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Bounds on the Kolmogorov complexity function for infinite words

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Abstract

The Kolmogorov complexity function of an infinite word ξ maps a natural number to the complexity $K(\xi\restriction n)$ of the n-length prefix of ξ. We investigate the maximally achievable complexity function if ξ is taken from a constructively describable set of infinite words. Here we are interested in linear upper bounds where the slope is the Hausdorff dimension of the set.

As sets we consider Π_1-definable sets obtained by dilution and sets obtained from constructively describable infinite iterated function systems. In these cases, for a priori and monotone complexity, the upper bound coincides (up to an additive constant) with the lower bound, thus verifying the existence of oscillation-free maximally complex infinite words.

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1 Introduction

The *Kolmogorov complexity* of a finite word w, $K(w)$, is, roughly speaking, the length of a shortest input (program) π for which a universal algorithm prints w.[1] For infinite words (ω-words) its asymptotic Kolmogorov complexity might be thought of as

$$\lim_{n\to\infty} \frac{K(\xi\upharpoonright n)}{n} \quad , \tag{1}$$

where $\xi\upharpoonright n$ denotes the prefix of length n of ξ.

Since this limit need not exist, the quantities

$$\underline{\kappa}(\xi) := \liminf_{n\to\infty} \frac{K(\xi\upharpoonright n)}{n} \text{ and } \kappa(\xi) := \limsup_{n\to\infty} \frac{K(\xi\upharpoonright n)}{n} \ .$$

[1] We require that w and π be words over the same finite (not necessarily binary) alphabet X of cardinality ≥ 2.

were considered (see [Rya86, Sta93, CH94, Sta98]).
These limits are also known as constructive dimension or constructive strong dimension, respectively, introduced in [Lut03] and [AHLM07], respectively. For more details about this see [Sta05a].

In these papers mainly the following approach was pursued: Given a set of infinite words (a so-called ω-language) F, bound the maximum possible Kolmogorov complexity $\underline{\kappa}(\xi)$ or $\kappa(\xi)$ for $\xi \in F$. In the present paper we are not only interested in this asymptotic case but also in bounds on the Kolmogorov complexity function $K(\xi \restriction n)$ of maximally complex infinite words (ω-words) $\xi \in F$.

In this asymptotic case, Ryabko [Rya86] showed that for arbitrary ω-languages F the Hausdorff dimension, $\dim F$, is a lower bound to $\underline{\kappa}(F) := \sup\{\underline{\kappa}(\xi) : \xi \in F\}$, but Example 3.18 of [Sta93] shows that already for simple computable ω-languages the Hausdorff dimension is not an upper bound to $\kappa(F) := \sup\{\kappa(\xi) : \xi \in F\}$ in general. In [Sta93, Sta98], we showed that for restricted classes of computably definable ω-languages F its Hausdorff dimension is also an upper bound to $\underline{\kappa}(F)$, thus giving a partial completion to Ryabko's lower bound.

The present paper focuses on a more detailed consideration of the Kolmogorov complexity function $K(\xi \restriction n)$ of infinite words in ω-languages F. Thus, in contrast to the asymptotic case where it is not relevant which kind of complexity is used, in the case of the Kolmogorov complexity function it matters which one of the complexities we consider.

Lower and upper bounds on the Kolmogorov complexity function are closely related to partial randomness. Partial randomness was investigated in the papers [Tad02] and [CST06]. It is a linear generalisation of the concept of random sequences (see the textbooks [Cal02, DH10]). The concept of partial randomness tries to specify sequences as random to some degree ε, $0 < \varepsilon \leq 1$, where the case $\varepsilon = 1$ coincides with usual randomness. In [Tad02] and [CST06] several different generalisations of the concepts for partially random sequences were given.

A simple idea what could be an example of a binary $\frac{1}{2}$-random infinite word is the following. Take $\xi = x_1 x_2 \cdots x_i \cdots$ to be a (1-)random infinite word and dilute it by inserting zeros at every other position to obtain $\xi' = x_1 0 x_2 0 \cdots x_i 0 \cdots$ (cf. [Sta93]).

As one observes easily the description complexity of the n-length prefix of a diluted word ξ' is about the complexity of the $\varepsilon \cdot n$-length prefix of the original word ξ where ε is the dilution coefficient (e.g. $\varepsilon = \frac{1}{2}$ in the above example). This was a motivation to consider the asymptotic complexity of an infinite word as the the limit of the quotient of the complexity of the n-length prefix and the length n (see [Rya86, Sta93]).

The aim of our paper is to survey several results concerning the Kolmogorov complexity function $K(\xi\restriction n)$ of infinite words contained in computably describable ω-languages. Here we investigate under which conditions certain simple construction principles yield ω-languages having maximally complex elements ξ with a linear complexity slope, that is, $K(\xi\restriction n) = \gamma\cdot n + O(1)$. As complexities we consider besides the usual plain complexity also variants like *a priori* complexity and monotone complexity. The construction principles presented here are dilution, e.g. as described above and infinite concatenation. Infinite concatenation is closely related to self-similarity. Instead of producing infinite words as products of finite ones (taken from a fixed language) one may regard this also as a process of shrinking the set of all infinite words successively by application of metric similitudes related to the words in the fixed language. This brings into play as a second dimension the similarity dimension known from Fractal Geometry. It turns out that under certain conditions – like in Fractal Geometry – similarity dimension and Hausdorff dimension coincide.

The paper is organised as follows. After introducing some notation, Cantor space and Hausdorff dimension the subsequent Section 3 introduces the plain, *a priori* and monotone complexity of finite words. The last part of this section presents the concept of of lower bounding the slope of the Kolmogorov complexity functions by Hausdorff dimension. The fourth section deals with dilution. As a preparation we investigate expansive prefix-monotone functions (see [CH94]) and connect them to Hölder conditions known from Fractal geometry (see [Fal90]). The last section is devoted to self-similar sets of infinite words and maximally achievable complexity functions of their elements. Here we present mainly results from [MS09] which show tight linear upper bounds on these complexity functions.

2 Notation and Preliminary Results

Next we introduce the notation used throughout the paper. By $\mathbb{N} = \{0,1,2,\ldots\}$ we denote the set of natural numbers, $\mathbb{Q}$ is the set of rational numbers and $\mathbb{R}_+$ is the set of non-negative reals. Let X, Y be alphabets of cardinality $|X|,|Y| \geq 2$. Usually we will denote the cardinality of X by $|X| = r$. X^* is the set (monoid) of words on X, including the *empty word* e, and X^ω is the set of infinite sequences (ω-words) over X. $|w|$ is the *length*[2] of the word $w \in X^*$ and $\mathbf{pref}(B)$ is the set of all finite prefixes of strings in $B \subseteq X^* \cup X^\omega$. We shall abbreviate $w \in \mathbf{pref}(\eta)$ ($\eta \in X^* \cup X^\omega$) by $w \sqsubseteq \eta$.

For $w \in X^*$ and $\eta \in X^* \cup X^\omega$ let $w \cdot \eta$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$. Thus X^n is the set of words of length n over X, and we use $X^{\leq c}$ as an abbreviation for $\{w : w \in X^* \wedge |w| \leq c\}$. For a language W let $W^* := \bigcup_{i \in \mathbb{N}} W^i$ be the *submonoid* of X^* generated by W, and $W^\omega := \{w_1 \cdots w_i \cdots : w_i \in W \setminus \{e\}\}$ is the set of infinite strings formed by concatenating words in W.

A language $V \subseteq X^*$ is called *prefix-free* provided for arbitrary $w, v \in V$ the relation $w \sqsubseteq v$ implies $w = v$.

We consider the set X^ω as a metric space (Cantor space) (X^ω, ρ) of all ω-words over the alphabet X, $|X| = r$, where the metric ρ is defined as follows.

$$\rho(\xi,\eta) := \inf\{r^{-|w|} : w \sqsubset \xi \wedge w \sqsubset \eta\}$$

The open balls in this space are sets of the form $w \cdot X^\omega$, their diameter is $\mathrm{diam}(w \cdot X^\omega) = r^{-|w|}$, and $\mathcal{C}(F) := \{\xi : \mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)\}$ is the *closure* of the set F (smallest closed subset containing F) in (X^ω, ρ).

Another way to describe ω-languages (sets of infinite words) by languages $W \subseteq X^*$ is the δ-limit $W^\delta := \{\xi : \xi \in X^\omega \wedge |\mathbf{pref}(\xi) \cap W| = \infty\}$ (see [Sta87, Sta07]).

The mapping $\Phi_w(\xi) := w \cdot \xi$ is a contracting similitude if only $w \neq e$. Thus a language $W \subseteq X^* \setminus \{e\}$ defines a possibly infinite IFS (IIFS) in (X^ω, ρ). Its (maximal) *fixed point* is the ω-power W^ω of the language W. It was observed in [Sta97] that, in general, the IIFS $(\Phi_w)_{w \in W}$ has a great variety of fixed points, that is, solutions of the equation

[2] If there is no danger of confusion, for a set M we use the same notation $|M|$ to denote its *cardinality*.

$\bigcup_{w\in W}\Phi_w(F)=F$. All of these fixed points are contained in W^ω, and, except for the empty set $\emptyset$, their closure equals $\mathcal{C}(W^\omega)$, which is the *attractor* of $(\Phi_w)_{w\in W}$.

Next we recall the definition of the Hausdorff measure and Hausdorff dimension of a subset of (X^ω,ρ) (see [Edg08, Fal90]). In the setting of languages and ω-languages this can be read as follows (see [Sta93]). For $F\subseteq X^\omega$ and $0\leq\alpha\leq 1$ the equation

$$\mathbb{L}_\alpha(F):=\lim_{l\to\infty}\inf\Big\{\sum_{w\in W}r^{-\alpha\cdot|w|}:F\subseteq W\cdot X^\omega\wedge\forall w(w\in W\to|w|\geq l)\Big\} \tag{2}$$

defines the α-dimensional metric outer measure on X^ω. The measure $\mathbb{L}_\alpha$ satisfies the following.

Corollary 1 *If* $\mathbb{L}_\alpha(F)<\infty$ *then* $\mathbb{L}_{\alpha+\varepsilon}(F)=0$ *for all* $\varepsilon>0$.

Then the *Hausdorff dimension* of F is defined as

$$\dim F:=\sup\{\alpha:\alpha=0\vee\mathbb{L}_\alpha(F)=\infty\}=\inf\{\alpha:\mathbb{L}_\alpha(F)=0\}\,.$$

It should be mentioned that $\dim$ is countably stable and shift invariant, that is,

$$\dim\bigcup_{i\in\mathbb{N}}F_i=\sup\{\dim F_i:i\in\mathbb{N}\}\quad\text{and}\quad\dim w\cdot F=\dim F\,. \tag{3}$$

3 Description Complexity of Finite Words

In this section we briefly recall the concept of description complexity of finite words. For a more comprehensive introduction see the textbooks [Cal02, DH10, Nie09] and the paper [US96]. We start with plain and prefix-free complexity.

3.1 Plain Complexity

Recall that the plain complexity (Kolmogorov) of a string $w\in X^*$ w.r.t. a partial computable function $\varphi:X^*\to X^*$ is $K_\varphi(w)=\inf\{|\pi|:\varphi(\pi)=w\}$. It is well-known that there is a universal partial computable function $U:X^*\to X^*$ such that

$$K_U(w)\leq K_\varphi(w)+c_\varphi$$

holds for all strings $w \in X^*$. Here the constant c_φ depends only on U and φ but not on the particular string $w \in X^*$. We will denote the complexity K_U simply by K.

Plain complexity satisfies the following property.

Proposition 1 *If $\varphi : X^* \to X^*$ is a partial computable function then there is a constant c such that*

$$\mathrm{K}(\varphi(w)) \leq \mathrm{K}(w) + c \textit{ for all } w \in X^* . \tag{4}$$

We conclude this section by a generalisation of Theorem 2.9 of [Sta93].

Theorem 1 *Let $W \subseteq X^*$ or $X^* \setminus W$ be computably enumerable, ε, $0 < \varepsilon < 1$, be a computable real number and let $\sum_{i=0}^{m} |W \cap X^i| \leq c \cdot r^{\varepsilon \cdot m}$ for some constant $c > 0$ and all $m \in \mathbb{N}$. Then*

$$\exists c' \big(c' > 0 \wedge \forall w (w \in W \to \mathrm{K}(w) \leq \varepsilon \cdot |w| + c') \big)$$

Proof. Let $c \leq r^{\varepsilon \cdot m_0}$. Then $\sum_{i=0}^{m} |W \cap X^i| \leq |X^l|$ for $l \geq \varepsilon \cdot (m + m_0)$.

If W is computably enumerable define a partial computable function $\varphi : X^* \to X^*$ as follows.

$$\varphi(\pi) := \text{ the } \pi\text{th word of length} \leq \lceil \tfrac{|\pi| - m_0}{\varepsilon} \rceil \text{ in the enumeration of } W . \tag{5}$$

Here we interpret a word $\pi \in X^n$ as a number between 0 and $r^n - 1$.

If $w \in X^*$, $|w| = m$, choose the smallest $l \in \mathbb{N}$, $l_{\min}$ say, such that $|X^l| \geq |W \cap X^{\leq m}|$. This l_{min} satisfies $\lfloor \frac{l_{\min}}{\varepsilon} \rfloor - m_0 = m$ and thus $l_{\min} \leq \varepsilon \cdot (m + m_0) + 1$. By the above remark there is a $\pi \in X^{l_{\min}}$ such that $\varphi(\pi) = w$. Consequently, $K_\varphi(w) \leq l_{\min} \leq \varepsilon \cdot |w| + \varepsilon \cdot m_0 + 1$.

If $X^* \setminus W$ is computably enumerable define a partial computable function $\psi : X^* \to X^*$ as follows.

Set $m := \lfloor \frac{|\pi|}{\varepsilon} \rfloor - m_0$ and enumerate $X^* \setminus W$ until $\sum_{i=0}^{m} r^i - r^{|\pi|}$ elements of length $\leq m$ are enumerated. Then take from the rest the πth word of length $\leq m$ as $\psi(\pi)$.

If $w \in W$, $|w| = m$, again choose the smallest $l_{\min} \in \mathbb{N}$ such that $|X^{l_{\min}}| \geq \sum_{i=0}^{m} |W \cap X^i|$. Observe that in view of $|W \cap X^{\leq m}| \leq |X^{l_{\min}}|$ this rest contains $W \cap X^{\leq m}$. As in the above case when W was assumed to be computably enumerable we obtain that for $w \in W$ there is a π such that $\psi(\pi) = w$ and $|\pi| = l_{\min}$. Then the proof proceeds as above. ❑

3.2 Monotone and *a priori* Complexity

In this section we consider *a priori* and monotone complexity. We derive some elementary properties needed in the sequel.

We start with the notion of a continuous (cylindrical) semi-measure on X^*. A *continuous (cylindrical) semi-measure* on X^* is a function $m : X^* \to \mathbb{R}_+$ which satisfies $m(e) \le 1$ and $m(w) \ge \sum_{x\in X} m(wx)$, for $w \in X^*$. If there is no danger of confusion, in the sequel we will refer to continuous (semi)-measures simply as measures.

If $m(w) = \sum_{x\in X} m(wx)$ the function m is called a *measure*. A semi-measure m has the following property.

Proposition 2 *If $C \subseteq w\cdot X^*$ is prefix-free then $m(w) \ge \sum_{v\in C} m(v)$.*

Thus, if $C \subseteq X^*$ is infinite and prefix-free, for every $\varepsilon > 0$, there is a word $v \in C$ such that $m(v) < \varepsilon$.

A function $m : X^* \to \mathbb{R}_+$ is referred to as *left-computable* or *approximable from below*, provided the set $\{(w,q) : w \in X^* \wedge Q \in \mathbb{Q} \wedge 0 \le q < m(w)\}$ is computably enumerable. *Right-computability* is defined analogously, and m is referred to as *computable* if it is right- and left-computable.

In [ZL70] the existence of a universal left-computable semi-measure $\mathbf{M}$ is proved: There is a left-computable semi-measure $\mathbf{M}$ which satisfies

$$\exists c_m > 0 \forall w \in X^* : m(w) \le c_m \cdot \mathbf{M}(w), \tag{6}$$

for all left-computable semi-measures m. $\mathbf{M}$ has the following property.

Proposition 3 *If $\xi \in X^\omega$ is a computable ω-word then there is a constant $c_\xi > 0$ such that $\mathbf{M}(w) \ge c_\xi$, for all $w \in \mathbf{pref}(\xi)$.*

Proof. If $\xi \in X^\omega$ is a computable ω-word then $\mathbf{pref}(\xi)$ is a computable subset of X^*. Construct a semi-measure m_ξ such that

$$m_\xi(w) = \begin{cases} 1, & \text{if } w \in \mathbf{pref}(\xi) \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Then m_ξ is a computable cylindrical measure and the assertion follows from Eq. (6). ❏

3.2.1 *a priori* complexity

The *a priori complexity* of a word $w \in X^*$ is defined as

$$\mathrm{KA}(w) := \left\lceil -\log_{|X|} \mathbf{M}(w) \right\rceil . \tag{7}$$

The properties of the semi-measure $\mathbf{M}$ imply $\mathrm{KA}(w) \leq \mathrm{KA}(w \cdot v)$ and $\sum_{v \in C} |X|^{-\mathrm{KA}(v)} \leq \mathbf{M}(e)$ when $C \subseteq X^*$ is prefix-free.

From Proposition 3 we obtain that KA does not satisfy the property of usual plain or prefix complexity that for arbitrary partial computable functions $\varphi : X^* \to X^*$ it holds $\mathrm{KA}(\varphi(w)) \leq \mathrm{KA}(w) + O(1)$.

Example 1 Let $X = \{0,1\}$ and define $\varphi(w) := w \cdot 1$. We consider the set $0^* = \mathbf{pref}(0^\omega) \subseteq X^*$. Then, in view of Proposition 3 $\mathrm{KA}(w) \leq c$ for all $w \in 0^*$ and some constant c. Now, the set $\varphi(0^*) = 0^* \cdot 1$ is prefix-free and according to Proposition 2 the complexity $\mathrm{KA}(\varphi(v))$, $v \in 0^* \cdot 1$, is unbounded. ❑

The aim of this section is to prove the fact that an property analogous to Proposition 1 holds for a subclass of partial computable functions.

Definition 1 *A partial mapping $\varphi : \subseteq X^* \to Y^*$ is referred to as* prefix-monotone *(or* sequential*) provided $w, v \in \mathrm{dom}(\varphi)$ and $w \sqsubseteq v$ imply $\varphi(w) \sqsubseteq \varphi(v)$.*

Let $\mathbf{U}_\varphi(w) := \mathrm{Min}_\sqsubseteq \{v : v \in \mathrm{dom}(\varphi) \wedge w \sqsubseteq \varphi(v)\}$ be the upper quasi-inverse for φ (see [Sta87]). Here $\mathrm{Min}_\sqsubseteq W$ is the set of all minimal elements w.r.t. the prefix ordering $\sqsubseteq$ in the language $W \subseteq X^*$. Then $\mathbf{U}_\varphi$ has the following properties.

Lemma 1 *Let φ be a prefix-monotone partial function mapping $\mathrm{dom}(\varphi) \subseteq X^*$ to Y^*. Then for $w \in Y^*, y, y' \in Y$ and $y \neq y'$ the following hold.*

1. $\mathbf{U}_\varphi(wy) \cap \mathbf{U}_\varphi(wy') = \emptyset$.
2. *$\bigcup_{y \in Y} \mathbf{U}_\varphi(wy)$ is prefix-free, and if $v' \in \mathbf{U}_\varphi(wy)$ then there is a $v \in \mathbf{U}_\varphi(w)$ such that $v \sqsubseteq v'$.*
3. *If μ is a cylindrical semi-measure on X^* then $\mu(\mathbf{U}_\varphi(w)) \geq \sum_{y \in Y} \mu(\mathbf{U}_\varphi(wy))$.*

Proof.

1. holds, since $wy, wy' \sqsubseteq \varphi(v)$ implies $y = y'$.

2. According the definition of $\mathbf{U}_\varphi$ and to 1. $\bigcup_{y\in Y} \mathbf{U}_\varphi(wy)$ is a union of pairwise disjoint prefix-free sets. Assume $u \in \mathbf{U}_\varphi(wy)$ and $v \in \mathbf{U}_\varphi(wy')$ where $u \sqsubset v$. Then $y \neq y'$ and $wy, wy' \sqsubseteq \varphi(v)$ which is impossible.

3. follows from 2. and Proposition 2. ❑

Lemma 2 *If φ is a prefix-monotone mapping and μ is a semi-measure then $\mu_\varphi : X^* \to \mathbb{R}_+$ defined by the equation*

$$\mu_\varphi(w) := \sum_{v\in\mathbf{U}_\varphi(w)} \mu(v) \tag{8}$$

is also a semi-measure.

Proof. We use from Lemma 1 the fact that the sets $\mathbf{U}_\varphi(w)$ are prefix-free and that for $y, y' \in Y$, $y \neq y'$ no pair words $v \in \mathbf{U}_\varphi(wy)$ and $u \in \mathbf{U}_\varphi(wy')$ satisfies $v \sqsubseteq u$ or $u \sqsubseteq v$. According to Lemma 1.2 $\bigcup_{y\in Y} \mathbf{U}_\varphi(wy)$ is a disjoint union and prefix-free.

Moreover, for every $v' \in \mathbf{U}_\varphi(wy)$ there is a $v \in \mathbf{U}_\varphi(w)$ such that $v \sqsubseteq v'$, and since μ is a semi-measure, $\mu(v) \geq \sum_{v'\in C} \mu(v')$ whenever $C \subseteq v \cdot X^*$ is prefix-free.

Consequently,

$$\mu_\varphi(w) = \sum_{v\in\mathbf{U}_\varphi(w)} \mu(v) \geq \sum_{x\in X} \sum_{v\in\mathbf{U}_\varphi(wx)} = \sum_{x\in X} \mu_\varphi(wx).$$ ❑

This will allow us to prove the following.

Theorem 2 *If $\mu : X^* \to \mathbb{R}_+$ is a left computable semi-measure and $\varphi : X^* \to X^*$ is a partial computable prefix-monotone mapping then $\mu_\varphi : X^* \to \mathbb{R}_+$ defined by Eq. (8) is a left-computable semi-measure.*

Proof. By Lemma 2 μ_φ is a semi-measure. It remains to show that μ_φ is left computable.

To this end we start with a computable monotone approximation $m(w,s)$ of μ satisfying (cf. the proof of Theorem 3.16.2 of [DH10])

1. $\mu(w) \geq m(w, s+1) \geq m(w,s)$ and

2. for all $w \in X^*$, we have $m(w,t) \geq \sum_{x\in X} m(wx, t)$.

It is obvious that every mapping $m(\cdot,s)$ is a computable semi-measure.

Moreover, we consider the partially defined prefix-monotone mapping $\varphi_t(w) := \varphi(w)$ if $w \in \mathrm{dom}^{(t)}(\varphi)$ where $\mathrm{dom}^{(t)}(\varphi)$ is the set consisting of the first t elements in a computable enumeration of $\mathrm{dom}(\varphi)$.[3]

Define $\mu_\varphi^{(t)}(w) := \sum_{v\in\mathbf{U}_{\varphi_t}} m(v,t)$ as in Lemma 2. Then $\mu_\varphi^{(t)}$ is a computable semi-measure.

Since for every $v' \in \mathbf{U}_{\varphi_t}(w)$ there is a $v \sqsubseteq v'$ such that $v \in \mathbf{U}_{\varphi_{t+1}}(w)$ and t for $v \in \mathbf{U}_{\varphi_{t+1}}(w)$ the set $\{v' : v \sqsubseteq v' \wedge v' \in \mathbf{U}_{\varphi_t}(w)\}$ is prefix-free, we obtain $\mu_\varphi^{(t)}(w) \le \mu_\varphi^{(t+1)}(w)$ for $t \in \mathbb{N}$ and $w \in X^*$ from Proposition 2.

Finally, we prove $\lim\limits_{t\to\infty} \mu_\varphi^{(t)}(w) = \mu_\varphi(w) = \sum_{v\in\mathbf{U}_\varphi(w)} \mu(v)$. To this end choose, for $\varepsilon > 0$, a finite subset $\{v_1,\ldots,v_\ell\} \subseteq \mathbf{U}_\varphi(w)$ with $\sum_{i=1}^{\ell} \mu(v_i) \ge \mu_\varphi(w) - \varepsilon$ a $t \in \mathbb{N}$ such that $\{v_1,\ldots,v_\ell\} \subseteq \mathrm{dom}^{(t)}(\varphi)$ and $m(v_i,t) \ge \mu(v_i) - \varepsilon \cdot 2^{-i}$. Then, clearly, $\mu_\varphi^{(t)}(w) \ge \mu_\varphi(w) - 2\varepsilon$. ❑

As a corollary we obtain the required inequality.

Corollary 2 *Let $\varphi : X^* \to X^*$ be a partial computable prefix-monotone function. Then there is a constant c_φ such that $\mathrm{KA}(\varphi(w)) \le \mathrm{KA}(w) + c_\varphi$ for all $w \in X^*$.*

Proof. Let $\mu := \mathbf{M}_\varphi$. Then μ is a left-computable semi-measure. Thus $\mu(w) = \mathbf{M}(\varphi(w)) \le c \cdot \mathbf{M}(w)$ for some constant c and all $w \in X^*$. ❑

3.2.2 Monotone complexity

In this section we introduce the monotone complexity along the lines of [She84] (see [US96]). To this end let $E \subseteq X^* \times X^*$ be a description mode (a computably enumerable set) universal among all description modes which satisfy the condition.

$$(\pi,w),(\pi',v) \in E \wedge \pi \sqsubseteq \pi' \to w \sqsubseteq v \vee v \sqsubseteq w \tag{9}$$

Then $K_E(w) := \inf\{|\pi| : \exists u(w \sqsubseteq u \wedge (\pi,u) \in E)\}$ is the *monotone complexity* of the word w. In the sequel we use the term $\mathrm{Km}(w)$.

Similar to KA the monotone complexity satisfies also an inequality involving partial computable sequential functions.

[3] If $|\mathrm{dom}(\varphi)| < t$ set $\mathrm{dom}^{(t)}(\varphi) := \mathrm{dom}(\varphi)$

Corollary 3 *Let* $\varphi : X^* \to X^*$ *be a partial computable prefix-monotone function. Then there is a constant* c_φ *such that* $\mathrm{Km}(\varphi(w)) \leq \mathrm{Km}(w) + c_\varphi$ *for all* $w \in X^*$.

Proof. Define $E_\varphi := \{(\pi', \varphi(w)) : \exists \pi(\pi \sqsubseteq \pi' \wedge (\pi, w) \in E)\}$. Then E_φ is computably enumerable and satisfies Eq. (9).

Since E is a universal description mode satisfying Eq. (9), we have $K_{E_\varphi}(\varphi(w)) \leq K_E(w) + c_\varphi$. ❑

Like *a priori* complexity monotone complexity has also relations to semi-measures [USS90, US96].

Proposition 4 *Let* $\mu : X^* \to \mathbb{R}_+$ *be a computable continuous semi-measure. Then* $\mathrm{Km}(w) \leq -\log \mu(w) + O(1)$.

Finally we mention some relations between the complexities K, KA and Km (see [DH10, US96]).

$$\mathrm{KA}(w) \leq \mathrm{Km}(w) + O(1), \tag{10}$$

$$|K_1(w) - K_2(w)| \leq O(\log |w|) \text{ for } K_i \in \{\mathrm{K}, \mathrm{KA}, \mathrm{Km}\} \tag{11}$$

3.3 Bounds via Hausdorff Dimension

In this section two bounds on the Kolmogorov complexity function from [Sta93] and [Mie08] are presented. Both are lower bounds which illustrate the principle that large sets contain complex elements. The first bound is for plain complexity K. Moreover, we present an asymptotic upper bound for some computably describable ω-languages from [Sta98].

Lemma 3 ([Sta93, Lemma 3.13]) *Let* $F \subseteq X^\omega$ *and* $\mathbb{L}_\alpha(F) > 0$. *Then for every* $f : \mathbb{N} \to \mathbb{N}$ *satisfying* $\sum_{n \in \mathbb{N}} r^{-f(n)} < \infty$ *there is a* $\xi \in F$ *such that* $\mathrm{K}(\xi \upharpoonright n) \geq_{\mathrm{ae}} \alpha \cdot n - f(n)$.

As a consequence we obtain Ryabko's bound [Rya86].

$$\underline{\kappa}(F) \geq \dim F \tag{12}$$

For the next lemma we mention that $\sum_{v \in C} r^{-\mathrm{KA}(v)} \leq \mathbf{M}(e)$ for every prefix-free language $C \subseteq X^*$.

Lemma 4 ([Mie08, Theorem 4.6]) *Let $F \subseteq X^\omega$ and $\mathbb{L}_\alpha(F) > r^{-c} \cdot \mathbf{M}(e)$. Then there is a $\xi \in F$ such that $\mathrm{KA}(\xi \upharpoonright n) \geq_{\mathrm{ae}} \alpha \cdot n - c$.*

Proof. It is readily seen that the set of infinite words not fulfilling the asserted inequality is the δ-limit of $W_c = \{w : \mathrm{KA}(w) \leq \alpha \cdot n - c\}$.

Let $V_m = \mathrm{Min}_{\sqsubseteq}(W_c \cap X^m \cdot X^*)$. Then V_m is prefix-free and $W_c^\delta \subseteq V_m X^\omega$ for all $m \in \mathbb{N}$. Consequently, $\mathbb{L}_\alpha(W_c^\delta) \leq \sum_{v \in V_m} r^{-\alpha \cdot |v|} \leq \sum_{v \in V_m} r^{-\mathrm{KA}(v) - c} \leq r^{-c} \cdot \mathbf{M}(e)$. Then the inequality $\mathbb{L}_\alpha(F) > \mathbb{L}_\alpha(W_c^\delta)$ shows the assertion $F \not\subseteq W_c^\delta$. ❏

Proposition 5 *If $F_i \subseteq X^\omega$ and $X^* \setminus \mathbf{pref}(F_i)$ is computably enumerable then $\underline{\kappa}(\bigcup_{i \in \mathbb{N}} F_i) = \dim \bigcup_{i \in \mathbb{N}} F_i$.*

4 Dilution

It is evident that inserting fixed letters, e.g. zeroes in a computable way into complex infinite words results in infinite words of lower complexity (see [Dal74]). This effect – called dilution – was used to obtain partially random infinite words (see [Sta93, CST06]). Here we are interested in the result of diluting sets of infinite words via computable mappings. Dilution can be seen locally as the application of a (computable) prefix-monotone mapping $\varphi : X^* \to X^*$ to (the prefixes of) an infinite word.

The extension of this mapping to a partial mapping $\overline{\varphi} : \mathrm{dom}(\overline{\varphi}) \to X^\omega$ is given by a limit process: $\overline{\varphi}(\xi)$ is the infinite word having infinitely many prefixes of the form $\varphi(w), w \sqsubset \xi$. If the set $\{\varphi(w) : w \sqsubset \xi\}$ is finite we say that $\overline{\varphi}(\xi)$ is not defined, otherwise the identity $\mathbf{pref}(\overline{\varphi}(\xi)) = \mathbf{pref}(\varphi(\mathbf{pref}(\xi)))$ holds.

Using the δ-limit this process can be formulated as $\{\overline{\varphi}(\xi)\} = \{\varphi(w) : w \sqsubset \xi\}^\delta$ (see [Sta87]). The mapping $\overline{\varphi}(\xi)$, however, need not be continuous on $\mathrm{dom}(\overline{\varphi})$. For more detailed results on the extension $\overline{\varphi}$ of (partially defined) prefix-monotone mappings $\varphi : X^* \to Y^*$ see [Sta87]. We mention here only the following.

$$\overline{\varphi}(W^\delta) \subseteq \varphi(W)^\delta \tag{13}$$

Proof. It is readily seen that $\{\overline{\varphi}(\xi)\} = \varphi(V_\xi)^\delta$ where V_ξ is any infinite subset of $\mathbf{pref}(\xi)$. Thus choose $V_\xi := \mathbf{pref}(\xi) \cap W$ for $\xi \in W^\delta$. Then $\bigcup_{\xi \in W^\delta} V_\xi \subseteq W$ implies $\overline{\varphi}(W^\delta) = \bigcup_{\xi \in W^\delta} \{\overline{\varphi}(\xi)\} = \bigcup_{\xi \in W^\delta} \varphi(V_\xi)^\delta \subseteq \varphi(W)^\delta$. ❏

4.1 Expansiveness and Hausdorff Dimension

In an addendum to Section 3 of [CH94] a relation between the growth of the quotient $\frac{|w|}{|\varphi(w)|}$ on the prefixes of $\xi \in F$ and the Hausdorff dimension of the image $\overline{\varphi}(F), F \subseteq X^\omega$, was established. To this end we introduce the following.

Definition 2 ([CH94, Rei04]) *A prefix-monotone mapping* $\varphi : X^* \to Y^*$ *is called* γ-expansive *on* $\xi \in X^\omega$ *provided* $\liminf_{w\to\xi} \frac{|\varphi(w)|}{|w|} \geq \gamma$.

We say that $\varphi : X^* \to Y^*$ *is* γ-expansive *on* $F \subseteq X^\omega$ *if it is* γ*-expansive on every* $\xi \in F$.

Remark 1 Cai and Hartmanis [CH94] used $\limsup_{w\to\xi} \frac{|w|}{|\varphi(w)|} = \gamma$ as defining equation. This results in replacing γ by γ^{-1}.

Then Conjecture C of [CH94] claims the following.

Claim 1 *Let* $\dim F = \alpha$ *and let* φ *satisfy* $\limsup_{w\to\xi} \frac{|w|}{|\varphi(w)|} = \gamma$ *for all* $\xi \in F$. *Then* $\dim \overline{\varphi}(F) \leq \alpha \cdot \gamma$.

Moreover, if φ *and* $\overline{\varphi}$ *are one-one functions then* $\dim \overline{\varphi}(F) = \alpha \cdot \gamma$.

We can prove here only the first part of this conjecture, for the second part we derive a counter-example.

Example 2 Let $X = \{0,1\}$, $m_i := \sum_{j=0}^{2i} j!$ a sequence of rapidly growing natural numbers and define the prefix-monotone mapping $\varphi : \{0,1\}^* \to \{0,1\}^*$ as follows ($w \in \{0,1\}^*, x \in \{0,1\}$).

$$\begin{aligned} \varphi(e) &:= e \\ \varphi(wx) &:= \begin{cases} \varphi(w)x & \text{, if } |\varphi(w)x| \notin \{m_i : i \in \mathbb{N}\} \text{ and} \\ \varphi(w)x0^{(2i+1)!} & \text{, if } |\varphi(w)x| = m_i. \end{cases} \end{aligned},$$

that is φ dilutes the input by inserting sparsely long blocks of zeros. Then $\overline{\varphi}(\{0,1\}^\omega) = \prod_{i=0}^{\infty} \{0,1\}^{(2i)!} \cdot 0^{(2i+1)!}$ whence $\dim \overline{\varphi}(\{0,1\}^\omega) = 0$ (cf. Example 3.18 of [Sta93]).

On the other hand, $\limsup_{w\to\xi} \frac{|w|}{|\varphi(w)|} \leq \gamma$ implies $\gamma \geq 1$. ❑

For a proof of the first part we derive several auxiliary lemmas.

Lemma 5 *Let $\varphi : X^* \to Y^*$ be a prefix-monotone mapping and let $V \subseteq X^*$. If $c > 0$ and for almost all $v \in V$ the relation $|v| \leq c \cdot |\varphi(v)|$ holds then there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\varphi\left(V \cap X^{\leq c\cdot n}\right) \supseteq \varphi(V) \cap Y^{\leq n}$.*

Proof. Let $v_0 \in V$ be a longest word such that $|v_0| > c \cdot |\varphi(v_0)|$, and let $n \geq \frac{|v_0|}{c}$.

Let $w \in \varphi(V)$ and $|w| \leq n$. Then there is a $v \in V$ such that $\varphi(v) = w$. If $|v| \leq |v_0|$ then $|v| \leq c \cdot n$ and if $|v| > |v_0|$ then $|v| \leq c \cdot |\varphi(v)| \leq c \cdot n$. In both cases $w \in \varphi\left(V \cap X^{\leq c\cdot n}\right)$. ❑

Moreover, we use the fact that the Hausdorff dimension of an ω-language $F \subseteq X^\omega$ can be described via the entropy of languages $W \subseteq X^*$. Here for a language $W \subseteq X^*$ we define its *entropy* as usual[4] (cf. [Kui70, Sta93]).

$$\mathbf{H}_W = \limsup_{n\to\infty} \frac{\log_r(|W \cap X^n|+1)}{n} = \limsup_{n\to\infty} \frac{\log_r(|W \cap X^{\leq n}|+1)}{n} \tag{14}$$

The following identity gives a relation between Hausdorff dimension and entropy of languages (see [Sta93, Eq. (3.11)]).

$$\dim F = \inf\{\mathbf{H}_W : W \subseteq X^* \wedge W^\delta \supseteq F\} \tag{15}$$

Lemma 6 *Let $\varphi : X^* \to Y^*$ be a prefix-monotone mapping which satisfies $\varphi\left(V \cap X^{\leq c\cdot n}\right) \supseteq \varphi(V) \cap Y^{\leq n}$ for almost all $n \in \mathbb{N}$. Then $\mathbf{H}_{\varphi(V)} \leq c \cdot \log_{|Y|}|X| \cdot \mathbf{H}_V$.*

Proof. If $\varphi(V)$ is finite the inequality is obvious. Let $\varphi(V)$ be infinite. Then using Lemma 5 we obtain

$$\begin{aligned}
\mathbf{H}_{\varphi(V)} &= \limsup_{n\to\infty} \frac{\log_{|Y|}\left|\varphi(V)\cap Y^{\leq n}\right|}{n} \leq \limsup_{n\to\infty} \frac{\log_{|Y|}\left|\varphi(V \cap X^{\leq c\cdot n})\right|}{n} \\
&\leq \limsup_{n\to\infty} \frac{\log_{|Y|}\left|V \cap X^{\leq c\cdot n}\right|}{n} = c \cdot \log_{|Y|}|X| \cdot \limsup_{n\to\infty} \frac{\log_{|X|}\left|V \cap X^{\leq c\cdot n}\right|}{c\cdot n} \\
&\leq c \cdot \log_{|Y|}|X| \cdot \limsup_{n\to\infty} \frac{\log_{|X|}\left|V \cap X^{\leq n}\right|}{n}\,.
\end{aligned}$$

❑

Then it holds (see also [CH94, Conjecture C] and [Rei04, Proposition 1.19]).

Theorem 3 *Let $\varphi : X^* \to Y^*$ be a prefix-monotone mapping such that for all $\xi \in F$ the inequality $\liminf_{w\to\xi} \frac{|\varphi(w)|}{|w|} \geq \gamma$ is true. Then $\dim \overline{\varphi}(F) \leq \frac{1}{\gamma} \cdot \log_{|Y|}|X| \cdot \dim F$.*

[4] The $+1$ in the numerator is added to avoid $\mathbf{H}_W = -\infty$ for finite W.

Proof. Using Eq. (15) it suffices to show that for every $c > \frac{1}{\gamma}$ the inequality $\dim\overline{\varphi}(F) \le c \cdot \log_{|Y|}|X| \cdot \dim F$ holds true.

Let $W_c := \left\{w : \frac{|\varphi(w)|}{|w|} \ge \frac{1}{c}\right\}$. Since $c > \frac{1}{\gamma}$, for every $\xi \in F$ the set $\mathbf{pref}(\xi) \setminus W_c$ is finite. Now, consider a $W \subseteq X^*$ such that $W^\delta \supseteq F$. Then for $\zeta \in F$ the set $(\mathbf{pref}(\zeta) \cap W) \setminus W_c$ is finite, too, whereas $(\mathbf{pref}(\zeta) \cap W)$ is infinite. Hence $F \subseteq (W \cap W_c)^\delta$.

Now Eq. (13) implies $\overline{\varphi}(F) \subseteq \overline{\varphi}((W \cap W_c)^\delta) \subseteq \varphi(W \cap W_c)^\delta$.

Then Lemma 6 yields $\mathbf{H}_{\varphi(W\cap W_c)} \le c \cdot \log_{|Y|}|X| \cdot \mathbf{H}_{W\cap W_c}$ and thus we obtain $\dim\overline{\varphi}(F) \le \mathbf{H}_{\varphi(W\cap W_c)} \le c \cdot \log_{|Y|}|X| \cdot \mathbf{H}_{W\cap W_c}$. Taking the infimum on the right hand side yields the assertion. ❏

4.2 Uniform Dilution and Hölder Condition

Our Theorem 3 is closely related to the Hölder condition (see [Fal90, Prop. 2.2 and 2.3]). Whereas Theorem 3 holds also for not necessarily continuous mappings $\overline{\varphi} : \mathrm{dom}(\overline{\varphi}) \to Y^\omega$ the Hölder condition implies continuity but yields also bounds on Hausdorff measure.

Theorem 4 *Let $F \subseteq X^\omega$ and $\Phi : F \to Y^\omega$ be a mapping such that for $c > 0$ and $\gamma > 0$ the condition $\forall\xi\forall\eta(\xi,\eta \in F \to \rho(\Phi(\xi),\Phi(\eta)) \le c \cdot \rho(\xi,\eta)^\gamma)$ is fulfilled. Then $\mathbb{L}_{\alpha/\gamma}(\Phi(F)) \le c^{\alpha/\gamma} \cdot \mathbb{L}_\alpha(F)$ for all $\alpha \in [0,1]$.*

The condition $\exists c\big(c > 0 \wedge \forall\xi\forall\eta(\xi,\eta \in F \to \rho(\Phi(\xi),\Phi(\eta)) \le c \cdot \rho(\xi,\eta)^\gamma)\big)$ is also known as Hölder condition of exponent γ. The following lemma gives a connection between Hölder condition of exponent γ and γ-expansive prefix-monotone mappings[5].

Lemma 7 *Let $F \subseteq X^\omega$ and $\Phi : F \to Y^\omega$ be a mapping such that for $c > 0$ and $\gamma > 0$ the condition $\forall\xi\forall\eta(\xi,\eta \in F \to \rho(\Phi(\xi),\Phi(\eta)) \le c \cdot \rho(\xi,\eta)^\gamma)$ is fulfilled.*

Then there is a γ-expansive prefix-monotone mapping $\varphi : X^* \to Y^*$ such that $\overline{\varphi}(\xi) = \Phi(\xi)$ for all $\xi \in F$.

Proof. Let $m \in \mathbb{N}$ be chosen such that $c \le r^m$. Then in view of $\rho(\Phi(w\cdot\xi),\Phi(w\cdot\eta)) \le c \cdot r^{-\gamma\cdot|w|} \le r^{-\gamma\cdot|w|+m}$ for $w\cdot\xi, w\cdot\eta \in F$, every $\zeta \in \Phi(w \cdot X^\omega \cap F)$

[5] This is also the reason why we altered the definition of [CH94].

has the same word v_w of length $\lceil\gamma\cdot|w|-m\rceil$ as prefix. Thus define $\varphi(w) := v_w$ for $w \in \mathbf{pref}(F)$ and φ is γ-expansive. ❑

From the proof of Lemma 7 we see that a Hölder condition of exponent γ puts a more restrictive requirement on a prefix-monotone mapping than the mere γ-expansiveness.

If, for some strictly increasing function $g : \mathbb{N} \to \mathbb{N}$, the prefix-monotone mapping $\varphi : X^* \to X^*$ satisfies the conditions $|\varphi(w)| = g(|w|)$ and for every $v \in \mathbf{pref}(\varphi(X^*))$ there are $w_v \in X^*$ and $x_v \in X$ such that

$$\varphi(w_v) \sqsubset v \sqsubseteq \varphi(w_v\cdot x_v) \wedge \forall y\big(y \in X \wedge y \neq x_v \to v \not\sqsubseteq \varphi(w_v\cdot y)\big) \tag{16}$$

then we call φ a *dilution function* with modulus g. If φ is a dilution function then $\overline{\varphi}$ is a one-to-one mapping. The condition of Eq. (16) is equivalent to the fact that for every $w \in X^*$ and every pair of letters $x, y \in X, x \neq y$, the words $\varphi(w\cdot x)$ and $\varphi(w\cdot y)$ differ in the letter immediately after $\varphi(w)$, in particular, the words $\varphi(w\cdot x)$ and $\varphi(w\cdot y)$ are incomparable w.r.t. $\sqsubseteq$.

As an illustration we consider the following dilution functions. Let $0 < \gamma < 1$ and define $g(n) := \lceil n/\gamma\rceil$ and $\varphi : X^* \to X^*$ via $(w \in X^*, x \in X)$

$$\begin{aligned}
\varphi(e) &:= e, \text{ and}\\
\varphi(wx) &:= \varphi(w)\cdot x^{g(n+1)-g(n)}.
\end{aligned}$$

Since $g(n)-1 < \frac{n}{\gamma} \leq g(n) < \frac{n}{\gamma}+1$ and, consequently, $\gamma\cdot g(n)-\gamma < n \leq \gamma\cdot g(n) < n+\gamma$, the dilution function φ satisfies $\rho(\overline{\varphi}(\xi),\overline{\varphi}(\eta)) = r^{-g(n)} \iff \rho(\xi,\eta) = r^{-n}$, that is, $\overline{\varphi}$ and $\overline{\varphi}^{-1}$ satisfy the following Hölder conditions on X^ω or $\overline{\varphi}(X^\omega)$, respectively.

$$\begin{aligned}
\rho(\overline{\varphi}(\xi),\overline{\varphi}(\eta)) &\leq r^{-n/\gamma} = \rho(\xi,\eta)^{1/\gamma}, && \text{and}\\
\rho(\xi,\eta) &\leq r^{-\gamma\cdot n+\gamma} = r^{\gamma}\cdot\rho(\overline{\varphi}(\xi),\overline{\varphi}(\eta))^{\gamma}.
\end{aligned}$$

As a consequence we obtain $\mathbb{L}_\gamma(\overline{\varphi}(X^\omega)) \leq \mathbb{L}_1(X^\omega) = 1 \leq r\cdot\mathbb{L}_\gamma(\overline{\varphi}(X^\omega))$ setting $\alpha = 1$ or $\alpha = \gamma$ in Theorem 4.

Thus we can state the following properties of computably diluted ω-languages.

Proposition 6 *Let $1 > \gamma > 0$ be a computable real number, $g(n) := \lceil\frac{n}{\gamma}\rceil$ and $\varphi : X^* \to X^*$ a computable dilution function with with modulus g.*

Then $\dim \overline{\varphi}(X^\omega) = \gamma$*,* $\mathbb{L}_\gamma(\overline{\varphi}(X^\omega)) > 0$ *and there is a* $\xi \in \overline{\varphi}(X^\omega)$ *such that* $\mathrm{KA}(\xi \upharpoonright n) \geq \gamma \cdot n - c$ *for some constant* c*.*

Moreover, for $K \in \{\mathrm{K}, \mathrm{KA}, \mathrm{Km}\}$ *the complexity functions* $K(\xi \upharpoonright n)$ *are bounded by* $\gamma \cdot n + c'$ *for some constant* c'*.*

Proof. The first assertion is shown above, and the lower bound follows from Lemma 4.

Using Proposition 1 and Corollaries 2 and 3 we have the bounds $K(\varphi(w)) \leq \gamma \cdot |w| + c$. Since the differences $|\varphi(wx)| - |\varphi(w)|$ are bounded by $\frac{1+\gamma}{\gamma}$, the intermediate values $K(v), w \sqsubseteq v \sqsubseteq wx$, cannot exceed the value $\gamma \cdot |v|$ too much. ❑

The most complex ω-words in $\overline{\varphi}(X^\omega)$ satisfy $|\mathrm{KA}(\xi \upharpoonright n) - \gamma \cdot n| = O(1)$, a behaviour which for $\gamma = 1$ characterises random ω-words. Thus their behaviour might be seen as a scaled down by factor γ randomness, a case of partial randomness. Partial randomness allows for oscillations above the slope $\gamma \cdot n$ (see [CST06, Tad02]). The partially random ω-words in $\overline{\varphi}(X^\omega)$, however, exhibit an oscillation-free randomness (see also [Sta08, MS09]).

5 Infinite Products and Self-Similarity

Another way to describe sets of infinite words is to concatenate them as infinite products with factors chosen from a given set of finite words V. This resembles one of defining a subset $F \subseteq X^\omega$ via the recurrence $F = V \cdot F$. Since the mappings $\Phi_w(\xi) := w \cdot \xi$ of the space (X^ω, ρ) into itself are metric similarities, the sets

$$F = V \cdot F = \bigcup_{w \in V} \Phi_w(F) \tag{17}$$

are self-similar sets in the space (X^ω, ρ). An equation like Eq. (17) may, however, have a great variety of solutions (see [Sta97]). Fortunately, there is a unique maximal w.r.t. set inclusion solution which is the ω-power V^ω of the language V. Relations between self-similarity and ω-power languages were investigated e.g. in [FS01, Sta96, Tad02].

In this section we focus on iterated function systems $(\Phi_v)_{v \in V}$ where $V \subseteq X^*$ is prefix-free. Thus the mappings Φ_v map the space (X^ω, ρ) into pairwise disjoint parts. A special rôle here plays the similarity dimension of the system $(\Phi_v)_{v\in V}$. It turns out that it coincides with the Hausdorff dimension of the infinite product V^ω.

5.1 Dimension and Asymptotic Complexity

In this part we review some results on the Hausdorff dimension of ω-power languages and their asymptotic complexities.

We start with some results on the Hausdorff dimension of ω-power languages W^ω (see [Sta93] or, in a more general setting [FS01]).

Eq. (6.2) of [Sta93] yields the following connection between the entropy of W^* and the Hausdorff dimension of W^ω.

$$\dim W^\omega = \mathbf{H}_{W^*} \tag{18}$$

Next we review some results on the upper and lower asymptotic complexity for ω-power languages W^ω.

Proposition 7 ([Sta93, Lemma 6.7]) *If the language $W^* \subseteq X^*$ or its complement $X^* \setminus W^*$ are computably enumerable then $\underline{\kappa}(W^\omega) = \underline{\kappa}\big((W^*)^\delta\big) = \dim W^\omega$.*

Moreover W^ω contains always an ω-word of highest upper asymptotic complexity (see [Sta93]) and, moreover, its closure $\mathcal{C}(W^\omega)$ in (X^ω, ρ) has the same upper bound (see Corollary 6.11 and Eq (6.13) of [Sta93]).

Lemma 8 $\kappa(W^\omega) = \kappa(\mathcal{C}(W^\omega)) = \max\{\kappa(\xi) : \xi \in W^\omega\} \geq \mathbf{H}_{\mathbf{pref}(W^*)}$.

For computably enumerable languages W we have the following exact bound.

Proposition 8 ([Sta93, Proposition 6.15]) *If $W \subseteq X^*$ is computably enumerable then $\kappa(W^\omega) = \mathbf{H}_{\mathbf{pref}(W^*)}$*

5.2 Similarity Dimension

Let $V \subseteq X^*$ and $\mathbf{t}_1(V) := \sup\{t : t \geq 0 \wedge \sum_{i\in\mathbb{N}} |V \cap X^i| \cdot t^i \leq 1\}$. Then the parameter $-\log_r \mathbf{t}_1(V)$ is the *similarity dimension* of the system $(\Phi_v)_{v\in V}$.

If V is finite $(\Phi_v)_{v\in V}$ is a usual IFS and $-\log_r \mathbf{t}_1(V)$ is the (unique) solution of the equation $\sum_{v\in V} r^{-\alpha\cdot|v|} = 1$, so one may replace $\leq$ by $=$ in the definition of $\mathbf{t}_1(V)$.

The value $\mathbf{t}_1(V)$ fulfils the following (see [Kui70, Sta93]).

Lemma 9 *If $V \subseteq X^*$ is prefix-free then $-\log_r \mathbf{t}_1(V) = \mathbf{H}_{V^*}$.*

Thus, for prefix-free languages $V \subseteq X^*$ Eq. (18) and Lemma 9 imply $\sum_{v\in V} r^{-\alpha\cdot|v|} \leq 1$ for $\alpha = \dim V^\omega$.

In [Sta05b] for certain ω-power languages a necessary and sufficient condition to be of non-null α-dimensional Hausdorff measure was derived. In this respect, for a language $V \subseteq X^*$, the *α-residue of V derived by w*, the value $\mathrm{res}_\alpha(V,w) := r^{-\alpha|w|}\sum_{v\in V} r^{-\alpha|v|} = \sum_{wv\in V} r^{-\alpha|v|}$ for $w \in \mathbf{pref}(V)$ plays a special rôle.

Theorem 5 ([Sta05b]) *Let $V \subseteq X^*$ be prefix-free and $\sum_{v\in V} r^{-\alpha|v|} = 1$. Then $\alpha = \dim V^\omega$, and, moreover $\mathbb{L}_\alpha(V^\omega) > 0$ if and only if the α-residues $\mathrm{res}_\alpha(V,w)$ of V are bounded from above.*

Thus in view of Lemma 4 such V^ω contain sequences ξ having a linear lower complexity bound $\alpha\cdot n - c$ for *a priori* complexity. It is interesting to observe that bounding the α-residues of V from below yields a linear upper bound on the slope α on the complexity of ω-words in the closure $\mathcal{C}(V^\omega)$.

5.3 Plain Complexity

First we show that bounding the α-residues of V from below results in an upper bound on the number of prefixes of V^ω. Then we apply Theorem 1 to computably enumerable prefix-free languages V to show that a positive lower bound to the α-residues of V implies a linear upper bound on the complexity function $\mathrm{K}(\xi\upharpoonright n)$ for $\xi \in \mathcal{C}(V^\omega)$.

Lemma 10 *Let $V \subseteq X^*$ be prefix-free, $\sum_{v\in V} r^{-\alpha|v|} \leq 1$ and $\sum_{wv\in V} r^{-\alpha|v|} \geq c' > 0$ for all $w \in \mathbf{pref}(V)$. Then $|\mathbf{pref}(V^*)\cap X^l| \leq c\cdot r^{\alpha\cdot l}$ for some constant $c > 0$ and all $l \in \mathbb{N}$.*

Proof. We have $\mathbf{pref}(V^*)\cap X^l = \mathbf{pref}(V^l)\cap X^l$ for $l \in X^*$. Let $a := \sum_{v\in V} r^{-\alpha|v|}$. Since V is prefix-free, $a^l = \sum_{v\in V^l} r^{-\alpha|v|} = \sum_{|w|=l, w\in\mathbf{pref}(V^*)}\left(r^{-\alpha\cdot l}\cdot\sum_{wv\in V^l} r^{-\alpha|v|}\right)$.

If $w = v_1 \cdots v_{i_w-1} \cdot w'$ with $v_j \in V$ and $w' \in \mathbf{pref}(V)$ then $\{v : wv \in V^l\} \supseteq \{v : w'v \in V^{l-i_w+1}\} \supseteq \{v : w'v \in V\} \cdot V^{l-i_w}$.

Thus, $\sum_{wv \in V^l} r^{-\alpha|v|} \geq \sum_{w'v \in V} r^{-\alpha|v|} \cdot a^{l-i_w} \geq c' \cdot a^{l-i_w} \geq c' \cdot a^l$ and we obtain $a^l \geq r^{-\alpha \cdot l} \cdot |\mathbf{pref}(V^*) \cap X^l| \cdot c' \cdot a^l$ which proves our assertion. ❏

Now, the fact that $\mathbf{pref}(V^*)$ is computably enumerable if only V is computably enumerable yields our result.

Lemma 11 ([MS09, Lemma 7]) *Let $V \subseteq X^*$ be a computably enumerable and prefix-free, α be right-computable and $\sum_{v \in V} r^{-\alpha \cdot |v|} = a \leq 1$.*

If there Is a $c > 0$ such that $\sum_{wv \in V} r^{-\alpha \cdot |v|} \geq c$ for all $w \in \mathbf{pref}(V)$ then there is a constant c such that

$$\mathrm{K}(\xi \upharpoonright n) \leq \alpha \cdot n + c \textit{ for every } \xi \in \mathcal{C}(V^\omega).$$

5.4 *A priori* and Monotone Complexity

Lemma 10, however is not applicable to *a priori* and monotone complexity. To this end we construct, for prefix-free languages $V \subseteq X^*$ and values $\alpha \in \mathbb{R}_+$ such that $\sum_{v \in V} r^{-\alpha \cdot |v|} \leq 1$ a continuous semi-measure μ satisfying $\mu(w) = r^{-\alpha \cdot |w|}$ for $w \in V^*$ (see the proof of [Sta08, Lemma 3.9]).

Proposition 9 *Let $V \subseteq X^*$ be prefix-free, $\alpha > 0$ and $\sum_{v \in V} r^{-\alpha \cdot |v|} \leq 1$. Then $\mu : X^* \to \mathbb{R}_+$ where*

$$\mu(w) = \begin{cases} 0 & \text{, if } w \notin \mathbf{pref}(V^*) \\ r^{-\alpha \cdot |w|} & \text{, if } w \in V^* \\ \sum_{wv \in V} r^{-\alpha|wv|} & \text{, if } w \in \mathbf{pref}(V) \setminus \{e\} \\ \mu(u) \cdot \mu(v) & \begin{array}{l} \text{, if } w = u \cdot v \\ \text{ with } u \in V \cdot V^* \wedge v \in \mathbf{pref}(V) \setminus V \end{array} \end{cases} \tag{19}$$

is a continuous semi-measure on X^*. If, moreover, $\sum_{v \in V} r^{-\alpha \cdot |v|} = 1$ then μ is a continuous measure.

Proof. We have to show that $\mu(w) \geq \sum_{x \in X} \mu(wx)$. We prove this by induction.

The equation $\mu(w) \geq \sum_{x \in X} \mu(wx)$ for $w \in \mathbf{pref}(V) \setminus V$ follows directly from the requirement $\sum_{v \in V} r^{-\alpha \cdot |v|} \leq 1$ and the third line of the construction. Observe that $\mu(e) > \sum_{x \in X} \mu(x) = \sum_{v \in V} r^{-\alpha \cdot |v|}$ if $\sum_{v \in V} r^{-\alpha \cdot |v|} < 1$.

Let $w \in V \cdot \mathbf{pref}(V^*)$. Since V is prefix-free, the decomposition in the last line of the construction is unique. Thus $w = u \cdot v$ where $u \in V \cdot V^*$ and $v \in \mathbf{pref}(V) \setminus V$. Consequently, $\mu(w) = \mu(u) \cdot \mu(v) \geq \mu(u) \cdot \sum_{x \in X} \mu(vx) = \sum_{x \in X} \mu(wx)$.

Finally, if $w \notin V \cdot \mathbf{pref}(V^*) \cup \mathbf{pref}(V) = \mathbf{pref}(V^*)$ then also $wx \notin \mathbf{pref}(V^*)$, and the inequality is trivially satisfied.

If $\sum_{v \in V} r^{-\alpha \cdot |v|} = 1$ then $\mu(w) = \sum_{x \in X} \mu(wx)$ for $w \in \mathbf{pref}(V) \setminus V$ and the identity $\mu(w) = \sum_{x \in X} \mu(wx)$ follows by induction. ❑

Now we can prove the announced bounds.

Lemma 12 ([MS09, Lemma 3]) *Let $V \subseteq X^*$ be a computably enumerable and prefix-free, α be right-computable such that $\sum_{v \in V} r^{-\alpha \cdot |v|} = a \leq 1$ and the α-residues $\mathrm{res}_\alpha(V, w) := \sum_{wv \in V} r^{-\alpha |v|}$ of V derived by $w \in \mathbf{pref}(V)$ be bounded from below. Then there is a constant c such that for every $\xi \in \mathcal{C}(V^\omega)$*

$$\mathrm{KA}(\xi \upharpoonright n) \leq \alpha \cdot n + c\,.$$

Proof. First we show that the semi-measure μ constructed in the previous proposition is left-computable.

To show that μ is left-computable we successively approximate the value $\mu(w)$ from below. Let V_i be the set of the first i elements in the enumeration of V and α_i the ith approximation of α from the right. We start with $\mu^{(j)}(e) = 1$ for $j > 0$ and $\mu^{(0)}(w) := 0$ for $w \neq e$. Suppose that the jth approximation $\mu^{(j)}$ for all words shorter than w is already computed. If there is a $v \in V_j$ with $w = v \cdot w'$, $w' \neq e$, then $\mu^{(j)}(w')$ is defined and we set $\mu^{(j)}(w) = \mu^{(j)}(v) \cdot \mu^{(j)}(w')$. Otherwise, if $w \in \mathbf{pref}(V_j)$ we set $\mu^{(j)}(w) = \sum_{v \in V_j \wedge w \sqsubseteq v} r^{-\alpha_j \cdot |v|}$. If $w \notin \mathbf{pref}(V_j) \cup V_j \cdot X^*$ then $\mu^{(j)}(w) = 0$.

From the construction in Proposition 9 we obtain that $\mu(w) = r^{-\alpha |u|} \cdot \sum_{vv' \in V} r^{-\alpha |vv'|} = r^{-\alpha |w|} \cdot \mathrm{res}_\alpha(V, v)$ when $w = u \cdot v$ is the unique decomposition of $w \in \mathbf{pref}(V^*)$ into factors $u \in V^*$ and $v \in \mathbf{pref}(V) \setminus V$.

Let $c_{\inf} := \inf\{\mathrm{res}_\alpha(V, v) : v \in \mathbf{pref}(V)\}$. Since μ is a left-computable semi-measure, the following inequality holds true.

$$\mathbf{M}(w) \cdot c_\mu \geq \mu(w) \geq r^{-\alpha |w|} \cdot c_{\inf}$$

Taking the negative logarithm on both sides of the inequality we obtain $\mathrm{KA}(w) \leq \alpha \cdot |w| + \log \frac{c_\mu}{c_{\inf}}$ for every $w \in \mathbf{pref}(V^*)$. ❑

The following example shows, that in Lemma 12 we cannot omit the condition that the α-residues are bounded from below. To this end we use a computable prefix-free language constructed in Example (6.4) of [Sta93].

Example 3 Let X=$\{0,1\}$ and consider $W := \bigcup_{i\in\mathbb{N}} 0^{i+1}\cdot 1\cdot X^{i+1}\cdot 0^{4\cdot i+3}$. The language W is a prefix-free. Its ω-power, W^ω, satisfies $\alpha = \dim W^\omega = \dim \mathcal{C}(W^\omega) = \frac{1}{3}$ and $\mathbb{L}_\alpha(W^\omega) = \mathbb{L}_\alpha(\mathcal{C}(W^\omega))$. For every $w \in \bigcup_{i\in\mathbb{N}} 0^{i+1}\cdot 1\cdot X^{i+1}$ we have $W \cap w\cdot X^* = w\cdot\{0^{4\cdot i+3}\}$. Thus $\sum_{wv\in W} r^{-\alpha\cdot|v|} = r^{-\alpha\cdot(4\cdot i+3)}$ and, consequently, $\inf\{\sum_{wv\in W} r^{-\alpha\cdot|v|} : w \in \mathbf{pref}(W)\} = 0$.

Since $\mathbf{pref}(W) \supseteq \bigcup_{i\in\mathbb{N}} 0^{i+1}\cdot 1\cdot X^{i+1}$, we have $\mathbf{H}_{\mathbf{pref}(W)^*} \geq \frac{1}{2}$. Now Proposition 8 shows $\sup\limits_{\xi\in W^\omega} \limsup\limits_{n\to\infty} \frac{\mathrm{KA}(\xi\upharpoonright n)}{n} = \kappa(W^\omega) \geq \frac{1}{2} > \frac{1}{3} = \dim W^\omega$. ❑

In connection with Theorem 5 our Lemma 12 yields a sufficient condition for ω-powers V^ω to contain ω-words ξ satisfying $|\mathrm{KA}(\xi\upharpoonright n) - \alpha\cdot n| = O(1)$.

Corollary 4 *Let $V \subseteq X^*$ be a computably enumerable prefix-free language and α right-computable such that $\sum_{v\in V} r^{-\alpha\cdot|v|} = 1$ and the α-residues $\mathrm{res}_\alpha(V,w)$ of V derived by $w \in \mathbf{pref}(V)$ are bounded from above and below. Then there is a $\xi \in V^\omega$ such that $|\mathrm{KA}(\xi\upharpoonright n) - \alpha\cdot n| = O(1)$.*

The results of Section 3.2 of [Sta08] show that Corollary 4 is valid for prefix-free languages definable by finite automata. The subsequent example verifies that there are also non-regular prefix-free languages which satisfy the hypotheses of Corollary 4.

Example 4 Let $X = \{0,1\}$ and consider the Łukasiewicz language L defined by the identity $L = 0 \cup 1\cdot L^2$. This language is prefix-free and Kuich [Kui70] showed that $\sum_{w\in L} 2^{-|w|} = 1$. Thus the language V defined by $V = 00 \cup 11\cdot V^2$ is also prefix-free and satisfies $\sum_{v\in V} 2^{-\frac{1}{2}\cdot|w|} = 1$. By induction one shows that for $v \in \mathbf{pref}(V)$ we have $V/v = w'\cdot V^k$ for suitable $k \in \mathbb{N}$ and $|w'| \leq 1$. Therefore the α-residues of V derived by $v \in \mathbf{pref}(V)$ are bounded from above and below. ❑

For the monotone complexity Km a result similar to Lemma 12 can be obtained for a smaller class of ω-languages. We start with an auxiliary result.

Proposition 10 *Let $V \subseteq X^*$ be computably enumerable.*

1. *If* $\sum_{v\in V} r^{-\alpha|v|} = 1$ *then* α *is left-computable.*
2. *If* $\sum_{v\in V} r^{-\alpha|v|} = 1$ *and* α *is right-computable then* V *is computable.*

Proof. The proof of part 1 is obvious. To prove part 2 we present an algorithm to decide whether a word w is in V or not.

Let V_j be the set of the first j elements in the enumeration of V and α_j the jth approximation of α from the right.

Input w
$j := 0$
  repeat
    $j := j+1$
    if $w \in V_j$ then accept and exit
  until $r^{-\alpha_j|w|} + \sum_{v\in V_j} r^{-\alpha_j|v|} > 1$
reject

If $w \notin V$ then the repeat until loop terminates as soon as $\sum_{v\in V_j} r^{-\alpha_j|v|} > 1 - r^{-\alpha_j|w|} \geq 1 - r^{-\alpha|w|}$ because $\sum_{v\in V_j} r^{-\alpha_j|v|} \to 1$ for $j \to \infty$. ❏

Now we can prove our result on monotone complexity.

Lemma 13 ([MS09, Lemma 4]) *Let $V \subseteq X^*$ be a computably enumerable prefix-free language. If α is right-computable such that $\sum_{v\in V} r^{-\alpha\cdot|v|} = 1$ and the α-residues $\mathrm{res}_\alpha(V,w)$ derived by $w \in \mathbf{pref}(V)$ are bounded from below then there is a constant c such that $\mathrm{Km}(\xi\upharpoonright n) \leq \alpha\cdot n + c$ for every $\xi \in \mathcal{C}(V^\omega)$.*

Proof. We construct μ as in Proposition 9. Then $\sum_{v\in V} r^{-\alpha\cdot|v|} = 1$ implies that μ is a measure and Lemma 12 shows that μ is left-computable.

Because of Proposition 10 we can assume that α is a computable real number and V is computable. Then for every $v \in V^*$ the number $\mu(v) =$ is computable. Since V is a computable prefix-free language, for every $w \in X^*$ we can compute the unique decomposition $w = v\cdot w'$ with $v \in V^*$ and $w' \notin V\cdot X^*$. Now

$$\mu(w) = \mu(v)\cdot\left(1 - \sum_{v'\in V \wedge w \not\sqsubseteq vv'} r^{-\alpha|v'|}\right)$$

shows that μ is also right-computable. If $w' \notin \mathbf{pref}(V)$ then the last factor is zero.

Again let $c_{\inf} := \inf\left\{\sum_{wv\in V} r^{-\alpha\cdot|v|} : w \in \mathbf{pref}(V)\right\}$. In view of Proposition 4 we get the bound

$$\mathrm{Km}(w) \leq -\log\mu(w) + c_\mu \leq \alpha\cdot|w| + c_\mu - \log c_{\inf}$$

for every $w \in \mathbf{pref}(V^*)$. ❑

As for Lemma 12 we obtain a sufficient condition for ω-powers V^ω to contain ω-words ξ satisfying $|\mathrm{Km}(\xi\upharpoonright n) - \alpha\cdot n| = O(1)$.

Corollary 5 *Let $V \subseteq X^*$ be a computably enumerable prefix-free language and α right-computable such that $\sum_{v\in V} r^{-\alpha\cdot|v|} = 1$ and the α-residues $\mathrm{res}_\alpha(V,w)$ of V derived by $w \in \mathbf{pref}(V)$ are bounded from above and below. Then there is a $\xi \in V^\omega$ such that $|\mathrm{Km}(\xi\upharpoonright n) - \alpha\cdot n| = O(1)$.*

Concluding Remark

Proposition 6 and the Lemmata 12 and 13 show that in certain computably describable ω-languages the maximally complex strings have (up to an additive constant) linear oscillation-free complexity functions w.r.t. a priori and monotone complexity. Though in the case of plain complexity we have also linear upper bounds Theorems 4.8 and 4.12 of [Sta93] show that maximally complex infinite strings in ω-languages definable by finite automata (in particular, those of the form V^ω with V definable by a finite automaton) exhibit complexity oscillations similar to random infinite strings (cf. Theorem 6.10 of [Cal02] or Lemma 3.11.1 in [DH10]).

For prefix complexity (see [Cal02, Section 4.2], [DH10, Section 3.5] or [Nie09, Section 2.2]), however, it seems to be not as simple to obtain linear upper bounds on the complexity function (see [CHS11, Tad10]) let alone to detect an oscillation-free behaviour as mentioned above. In fact, Theorem 5 of [Sta12] shows that the oscillation-free behaviour w.r.t. to prefix complexity differs substantially from the one of a priori complexity.

References

[AHLM07] Krishna B. Athreya, John M. Hitchcock, Jack H. Lutz, and Elvira Mayordomo. Effective strong dimension in algorithmic information and computational complexity. *SIAM J. Comput.*, 37(3):671–705, 2007.

[BDGK08] V. Brattka, R. Dillhage, T. Grubba, and A. Klutsch, editors. *Proceedings of the Fifth International Conference on Computability and Complexity in Analysis (CCA 2008)*, volume 221 of *Electronic Notes in Theoretical Computer Science*. Elsevier, Amsterdam, 2008.

[Cal02] Cristian S. Calude. *Information and randomness*. Texts in Theoretical Computer Science. An EATCS Series. Springer-Verlag, Berlin, second edition, 2002.

[CH94] Jin-Yi Cai and Juris Hartmanis. On Hausdorff and topological dimensions of the Kolmogorov complexity of the real line. *J. Comput. System Sci.*, 49(3):605–619, 1994.

[CHS11] Cristian S. Calude, Nicholas J. Hay, and Frank Stephan. Representation of left-computable ε-random reals. *J. Comput. System Sci.*, 77(4):812–819, 2011.

[CST06] Cristian S. Calude, Ludwig Staiger, and Sebastiaan A. Terwijn. On partial randomness. *Ann. Pure Appl. Logic*, 138(1-3):20–30, 2006.

[Dal74] Richard P. Daley. The extent and density of sequences within the minimal-program complexity hierarchies. *J. Comput. System Sci.*, 9:151–163, 1974.

[DH10] Rodney G. Downey and Denis R. Hirschfeldt. *Algorithmic randomness and complexity*. Theory and Applications of Computability. Springer-Verlag, New York, 2010.

[Edg08] Gerald Edgar. *Measure, topology, and fractal geometry*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, second edition, 2008.

[Fal90] Kenneth Falconer. *Fractal geometry*. John Wiley & Sons Ltd., Chichester, 1990.

[FS01] Henning Fernau and Ludwig Staiger. Iterated function systems and control languages. *Inform. and Comput.*, 168(2):125–143, 2001.

[Kui70] Werner Kuich. On the entropy of context-free languages. *Information and Control*, 16:173–200, 1970.

[Lut03] Jack H. Lutz. The dimensions of individual strings and sequences. *Inform. and Comput.*, 187(1):49–79, 2003.

[Mie08] Jöran Mielke. Refined bounds on Kolmogorov complexity for ω-languages. In Brattka et al. [BDGK08], pages 181–189.

[MS09] Jöran Mielke and Ludwig Staiger. On oscillation-free ε-random sequences II. In Andrej Bauer, Peter Hertling, and Ker-I Ko, editors, *Computability and Complexity in Analysis*, volume 09003 of *Dagstuhl Seminar Proceedings*. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, Germany, 2009.

[Nie09] André Nies. *Computability and randomness*, volume 51 of *Oxford Logic Guides*. Oxford University Press, Oxford, 2009.

[Rei04] Jan Reimann. *Computability and Fractal Dimension*. PhD thesis, Ruprecht-Karls-Universität Heidelberg, 2004.

[Rya86] Boris Ya. Ryabko. Noise-free coding of combinatorial sources, Hausdorff dimension and Kolmogorov complexity. *Problemy Peredachi Informatsii*, 22(3):16–26, 1986.

[She84] Aleksander Kh. Shen´. Algorithmic variants of the notion of entropy. *Dokl. Akad. Nauk SSSR*, 276(3):563–566, 1984.

[Sta87] Ludwig Staiger. Sequential mappings of ω-languages. *RAIRO Inform. Théor. Appl.*, 21(2):147–173, 1987.

[Sta93] Ludwig Staiger. Kolmogorov complexity and Hausdorff dimension. *Inform. and Comput.*, 103(2):159–194, 1993.

[Sta96] Ludwig Staiger. Codes, simplifying words, and open set condition. *Inform. Process. Lett.*, 58(6):297–301, 1996.

[Sta97] Ludwig Staiger. On ω-power languages. In Gheorghe Păun and Arto Salomaa, editors, *New trends in formal languages*, volume 1218 of *Lecture Notes in Computer Science*, pages 377–394. Springer-Verlag, Berlin, 1997.

[Sta98] Ludwig Staiger. A tight upper bound on Kolmogorov complexity and uniformly optimal prediction. *Theory Comput. Syst.*, 31(3):215–229, 1998.

[Sta05a] Ludwig Staiger. Constructive dimension equals Kolmogorov complexity. *Inform. Process. Lett.*, 93(3):149–153, 2005.

[Sta05b] Ludwig Staiger. Infinite iterated function systems in Cantor space and the Hausdorff measure of ω-power languages. *Internat. J. Found. Comput. Sci.*, 16(4):787–802, 2005.

[Sta07] Ludwig Staiger. The Kolmogorov complexity of infinite words. *Theoret. Comput. Sci.*, 383(2-3):187–199, 2007.

[Sta08] Ludwig Staiger. On oscillation-free ε-random sequences. In Brattka et al. [BDGK08], pages 287–297.

[Sta12] Ludwig Staiger. On oscillation-free Chaitin h-random sequences. In Michael J. Dinneen, Bakhadyr Khoussainov, and André Nies, editors, *Computation, Physics and Beyond*, volume 7160 of *Lecture Notes in Computer Science*, pages 194–202, Springer-Verlag, Heidelberg, 2012.

[Tad02] Kohtaro Tadaki. A generalization of Chaitin's halting probability Ω and halting self-similar sets. *Hokkaido Math. J.*, 31(1):219–253, 2002.

[Tad10] Kohtaro Tadaki. A new representation of Chaitin Ω number based on compressible strings. In Cristian S. Calude, Masami Hagiya, Kenichi Morita, Grzegorz Rozenberg, and Jon Timmis, editors, *Unconventional Computation*, volume 6079 of *Lecture Notes in Computer Science*, pages 127–139. Springer-Verlag, Heidelberg, 2010.

[US96] Vladimir A. Uspensky and Alexander Shen. Relations between varieties of Kolmogorov complexities. *Math. Systems Theory*, 29(3):271–292, 1996.

[USS90] Vladimir A. Uspenskiĭ, Aleksei L. Semenov, and Aleksander Kh. Shen´. Can an (individual) sequence of zeros and ones be random? *Uspekhi Mat. Nauk*, 45(1(271)):105–162, 1990.

[ZL70] Alexander K. Zvonkin and Leonid A. Levin. The complexity of finite objects and the basing of the concepts of information and randomness on the theory of algorithms. *Uspehi Mat. Nauk*, 25(6(156)):85–127, 1970.

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