Sums of read-once formulas: How many summands suffice?

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Abstract. An arithmetic read-once formula (ROF) is a formula (circuit of fan-out 1) over $+, \times$ where each variable labels at most one leaf. Every multilinear polynomial can be expressed as the sum of ROFs. In this work, we prove, for certain multilinear polynomials, a tight lower bound on the number of summands in such an expression.

1   Introduction

Read-once formulae (ROF) are formulae (circuits of fan-out 1) in which each variable appears at most once. A formula computing a polynomial that depends on all its variables must read each variable at least once. Therefore, ROFs compute some of the simplest possible functions that depend on all of their variables. The polynomials computed by such formulas are known as read-once polynomials (ROPs). Since every variable is read at most once, ROPs are multilinear [1]. But not every multilinear polynomial is a ROP. For example, $x_1 x_2 + x_2 x_3 + x_1 x_3$.

We investigate the following question: Given an n-variate multilinear polynomial, can it be expressed as a sum of at most k ROPs? It is easy to see that every bivariate multilinear polynomial is an ROP. Any tri-variate multilinear polynomial can be expressed as a sum of 2 ROPs. With a little thought, we can obtain a sum-of-3-ROPs expression for any 4-variate multilinear polynomial. An easy induction on n then shows that any n-variate multilinear polynomial, for $n \geq 4$, can be written as a sum of at most $3 \times 2^{n-4}$ ROPs. We ask the following question: Does there exist a strict hierarchy among k-sums of ROPs? Formally,

Problem 1. Consider the family of n-variate multilinear polynomials. For $1 < k \leq 3 \times 2^{n-4}$, is $\sum^k \cdot \text{ROP}$ strictly more powerful than $\sum^{k-1} \cdot \text{ROP}$? If so, what explicit polynomials witness the separations?

We answer this affirmatively for $k \leq \lceil n/2 \rceil$. In particular, for $k = \lceil n/2 \rceil$, there exists an explicit n-variate multilinear polynomial which cannot be written as a sum of less than k ROPs but it admits a sum-of-k-ROPs representation.

Note that n-variate ROPs are computed by linear sized formulas. Thus if an n-variate polynomial p is in $\sum^k \cdot \text{ROP}$, then p is computed by a formula of size

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[1] A polynomial is said to be multilinear if the individual degree of each variable is at most one.

$O(kn)$ where every intermediate node computes a multilinear polynomial. Since superpolynomial lower bounds are already known for the model of multilinear formulas [7], we know that for those polynomials (including the determinant and the permanent), a $\sum^k \cdot$ROP expression must have k at least quasi-polynomial in n. However the best upper bound on k is only exponential in n, leaving a big gap between the lower and upper bound. On the other hand, our lower bound is provably tight.

A natural question to ask is whether stronger lower bounds than the above result can be proven. In particular, to separate $\sum^{k-1} \cdot$ROP from $\sum^k \cdot$ROP, how many variables are needed? The above hierarchy result says that $2k-1$ variables suffice, but there may be simpler polynomials (with fewer variables) witnessing this separation. We demonstrate another technique which improves upon the previous result for $k = 3$, showing that 4 variables suffice. In particular, we show that over the field of reals, there exists an explicit multilinear 4-variate multilinear polynomial which cannot be written as a sum of 2 ROPs. This lower bound is again tight, as there is a sum of 3 ROPs representation for every 4-variate multilinear polynomial.

Our results and techniques

We now formally state our results.

Theorem 1. *For each $n \geq 1$, the n-variate degree $n-1$ symmetric polynomial S_n^{n-1} cannot be written as a sum of less than $\lceil n/2 \rceil$ ROPs, but it can be written as a sum of $\lceil n/2 \rceil$ ROPs.*

The idea behind the lower bound is that if g can be expressed as a sum of less than $\lceil n/2 \rceil$ ROFs, then one of the ROFs can be eliminated by taking partial derivative with respect to one variable and substituting another by a field constant. We then use the inductive hypothesis to arrive at a contradiction. This approach necessitates a stronger hypothesis than the statement of the theorem, and we prove this stronger statement in Theorem 6.

Theorem 2. *There is an explicit 4-variate multilinear polynomial f which cannot be written as the sum of 2 ROPs over $\mathbb{R}$.*

The proof of this theorem mainly relies on a structural lemma (Lemma 6) for sum of 2 read-once formulas. In particular, we show that if f can be written as a sum of 2 ROPs then one of the following must be true: 1. Some 2-variate restriction is a linear polynomial. 2. There exist variables $x_i, x_j \in \mathrm{Var}(f)$ such that the polynomials $x_i, x_j, \partial_{x_i}(f), \partial_{x_j}(f)$ are affinely dependent. 3. We can represent f as $f = l_1 \cdot l_2 + l_3 \cdot l_4$ where (l_1, l_2) and (l_3, l_4) are variable-disjoint linear forms. Checking the first two conditions is easy. For the third condition we use the commutator of f, introduced in [10], to find one of the l_i's. The knowledge of one of the l_i's suffices to determine all the linear forms. Finally, we construct a 4-variate polynomial which does not satisfy any of the above mentioned conditions. This construction does not work over algebraically closed fields. We do not yet know

how to construct an explicit 4-variate multilinear polynomial not expressible as the sum of 2 ROPs over such fields, or even whether such polynomials exist.

Related work

Despite their simplicity, ROFs have received a lot of attention both in the arithmetic as well as in the Boolean world [4, 3, 1, 2, 8, 10]. The most fundamental question that can be asked about polynomials is the polynomial identity testing (PIT) problem: Given an arithmetic circuit $\mathcal{C}$, is the polynomial computed by $\mathcal{C}$ identically zero or not. PIT has a randomized polynomial time algorithm: Evaluate the polynomial at random points. It is not known whether PIT has a deterministic polynomial time algorithm. In 2004, Kabanets and Impagliazzo established a connection between PIT algorithms and proving general circuit lower bounds [5]. However, for restricted arithmetic circuits, no such result is known. For instance, consider the case of multilinear formulas. Even though strong lower bounds are known for this model, there is no efficient deterministic PIT algorithm. For this reason, PIT was studied for the weaker model of sum of read-once formulas. Notice that multilinear depth 3 circuits are a special case of this model.

Shpilka and Volkovich gave a deterministic PIT algorithm for the sum of a small number of ROPs [8]. Interestingly, their proof uses a lower bound for a weaker model, that of 0-justified ROFs. In particular, they show that the polynomial $\mathcal{M}_n = x_1 x_2 \cdots x_n$, consisting of just a single monomial, cannot be represented as a sum of less than $n/3$ weakly justified ROPs. More recently, Kayal showed that if $\mathcal{M}_n$ is represented as a sum of powers of low degree (at most d) polynomials, then the number of summands is at most $\exp(\Omega(n/d))$ [6]. He used this lower bound to give a PIT algorithm. Our lower bound from Theorem 1 is orthogonal to both these results and is provably tight. An interesting question is whether it can be used to give a PIT algorithm.

1.1 Organization

The paper is organized as follows. In Section 2 we give the basic definitions and notations. In Section 3, we establish Theorem 1. showing that the hierarchy of k-sums of ROPs is proper. In Section 4 we establish Theorem 2, showing an explicit 4-variate multilinear polynomial that is not expressible as the sum of two ROPs. We conclude in Section 5 with some further questions that are still open.

2 Preliminaries

For a positive integer n, we denote $[n] = \{1, 2, \ldots, n\}$. For a polynomial f, by $\mathrm{Var}(f)$ we mean the set of variables occurring in f. For a polynomial $f(x_1, x_2, \ldots, x_n)$, a variable x_i and a field element α, we denote by $f \mid_{x_i = \alpha}$ the polynomial resulting from setting $x_i = \alpha$. Let f be an n-variate polynomial. We say that g is a

k-variate restriction of f if g is obtained by setting some variables in f to field constants and $|\mathrm{Var}(g)| \leq k$. A set of polynomials $f_1, f_2, \ldots, f_k$ over the field $\mathbb{F}$ is said to be affinely dependent if there exist constants $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that

$$\sum_{i \in [k]} \alpha_i f_i = 0.$$

The n-variate degree k elementary symmetric polynomial, denoted S_n^k, is defined as follows:

$$S_n^k(x_1, \ldots, x_n) = \sum_{A \subseteq [n], |A| = k} \prod_{i \in A} x_i.$$

A circuit is a directed acyclic graph with variables and field constants labeling the leaves, field operations $+, \times$ labeling internal nodes, and a designated sink node. Each node naturally computes a polynomial; the polynomial at the designated sink node is the polynomial computed by the circuit. If the underlying undirected graph is also acyclic, then the circuit is called a formula. A formula is said to be read-k if each variable appears as a leaf label at most k times.

For read-once formulas, it is more convenient to use the following "normal form" from [8].

Definition 1 (Read-once formulas [8]). *A read-once arithmetic formula (ROF) over a field $\mathbb{F}$ in the variables $\{x_1, x_2, \ldots, x_n\}$ is a binary tree as follows. The leaves are labeled by variables and internal nodes by $\{+, \times\}$. In addition, every node is labeled by a pair of field elements $(\alpha, \beta) \in \mathbb{F}^2$. Each input variable labels at most once leaf. The computation is performed in the following way. A leaf labeled by x_i and (α, β) computes $\alpha x_i + \beta$. If a node v is labeled by $\star \in \{+, \times\}$ and (α, β) and its children compute the polynomials f_1 and f_2, then v computes $\alpha(f_1 \star f_2) + \beta$.*

We say that f is a read-once polynomial (ROP) if it can be computed by a ROF, and is in $\sum^k \cdot$ROP if it can be expressed as the sum of at most k ROPs.

Proposition 1. *For every n, every n-variate multilinear polynomial can be written as the sum of at most $\lceil 3 \times 2^{n-4} \rceil$ ROPs.*

Proof. For $n = 1, 2, 3$ this is easy to see.

For $n = 4$, let $f(X)$ be given by the expression $\sum_{S \subseteq [4]} A_S x_S$, where x_S denotes the monomial $\prod_{i \in S} x_i$. We want to express f as $f_1 + f_2 + f_3$, where each f_i is an ROP. If there are no degree 2 terms, we use the following:

$$f_1 = A_\emptyset + A_1 x_1 + A_2 x_2 + A_3 x_3 + A_4 x_4$$
$$f_2 = x_1 x_2 (A_{123} x_3 + A_{124} x_4)$$
$$f_3 = x_3 x_4 (A_{134} x_1 + A_{234} x_2 + A_{1234} x_1 x_2)$$

Otherwise, assume wlog that $A_{13} \neq 0$. Then define

$$f_1 = \left[\sum_{S \subseteq [2]} A_S \prod_{i \in S} x_i \right] + \left[\sum_{\emptyset \neq S \subseteq \{3,4\}} A_S \prod_{i \in S} x_i \right]$$

$$f_2 = (A_{13}x_1 + A_{23}x_2 + A_{123}x_1x_2) \cdot \left(\frac{A_{14}}{A_{13}} x_4 + x_3 + \frac{A_{134}}{A_{13}} x_3 x_4 \right)$$

$$f_3 = x_2 x_4 \left[\left(A_{24} - \frac{A_{14}A_{23}}{A_{13}} \right) + x_1 \left(A_{124} - \frac{A_{14}A_{123}}{A_{13}} \right) \right.$$

$$\left. + x_3 \left(A_{234} - \frac{A_{134}A_{23}}{A_{13}} \right) + x_1 x_3 \left(A_{1234} - \frac{A_{134}A_{123}}{A_{13}} \right) \right]$$

Since any bivariate multilinear polynomial is a ROP, each f_i is indeed an ROP.

For $n > 4$, express f as $x_n g + h$ where $g = \partial_{x_n} f$ and $h = f \mid_{x_n=0}$, and use induction, along with the fact that g does not have variable x_n. □

Fact 3 (Useful Fact about ROPs [9]) *The partial derivatives of ROPs are also ROPs.*

Proposition 2 (3-variate ROPs). *Let $f \in \mathbb{F}[x_1, x_2, x_3]$ be a 3-variate ROP. Then there exists $i \in [3]$ and $A \in \mathbb{F}$ such that $\deg(f \mid_{x_i=A}) \leq 1$.*

Proof. Assume without loss of generality that $f = f_1(x_1) \star f_2(x_2, x_3) + c$ where $\star \in \{+, \times\}$ and $c \in \mathbb{F}$. If $\star = +$, then for all $A \in \mathbb{F}$, $\deg(f \mid_{x_2=A}) \leq 1$. If $\star = \times$, $\deg(f \mid_{f_1=0}) \leq 1$. □

We will also be dealing with a special case of ROFs called multiplicative ROFs defined below:

Definition 2 (Multiplicative Read-once formulas). *A ROF is said to be a multiplicative ROF if it does not contain any addition gates. We say that f is a multiplicative ROP if it can be computed by a multiplicative ROF.*

Multiplicative ROPs have the following useful property, observed in [9]. (See Lemma 5.13 in [9]. For completeness, and since we refer to the proof later, we include a proof sketch here.)

Lemma 1 ([9]). *Let g be a multiplicative ROP with $|\mathrm{Var}(g)| \geq 2$. For every $x_i \in \mathrm{Var}(g)$, there exists $x_j \in \mathrm{Var}(g) \setminus \{x_i\}$ and $\gamma \in \mathbb{F}$ such that $\partial_{x_j}(g) \mid_{x_i=\gamma} = 0$.*

Proof. Let φ be a multiplicative ROF computing g. As $|\mathrm{Var}(\varphi)| = |\mathrm{Var}(g)| \geq 2$, φ has at least one gate. Let v be the unique neighbour (parent) of the leaf labeled by x_i. We denote by $P_v(\bar{x})$ the ROP that is computed by v. Assume wlog that $\mathrm{Var}(P_v) = \{x_1, x_2, \ldots, x_{i-1}, x_i\}$. We show that there exists some ROP Q such that $Q(P_v, x_{l+1}, x_{l+2}, \ldots, x_n) \equiv f_i$ where Q and P_v are variable-disjoint ROPs. Consider the ROF φ_i computing f_i. Denote with ψ_i the subformula rooted at v. The output of ψ_i is wired as one of the inputs to φ_i. Let the resulting polynomial

computed by φ_i be denoted as Q. It follows that $Q(P_v, \bar{x}) \equiv f_i$. Also Q and P_v are variable-disjoint as they are computed by different parts of the same ROP.

Since v is a multiplication gate (recall that φ is a multiplicative ROF) and neighbor of the leaf labeled by x_i, P_v can be written as $P_v(\bar{x}) = (x_i - \alpha)h(\bar{x}) + c$ for some ROP h such that $\mathrm{Var}(h) \neq \emptyset$ and $x_i \notin \mathrm{Var}(h)$

Finally, by the chain rule, for every variable $x_j \in \mathrm{Var}(h)$ we have that:

$$\partial_{x_j}(f_i) = \partial_y(Q) \cdot \partial_{x_j}(P_v) = \partial_y(Q) \cdot (x_i - \alpha) \cdot \partial_{x_j}(h)$$

It follows that $\partial_{x_j}(g) \mid_{x_i = \alpha} = 0$. $\square$

Along with partial derivatives, another operator that we will find useful is the commutator of a polynomial. The commutator of a polynomial has previously been used for polynomial factorization and in reconstruction algorithms for read-once formulas, see [10].

Definition 3 (Commutator [10]). *Let $P \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ be a multilinear polynomial and let $i, j \in [n]$. The commutator between x_i and x_j, denoted $\triangle_{ij}P$, is defined as follows.*

$$\triangle_{ij}P = \left(P \mid_{x_i=0,x_j=0}\right) \cdot \left(P \mid_{x_i=1,x_j=1}\right) - \left(P \mid_{x_i=0,x_j=1}\right) \cdot \left(P \mid_{x_i=1,x_j=0}\right)$$

The following property of the commutator will be useful to us.

Lemma 2. *Let $f = l_1(x_1, x_2) \cdot l_2(x_3, x_4) + l_3(x_1, x_3) \cdot l_4(x_2, x_4)$ where the l_i's are linear polynomials. Then l_2 divides $\triangle_{12}(f)$.*

Proof. First, we show that $\triangle_{12}(l_3 \cdot l_4) = 0$. Assume $l_3 = Cx_1 + m$ and $l_4 = Dx_2 + n$ where $C, D \in \mathbb{F}$ and m, n are linear polynomials in x_3, x_4 respectively. By definition, $\triangle_{12}(l_3 \cdot l_4) = mn(C + m)(D + n) - m(D + n)(C + m)n = 0$.

Now we write $\triangle_{12}f$ explicitly. Let $l_1 = ax_1 + bx_2 + c$. By definition,

$$\begin{aligned}
\triangle_{12}f &= \triangle_{12}(l_1 l_2 + l_3 l_4) \\
&= (cl_2 + mn)((a + b + c)l_2 + (C + m)(D + n)) - \\
&\quad ((b + c)l_2 + m(D + n)) \cdot ((a + c)l_2 + n(C + m)) \\
&= l_2^2(c(a + b + c) - (a + c)(b + c)) \\
&\quad + l_2(c(C + m)(D + n) + mn(a + b + c) - n(b + c)(C + m) - m(a + c)(D + n))
\end{aligned}$$

It follows that l_2 divides $\triangle_{12}f$. $\square$

3   A proper hierarchy in $\sum^k \cdot \mathrm{ROP}$

This section is devoted to proving Theorem 1.

We prove the lower bound for S_n^{n-1} by induction. This necessitates a stronger induction hypothesis, so we will actually prove the lower bound for a larger class

of polynomials. For any $\alpha, \beta \in \mathbb{F}$, we define the polynomial $\mathcal{M}_n^{\alpha,\beta} = \alpha S_n^n + \beta S_n^{n-1}$. We note the following recursive structure of $\mathcal{M}_n^{\alpha,\beta}$:

$$(\mathcal{M}_n^{\alpha,\beta})\mid_{x_n=\gamma} = \mathcal{M}_{n-1}^{\alpha\gamma+\beta,\beta\gamma} \quad .$$
$$\partial_{x_n}(\mathcal{M}_n^{\alpha,\beta}) = \mathcal{M}_{n-1}^{\alpha,\beta} \quad .$$

We show below that each $\mathcal{M}_n^{\alpha,\beta}$ is expressible as the sum of $\lceil n/2 \rceil$ ROPs (Lemma 4); however, for any non-zero $\beta \neq 0$, $\mathcal{M}_n^{\alpha,\beta}$ cannot be written as the sum of fewer than $\lceil n/2 \rceil$ ROPs (Lemma 3). At $\alpha = 0$, $\beta = 1$, we get S_n^{n-1}, the simplest such polynomials, establishing Theorem 1.

Lemma 3. *Let $\mathbb{F}$ be a field. For every $\alpha \in \mathbb{F}$ and $\beta \in \mathbb{F} \setminus \{0\}$, the polynomial $\mathcal{M}_n^{\alpha,\beta} = \alpha S_n^n + \beta S_n^{n-1}$ cannot be written as a sum of $k < n/2$ ROPs.*

Proof. The proof is by induction on n. The cases $n = 1, 2$ are easy to see. We now assume that $k \geq 1$ and $n > 2k$. Assume to the contrary that there are ROPs $f_1, f_2, \ldots, f_k$ over $\mathbb{F}[x_1, x_2, \ldots, x_n]$ such that $\sum_{m \in [k]} f_m = \mathcal{M}_n^{\alpha,\beta}$. The main steps in the proof are as follows:

1. Show using the inductive hypothesis that for all $m \in [k]$ and $a, b \in [n]$, $\partial_{x_a}\partial_{x_b}(f_m) \neq 0$.
2. Conclude that for all $m \in [k]$, f_m must be a multiplicative ROP. That is, the ROF computing f_m does not contain any addition gate.
3. Use the multiplicative property of f_k to show that f_k can be eliminated by taking partial derivative with respect to one variable and substituting another by a field constant. If this constant is non-zero, we contradict the inductive hypothesis.
4. Otherwise, use the sum of (multiplicative) ROPs representation of $\mathcal{M}_n^{\alpha,\beta}$ to show that the degree of f can be made at most $(n-2)$ by setting one of the variables to zero. This contradicts our choice of f since $\beta \neq 0$.

We now proceed with the proof.

Claim 4 *For all $m \in [k]$ and $a, b \in [n]$, $\partial_{x_a}\partial_{x_b}(f_m) \neq 0$.*

Proof. Suppose to the contrary that $\partial_{x_a}\partial_{x_b}(f_m) = 0$. Assume wlog that $a = n$, $b = n - 1$, $m = k$, so $\partial_{x_n}\partial_{x_{n-1}}(f_k) = 0$. Then,

$$\mathcal{M}_n^{\alpha,\beta} = \sum_{m=0}^{k} f_m \qquad \text{(by assumption)}$$

$$\partial_{x_n}\partial_{x_{n-1}}(\mathcal{M}_n^{\alpha,\beta}) = \sum_{m=0}^{k} \partial_{x_n}\partial_{x_{n-1}}(f_m) \quad \text{(by subadditivity of partial derivative)}$$

$$\mathcal{M}_{n-2}^{\alpha,\beta} = \sum_{m=0}^{k-1} \partial_{x_n}\partial_{x_{n-1}}(f_m) \quad \text{(by recursive structure of } \mathcal{M}_n,$$

$$\text{and since } \partial_{x_n}\partial_{x_{n-1}}(f_k) = 0)$$

Thus $\mathcal{M}_{n-2}^{\alpha,\beta}$ can be written as the sum of $k-1$ polynomials, each of which is an ROP (by Fact 3). By the inductive hypothesis, $2(k-1) \geq (n-2)$. Therefore, $k \geq n/2$ contradicting our assumption. $\qquad\square$

Let φ_m be the ROF computing f_m. The next step is to show that for $m \in [k]$, φ_m does not contain an addition gate. Suppose to the contrary that for some $m \in [k]$ and $p,q \in [n]$, the least common ancestor of x_p and x_q in φ_m is a $+$ gate. It follows that $\partial_{x_p}\partial_{x_q}(f_m) = 0$, contradicting Claim 4. We record this observation below.

Observation 5 *For all $m \in [k]$, f_m is a multiplicative ROP.*

Observation 5 and Lemma 1 together imply that for each $m \in [k]$ and $i \in [n]$, there exist $j \neq i \in [n]$ and $\gamma \in \mathbb{F}$ such that $\partial_{x_j}(f_k)\,|_{x_i=\gamma}= 0$. There are two cases to consider.

First, consider the case when for some m, i and the corresponding j, γ, it turns out that $\gamma \neq 0$. Assume without loss of generality that $m = k$, $i = n-1$, $j = n$, so that $\partial_{x_n}(f_k)\,|_{x_{n-1}=\gamma}= 0$. (For other indices the argument is symmetric.) Then

$$\mathcal{M}_n^{\alpha,\beta} = \sum_{i\in[k]} f_i \qquad\qquad \text{(by assumption)}$$

$$\partial_{x_n}(\mathcal{M}_n^{\alpha,\beta})\,|_{x_{n-1}=\gamma} = \sum_{i\in[k]} \partial_{x_n}(f_i)\,|_{x_{n-1}=\gamma} \qquad \text{(by subadditivity of partial derivative)}$$

$$\mathcal{M}_{n-1}^{\alpha,\beta}\,|_{x_{n-1}=\gamma} = \sum_{i\in[k-1]} \partial_{x_n}(f_i)\,|_{x_{n-1}=\gamma} \quad \text{(since } \gamma \text{ is chosen as per Lemma 1)}$$

$$\mathcal{M}_{n-2}^{\alpha\gamma+\beta,\beta\gamma} = \sum_{i\in[k-1]} \partial_{x_n}(f_i)\,|_{x_{n-1}=\gamma} \quad \text{(recursive structure of } \mathcal{M}_n\text{)}$$

Therefore, $\mathcal{M}_{n-2}^{\alpha\gamma+\beta,\beta\gamma}$ can be written as a sum of at most $k-1$ polynomials, each of which is an ROP (Fact 3). By the inductive hypothesis, $2(k-1) \geq n-2$ implying that $k \geq n/2$ contradicting our assumption.

(Note: the term $\mathcal{M}_{n-2}^{\alpha\gamma+\beta,\beta\gamma}$ is what necessitates a stronger induction hypothesis than working with just $\alpha = 0, \beta = 1$.)

It remains to handle the case when for all $m \in [k]$ and $i \in [n]$, the corresponding value of γ to some x_j (as guaranteed by Lemma 1) is 0. Examining the proof of Lemma 1, this implies that each leaf node in any of the ROFs can be made zero only by setting the corresponding variable to zero. That is, the linear forms at all leaves are of the form $a_i x_i$.

Since each φ_m is a multiplicative ROP, setting $x_n = 0$ makes the variables in the polynomial computed at the sibling of the leaf node $a_n x_n$ redundant. Hence setting $x_n = 0$ reduces the degree of each f_m by at least 2. That is, $\deg(f\,|_{x_n=0}) \leq n - 2$. But $f\,|_{x_n=0}$ equals $\mathcal{M}_{n-1}^{\beta,0} = \beta S_{n-1}^{n-1}$, which has degree $n - 1$, a contradiction. $\qquad\square$

The following lemma shows that the above lower bound is indeed optimal.

Lemma 4. *For any field $\mathbb{F}$ and $A, B \in \mathbb{F}$, the polynomial $f = AS_n^n + BS_n^{n-1}$ can be written as a sum of at most $\lceil n/2 \rceil$ ROPs.*

Proof. Define $f_i := (x_{2i-1} + x_{2i}) \cdot \left(\prod_{\substack{k \in [n] \\ k \neq 2i, 2i-1}} x_k \right)$. Notice that each f_i is a ROP.

Depending on the parity of n, we consider two cases:
Case 1: n is even; $n = 2k$. Then, defining

$$f_k' = (Bx_{2k-1} + Bx_{2k} + Ax_{2k-1}x_{2k}) \cdot \left(\prod_{\substack{m \in [n] \\ k \neq 2k, 2k-1}} x_m \right),$$

we have $f = B(f_1 + f_2 + \ldots + f_{k-1}) + f_k'$. Note that f_k' is also an ROP; the factor involving x_{2k-1} and x_{2k} is bivariate multilinear and hence an ROP.
Case 2: n is odd; $n = 2k + 1$. Then, defining

$$f_{k+1}' = x_1 x_2 \cdots x_{2k}(B + Ax_{2k+1}),$$

we have $f = B(f_1 + f_2 + \ldots + f_k) + f_{k+1}'$. Note that f_{k+1}' is also an ROP.
In either case, since all the polynomials f_i, f_k', f_{k+1}' are ROPs, we have a representation of f as a sum of at most $\lceil n/2 \rceil$ ROPs. $\qquad \square$

Combining the results of Lemma 3 and Lemma 4, we obtain the following theorem. At $\alpha = 0, \beta = 1$, it yields Theorem 1.

Theorem 6. *For each $n \geq 1$, any $\alpha \in \mathbb{F}$ and any any $\beta \in \mathbb{F} \backslash \{0\}$, the polynomial $\alpha S_n^n + \beta S_n^{n-1}$ is in $\sum^m \cdot ROP$ but not in $\sum^{m-1} \cdot ROP$, where $m = \lceil n/2 \rceil$.*

4   A 4-variate multilinear polynomial not in $\sum^2 \cdot$ROP

This section is devoted to proving Theorem 2. We want to find an explicit 4-variate multilinear polynomial that is not expressible as the sum of 2 ROPs.

Note that the proof of Theorem 1 does not help here, since the polynomials separating $\sum^2 \cdot$ROP from $\sum^3 \cdot$ROP have 5 or 6 variables. One obvious approach is to consider other combinations of the symmetric polynomials. This fails too; we can show that all such combinations are in $\sum^2 \cdot$ROP.

Proposition 3. *For every choice of field constants a_i for each $i \in \{0, 1, 2, 3, 4\}$, the polynomial $\sum_{i=0}^4 a_i S_4^i$ can be expressed as the sum of two ROPs.*

Proof. Let $g = \sum_i a_i S_4^i$. We obtain the expression for g in different ways in 4 different cases.

Case	Expression
$a_2 = a_3 = 0$	$g = a_0 + a_1 S_4^1 + a_4 S_4^4$
$a_2 = 0$; $a_3 \neq 0$	$g = \left(a_1 + a_3 x_1 x_2)(x_3 + x_4 + \frac{a_4}{a_3} x_3 x_4)\right)$ $+ \left((a_1 + a_3 x_3 x_4)(x_1 + x_2 - \frac{a_1 a_4}{a_3^2})\right) + c$
$a_2 \neq 0$; $a_2 a_4 = a_3^2$	$a_2 g = (a_1 + a_2(x_1 + x_2) + a_3 x_1 x_2)(a_1 + a_2(x_3 + x_4) + a_3 x_3 x_4)$ $+ \left(a_2^2 - a_1 a_3)(x_1 x_2 + x_3 x_4)\right) + c$
$a_2 \neq 0$; $a_2 a_4 \neq a_3^2$	$a_2 g = (a_1 + a_2(x_1 + x_2) + a_3 x_1 x_2)(a_1 + a_2(x_3 + x_4) + a_3 x_3 x_4))$ $+ \left(x_1 x_2 + \frac{a_2^2 - a_1 a_3}{a_2 a_4 - a_3^2})((a_2 a_4 - a_3^2)x_3 x_4 + a_2^2 - a_1 a_3) + c\right)$

In the above, c is an appropriate field constant, and can be added to any ROP. Notice that the first expression is a sum of two ROPs since it is the sum of a linear polynomial and a single monomial. All the other expressions have two summands, each of which is a product of variable-disjoint bivariate polynomials (ignoring constant terms). Since every bivariate polynomial is a ROP, these representations are also sums of 2 ROPs. □

Instead, we define a polynomial that gives carefully chosen weights to the monomials of S_4^2. Let $f^{\alpha,\beta,\gamma}$ denote the following polynomial:

$$f^{\alpha,\beta,\gamma} = \alpha \cdot (x_1 x_2 + x_3 x_4) + \beta \cdot (x_1 x_3 + x_2 x_4) + \gamma \cdot (x_1 x_4 + x_2 x_3).$$

To keep notation simple, we will omit the superscript when it is clear from the context. In the theorem below, we obtain necessary and sufficient conditions on α, β, γ under which f can be expressed as a sum of two ROPs.

Theorem 7 (Hardness of representation for sum of 2 ROPs). *Let f be the polynomial $f^{\alpha,\beta,\gamma} = \alpha \cdot (x_1 x_2 + x_3 x_4) + \beta \cdot (x_1 x_3 + x_2 x_4) + \gamma \cdot (x_1 x_4 + x_2 x_3)$. The following are equivalent:*

1. *f is not expressible as the sum of two ROPs.*
2. *α, β, γ satisfy all the three conditions C1, C2, C3 listed below.*
   C1: *$\alpha\beta\gamma \neq 0$.*
   C2: *$(\alpha^2 - \beta^2)(\beta^2 - \gamma^2)(\gamma^2 - \alpha^2) \neq 0$.*
   C3: *None of the equations $X^2 - D_i = 0$, $i \in [3]$, has a root in $\mathbb{F}$, where*

$$D_1 = (+\alpha^2 - \beta^2 - \gamma^2)^2 - (2\beta\gamma)^2$$
$$D_2 = (-\alpha^2 + \beta^2 - \gamma^2)^2 - (2\alpha\gamma)^2$$
$$D_3 = (-\alpha^2 - \beta^2 + \gamma^2)^2 - (2\alpha\beta)^2$$

Remark 1. 1. It follows, for instance, that $2(x_1 x_2 + x_3 x_4) + 4(x_1 x_3 + x_2 x_4) + 5(x_1 x_4 + x_2 x_3)$ cannot be written as a sum of 2 ROPs over reals, yielding Theorem 2.

2. If $\mathbb{F}$ is an algebraically closed field, then for every α, β, γ, condition C3 fails, and so every $f^{\alpha,\beta,\gamma}$ can be written as a sum of 2 ROPs. However we do not know if there are other examples, or whether all multilinear 4-variate polynomials are expressible as the sum of two ROPs.
3. Even if $\mathbb{F}$ is not algebraically closed, condition C3 fails if for each $a \in \mathbb{F}$, the equation $X^2 = a$ has a root.

To prove Theorem 7, we first consider the easier direction, $1 \Rightarrow 2$, and prove the contrapositive.

Lemma 5. *If α, β, γ do not satisfy all of C1,C2,C3, then the polynomial f can be written as a sum of 2 ROPs.*

Proof. **C1 false:** If any of α, β, γ is zero, then by definition f is the the sum of at most two ROPs.
C2 false: Without loss of generality, assume $\alpha^2 = \beta^2$, so $\alpha = \pm\beta$. Then f is computed by $f = \alpha \cdot (x_1 \pm x_4)(x_2 \pm x_3) + \gamma \cdot (x_1 x_4 + x_2 x_3)$.
C1 true; C3 false: Without loss of generality, the equation $X^2 - D_1 = 0$ has a root τ. We try to express f as

$$\alpha(x_1 - Ax_3)(x_2 - Bx_4) + \beta(x_1 - Cx_2)(x_3 - Dx_4).$$

The coefficients for $x_3 x_4$ and $x_2 x_4$ force $AB = 1$, $CD = 1$, giving the form

$$\alpha(x_1 - Ax_3)(x_2 - \frac{1}{A}x_4) + \beta(x_1 - Cx_2)(x_3 - \frac{1}{C}x_4).$$

Comparing the coefficients for $x_1 x_4$ and $x_2 x_3$, we obtain the constraints

$$-\frac{\alpha}{A} - \frac{\beta}{C} = \gamma; \qquad -\alpha A - \beta C = \gamma$$

Expressing A as $\frac{-\gamma-\beta C}{\alpha}$, we get a quadratic constraint on C; it must be a root of the equation

$$Z^2 + \frac{-\alpha^2 + \beta^2 + \gamma^2}{\beta\gamma} Z + 1 = 0.$$

Using the fact that $\tau^2 = D_1 = (-\alpha^2 + \beta^2 + \gamma^2)^2 - (2\beta\gamma)^2$, we see that indeed this equation does have roots. The left-hand size splits into linear factors, giving

$$(Z - \delta)(Z - \frac{1}{\delta}) = 0 \quad \text{where} \quad \delta = \frac{\alpha^2 - \beta^2 - \gamma^2 + \tau}{2\beta\gamma}.$$

It is easy to verify that $\delta \neq 0$ and $\delta \neq -\frac{\gamma}{\beta}$ (since $\alpha \neq 0$). Further, define $\mu = \frac{-(\gamma+\beta\delta)}{\alpha}$. Then μ is well-defined (because $\alpha \neq 0$) and is also non-zero. Now setting $C = \delta$ and $A = \mu$, we have satisfied all the constraints and so we can write f as the sum of 2 ROPs as follows:

$$f = \alpha(x_1 - \mu x_3)(x_2 - \frac{1}{\mu}x_4) + \beta(x_1 - \delta x_2)(x_3 - \frac{1}{\delta}x_4).$$

$\square$

Now we consider the harder direction: $2 \Rightarrow 1$. Again, we consider the contrapositive. We first show (Lemma 6) a structural property satisfied by every polynomial in $\sum^2 \cdot \text{ROP}$: it must satisfy at least one of the three properties $C1', C2', C3'$ described in the lemma. We then show (Lemma 7) that under the conditions $C1, C2, C3$ from the theorem statement, f does not satisfy any of $C1', C2', C3'$; it follows that f is not expressible as the sum of 2 ROPs.

Lemma 6. *Let g be a 4-variate multilinear polynomial over the field $\mathbb{F}$ which can be expressed as a sum of 2 ROPs. Then at least one of the following conditions is true:*

C1': *There exist $i, j \in [4]$ and $A, B \in \mathbb{F}$ such that $g \mid_{x_i = A, x_j = B}$ is linear.*
C2': *There exist $i, j \in [4]$ such that $x_i, x_j, \partial_{x_i}(g), \partial_{x_j}(g)$ are affinely dependent.*
C3': *$g = l_1 \cdot l_2 + l_3 \cdot l_4$ where l_is are variable disjoint linear forms.*

Proof. Let φ be a sum of 2 ROFs computing g. Let v_1 and v_2 be the children of the topmost $+$ gate. The proof is in two steps. First, we reduce to the case when $|\text{Var}(v_1)| = |\text{Var}(v_2)| = 4$. Then we use a case analysis to show that at least one of the aforementioned conditions hold true. In both steps, we will repeatedly use Proposition 2, which showed that any 3-variate ROP can be reduced to a linear polynomial by substituting a single variable with a field constant. We now proceed with the proof.

Suppose $|\text{Var}(v_1)| \leq 3$. Applying Proposition 2 first to v_1 and then to the resulting restriction of v_2, one can see that there exist $i, j \in [4]$ and $A, B \in \mathbb{F}$ such that $g \mid_{x_i = A, x_j = B}$ is a linear polynomial. So condition $C1'$ is satisfied.

Now assume that $|\text{Var}(v_1)| = |\text{Var}(v_2)| = 4$. Depending on the type of gates of v_1 and v_2, we consider 3 cases.

Case 1: Both v_1 and v_2 are $\times$ gates. Then g can be represented as $M_1 \cdot M_2 + M_3 \cdot M_4$ where (M_1, M_2) and (M_3, M_4) are variable-disjoint ROPs.

Suppose that for some i, $|\text{Var}(M_i)| = 1$. Then, $g \mid_{M_i \to 0}$ is a 3-variate restriction of f and is clearly an ROP. Applying Proposition 2 to this restriction, we see that condition $C1'$ holds.

Otherwise each M_i has $|\text{Var}(M_i)| = 2$.

Suppose (M_1, M_2) and (M_3, M_4) define distinct partitions of the variable set. Assume wlog that $g = M_1(x_1, x_2) \cdot M_2(x_3, x_4) + M_3(x_1, x_3) \cdot M_4(x_2, x_4)$. If all M_is are linear forms, it is clear that condition $C3'$ holds. If not, assume that M_1 is of the form $l_1(x_1) \cdot m_1(x_2) + c_1$ where l_1, m_1 are linear forms and $c_1 \in \mathbb{F}$. Now $g \mid_{l_1 \to 0} = c_1 \cdot M_2(x_3, x_4) + M_3'(x_3) \cdot M_4(x_2, x_4)$. Either set x_3 to make M_3' zero, or, if that is not possible because M_3' is a non-zero field constant, then set $x_4 \to B$ where $B \in \mathbb{F}$. In both cases, by setting at most 2 variables, we obtain a linear polynomial, so $C1'$ holds.

Otherwise, (M_1, M_2) and (M_3, M_4) define the same partition of the variable set. Assume wlog that $g = M_1(x_1, x_2) \cdot M_2(x_3, x_4) + M_3(x_1, x_2) \cdot M_4(x_3, x_4)$. If one of the M_is is linear, say wlog that M_1 is a linear form, then $g \mid_{M_4 \to 0}$ is a 2-variate restriction which is also a linear form, so $C1'$ holds. Otherwise, none of the M_is is a linear form. Then each M_i can be represented as $l_i \cdot m_i + c_i$ where

l_i, m_i are univariate linear forms and $c_i \in \mathbb{F}$. We consider a 2-variate restriction which sets l_1 and m_4 to 0. (Note that $\mathrm{Var}(l_1) \cap \mathrm{Var}(m_4) = \emptyset$.) Then the resulting polynomial is a linear form, so $C1'$ holds.

Case 2: Both v_1 and v_2 are $+$ gates. Then g can be written as $f = M_1 + M_2 + M_3 + M_4$ where (M_1, M_2) and (M_3, M_4) are variable-disjoint ROPs.

Suppose (M_1, M_2) and (M_3, M_4) define distinct partitions of the variable set.

Suppose further that there exists M_i such that $|\mathrm{Var}(M_i)| = 1$. Wlog, $\mathrm{Var}(M_1) = \{x_1\}$, $\{x_1, x_2\} \subseteq \mathrm{Var}(M_3)$, and $x_3 \in \mathrm{Var}(M_4)$. Any setting to x_2 and x_4 results in a linear polynomial, so $C1'$ holds.

So assume wlog that $g = M_1(x_1, x_2) + M_2(x_3, x_4) + M_3(x_1, x_3) + M_4(x_2, x_4)$. Then for $A, B \in \mathbb{F}$, $g\mid_{x_1 = A, x_4 = B}$ is a linear polynomial, so $C1'$ holds.

Otherwise, (M_1, M_2) and (M_3, M_4) define the same partition of the variable set. Again, if say $|\mathrm{Var}(M_1)| = 1$, then setting two variables from M_2 shows that $C1'$ holds. So assume wlog that $g = M_1(x_1, x_2) + M_2(x_3, x_4) + M_3(x_1, x_2) + M_4(x_3, x_4)$. Then for $A, B \in \mathbb{F}$, $g\mid_{x_1 = A, x_3 = B}$ is a linear polynomial, so again $C1'$ holds.

Case 3: One of v_1, v_2 is a $+$ gate and the other is a $\times$ gate. Then g can be written as $g = M_1 + M_2 + M_3 \cdot M_4$ where (M_1, M_2) and (M_3, M_4) are variable-disjoint ROPs. Suppose that $|\mathrm{Var}(M_3)| = 1$. Then $g\mid_{M_3 \to 0}$ is a 3-variate restriction which is a ROP. Using Proposition 2, we get a 2-variate restriction of g which is also linear, so $C1'$ holds. The same argument works when $|\mathrm{Var}(M_4)| = 1$. So assume that M_3 and M_4 are bivariate polynomials.

Suppose that (M_1, M_2) and (M_3, M_4) define distinct partitions of the variable set. Assume wlog that $g = M_1 + M_2 + M_3(x_1, x_2) \cdot M_4(x_3, x_4)$, and x_3, x_4 are separated by M_1, M_2. Then $g\mid_{M_3 \to 0}$ is a 2-variate restriction which is also linear, so $C1'$ holds.

Otherwise (M_1, M_2) and (M_3, M_4) define the same partition of the variable set. Assume wlog that $g = M_1(x_1, x_2) + M_2(x_3, x_4) + M_3(x_1, x_2) \cdot M_4(x_3, x_4)$. If M_1 (or M_2) is a linear form, then consider a 2-variate restriction of g which sets M_4 (or M_3) to 0. The resulting polynomial is a linear form. Similarly if M_3 (or M_4) is of the form $l \cdot m + c$ where l, m are univariate linear forms, then we consider a 2-variate restriction which sets l to 0 and some $x_i \in \mathrm{Var}(M_4)$ to a field constant. The resulting polynomial again is a linear form. In all these cases, $C1'$ holds.

The only case that remains is that M_3 and M_4 are linear forms while M_1 and M_2 are not. Assume that $M_1 = (A_1 x_1 + B_1)(A_2 x_2 + B_2) + C$ and $M_3 = A_3 x_1 + B_3 x_2 + C_3$. Then $\partial_{x_1}(g) = A_1(A_2 x_2 + B_2) + A_3 M_4$ and $\partial_{x_2}(g) = (A_1 x_1 + B_1)A_2 + B_3 M_4$. It follows that $B_3 \cdot \partial_{x_1}(g) - A_3 \cdot \partial_{x_2}(g) + A_1 A_2 A_3 x_1 - A_1 A_2 B_3 x_2 \in \mathbb{F}$, and hence the polynomials $x_1, x_2, \partial_{x_1}(g), \partial_{x_2}(g)$ are affinely dependent. Therefore, condition $C2'$ of the lemma is satisfied. □

Lemma 7. *If α, β, γ satisfy conditions $C1, C2, C3$ from the statement of Theorem 7, then the polynomial $f^{\alpha, \beta, \gamma}$ does not satisfy any of the properties $C1', C2', C3'$ from Lemma 6.*

Proof. **C1** $\Rightarrow \neg$**C1'**: Since $\alpha\beta\gamma \neq 0$, f contains all possible degree 2 monomials. Hence after setting $x_i = A$ and $x_j = B$, the monomial $x_k x_l$ where $k, l \in [4]\backslash\{i, j\}$ still survives.

C2 $\Rightarrow \neg$**C2'**: The proof is by contradiction. Assume to the contrary that for some i, j, wlog say for $i = 1$ and $j = 2$, the polynomials $x_1, x_2, \partial_{x_1}(f), \partial_{x_2}(f)$ are affinely dependent. Note that $\partial_{x_1}(f) = \alpha x_2 + \beta x_3 + \gamma x_4$ and $\partial_{x_2}(f) = \alpha x_1 + \gamma x_3 + \beta x_4$. This implies that the vectors $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, \alpha, \beta, \gamma)$ and $(\alpha, 0, \gamma, \beta)$ are linearly dependent. This further implies that the vectors (β, γ) and (γ, β) are linearly dependent. Therefore, $\beta = \pm\gamma$, contradicting C2.

C1 $\wedge$ **C2** $\wedge$ **C3** $\Rightarrow \neg$**C3'**: Suppose, to the contrary, that $C3'$ holds. That is, f can be written as $f = l_1 \cdot l_2 + l_3 \cdot l_4$ where (l_1, l_2) and (l_3, l_4) are variable-disjoint linear forms. By the preceding arguments, we know that f does not satisfy $C1'$ or $C2'$.

First consider the case when (l_1, l_2) and (l_3, l_4) define the same partition of the variable set. Assume wlog that $\mathrm{Var}(l_1) = \mathrm{Var}(l_3)$, $\mathrm{Var}(l_2) = \mathrm{Var}(l_4)$, and $|\mathrm{Var}(l_1)| \leq 2$. Setting the variables in l_1 to any field constants yields a linear form, so f satisfies C1', a contradiction.

Hence it must be the case that (l_1, l_2) and (l_3, l_4) define different partitions of the variable set. Since all degree-2 monomials are present in f, each pair x_i, x_j must be separated by at least one of the two partitions. This implies that both partitions have exactly 2 variables in each part. Assume without loss of generality that $f = l_1(x_1, x_2) \cdot l_2(x_3, x_4) + l_3(x_1, x_3) \cdot l_4(x_2, x_4)$.

At this point, we use properties of the commutator of f; recall Definition 3. By Lemma 2, we know that l_2 divides $\triangle_{12} f$. We compute $\triangle_{12} f$ explicitly for our candidate polynomial:

$$\begin{aligned}
\triangle_{12} f &= (\alpha x_3 x_4)(\alpha + (\beta + \gamma)(x_3 + x_4) + \alpha x_3 x_4) \\
&\quad - (\beta x_4 + \gamma x_3 + \alpha x_3 x_4)(\beta x_3 + \gamma x_4 + \alpha x_3 x_4) \\
&= -\beta\gamma(x_3^2 + x_4^2) + (\alpha^2 - \beta^2 - \gamma^2)x_3 x_4
\end{aligned}$$

Since l_2 divides $\triangle_{12} f$, $\triangle_{12} f$ is not irreducible but is the product of two linear factors. Since $\triangle_{12} f(0, 0) = 0$, at least one of the linear factors of $\triangle_{12} f$ must vanish at $(0, 0)$. Let $x_3 - \delta x_4$ be such a factor. Then $\triangle_{12}(f)$ vanishes not only at $(0, 0)$, but whenever $x_3 = \delta x_4$. Substituting $x_3 = \delta x_4$ in $\triangle_{12} f$, we get

$$-\delta^2 \beta\gamma - \beta\gamma + \delta(\alpha^2 - \beta^2 - \gamma^2) = 0$$

Hence δ is of the form

$$\delta = \frac{-(\alpha^2 - \beta^2 - \gamma^2) \pm \sqrt{(\alpha^2 - \beta^2 - \gamma^2)^2 - 4\beta^2\gamma^2}}{-2\beta\gamma}$$

Hence $2\beta\gamma\delta - (\alpha^2 - \beta^2 - \gamma^2)$ is a root of the equation $X^2 - D_1 = 0$, contradicting the assumption that C3 holds.

Hence it must be the case that $C3'$ does not hold. $\qquad\square$

With this, the proof of Theorem 7 is complete.

The conditions imposed on α, β, γ in Theorem 7 are tight and irredundant. Below we give some explicit examples over the field of reals.

1. $f = 2(x_1 x_2 + x_3 x_4) + 2(x_1 x_3 + x_2 x_4) + 3(x_1 x_4 + x_2 x_3)$ satisfies conditions C1 and C3 from the Theorem but not C2; $\alpha = \beta$. A $\sum^2 \cdot$ROP representation for f is $f = 2(x_1 + x_4)(x_2 + x_3) + 3(x_1 x_4 + x_2 x_3)$.
2. $f = 2(x_1 x_2 + x_3 x_4) - 2(x_1 x_3 + x_2 x_4) + 3(x_1 x_4 + x_2 x_3)$ satisfies conditions C1 and C3 but not C2; $\alpha = -\beta$. A $\sum^2 \cdot$ROP representation for f is $f = 2(x_1 - x_4)(x_2 - x_3) + 3(x_1 x_4 + x_2 x_3)$.
3. $f = (x_1 x_2 + x_3 x_4) + 2(x_1 x_3 + x_2 x_4) + 3(x_1 x_4 + x_2 x_3)$ satisfies conditions C1 and C2 but not C3. A $\sum^2 \cdot$ROP representation for f is $f = (x_1 + x_3)(x_2 + x_4) + 2(x_1 + x_2)(x_3 + x_4)$.

5   Conclusions

1. We have seen in Proposition 1 that every n-variate multilinear polynomial $(n \geq 4)$ can be written as the sum of $3 \times 2^{n-4}$ ROPs. We have also shown in Lemma 3 that there are n-variate multilinear polynomials that require $\lceil n/2 \rceil$ ROPs in a sum-of-ROPs representation. Between $\lceil n/2 \rceil$ and $3 \times 2^{n-4}$, what is the true tight bound?
2. We have shown in Theorem 1 that for each k, $\sum^k \cdot$ROP can be separated from $\sum^{k-1} \cdot$ROP by a polynomial on $2k-1$ variables. Can we separate these classes with fewer variables? Note that any separating polynomial must have $\Omega(\log k)$ variables.
3. In particular, can 4-variate multilinear polynomials separate sums of 3 ROPs from sums of 2 ROPs over every field? If not, what is an explicit example?

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