(Q(f|_{S \cap P})^2 \text{Bal}(f|_{S \cap P})) = O(Q(f|_P)^2 \text{Bal}(f|_S)) = O(Q(f|_P)^2 H(C_f^2))$$

queries to compute $f|_{S \cap P}$. Running this algorithm allows us to compute $f|_P$. The total number of queries used was

$$O(Q(f|_P)^2 H(C_f^2) + H(C_f^2)) = O(Q(f|_P)^2 H(C_f^2)).$$

□

Note that [Theorem 12](#) completes the first part of the proof of [Theorem 2](#), since $H(C_f^2) \leq H(C_f)^2$. It is natural to wonder whether [Theorem 12](#) is always at least as strong as [Corollary 11](#). In [Theorem 22](#), we will show that it is, up to a quadratic factor and a $\log N$ factor.

4 Sculpting from Communication Complexity

In this section, we show that if a function f has many inputs with large randomized certificate complexity then it *can* be sculpted: there is a promise P so that $f|_P$ exhibits a large quantum speedup. This means that if $H(\text{RC}_f)$ is large, the function f can be sculpted. In [Section 5](#), we will relate $H(\text{RC}_f)$ to $H(C_f)$, thereby completing the proof of [Theorem 2](#).

Our sculptability proof will rely on the solution to a problem we call the “extended queries problem,” which might be of independent interest. The solution to this problem will in turn use results from communication complexity.

4.1 The Extended Queries Problem

We usually let an algorithm for computing a (possibly partial) function $f : \{0, 1\}^N \rightarrow \{0, 1\}$ query the bits of the input x . But what happens if we let the algorithm make other types of queries? For example, if x is a Boolean string, we can let the algorithm query the parity of x . How does this extra power affect the query complexity of f ? In particular, is there some special set of additional queries such that if a randomized algorithm is allowed to make the special queries, it can simulate *any* quantum algorithm? If so, how many special queries suffice for this property to hold?

To formalize this question, we need a few definitions.

Definition 13. *An extension function with extension G is an injective total function $\phi : \{0, 1\}^N \rightarrow \{0, 1\}^G$ (in particular, we need $G \geq N$).*

An extension function specifies, for each input $x \in \{0, 1\}^N$, the types of queries an algorithm is allowed to make on x . In other words, we will let algorithms query from $\phi(x)$ instead of from x . Note that the extension function may provide easy access to information about x that is hard to obtain otherwise (such as its parity).

Definition 14. Let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be a partial function, and let ϕ be an extension function. The extended version of f with respect to ϕ is the partial function $f^\phi : \phi(\text{Dom}(f)) \rightarrow \{0, 1\}$ defined by $f^\phi(x) = f(\phi^{-1}(x))$.

Note that f^ϕ is a partial function from $\{0, 1\}^G$ to $\{0, 1\}$. We can consider $D(f^\phi)$, $R(f^\phi)$, $Q(f^\phi)$, and so on. To pose the extended queries problem, we will need a notion of the complexity of a set of functions, defined as the maximum complexity of any function in that set.

Definition 15. For any set of functions S , we define $D(S) := \max_{f \in S} D(f)$. We define $R(S)$, $Q(S)$, etc. similarly. Further, we define $D^G(S)$, the extended query complexity of S with extension G , to be the minimum, over all extension functions ϕ with extension G , of $\max_{f \in S} D(f^\phi)$. We define $R^G(S)$, $Q^G(S)$, etc. similarly.

In other words, for any set of functions, the extended query complexity of the set with G extension is the number of queries required to compute all functions in the set given the best possible extension. We observe that if $G \geq |S|$, the extended query complexity $D^G(S)$ is 1, since the extension $\phi(x)$ for a given input x could simply specify the values of all the functions in S on x . We also observe that for all $G \geq N$, we have $D^G(S) \leq D(S)$, since the identity function is always a valid extension function. Moreover, the extended query complexity of a set is decreasing in G . We now ask the following question.

The Extended Queries Problem. Is there a set of functions S for which $Q(S)$ is small but $R^G(S)$ is large, even when the extension G is exponentially large in the input size N ? We can also ask this question for other complexity measures, such as $R(S)$ vs. $R_0^G(S)$ or $R_0(S)$ vs. $D^G(S)$.

It turns out that a negative solution to the extended queries problem implies a sculptability result in terms of $H(\text{RC}_f)$, as the following theorem shows.

Theorem 16. Let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be a total function. Let $A = \frac{H(\text{RC}_f)}{4 \log N}$, and let S be any set of partial functions from $\{0, 1\}^A$ to $\{0, 1\}$. Then there is a promise $P \subseteq \{0, 1\}^N$ such that

$$Q(f|_P) = O(Q(S)), \quad R(f|_P) = \Omega(R^N(S)).$$

Analogous statements hold for other pairs of complexity measures, such as D and R_0 or R_0 and R .

We delay the proof of [Theorem 16](#) to [Section 4.3](#). First, we settle the extended queries problem for R vs. Q : [Theorem 18](#) will provide an exponential lower bound on G by reducing the extended queries problem to a problem in communication complexity.

4.2 Reducing Extension to Communication Complexity

For a partial function $f : \{0, 1\}^{N_1} \times \{0, 1\}^{N_2} \rightarrow \{0, 1\}$, we will denote the communication complexities of f by $D^{\text{CC}}(f)$, $R^{\text{CC}}(f)$, $Q^{\text{CC}}(f)$, and $R_0^{\text{CC}}(f)$. We will use the following definition.

Definition 17. Let $f : \{0, 1\}^{N_1} \times \{0, 1\}^{N_2} \rightarrow \{0, 1\}$ be a partial function. For any $x \in \text{Dom}(f)$, we write $x = x_A x_B$, with $x_A \in \{0, 1\}^{N_1}$ and $x_B \in \{0, 1\}^{N_2}$. Let $\text{Dom}_A(f) = \{x_A : x \in \text{Dom}(f)\}$ and $\text{Dom}_B(f) = \{x_B : x \in \text{Dom}(f)\}$. For any $a \in \text{Dom}_A(f)$, we define the marginal of f with respect to a to be the partial function $f_a : \{0, 1\}^{N_2} \rightarrow \{0, 1\}$ defined by $f_a(b) := f(a, b)$ for all $b \in \{0, 1\}^{N_2}$ such that $(a, b) \in \text{Dom}(f)$. We define $\text{Mar}(f) = \{f_a : a \in \text{Dom}_A(f)\}$ to be the set of all marginal functions for f .

We now connect communication complexity to the extended queries problem.

Theorem 18. *Let $f : \{0, 1\}^{N_1} \times \{0, 1\}^{N_2} \rightarrow \{0, 1\}$ be a partial function. Then*

$$R^G(\text{Mar}(f)) = \Omega\left(\frac{R^{\text{CC}}(f)}{\log G}\right).$$

Similarly, we also have $D^G(\text{Mar}(f)) = \Omega(D^{\text{CC}}(f)/\log G)$, $R_0^G(\text{Mar}(f)) = \Omega(R_0^{\text{CC}}(f)/\log G)$, and $Q^G(\text{Mar}(f)) = \Omega(Q^{\text{CC}}(f)/\log G)$.

Proof. We prove the theorem for R . The statements for D , R_0 , and Q will follow analogously. Let $\phi : \{0, 1\}^{N_2} \rightarrow \{0, 1\}^G$ be the best possible extension function, so that $R^G(\text{Mar}(f)) = \max_{g \in \text{Mar}(f)} R(g^\phi)$.

We now describe a randomized communication protocol for computing f . Alice receives a string a , and must compute $f(a, b)$, where b is Bob's string. This is equivalent to computing $f_a(b)$. Since Alice knows f_a , she also knows f_a^ϕ . Let R a randomized algorithm that computes f_a^ϕ using at most $R^G(\text{Mar}(f))$ queries. Alice will run this algorithm, and for each query, she will send the index of that query to Bob (as a number between 1 and G). Bob will reply with the corresponding bit of $\phi(y)$ (as a bit in $\{0, 1\}$). This allows Alice to compute $f_a(b) = f(a, b)$.

The total communication in this protocol is at most $(\lceil \log G \rceil + 1)R^G(\text{Mar}(f))$. Since this upper-bounds the randomized communication complexity of f (using private coins), the desired result follows. \square

[Theorem 18](#) allows us to use communication complexity as a tool for lower-bounding the extended query complexity of certain sets of functions. To use it to solve the extended queries problem, we need a function f that has large randomized communication complexity but for which $Q(\text{Mar}(f))$ is small. To construct such a function, we start from a simple function that was recently shown to separate randomized from quantum communication complexity, called the Vector in Subspace problem.

The Vector in Subspace Problem. In this problem, Bob gets a unit vector $v \in \mathbb{R}^n$, and Alice gets a subspace H of \mathbb{R}^n of dimension $n/2$. It is promised that either $v \in H$ or $v \in H^\perp$; the task is to determine which is the case. We assume for simplicity that n is a power of 2.

This problem was first studied in [\[Kre95\]](#) and was also described in [\[Raz99\]](#). Klartag and Regev [\[KR11\]](#) showed that this problem has randomized communication complexity $\Omega(n^{1/3})$. In addition, it is easy to see that the one-way quantum communication complexity of the problem is at most $\log n$: Bob can send a superposition over $\log n$ bits with amplitudes determined by v ; Alice can then apply the projective measurement given by (H, H^\perp) .

To apply this function to the extended queries problem, we need a few modifications. First, we need a discrete version of the problem. [\[KR11\]](#) showed that a lower bound of $\Omega(n^{1/3})$ for randomized communication complexity applies to a discrete version of the problem in which each real number is described using $O(\log n)$ bits; that is, Alice's subspace is given using $n/2$ vectors of length n , whose entries are specified using $O(\log n)$ bits, and Bob's vector is specified using n real numbers of $O(\log n)$ bits each.

$\text{Mar}(f)$ is the set of functions where we know Alice's subspace H , and are allowed to query from Bob's input vector. However, phrased this way, it is not clear how to use a quantum algorithm to compute such functions using few queries. To solve this problem, we modify the way Bob's input

is specified. Instead of specifying only the entries to the vector, Bob’s input string also lists some “partial sums” of the vector entries.

The idea is for Bob’s vector to allow Alice to use the following algorithm to construct the state with amplitudes specified by v . We interpret v as specifying a superposition over strings of length $\log n$. Alice starts by querying the probability p that the first bit of this string is 0 when this state is measured. Alice will now place a \sqrt{p} amplitude on querying the probability that the second bit is 0 conditioned on the first bit being 0, and a $1 - \sqrt{p}$ amplitude on querying the probability that the second bit is 0 conditioned on the first bit being 1. Alice keeps going in this way, until she gets to the final bit of the string of length $\log n$, at which point she queries the phase. This allows her to construct the state determined by the amplitudes in v .

Of course, for this to work, Bob’s input must provide all of these conditional probabilities. There is one such probability to specify for the first bit, two for the second, four for the third, and so on. Since there are $\log n$ bits, Bob’s input needs to specify only $O(n)$ probabilities. Each can be specified with $O(\log n)$ precision, so Bob’s total input size is $O(n \log n)$. Moreover, Alice constructs the desired state after $O(\log n)$ queries to the probabilities, or $O(\log^2 n)$ queries to the bits of Bob’s input.

We thus get the following theorem.

Theorem 19. *For all $A \in \mathbb{N}$, there is a set S of partial functions from $\{0, 1\}^A$ to $\{0, 1\}$ such that for all $G \geq A$,*

$$Q(S) = O(\log^2 A), \quad R^G(S) = \Omega\left(\frac{A^{1/3}}{\log^{1/3} A \cdot \log G}\right).$$

Proof. Let f be the function described above with $n = A/\log A$, and let $S = \text{Mar}(f)$. Then $Q(S) = O(\log^2 n) = O(\log^2 A)$ and $R^{\text{CC}}(f) = \Omega(n^{1/3}) = \Omega(A^{1/3}/\log^{1/3} A)$. By [Theorem 18](#), we get $R^G(S) = \Omega(A^{1/3}/(\log^{1/3} A \cdot \log G))$. \square

Together with [Theorem 16](#), this implies that any function with large $H(\text{RC}_f)$ can be sculpted, simply by plugging S from [Theorem 19](#) into [Theorem 16](#) and setting $G = N$.

4.3 Reducing Sculpting to Extended Query Complexity

We now prove [Theorem 16](#), restated here for convenience.

Theorem 16. *Let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be a total function. Let $A = \frac{H(\text{RC}_f)}{4 \log N}$, and let S be any set of partial functions from $\{0, 1\}^A$ to $\{0, 1\}$. Then there is a promise $P \subseteq \{0, 1\}^N$ such that*

$$Q(f|_P) = O(Q(S)), \quad R(f|_P) = \Omega(R^N(S)).$$

Analogous statements hold for other pairs of complexity measures, such as D and R_0 or R_0 and R .

Proof. There are at least $2^{H(\text{RC}_f \cdot 2 \log N)}$ inputs x with $\text{RC}_f(x) \geq H(\text{RC}_f \cdot 2 \log N)/(2 \log N)$. Let the set of such inputs be C . By [Lemma 8](#), if $N \geq 2$, there is a set B of

$$\frac{H(\text{RC}_f \cdot 2 \log N)}{2 \log N} \geq \frac{H(\text{RC}_f)}{2 \log N}$$

indices in $\{1, 2, \dots, N\}$ that is shattered by the inputs in C . We’ll restrict B to have size at most $H(\text{RC}_f)/(4 \log N)$, so $|B| = A$. Let $\phi : \{0, 1\}^A \rightarrow \{0, 1\}^N$ be defined by mapping each string

$x \in \{0, 1\}^A$ to a string z in C such that restricting z to A gives x . This is an injective mapping, so ϕ is an extension function with extension size N .

Next, consider the set S of partial Boolean functions from $\{0, 1\}^A$ to $\{0, 1\}$. Let $S^\phi = \{g^\phi : g \in S\}$. Then $R(S^\phi) \geq R^N(S)$. It follows that there is some function $g^\phi \in S^\phi$ such that $R(g^\phi) \geq R^N(S)$.

We will use the function g^ϕ to define the desired promise P . The domain of g^ϕ is contained in C . Let x be in this domain, so that $\text{RC}_f(x) \geq H(\text{RC}_f \cdot 2 \log N) / (2 \log N) \geq 2A$. Let μ_x be a distribution over inputs y such that $f(x) \neq f(y)$, with the property that for any bit i , $\Pr[y_i \neq x_i] \leq 1/\text{RC}_f(x) \leq 1/(2A)$. Then for all $x \in C$, a randomized algorithm has a hard time distinguishing between x and μ_x . For each such x , let μ'_x be the distribution μ_x conditioned on the sampled input agreeing with x on the bits in B . Since the probability of an input sampled from μ_x disagreeing with x on B is at most $|B| \cdot 1/(2A) \leq 1/2$, the distribution μ'_x is not too far from μ_x . In particular, any randomized algorithm that finds a disagreement with x on an input sampled from μ'_x with probability p will also find a disagreement with x on an input sampled from μ_x with probability at least $p/2$. It follows that a randomized algorithm must use $\Omega(A)$ queries to distinguish x from μ'_x .

We construct the promise P as follows. Start with $P = \emptyset$. For each $x \in \text{Dom}(g^\phi)$, we add x to P if $f(x) = g^\phi(x)$; otherwise, we add the support of μ'_x to P .

It remains to lower-bound $R(f|_P)$ and to upper-bound $Q(f|_P)$. We start with the upper bound. Let $y \in P$, and consider the value of y on B . If x is an input of the domain of g^ϕ that caused y to be added, then x and y agree on B . Further, the values of x on B are simply $\phi^{-1}(x) \in \{0, 1\}^{|B|}$, and $g(\phi^{-1}(x)) = g^\phi(x) = f(y)$. This means $g(y|_B) = f(y)$. We now have the quantum algorithm work only with the bits of $y|_B$, ignoring the rest. The algorithm need only compute $g(y|_B)$. Since $g \in S$, we get $Q(f|_P) \leq Q(g) \leq Q(S)$, as desired. A similar argument would upper-bound other complexity measures, such as R , R_0 , or D .

For the lower bound, consider the hard distribution μ on inputs to g^ϕ obtained from Yao's minimax principle [Yao77]. This distribution has the property that any randomized algorithm for g^ϕ that succeeds with probability at least $2/3$ on inputs sampled from μ must use $R(g^\phi)$ queries. We construct a new distribution μ' over P by generating an element $x \in \text{Dom}(g^\phi)$ according to μ , and then outputting either x or a sample from μ'_x , depending on which of them was added to P . We lower-bound the number of queries a randomized algorithm requires to compute f on an input sampled from μ' by a reduction from either computing g^ϕ on inputs sampled from μ , or else distinguishing x from μ'_x .

Let R be a randomized algorithm for $f|_P$. Let $x \sim \mu$. We wish to compute $g^\phi(x)$. Although x may not be in P , consider running R on x anyway. The algorithm will correctly output $g^\phi(x)$ with some probability p , depending on both the internal randomness of R and on μ . If $p \geq 3/5$, we could amplify R a constant number of times to turn it into an algorithm for g that works on inputs sampled from the hard distribution μ , which means R must use $\Omega(R(g^\phi)) = \Omega(R^N(S))$ queries. So suppose that $p \leq 3/5$.

Next, given $x \sim \mu$, we let y_x be either x or a sample from μ'_x , as μ' dictates. Then running R on y_x gives $f(y_x) = g^\phi(x)$ with probability at least $2/3$. On the other hand, running R on x gives output $g^\phi(x)$ with probability at most $3/5$. That is, we have

$$\Pr_{R, x \sim \mu} [R(x) = g^\phi(x)] = \mathbb{E}_{x \sim \mu} \left[\Pr_R [R(x) = g^\phi(x)] \right] \leq 3/5$$

$$\Pr_{R, x \sim \mu, y_x} [R(y_x) = g^\phi(x)] = \mathbb{E}_{x \sim \mu} \left[\Pr_{R, y_x} [R(y_x) = g^\phi(x)] \right] \geq 2/3$$

From which it follows that

$$\mathbb{E}_{x \sim \mu} \left[\Pr_{R, y_x} [R(y_x) = g^\phi(x)] - \Pr_R [R(x) = g^\phi(x)] \right] \geq 1/15.$$

This means there must be some specific input \hat{x} such that the probability of R outputting $g^\phi(\hat{x})$ when run on $y_{\hat{x}}$ is at least $1/15$ more than the probability of R outputting $g^\phi(\hat{x})$ when run on \hat{x} . In particular, we must have $y_{\hat{x}} \neq \hat{x}$, so $y_{\hat{x}}$ is a sample from $\mu'_{\hat{x}}$. Therefore, R distinguishes \hat{x} from $\mu'_{\hat{x}}$ with constant probability, so it uses at least $\Omega(A)$ queries.

We conclude that $R(f|_P) = \Omega(\min\{R^N(S), A\})$. Since the domain of the functions in S is $\{0, 1\}^A$, their query complexity is at most A . Thus $R(f|_P) = \Omega(R^N(S))$, as desired. A similar argument lower-bounds other complexity measures, such as R_0 or D . \square

This proof uses the fact that RC lower-bounds R , so it would not work on complexity measures that are not lower-bounded by RC (for example, $C^{(1)}$). For Q , it might be possible to use a similar argument and suffer a quadratic loss, since Q is lower-bounded by \sqrt{RC} . However, since there is no hard distribution for a quantum query complexity problem, this might be trickier to prove (we will not need it in this paper).

We can use the previous theorems to get a sculptability result for R vs. Q in terms of the H-index of randomized certificate complexity.

Corollary 20. *Let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be a total function. Then there is a promise $P \subseteq \{0, 1\}^N$ such that*

$$Q(f|_P) = O(\log^2 H(\text{RC}_f)), \quad R(f|_P) = \Omega\left(\frac{H(\text{RC}_f)^{1/3}}{\log^{5/3} N}\right).$$

There is also a promise $P' \subseteq \{0, 1\}^N$ such that

$$R_0(f|_{P'}) = O(1), \quad D(f|_{P'}) = \Omega\left(\frac{H(\text{RC}_f)}{\log^2 N}\right).$$

Proof. This follows from [Theorem 16](#) together with [Theorem 19](#) and [Theorem 26](#). \square

To complete the proof of [Theorem 2](#), all that remains is relating $H(\text{RC}_f)$ to $H(C_f)$.

5 Relating $H(C_f)$, $H(\text{RC}_f)$, and $H(\text{bs}_f)$

In this section, we relate $H(C_f)$ to $H(\text{RC}_f)$, completing the characterization of sculpting. Actually, we will prove a relationship between $H(C_f)$ and $H(\text{bs}_f)$, which implies the desired relationship since $H(\text{bs}_f) \leq H(\text{RC}_f)$. The proof is somewhat involved, but splits naturally into three parts. In [Lemma 21](#), we show a relationship between $C_f(x)$ and $\text{RC}_f(x)$ in terms of the number of 0- and 1-inputs of f . In [Theorem 22](#), we show that $H(C_f) = O(\text{Bal}(f) \log N)$. Finally, [Theorem 23](#) gives the desired relationship between $H(C_f)$ and $H(\text{bs}_f)$.

Lemma 21. *Let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be a partial function, and let $x \in \text{Dom}(f)$. If $f(x) = 0$, then*

$$C_f(x) \leq \text{RC}_f(x)(1 + \log |f^{-1}(1)|)$$

and if $f(x) = 1$, then

$$C_f(x) \leq \text{RC}_f(x)(1 + \log |f^{-1}(0)|).$$

Proof. For $x \in f^{-1}(1)$, we wish to upper-bound $C_f(x)$ in terms of $\text{RC}_f(x)$, assuming $f^{-1}(0)$ is small. A certificate for x consists of a partial assignment of x that contradicts all the elements of $f^{-1}(0)$.

Consider the greedy strategy for certifying x , which works by repeatedly choosing the bit of x that contradicts as many of the 0-inputs as possible, and adding it to the certificate. By definition, this strategy produces a certificate for x of size at least $C_f(x)$.

Let p_i be the fraction of the remaining inputs which are contradicted by the i -th bit of the greedy algorithm. The number of remaining inputs during the run of the greedy algorithm is then

$$|f^{-1}(0)|, |f^{-1}(0)|(1 - p_1), |f^{-1}(0)|(1 - p_1)(1 - p_2), \dots$$

The number of remaining inputs in the greedy algorithm will be upper-bounded by a geometric sequence that starts at $|f^{-1}(0)|$ and has ratio $1 - \min_i p_i$. Such a sequence decreases to 1 after at most

$$\frac{-1}{\log(1 - \min_i p_i)} (1 + \log |f^{-1}(0)|) \leq \frac{1 + \log |f^{-1}(0)|}{\min_i p_i}$$

steps. It follows that

$$C_f(x) \leq \frac{1 + \log |f^{-1}(0)|}{\min_i p_i}.$$

It remains to show that $\text{RC}_f(x) = \Omega(1/\min_i p_i)$. Let j be the step of the greedy algorithm that achieves this minimum, i.e. $p_j = \min_i p_i$. Then before the j^{th} step of the algorithm, there is a non-empty set S of 0-inputs for f such that for any bit of the input, at most a p_j fraction of the elements of S disagree with x on that bit. In other words, x is entry-wise very close to the ‘‘average’’ of the elements of S . If we give each element of S weight $1/(p_j|S|)$, we would get a feasible set of fractional blocks with total weight $1/p_j$. Thus $\text{RC}_f(x) \geq 1/p_j$, so $C_f(x) \leq \text{RC}_f(x)(\log |f^{-1}(0)| + 1)$. An analogous argument works when x is a 0-input to f . \square

Theorem 22. *Let $N \geq 2$, and let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be a total function. Then*

$$H(C_f) \leq 10 \text{Bal}(f) \log N.$$

Proof. Without loss of generality, suppose $|f^{-1}(0)| \leq |f^{-1}(1)|$. The number of 0-inputs with large certificates is at most $|f^{-1}(0)| \leq 2^{\text{Bal}(f)}$. Let S be the set of 1-inputs with certificates of size greater than $5 \text{Bal}(f)$. We wish to show that S is small. [Lemma 8](#) implies there is a set $B = \{i_1, i_2, \dots, i_{|B|}\}$ of indices of the input of size at least $\log |S| / (2 \log N)$ that is shattered by S . Therefore, to show that S is small, it suffices to show that B is small.

From [Lemma 21](#), we have $C_f(x) \leq \text{RC}_f(x) \text{Bal}(f)$ for any 1-input x , so for all $x \in S$, we have $\text{RC}_f(x) \geq C_f(x) / \text{Bal}(f) > 5$. This means for all $x \in S$, there is a distribution μ_x over 0-inputs such that for each i , the probability that $y_i \neq x_i$ when y is sampled from μ_x is less than $1/5$.

Let μ_B be the uniform distribution over B . Let $\delta(b, c) = 1$ if $b \neq c$ and 0 otherwise. We then write

$$\frac{1}{5} > \mathbb{E}_{i \sim \mu_B} \left(\mathbb{E}_{y \sim \mu_x} \delta(x_i, y_i) \right) = \mathbb{E}_{y \sim \mu_x} \left(\mathbb{E}_{i \sim \mu_B} \delta(x_i, y_i) \right).$$

We can conclude that for any $x \in S$, there exists a 0-input y_x that differs from x in less than one fifth of the bits of B . In other words, the distance between $x|_B$ and $y_x|_B$ is less than $|B|/5$. The idea is now to upper-bound $|B|$ by using the fact that for every string in $\{0, 1\}^{|B|}$ there is a 0 input

y such that $y|_B$ is close to that string, and there are few 0-inputs overall. Indeed, the number of strings in $\{0, 1\}^{|B|}$ is $2^{|B|}$, and each 0-input can only be of distance less than $|B|/5$ from $2^{H(1/5)|B|}$ of them (where $H(1/5)$ denotes the entropy of $1/5$). Therefore, to cover all the strings in $\{0, 1\}^{|B|}$, there must be more than $2^{(1-H(1/5))|B|}$ 0-inputs. Then

$$\text{Bal}(f) \geq \log |f^{-1}(0)| > (1 - H(1/5))|B| \geq (1 - H(1/5)) \frac{\log |S|}{2 \log N},$$

so

$$\log |S| < \frac{2 \text{Bal}(f) \log N}{1 - H(1/5)} \leq 8 \text{Bal}(f) \log N.$$

This means there are less than $2^{8 \text{Bal}(f) \log N}$ 1-inputs with certificate size at least $5 \text{Bal}(f)$. There are also at most $2^{\text{Bal}(f)}$ 0-inputs with certificate size at least $5 \text{Bal}(f)$ (because there are at most that many 0-inputs in total). Thus the log of the total number of inputs with certificates larger than $5 \text{Bal}(f)$ is at most $10 \text{Bal}(f) \log N$. It follows that $H(C_f) \leq 10 \text{Bal}(f) \log N$. \square

Theorem 23. *Let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be a total function. Then*

$$H(C_f) = O(H(\text{bs}_f)^2 \log N).$$

Proof. Let A be the set of inputs that have certificate size more than $H(C_f)$. Let A_0 be the set of 0-inputs in A , and let A_1 be the set of 1-inputs in A . Let B be the set of inputs that have block sensitivity more than b , with $b = \sqrt{H(C_f)/2}$. Let B_0 be the set of 0-inputs in B , and let B_1 be the set of 1-inputs in B . Without loss of generality, assume $|A_0| \geq |A_1|$. Since $|A| \geq 2^{H(C_f)}$, we have $|A_0| \geq 2^{H(C_f)-1}$.

Now, let $g : \{0, 1\}^N \rightarrow \{0, 1\}$ be the total function defined by $g(x) = 1$ if and only if $x \in B_1$. Suppose x is an element of $A_0 \setminus B_0$. Consider certifying that x is a 0-input for g ; let p be the smallest such certificate. Then p is consistent with x but inconsistent with all the strings in B_1 . We claim that this certificate must be large: its size must be greater than $H(C_f) - b^2 = H(C_f)/2$. To show this, we show that we can turn p into a certificate for x with respect to f (instead of with respect to g) by adding only b^2 bits to it.

Let q be a minimal sensitive block of x (with respect to f) that is disjoint from p . Since x is a 0-input for f , x^q is a 1-input for f . Since q is disjoint from p , x^q is consistent with p , so $x^q \notin B_1$. Thus the block sensitivity of x^q is at most b . However, since q was a minimal sensitive block, the sensitivity of x^q is at least $|q|$; thus $|q| \leq b$. It follows that all minimal sensitive blocks of x that are disjoint from p must have size at most b .

In addition, since $x \in A_0 \setminus B_0$, the block sensitivity of x is at most b . We can now construct a certificate for x by taking a maximal set of minimal disjoint sensitive blocks for x , all of which are disjoint from p . There will be at most b such blocks, and each will have size at most b . Therefore, this certificate for x has size at most $|p| + b^2$. Since $x \in A_0$, we must have $|p| + b^2 > H(C_f)$, or $|p| > H(C_f) - b^2 = H(C_f)/2$. We have shown that the elements of $A_0 \setminus B_0$ all have certificate size greater than $H(C_f)/2$ even with respect to g .

Now, by [Theorem 22](#), the number of inputs x that have certificate size more than $10(1 + \log |B_1|) \log N$ with respect to g is at most $2^{10(1 + \log |B_1|) \log N}$. It follows that either $H(C_f)/2 \leq 10(1 + \log |B_1|) \log N$ (so that the theorem doesn't apply), or else $|A_0 \setminus B_0| \leq 2^{10(1 + \log |B_1|) \log N}$.

In the former case, we have

$$\log |B| \geq \log |B_1| \geq \frac{H(C_f)}{20 \log N} - 1.$$

In the latter case, we have

$$2^{\mathsf{H}(C_f)-1} \leq |A_0| \leq |B_0| + 2^{10(1+\log|B_1|)\log N} = |B_0| + (2|B_1|)^{10\log N} \leq (2|B|)^{10\log N},$$

so in that case,

$$\log|B| \geq \frac{\mathsf{H}(C_f) - 1}{10\log N} - 1.$$

Thus, in both cases,

$$\log|B| \geq \frac{\mathsf{H}(C_f) - 1}{20\log N} - 1 = \Omega\left(\frac{\mathsf{H}(C_f)}{\log N}\right).$$

This means there are $2^{\Omega(\mathsf{H}(C_f)/\log N)}$ inputs with block sensitivity more than $\sqrt{\mathsf{H}(C_f)/2}$. We thus have

$$\mathsf{H}(\text{bs}_f) = \Omega\left(\min\left\{\frac{\mathsf{H}(C_f)}{\log N}, \sqrt{\mathsf{H}(C_f)}\right\}\right) = \Omega\left(\sqrt{\frac{\mathsf{H}(C_f)}{\log N}}\right).$$

□

[Theorem 2](#) now follows from [Theorem 10](#) (the non-sculptability theorem in terms of $\mathsf{H}(C_f^2)$), [Corollary 20](#) (the sculptability result in terms of $\mathsf{H}(\text{RC}_f)$), and [Theorem 23](#) (relating $\mathsf{H}(\text{bs}_f)$ to $\mathsf{H}(C_f)$), together with the properties that $\mathsf{H}(C_f^2) \leq \mathsf{H}(C_f)^2$ and that $\mathsf{H}(\text{bs}_f) \leq \mathsf{H}(\text{RC}_f)$. We restate [Theorem 2](#) here for convenience.

Theorem 2. *For all total functions $f : \{0, 1\}^N \rightarrow \{0, 1\}$ and all promises $P \subseteq \{0, 1\}^N$, we have*

$$\mathsf{R}(f|_P) = O(\mathsf{Q}(f|_P)^2 \mathsf{H}(C_f)^2).$$

Conversely, for all total functions $f : \{0, 1\}^N \rightarrow \{0, 1\}$, there is a promise $P \subseteq \{0, 1\}^N$ such that

$$\mathsf{R}(f|_P) = \Omega\left(\frac{\mathsf{H}(C_f)^{1/6}}{\log^{13/6} N}\right) \quad \text{and} \quad \mathsf{Q}(f|_P) = O(\log^2 \mathsf{H}(C_f)).$$

[Theorem 1](#) follows as a corollary. This completes the proof of our main result.

6 Sculpting Randomized Speedups

Now that we've characterized sculpting quantum query complexity, we turn our attention to sculpting other measures. Recall that

$$\mathsf{Q}(f) \leq \mathsf{R}(f) \leq \mathsf{R}_0(f) \leq \mathsf{D}(f).$$

We showed that sculpting $\mathsf{R}(f)$ vs. $\mathsf{Q}(f)$ is possible if and only if f has a large number of large certificates. We now show that the exact same condition characterizes sculpting $\mathsf{D}(f)$ vs. $\mathsf{R}_0(f)$. On the other hand, we show that $\mathsf{R}_0(f)$ vs. $\mathsf{R}(f)$ behaves differently: it's *always* possible to sculpt a function f to a promise P such that $\mathsf{R}(f|_P)$ is constant and $\mathsf{R}_0(f|_P)$ is almost as large as $\mathsf{R}_0(f)$.

We start by characterizing sculpting for D vs. R_0 .

6.1 Sculpting D vs. R_0

The proof of this characterization will follow that of [Theorem 2](#). For the non-sculptability direction, we need an analogue of [Theorem 10](#), relating deterministic and zero-error randomized query complexities in terms of $\text{Bal}(f)$. We prove the following theorem.

Theorem 24. *Let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be a partial function. Then*

$$D(f) \leq 2R_0(f) \text{Bal}(f).$$

Proof. Consider the zero-error randomized algorithm that takes $R_0(f)$ expected queries to evaluate f . By Markov's inequality, if we let this algorithm make $2R_0(f)$ queries on input x , it will succeed in computing $f(x)$ (and provide a certificate for x) with probability at least $1/2$. This gives us a probability distribution μ over deterministic algorithms, each of which makes $2R_0(f)$ queries, such that for each input x the probability that an algorithm sampled from μ finds a certificate when run on x is at least $1/2$.

For a deterministic algorithm D and an input x , let $c(D, x) = 1$ if D finds a certificate for x , and $c(D, x) = 0$ otherwise. Let $Z \subseteq \{0, 1\}^N$. Then

$$\mathbb{E}_{D \sim \mu} \left[\sum_{x \in Z} c(D, x) \right] = \sum_{x \in Z} \mathbb{E}_{D \sim \mu} [c(D, x)] \geq \sum_{x \in Z} (1/2) = \frac{|Z|}{2}.$$

It follows that there is a deterministic algorithm D_Z that makes $2R_0(f)$ queries and finds a certificate when run on half the inputs in Z .

Suppose without loss of generality that $|f^{-1}(0)| \leq |f^{-1}(1)|$. Now, on input x , set $Z = f^{-1}(0)$, and run D_Z on x . If it fails to find a certificate, then we have eliminated half of Z as possibilities for the input. Repeating this $\lceil \log |f^{-1}(0)| \rceil + 1 \leq \text{Bal}(f)$ times suffices to eliminate all of $f^{-1}(0)$ as possibilities for x , and hence to determine the value of $f(x)$. The total number of queries used is at most $2R_0(f) \text{Bal}(f)$. \square

Note that [Theorem 24](#) and [Theorem 10](#) together complete the proof of [Theorem 4](#).

Next, we turn [Theorem 24](#) into a non-sculptability theorem in terms of $H(C_f)$. The argument in [Theorem 12](#) follows verbatim, and we get the following sculpting lower bound.

Corollary 25. *Let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be a total function. For any promise $P \subseteq \{0, 1\}^N$, we have*

$$D(f|_P) = O(R_0(f|_P) H(C_f)^2).$$

We now prove the other direction: we show that sculpting is possible when $H(\text{RC}_f)$ is large. Using the arguments from [Section 4](#), it suffices to solve the extended queries problem for D vs. R_0 . We do this using the reduction to communication complexity in [Theorem 18](#).

Theorem 26. *For all $N \in \mathbb{N}$, there is a set of partial functions S from $\{0, 1\}^N$ to $\{0, 1\}$ such that for all $G \geq N$,*

$$R_0(S) = O(1), \quad D^G(S) = \Omega\left(\frac{N}{\log G}\right).$$

Proof. We start with EQUALITY, in which Alice and Bob are each given an n -bit string and wish to know if their strings are equal. This problem has deterministic query complexity $\Omega(n)$, but small randomized query complexity. To make the zero-error randomized query complexity small as well, we give Alice and Bob two strings each, with the promise that either their first strings are equal and the second strings are not, or vice versa. The goal will be to determine which is the case. It is not hard to see that the deterministic communication complexity of this problem is still $\Omega(n)$.

We need to get the zero-error randomized query complexity of the marginal functions to be small. To do this, we introduce another modification: we encode each of Bob's strings using a fixed random code of length $3n$. This code will have the property that the distance between any pair of codewords is $\Omega(n)$. To compute a marginal function f_{a_1, a_2} indexed by Alice's strings, we can simply randomly sample from each of Bob's strings; after $O(1)$ samples, we will discover which of his strings do not match the codeword corresponding to a_1 and a_2 .

This construction gives us a function $f : \{0, 1\}^{2n} \times \{0, 1\}^{6n} \rightarrow \{0, 1\}$ such that $D^{\text{CC}}(f) = \Omega(n)$ and $R_0(\text{Mar}(f)) = O(1)$. Setting $N = 6n$ and using [Theorem 18](#) finishes the proof. \square

Putting this together, we get the following sculpting theorem which, together with [Corollary 25](#), is analogous to [Theorem 2](#).

Theorem 27. *For all total functions $f : \{0, 1\}^N \rightarrow \{0, 1\}$, there is a promise $P \subseteq \{0, 1\}^N$ such that*

$$D(f|_P) = \Omega\left(\frac{\sqrt{H(C_f)}}{\log^{5/2} N}\right) \quad \text{and} \quad R_0(f|_P) = O(1).$$

Proof. This follows from [Theorem 26](#) together with [Theorem 16](#) and [Theorem 23](#). \square

We also get the following corollary, analogous to [Theorem 1](#).

Corollary 28. *Let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be a total function. Then there is a promise $P \subseteq \{0, 1\}^N$ such that $D(f|_P) = N^{\Omega(1)}$ and $R_0(f|_P) = N^{o(1)}$, if and only if $H(C_f) = N^{\Omega(1)}$. Furthermore, in this case we also have $R_0(f|_P) = O(1)$.*

6.2 Sculpting R_0 vs. R

While it is possible to use the above argument to get a sculptability result for R_0 vs. R , we can get a stronger result by a direct argument. In fact, unlike R vs. Q or D vs. R_0 , sculpting R_0 vs. R is *always* possible (there is no dependence on any H-index).

Theorem 29. *Let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be a non-constant total function. Then there is a promise $P \subseteq \{0, 1\}^N$ such that*

$$R(f|_P) = 1, \quad R_0(f|_P) \geq \frac{R_0(f)^{1/3}}{6}.$$

Proof. We actually prove a stronger result, finding a promise P such that $R(f|_P) = 1$ and $R_0(f|_P) \geq \text{bs}(f)/6$. We then use the known relationship $R_0(f) \leq \text{bs}(f)^3$ for total functions to get the desired result. Note that finding P with $R(f|_P) = 1$ and $R_0(f|_P) \geq \text{bs}(f)/6$ is trivial when $\text{bs}(f) \leq 6$; thus we assume $\text{bs}(f) > 6$.

Let $x \in \{0, 1\}^N$ be such that $\text{bs}_f(x) = \text{bs}(f)$. Assume without loss of generality that $f(x) = 0$. Let S_1 be the set of all 1-inputs with Hamming distance at least $(2/3)N$ from x . For any partial

assignment p consistent with x , let $S_1^p \subseteq S_1$ be the set of all inputs y in S_1 that are consistent with p .

There are two cases. If S_1^p is non-empty for all partial assignments p consistent with x of size less than $\text{bs}_f(x)/6$, then we can pick the promise to be $P = \{x\} \cup S_1$. It then follows that certifying that $f|_P$ is 0 on input x takes at least $\text{bs}_f(x)/6$ queries, whence $R_0(f|_P) \geq \text{bs}_f(x)/6$. On the other hand, a randomized algorithm can make 1 query to check if the input differs from x . Thus $R(f|_P) = 1$.

The other case is that there is some partial assignment p of size less than $\text{bs}_f(x)/6$ such that S_1^p is empty. We restrict our attention to inputs consistent with p . Since x has $\text{bs}_f(x)$ disjoint sensitive blocks, it has at least $(5/6)\text{bs}_f(x)$ disjoint sensitive blocks that do not overlap with p . We exclude blocks of size larger than $N/3$. Since there are at most 2 such blocks, this leaves at least $(5/6)\text{bs}_f(x) - 2$. Let B be the set of inputs we get by flipping one of these blocks of x . Then B contains only 1-inputs to f that are consistent with p , all of which have Hamming distance at most $N/3$ from x . Since $\text{bs}_f(x) = \text{bs}(f) > 6$, we have $B \neq \emptyset$.

Let S be the set of inputs consistent with p that have Hamming distance at least $(2/3)N$ from x . Since S_1^p is empty, S contains only 0-inputs to f . Let $P = B \cup S$. Now, consider certifying that an input y to $f|_P$ is a 1-input. Since all inputs of Hamming distance at least $(2/3)N$ from x that are consistent with p are 0-inputs, this requires showing at least $N/3 - |p|$ bits of y . Since $|p| < \text{bs}_f(x)/6 \leq N/6$, this is at least $N/6$. Thus $R_0(f|_P) \geq N/6 \geq \text{bs}(f)/6$.

On the other hand, a bounded-error randomized algorithm can simply query a bit of the input at random, and check for agreement with x . If the bit agrees, the algorithm can output 1, and if the bit disagrees, the algorithm can output 0. This works because 0-inputs have distance at least $(2/3)N$ from x , while all 1-inputs have distance at most $N/3$ from x (since the sensitive blocks used to construct B were of size at most $N/3$). Thus $R(f|_P) = 1$. \square

7 Other Query Complexity Results

We can use some of the tools introduced in the previous sections to prove some new relations in query complexity. In [Section 7.1](#), we prove a quadratic relationship between $D(f)$ and $Q(f)$ for partial functions f that have small domain. In [Section 7.2](#), we prove a quadratic relationship between $D(f)$ and $Q(f)$ for total functions f for which $H(C_f)$ is small.

7.1 Query Complexity on Small Promises

We prove [Theorem 3](#), which we restate for convenience.

Theorem 3. *Let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be a partial function, and let $\text{Dom}(f)$ denote the domain of f . Then*

$$Q(f) = \Omega\left(\frac{\sqrt{D(f)}}{\log |\text{Dom}(f)|}\right).$$

Proof. We follow the proof of [Theorem 10](#). The randomized algorithm used in that proof relies only on the existence of a randomized algorithm distinguishing a string $a \in \{0, 1\}^N$ from either $f^{-1}(0)$ or $f^{-1}(1)$, which is in turn guaranteed by [Lemma 9](#). To make that algorithm deterministic, we only need to turn this distinguishing algorithm into a deterministic one. By [Lemma 21](#), we have $C_f(x) = O(\text{RC}_f(x) \log |\text{Dom}(f)|)$. On the task of distinguishing a single input from a set

of inputs, certificate complexity equals deterministic query complexity. Using this observation, we can modify the proof of [Theorem 10](#) to get the result

$$D(f) = O(Q(f)^2 \text{Bal}(f) \log |\text{Dom}(f)|) = O(Q(f)^2 \log^2 |\text{Dom}(f)|),$$

from which the desired result follows. \square

7.2 Relationship for Total Functions

We can use H-indices to improve some of the relationships between complexity measures on total functions, proving [Theorem 5](#). Recall that for total functions, we have $D(f) \leq C(f) \text{bs}(f)$ and $\text{bs}(f) = O(Q(f)^2)$, from which we have $D(f) = O(Q(f)^2 C(f))$. We strengthen this result to $D(f) = O(Q(f)^2 H(\sqrt{C_f})^2)$ for total Boolean functions. Since $H(\sqrt{C_f}) \leq \sqrt{C(f)}$, this result is always stronger. In addition, since $C(\text{OR}) = n$ and $H(C_{\text{OR}}) = 1$, this improvement is sometimes very strong.

We restate [Theorem 5](#) for convenience.

Theorem 5. *Let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be a total function. Then*

$$D(f) = O(Q(f)^2 H(\sqrt{C_f})^2).$$

Proof. The proof follows the proof that $D(f) \leq C(f) \text{bs}(f)$ [[BBC⁺01](#)]. We start by reviewing this proof. The deterministic algorithm repeatedly picks possible 0-certificates that are consistent with the input observed so far, and queries the entries of these certificates. If the queried entries match the 0-certificate, the algorithm is done (the value of $f(x)$ is known to be 0). If ever there are no additional 0-certificates consistent with the observed part of the input, the value of the function is known to be 1.

The key insight is that if this process repeats k times, then the block sensitivity of the function is at least k . Indeed, let p be the partial assignment revealed after k iterations. Pick a 1-input y for f that is consistent with p . Let B_i be the set of entries queried in the i -th iteration of the algorithm. Then for each i , there is a way to change only the variables in B_i to form a 0-certificate for f . It follows that each B_i contains a sensitive block for y . Since the B_i sets are disjoint, we get $\text{bs}_f(y) \geq k$, so $\text{bs}(f) \geq k$.

We modify the algorithm as follows. In each step, we only allow the algorithm to pick 0-certificates that are of size at most $H(\sqrt{C_f})^2$. Thus the algorithm uses at most $\text{bs}(f) H(\sqrt{C_f})^2$ queries before it gets stuck. When it gets stuck, either the value of f on the input is determined, or else there are no more 0-certificates that are small enough.

Next, we repeat the same process with 1-certificates instead of 0-certificates. If the value of f is not yet determined, it means that the input is not consistent with any small enough certificates, so the certificate complexity of the input x is greater than $H(\sqrt{C_f})^2$. This gives $\sqrt{C_f(x)} > H(\sqrt{C_f})$.

By definition of the H-index, there are now at most $2^{H(\sqrt{C_f})}$ possibilities for the input. We've therefore restricted f to a small domain P . We now use [Theorem 3](#) to evaluate f using

$$O(Q(f)^2 \log^2 |\text{Dom}(f|_P)|) = O(Q(f)^2 H(\sqrt{C_f})^2)$$

deterministic queries. This is added to the $\text{bs}(f) H(\sqrt{C_f})^2$ queries from before. Using $\text{bs}(f) = O(Q(f)^2)$, we get

$$D(f) = O(Q(f)^2 H(\sqrt{C_f})^2).$$

\square

8 Sculpting in the Computational Complexity Model

In this section, we examine sculpting in the computational complexity model. We start with some notation. Given a language $L \subseteq \{0,1\}^*$, we let $L(x) \in \{0,1\}$ be its characteristic function. Also, given a language L together with a promise $P \subseteq \{0,1\}^*$, we let $L|_P$ be the promise problem of distinguishing the set $P \cap L$ from the set $P \setminus L$.

Now, we call the language L *sculptable* if there exists a promise P , such that the promise problem $L|_P$ is in PromiseBQP but not in PromiseBPP. We will use the following definition.

Definition 30 ([BH77]). *A language L is called paddable, if there exists a polynomial-time function $f(x,y)$ such that*

- (1) f is polynomial-time invertible, and
- (2) for all x,y , we have $x \in L \iff f(x,y) \in L$.

In other words, L is paddable if it is possible to “pad out” any input x with irrelevant information y , in an invertible way, without affecting membership in L .

The paddable languages were introduced by Berman and Hartmanis [BH77], as part of their exploration of whether all NP-complete languages are polynomial-time isomorphic: they showed that the answer was ‘yes’ for all *paddable* NP-complete languages. Under strong cryptographic assumptions, we now know that there exist NP-complete languages that are neither paddable nor isomorphic to each other [KMR95]. Nevertheless, it remains the case that almost all the languages that “naturally arise in complexity theory” are paddable.

Next, let us say that PromiseBQP is *hard on average* for P/poly if there exists a promise problem $H|_S \in$ PromiseBQP, as well as a family of distributions $\{\mathcal{D}_n\}_n$ with support on the promise set S , such that

- (1) \mathcal{D}_n is samplable in classical $\text{poly}(n)$ time, and
- (2) there is no family of classical circuits $\{C_n\}_n$, of size $\text{poly}(n)$, such that for all n ,

$$\Pr_{y \sim \mathcal{D}_n} [C_n(y) = H(y)] \geq \frac{3}{4}.$$

So for example, because of Shor’s algorithm [Sho97], combined with the worst-case/average-case equivalence of the discrete log problem, we can say that *if discrete log is not in P/poly, then PromiseBQP is hard on average for P/poly*.

We now prove [Theorem 6](#), which we restate here for convenience.

Theorem 6. *Assume PromiseBQP is hard on average for P/poly. Then every paddable language outside of BPP is sculptable.*

Proof. Let L be a paddable language, and let f be the padding function for L . Also, let $H|_S$ be any problem in PromiseBQP that is hard on average for P/poly, and let $\{\mathcal{D}_n\}_n$ be the associated family of hard distributions. Then we need to construct a promise, $P \subseteq \{0,1\}^*$, such that the promise problem $L|_P$ is in PromiseBQP but not in PromiseBPP.

Our promise P will simply consist of all inputs of the form $f(x,y,a)$ such that $y \in S$ and

$$L(x) \equiv H(y) + a \pmod{2}.$$

Here $a \in \{0, 1\}$ is a single bit, which we think of as concatenated onto the end of y .

Clearly, $L|_P$ is in PromiseBQP: just invert f to extract the “comment” (y, a) , then compute $H(y) + a \pmod{2}$.

We need to show that $L|_P$ is not in PromiseBPP. Suppose by contradiction that it was, and let \mathcal{A} be the algorithm such that $\mathcal{A}(x) = L(x)$ for all $x \in P$. Then we’ll show how to either

- (1) decide L in BPP (with no promise), or
- (2) decide H in P/poly, with high probability over \mathcal{D}_n .

Given an arbitrary input $x \in \{0, 1\}^n$, imagine we do the following: first sample $y \sim \mathcal{D}_n$, then run \mathcal{A} on the inputs $f(x, y, 0)$ and $f(x, y, 1)$. There are two cases: first suppose

$$\mathcal{A}(f(x, y, 0)) = \mathcal{A}(f(x, y, 1)).$$

Now, *one* of the two inputs $f(x, y, 0)$ and $f(x, y, 1)$ must belong to P . If $f(x, y, 0) \in P$, then $\mathcal{A}(f(x, y, 0)) = L(x)$, while if $f(x, y, 1) \in P$, then $\mathcal{A}(f(x, y, 1)) = L(x)$. Either way, then, we have learned whether $x \in L$, and we know we have learned this.

Second, suppose

$$\mathcal{A}(f(x, y, 0)) \neq \mathcal{A}(f(x, y, 1)).$$

Then assuming $y \in S$:

$$\begin{aligned} x \in L, y \in H &\implies \mathcal{A}(f(x, y, 0)) = 1 \implies \mathcal{A}(f(x, y, 1)) = 0, \\ x \in L, y \notin H &\implies \mathcal{A}(f(x, y, 1)) = 1 \implies \mathcal{A}(f(x, y, 0)) = 0, \\ x \notin L, y \in H &\implies \mathcal{A}(f(x, y, 1)) = 0 \implies \mathcal{A}(f(x, y, 0)) = 1, \\ x \notin L, y \notin H &\implies \mathcal{A}(f(x, y, 0)) = 0 \implies \mathcal{A}(f(x, y, 1)) = 1. \end{aligned}$$

Thus, regardless of whether $x \in L$, we have learned whether $y \in H$, and again we know we have learned this.

Now suppose there were an input $x \in \{0, 1\}^n$, such that running \mathcal{A} as above told us whether $y \in H$ with probability more than (say) $1/2$ over the choice of $y \sim \mathcal{D}_n$. Then let C_n be a polynomial-size circuit that hardwires x , and that given an input $y \in S$:

- Simulates both $\mathcal{A}(f(x, y, 0))$ and $\mathcal{A}(f(x, y, 1))$.
- Outputs $H(y)$ whenever it successfully learns the value of $H(y)$.
- Guesses a hardwired value for $H(y)$ (whichever of $\{0, 1\}$ is more probable) whenever it does not.

Then

$$\Pr_{y \sim \mathcal{D}_n} [C_n(y) = H(y)] \geq \frac{3}{4},$$

violating the assumption that no such circuit exists.

So we conclude that for every $x \in \{0, 1\}^n$, we must instead learn whether $x \in L$ with probability at least $1/2$ over the choice of $y \sim \mathcal{D}_n$. This, in turn, means that by simply generating y ’s randomly until we succeed, we can decide L in PromiseBPP. \square

Next, given a language $H \subseteq \{0, 1\}^*$, we say H is *BPP-bi-immune* if neither H nor its complement \overline{H} has any infinite subset in BPP. The notion of immunity was introduced by [FS74]. Here is a useful alternative characterization:

Lemma 31. *A language H is BPP-bi-immune if and only if there is no infinite set $S \in \text{BPP}$, such that the promise problem $H|_S$ is solvable in PromiseBPP.*

Proof. First, suppose H is not BPP-bi-immune, so that either H or \overline{H} has an infinite subset $S \in \text{BPP}$. Then clearly, S itself is an infinite set in BPP such that the promise problem $H|_S$ is trivial (the answer is either always 0 or always 1).

Conversely, suppose there exists an infinite set $S \in \text{BPP}$ such that $H|_S$ is solvable in polynomial time. Then clearly $S \cap H$ and $S \cap \overline{H}$ are both in BPP, and at least one of the two must be infinite. So H is not BPP-bi-immune. \square

We now suggest what, as far as we know, is a new conjecture in quantum complexity theory.

Conjecture 32. *There exists a BPP-bi-immune language in BQP.*

[Conjecture 32](#) is extremely strong. Note, in particular, that none of the “standard” BQP languages, such as languages based on factoring or discrete log, will be BPP-bi-immune, because they all have infinite special cases that are classically recognizable and easy (for example, the powers of 2, in the case of factoring). Nevertheless, we believe [Conjecture 32](#) is plausible. As a concrete candidate for a BPP-bi-immune language in BQP, let $g : \{0, 1\}^* \rightarrow \{0, 1\}^*$ be some strong pseudorandom generator. Then consider the language

$$L = \{x : g(x), \text{ interpreted as a positive integer, has an odd number of distinct prime factors}\}.$$

We now prove [Theorem 7](#), restated here for convenience.

Theorem 7. *Assume [Conjecture 32](#). Then every language outside of BPP is sculptable.*

Proof. Assume by way of contradiction that $L \notin \text{BPP}$ is non-sculptable. Also, let H be a BPP-bi-immune language in BQP. Then consider the set

$$S := \{x : L(x) = H(x)\}.$$

By our assumption, S is a promise on which no superpolynomial quantum speedup is possible for L , and \overline{S} is another such promise. Hence, there must be a BPP algorithm, call it \mathcal{A}_S , that solves the promise problem $H|_S$, which (by the definition of S) is equivalent to solving $L|_S$. And there must be another polynomial-time classical algorithm, call it $\mathcal{A}_{\overline{S}}$, that solves $H|_{\overline{S}}$, which (again by the definition of S) is equivalent to solving $\overline{L}|_{\overline{S}}$.

Now, given an input x , suppose we run both \mathcal{A}_S and $\mathcal{A}_{\overline{S}}$. Then as in the proof of [Theorem 6](#), there are two possibilities. If $\mathcal{A}_S(x) = \mathcal{A}_{\overline{S}}(x)$, then $x \in S$ implies $H(x) = \mathcal{A}_S(x)$ while $x \notin S$ implies $H(x) = \mathcal{A}_{\overline{S}}(x)$, so either way we have learned $H(x)$ (and we know that we have learned it). On the other hand, if $\mathcal{A}_S(x) \neq \mathcal{A}_{\overline{S}}(x)$, then $x \in S$ implies $L(x) = \mathcal{A}_S(x)$ while $x \notin S$ implies $L(x) = 1 - \mathcal{A}_{\overline{S}}(x)$. So, merely by seeing that $\mathcal{A}_S(x)$ and $\mathcal{A}_{\overline{S}}(x)$ are different, we have learned $L(x)$ (and we know that we have learned it).

In summary, there is a BPP algorithm \mathcal{B} that, for every input $x \in \{0, 1\}^*$, correctly outputs either $H(x)$ or $L(x)$, and that moreover tells us which one it outputs.

Now let Q be the set of all x such that $\mathcal{B}(x)$ outputs $H(x)$. Then there are two possibilities: if Q is finite, then \mathcal{B} decides L on all but finitely many inputs. Hence $L \in \text{BPP}$, contrary to assumption. If, on the other hand, Q is infinite, then $H|_Q$ is an infinite promise problem in PromiseBPP . So H was not BPP -bi-immune, again contrary to assumption. \square

In [Theorem 6](#) and [Theorem 7](#), there is almost nothing specific to the complexity classes BQP and BPP , apart from some simple closure properties. Thus, one can prove analogous sculpting theorems for many other pairs of complexity classes. In some cases, we do not even need an unproved conjecture. For example, we have:

Theorem 33. *For every language $L \notin \text{P}$, there exists a promise S such that $L|_S$ is solvable in exponential time, but is not solvable in polynomial time.*

Proof. The proof of [Theorem 7](#) follows through for P and EXP instead of BPP and BQP . In addition, it is known that there is a P -bi-immune language in EXP [[BH77](#)]. The desired result follows. \square

9 Concluding Remarks and Open Problems

In this work, we gave a full characterization of the class of Boolean functions f that can be sculpted into a promise problem with an exponential quantum speedup in query complexity. We similarly characterized sculptability for R_0 vs. R and D vs. R_0 . Along the way, we showed that Q is polynomially related (indeed, *quadratically* related) to D and R for a much wider set of promise problems than was previously known. Finally, we studied sculpting in *computational* complexity, giving a strong conjecture under which every language outside BPP is sculptable into a superpolynomial quantum speedup, and a weaker conjecture under which every *paddable* language outside BPP is sculptable.

One might object that many of our sculpted promise problems are somewhat artificial. This is particularly clear in the case of paddable languages, where (in essence) one uses the paddability to append to each instance x , as a “comment,” an instance of a hard BQP problem (such as factoring) that is promised to have the same answer as x . Even in the query complexity setting, however, one can observe by direct analogy that *the property of being sculptable is not closed under the removal of dummy variables*. So for example, we saw before that the N -bit OR function is not sculptable. By contrast, observe that the function

$$f(x_1, \dots, x_{2N}) := \text{OR}(x_1, \dots, x_N)$$

is sculptable. This follows as an immediate consequence of [Theorem 1](#): just by adding dummy variables to the OR function, we have vastly increased the number of inputs x that have large certificate complexity, from 1 to 2^N . However, an even simpler way to see why f is sculptable, is that we can embed (say) Simon’s problem into the variables x_{N+1}, \dots, x_{2N} , and then impose the promise that

$$\text{OR}(x_1, \dots, x_N) = \text{SIMON}(x_{N+1}, \dots, x_{2N})$$

(in addition to the Simon promise itself).

Of course, most Boolean functions do *not* contain such dummy variables, so the problems of sculpting them, and deciding whether they are sculptable at all, are much more complicated, as we saw in this paper.

Now, it might feel like “cheating” to sculpt a promise problem with a large quantum/classical gap by using dummy variables to encode a different, unrelated problem. If so, however, that points to an interesting direction for future research: namely, can we somehow formalize what we mean by a “natural” special case of a problem, and can we then understand which problems are “naturally” sculptable?

Here are some more specific open problems.

- Some of our inequalities could be off by polynomial factors; it would be nice to tighten them (or prove separations). For example, it may be possible to improve [Theorem 3](#) to $Q(f) = \Omega(\sqrt{D(f)}/\log |\text{Dom}(f)|)$, quadratically improving the $\log |\text{Dom}(f)|$ factor.
- Can our results – and specifically, [Theorem 5](#) – be used to improve the relation $D(f) = O(Q(f)^6)$ due to Beals et al. [[BBC⁺01](#)]?
- Can we give a characterization of the sculptable Boolean functions in *communication* complexity – analogous to this paper’s characterization of sculptability in query complexity?
- Is there any natural pair of complexity classes $\mathcal{C} \subseteq \mathcal{D}$, for which \mathcal{C} is known or believed to be strictly contained in \mathcal{D} , and yet it is plausible that no languages in \mathcal{D} are \mathcal{C} -bi-immune, and (related to that) there exist languages $L \notin \mathcal{C}$ that *cannot* be sculpted into a promise problem in $\mathcal{D} \setminus \mathcal{C}$?
- One can, of course, consider sculpting for many other pairs of computational models, besides R vs. Q or R_0 vs. R or D vs. R_0 . One interesting case is sculpting versus certificate complexity – for example, D vs. C. What is the correct characterization there?

We make some observations on the last problem. It’s easy to see that $D(\text{OR}|_P) = C(\text{OR}|_P)$ for any promise P , so sculpting D vs. C is not always possible. On the other hand, sculpting D vs. C is sometimes possible even when $H(C_f)$ is small. To see this, consider the function f with $f(x) = 1$ if and only if the Hamming weight of x is 1, and the single ‘1’ bit occurs on the left half of the input string. This function can be sculpted to $D(f|_P) = N/2$ and $C(f|_P) = 1$ by setting P to the set of inputs with Hamming weight 1. However, $H(C_f) = O(\log N)$ for this function, since all inputs with Hamming weight at least 2 have small certificates (just display two ‘1’ bits).

This means something qualitatively different happens with D vs. C than what was found in this paper.

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A Properties of H Indices

Lemma 34. *Let $g : \{0, 1\}^n \rightarrow [0, \infty)$. Define*

$$H(g) := \inf \left\{ h \in [0, \infty) : |\{x \in \{0, 1\}^n : g(x) > h\}| \leq 2^h \right\}.$$

Then

1. $H(g) \in [0, n]$
2. $H(g) \leq \max_x g(x)$
3. *The number of $x \in \{0, 1\}^n$ for which $g(x) > H(g)$ is at most $2^{H(g)}$ (equivalently, the infimum in the definition of $H(g)$ is actually a minimum)*
4. *If $g' : \{0, 1\}^n \rightarrow [0, \infty)$ is such that $g(x) \leq g'(x)$ for all $x \in \{0, 1\}^n$, then $H(g) \leq H(g')$*
5. *If $\alpha : [0, \infty) \rightarrow [0, \infty)$ is an increasing function, then $H(\alpha \circ g) \leq \max\{H(g), \alpha(H(g))\}$.*
6. *There are at least $2^{H(g)}$ inputs $x \in \{0, 1\}^n$ with $g(x) \geq H(g)$.*

Proof. Let $S_g(h) = \{x \in \{0, 1\}^n : g(x) > h\}$ and let $H_g = \{h \in [0, \infty) : |S_g(h)| \leq 2^h\}$. Then $H(g) = \inf H_g$. Part 1 follows from noticing that for all h , $S_g(h) \subseteq \{0, 1\}^n$, so $|S_g(h)| \leq 2^n$, whence $n \in H_g$. Part 2 follows from noticing that $S_g(\max_x g(x))$ is empty, so $\max_x g(x) \in H_g$.

To show 3, we show that H_g contains its infimum. Consider an infinite decreasing sequence $h_1, h_2, \dots \in H_g$ that converges to $H(g)$. Then the sequence $|S_g(h_1)|, |S_g(h_2)|, \dots$ is a non-decreasing sequence of integers which is bounded above by 2^n . In addition, $S_g(h_i) \subseteq S_g(h_{i+1})$ for all i . It follows that there is some ℓ such that $S_g(h_i) = S_g(h_\ell)$ for all $i \geq \ell$. For each $x \in S_g(h_\ell)$, we have $g(x) > h_\ell > H(g)$, and for each $x \notin S_g(h_\ell)$, we have $g(x) \leq h_i$ for all i . It follows that $g(x) \leq H(g)$ for each $x \notin S_g(h_\ell)$, so $S_g(H(g)) = \{x \in \{0, 1\}^n : g(x) > H(g)\} = S_g(h_i)$ for all $i \geq \ell$. Finally, since $h_i \in H_g$ for all i , we have $|S_g(H(g))| = |S_g(h_i)| \leq 2^{h_i}$ for all $i \geq \ell$. From this it follows that $|S_g(H(g))| \leq \lim_{i \rightarrow \infty} 2^{h_i} = 2^{H(g)}$, so $H(g) \in H_g$.

We now show 4. If g' is point-wise greater or equal to g , then $S_g(H(g')) \subseteq S_{g'}(H(g'))$. Since $H(g') \in H_g$, we have $|S_{g'}(H(g'))| \leq 2^{H(g')}$, so $|S_g(H(g'))| \leq 2^{H(g')}$. Thus $H(g') \in H_g$, so $H(g) = \inf H_g \leq H(g')$.

We prove 5. Let α be an increasing function. We have

$$S_g(H(g)) = \{x \in \{0, 1\}^n : g(x) > H(g)\} = \{x \in \{0, 1\}^n : \alpha \circ g(x) > \alpha(H(g))\} = S_{\alpha \circ g}(\alpha(H(g))).$$

Thus

$$|S_{\alpha \circ g}(\max\{H(g), \alpha(H(g))\})| \leq |S_{\alpha \circ g}(\alpha(H(g)))| = |S_g(H(g))| \leq 2^{H(g)} \leq 2^{\max\{H(g), \alpha(H(g))\}}$$

so $\max\{H(g), \alpha(H(g))\} \in H_{\alpha \circ g}$. Hence $H(\alpha \circ g) \leq \max\{H(g), \alpha(H(g))\}$.

Finally, we show 6. If it was false, there would be less than $2^{H(g)}$ inputs with $g(x) \geq H(g)$. Thus there is some $\epsilon > 0$ such that there are less than $2^{H(g)-\epsilon}$ inputs with $g(x) \geq H(g) > H(g) - \epsilon$. But this implies $H(g) - \epsilon \geq H(g)$, a contradiction. □

B Proof of Lemma 8

Lemma 8. *Let $S \subseteq \{0, 1\}^N$ be a collection of strings. Then there is a shattered set of indices of size at least*

$$\frac{\log |S|}{\log(N+1)}.$$

Proof. Let d be the size of the largest set that is shattered by S . Then the Sauer-Shelah lemma [Sau72] states

$$|S| \leq \sum_{i=0}^d \binom{N}{i}.$$

A well-known bound states

$$\sum_{i=0}^d \binom{N}{i} \leq 2^{\mathcal{H}(d/N)N},$$

where $\mathcal{H}(d/N)$ is the binary entropy of d/N . Then

$$\begin{aligned} \log_2 |S| &\leq \mathcal{H}(d/N)N = d \log_2(N/d) + (N-d) \log_2(1 + d/(N-d)) \\ &\leq d \log_2(N/d) + d \log_2 e = d \log_2 N - d \log_2(d/e) \leq d \log_2 N \end{aligned}$$

(if $d \geq e$). Thus

$$d \geq \frac{\log |S|}{\log N}$$

unless $d \leq 2$.

The Sauer-Shelah lemma implies $|S| \leq 1$ when $d = 0$ and $|S| \leq N^2$ when $d = 2$ (assuming $N \geq 2$). The only problematic case is $d = 1$ and $|S| = N + 1$. Thus, in all cases, we have $d \geq \log |S| / \log(N + 1)$, as desired. □