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Limitations of Linear Programming Techniques for Bounded Color Matchings

Georgios Stamoulis[1],[2]

[1] LAMSADE, CNRS UMR 7243, Universitè Paris-Dauphine
Paris, France
[2] Universitá della Svizzera Italiana (USI)
Lugano, Switzerland
stamoulis.georgios@dauphine.fr

Abstract. Given a weighted graph $G = (V, E, w)$, with weight function $w : E \to \mathbb{Q}^+$, a *matching* M is a set of pairwise non-adjacent edges. In the optimization setting, one seeks to find a matching of *maximum* weight. In the *multi-criteria* (or *multi-budgeted*) setting, we are also given ℓ length functions $\alpha_1, \dots, \alpha_\ell$ assigning positive numbers to each edge and ℓ numbers $\beta_1, \dots, \beta_\ell \in \mathbb{Q}^+$, each one associated with the corresponding length function. The goal is to find a maximum weight matching M (under function w) such that $\sum_{e \in M} \alpha_i(e) \le \beta_i$, $\forall i \in [\ell]$ (these are the budgets). In this paper we are interested in the case where each edge $e \in E$ belongs to a unique budget, i.e., it has a unique *color*. In [30] an $\frac{1}{2}$-approximation algorithm was given based on rounding the natural linear programming relaxation of the problem, and this is optimal modulo the integrality gap of the formulation. The purpose of this paper is to study to what extend linear programming methods help us design algorithms with a better performance guarantee. We prove the following *unconditional* inapproximability result: even a large family of linear programs, generated by a *logarithmic* number of rounds of the Sherali-Adams hierarchy [29] or a *linear* number of rounds of the BCC operator [1], does not suffice to change the integrality gap even the slightest.

1 Introduction And Problem Definition

In 1982, Papadimitriou & Yannakakis defined the *Exact Matching* (EM) problem: Given a bipartite graph B where some of its edges are painted red, and a positive integer k, decide if B contains a perfect matching with *exactly* k red edges. This is one of the very few problems whose complexity is not yet fully understood. On one hand, there exists a randomized **NC** algorithms [24] which suggests that EM is probably not **NP**-complete. Moreover, there exists an algorithm which, in polynomial time, returns a matching of maximum cardinality with at most $k + 1$ red edges, if such matching exists [32]. This last result puts EM as close to **P** as possible (unless of course EM $\in$ **P**).

To the best of our knowledge, the first time a generalization of the EM problem was studied, at least from an approximation point of view, was in [26]

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where the authors defined and studied the so-called *blue-red* matching: Given a graph G where each edge is either red or blue, find a maximum matching with at most w red and at most w blue edges (their result extends in cases where an edge is allowed to have more than one color). Their motivation was that blue-red matchings can be used to approximately solve the Directed Maximum Routing and Wavelength Assignment problem (DirMRWA) [25] in *rings* which is a fundamental network topology, see [26], [5] and they provided an $\mathbf{RNC^2}$ algorithm and a 3/4 combinatorial algorithm. They also noticed that the simple greedy procedure produces a 1/2-approximate solution.

In this paper we consider the following natural generalization of the EM (as well as the blue-red Matching) problem in *general graphs* (which we call *Bounded Color Matching-BCM*): We are given a (simple, non-directed) graph $G = (V, E)$. The edge set is partitioned into k sets $E_1 \cup E_2 \cup \cdots \cup E_k$ i.e., every edge e has color C_j if $e \in E_j$, a profit p_e and a *length* α_e both $\in \mathbb{Q}^+$. We are asked to find a maximum (weighted) matching M such that in M the sum of the lengths of edges of color C_j is at most $w_j \in \mathbb{Q}^+$ i.e. a matching M such that $\sum_{e\in M\cap E_j} \alpha(e) \leq w_j, \forall C_j$. Let $\mathcal{C} = \cup_{i=1,\dots,k} C_i$ be the collection of all colors. The problem is long known to be **NP**-hard even in bipartite graphs [13] (where the problem was mentioned as *multiple choice matching* and **APX**-hardness can be deduced even in 2-regular bipartite graphs from [23]. When the edges are allowed to have multiple colors (be part of an unrestricted number of budgets) the problem is equivalent to the Maximum Independent Set problem.

In [30] a $\frac{1}{2}$-approximation algorithm was given based on rounding the natural linear programming relaxation of the problem. The rounding procedure was based on the elegant technique of *approximate convex decompositions* by Parekh [27]. This gives an inductive process to write any basic feasible solution x^* of the relaxed LP (by dropping the integrality constraints) as an approximate *sparse* convex combination of integral solutions, i.e., $\alpha \cdot p^T x^* = \sum_{i\in I} \lambda_i \mu_i$ where the λ_i's are non-negative and sum up to one and each μ_i is a feasible matching for the Bounded Color Matching problem. Then, by selecting the most profitable among all the matchings μ_i we get an α-approximate solution. The bulk of the work is to show that it is always possible to select $\alpha = \frac{1}{2}$ giving the $\frac{1}{2}$-approximation guarantee. The initial result was for uniform lengths and integral bounds, but it can immediately be generalized to the more general setting with the same approximation guarantee. The result has been further generalized on uniform hypergraphs in [28]. This also matches the *integrality gap* of the natural relaxation of the problem and suggests that using this LP alone is not enough to achieve something better than that.

A natural questions occurs: a negative result based on a bad integrality gap instance, rules out the possibility of a good approximation ratio for the problem we want to study. But this holds for that particular linear (or even semi-definite) relaxation. What about other, more complicated, relaxations? For example, if we take the normal (degree-constraint) relaxation for the classical matching problem (which has integrality gap of 2/3) and enhance it with the *blossom inequalities*, we get an exact formulation of the convex hull of all integer points for the Matching problem. But more usually than not, this is not the case.

A large body of work has been dedicated on trying to find systematic techniques to enhance the quality of a given linear (or semi-definite) program with valid inequalities (inequalities that are satisfied by all *integral* points) with the hope that the part of the polyhedron being responsible for the bad integrality gap example will be eliminated. Many such "lift and project" methods have been proposed so far for example by Sherali and Adams [29], by Lovász and Schrijver [20], by Balas, Ceria and Cornuéjols [1], [2] and by Lasserre [18], [17] and the effect of such methods has been extensively studied for a host of combinatorial optimization problems, for example see [22], [16], [6] and the most recent articles [8], [9] and [10] (which is actually one of the few "positive" applications of lift-and-project methods), and the references therein.

Our Contribution: The main contribution of this paper is to provide the following unconditional (not based on any complexity theoretic assumption) inapproximability result: even a *logarithmic* number of rounds of the Sherali-Adams (SA) hierarchy applied to the natural linear programming relaxation for the Bounded Color Matching problem, is not enough to *increase* the integrality ratio above the $\frac{1}{2}$. This unconditionally rules out a large class of algorithms (running even in *quasi polynomial time*) hoping to achieve an approximation guarantee better than $1/2$, and thus the results of [30] are in some strong sense "optimal" (relative to linear programming techniques). We note that the effect of the SA hierarchy on the usual matching polytope was fully studied in [22]. Our result is using the "algebraic" (linear formulation) version of the SA hierarchy. We note that similar bounds and instances (uniform lengths/sizes, fractional bounds/capacities) have been used in the study of the Knapsack problem [16]. As in the Knapsack problem, the morality of this result is that it demonstrates a severe limitation of this more general computational model, i.e., even large linear programs cannot "realize" such relatively simple structured instances. Since SA subsumes the Lovász-Schriver hierarchy [20],[19], the same claim holds for the LS hierarchy (but not the LS+).

Moreover, we prove that for the **BCC** operator [1] a greater bound can be easily shown: even a *linear* (with respect to the size instance) number of rounds of the BCC operator is not enough to change the integrality gap of $\frac{1}{2}$ even the slightest. We believe that this, relatively easier, result is interesting for an extra particular reason: that the SA hierarchy applied to those instances cannot be "fooled" for more but a very small number of rounds, showing that SA is strictly stronger than BCC at least for the BCM setting.

To prove these results we show the existence of particular graph families (instances for the BCM problem) and demonstrate a feasible *fractional* solution for such instances with fractional value twice as much as the optimal integral one. The bulk of the work, at least for the SA hierarchy, is to show that such proposed vectors constitutes in fact a *feasible* solution.

Related Work: Budgeted versions of the maximum matching problem have been recently studied intensively. When G is bipartite there is a PTAS for the case where $\ell = 1$ [3] and the case where $\ell = \mathcal{O}(1)$ [14]. For general graphs there is a PTAS for the 2-budgeted maximum matching problem [15] and a bicriteria PTAS for $\ell = \mathcal{O}(1)$ [7] (where the returned solution might violate the budgets by a factor of $(1+\epsilon)$). This approach works also for *unbounded* number of budgets albeit a *logarithmic* overflow of the budgets. In [21, **?**] the BCM was considered from a bi-criteria point of view: given a parameter $\lambda \in [0,1]$ there is an $(\frac{2}{3+\lambda})$ approximation algorithm for BCM which might violate the budgets w_j by a factor of at most $(\frac{2}{1+\lambda})$.

Remark: We note that the BCM problem is a special case of the 3-hypergraph β-matching problem [27] or 3-set packing. The integrality gap obtained by the natural LP relaxation is $\frac{3}{7}$ and this is tight [27]. So, additional arguments are needed to provide the 1/2 approximation in [30]. For 3-set packing there exists a combinatorial $\frac{1}{2} - \epsilon$ approximation algorithm for the *weighted* case [4] and a recent $\frac{3}{4} - \epsilon$ for the non-weighted case [12] building on the work of [31]. We also note that this problem can be recast as a problem of maximizing a linear function subject to a matching constraint and a partition matroid constraint which enforces that at most w_j elements be chosen with color C_j. This constitutes a 3-system and hence a greedy algorithm gives $\frac{1}{3}$-approximation and this is tight by elementary examples. This is in contrast with the greedy approach on blue-red matching which gives $\frac{1}{2}$-approximation.

2 Limitations of Linear Programming Techniques

If we formulate BCM as a linear program, the polyhedron $\mathcal{M}_c$ containing all feasible matchings M for the BCM is

$$\mathcal{M}_c = \Big\{ \boldsymbol{y} \in \{0,1\}^{|E|} : \quad \boldsymbol{y} \in \mathcal{M} \quad \bigwedge \quad \sum_{e \in E_j} \alpha_e y_e \leq w_j, \quad \forall j \in [k] \Big\} \tag{1}$$

where $\mathcal{M}$ is the usual matching polyhedron: $\mathcal{M} = \{x \in \{0,1\}^{|E|} : \quad \sum_{e \in \delta(v)} x_e \leq 1, \ \forall v \in V\}$. We would like to find $\boldsymbol{y} \in \{0,1\}^{|E|}$ such that

$$\boldsymbol{y} = \max_{x \in \mathcal{M}_c} \Big\{ \boldsymbol{p}^T \boldsymbol{x} = \sum_{e \in E} p_e x_e \Big\}, \ \boldsymbol{p} \in \mathbb{Q}^{|E|}_{\geq 0}.$$

As usual, we relax the integrality constraints $\boldsymbol{y} \in \{0,1\}^{|E|}$ to $\boldsymbol{y} \in [0,1]^{|E|}$ and we solve the corresponding linear relaxation efficiently to obtain a *fractional* $|E|$-dimensional vector $\boldsymbol{y}$. It is not hard to show that the *integrality gap* of $\mathcal{M}_c$ is essentially 1/2 (for example 4-cycle with alternating red-blue edges where we want a maximum matching with at most one edge per color) and this is true even if we add the blossom inequalities [21] i.e., if instead of $\mathcal{M}$ as defined here, we use the well known Edmond's LP [11].

The Sherali-Adams Hierarchy: Sherali-Adams is one of the most widely used "lift and project" methods. Such methods constitute a systematic way to derive an integral formulation for binary optimization problems. The idea behind these methods is to consider the binary optimization problem in a higher dimensional space (this is the *lifting* phase), add there *valid* constraints for the binary optimization problem at hand (i.e. constrains that are satisfied by all integral points of the polyhedron) and them project back the new solution found in this higher dimensional space to the initial variable space (and this is the *projection* phase of the technique). By projection we mean the following:

Definition 1. *Let a polyhedron $P = \{(x, y) : \ Ax + By \leq \Gamma\}$ for matrices A, B, Γ of appropriate dimension. Then, the* ***projection*** *of P onto the set of the x variables is simply the following polyhedron:*

$$P_x = \Big\{x : \ \exists y \textit{ such that } (x, y) \in P\Big\}$$

We recall the definition of the Sherali-Adams hierarchy of progressively stronger relaxations of an integer polyhedron in the n-th dimensional hypercube $\{0, 1\}^n$. We use the original definitions [29] (see also [19]).

Let $F_0 = \{y \in [0, 1]^n : \quad \boldsymbol{\alpha_i}^T\boldsymbol{x} = \sum_{j\in[n]} \alpha_{ij}x_j \leq \beta_i, \quad \forall i \in [m]\}$ with $\alpha_{ij}, \beta_j \in \mathbb{Q}$, $\forall i \in [m], j \in [n]$ be an initial convex polyhedron in $[0, 1]^n$. Let $\mathcal{I} = \mathsf{conv}(F_0 \cap \{0, 1\}^n)$ be the convex hull of all integer points of F_0. The Sherali-Adams hierarchy, starting from F_0, constructs an hierarchy of progressively *non-weaker* relaxations $F_1, F_2, \ldots$ of $\mathcal{I}$ in the sense that $F_n \subseteq F_{n-1} \subseteq \cdots \subseteq F_0$. Let F_ψ be the polyhedron resulting after ψ iterations of the Sherali-Adams methods applied initially to F_0. After at most rounds n we will arrive at $\mathcal{I}$ i.e. $F_n = \mathcal{I}$. Sometimes n rounds are necessary in order to arrive at $\mathcal{I}$. Also, we can efficiently optimize any linear objective function over F_k, for any fixed k. At the ψ-th iteration, S-A hierarchy obtains F_ψ from $F_{\psi-1}$ as follows:

Let Γ, Δ be *disjoint* subsets of $[n]$ such that $|\Gamma| + |\Delta| \leq \min\{\psi + 1, n\}$. Multiply each constraint $\boldsymbol{\alpha_i}^T\boldsymbol{x}$ by *each* product $\prod_{\gamma\in\Gamma} x_\gamma \prod_{\delta\in\Delta}(1 - x_\delta)$ and produce a set of polynomial inequalities, for each Γ, Δ as above. Replace each square term x_i^2 by x_i (so that the constraints are multilinear) and linearize each product of monomials $\prod_{\zeta\in Z} x_\zeta$ by a new variable y_Z. Add to this set of inequalities all the constraints of the form

$$\prod_{\gamma\in\Gamma} x_\gamma \ \prod_{\delta\in\Delta}(1 - x_\delta) \equiv \sum_{\Lambda\subseteq\Delta}(-1)^{|\Lambda|} y_{\Gamma\cup\Lambda} \ \geq \ 0$$

Finally, Project F_ψ onto the original n-th dimensional space: $F_\psi^p = \{x \in [0, 1]^n : \exists y \in F_\psi^p, \ y_{\{i\}} = x_i\}$.

Now we will show that the integrality gap of $\mathcal{M}_c$ resists *any* constant number of rounds of the S-A operator by providing a particular family of graphs and a feasible solution vector for the t-level of the SA hierarchy with high fractional value with respect to the optimal integral one.

The integrality gap example: Consider the following graph $G = (V, E)$: for any $\ell \in \mathbb{N} \geq 3$, G has 2^ℓ vertices. It would help to visualize the vertices to be oriented clockwise on a circle. G has $2^{\ell-2}$ color classes $C_1, \ldots, C_{2^{\ell-2}}$. For each color class C_j we set the bound $w_j = 2(1-\epsilon)$, for some $\epsilon > 0$. Each color class C_j has 2ℓ edges (i.e. $|E_j| = 2\ell$). The total number of edges in G is $\ell \cdot 2^{\ell-1}$ and the degree of every vertex will be ℓ.

Now we define the edge sets E_j for each color class C_j. For each $j \in [2^{\ell-2}]$, the edges $\{\alpha 2^{\ell-2}+j, \alpha 2^{\ell-2}+j+1\}$ are in E_j, for each $\alpha \in \{0,1,2,3\}$. Basically we go around the circle and we consider the four chains of length $2^{\ell-2}$, and we paint consecutive edges with one of the $2^{\ell-2}$ available colors, in the same order for each chain. So, around the circle, each color appears in exactly four edges. Now we define the rest of the edges. For each vertex v_j with index $\alpha 2^{\ell-2}+j$ in each of the two first chains (i.e., the vertices from 1 till $2^{\ell-3}$ for the first chain and from $2^{\ell-3}+1$ to $2^{\ell-2}$ for the second chain) add $\ell-2$ edges as follows: Connect $\alpha 2^{\ell-2}+j$ with $\alpha 2^{\ell-1}+2^{\ell-1}+j-1+2i$, for the particular α and $i = 0,\ldots,\ell-3$. So, each vertex has degree $2+\ell-3+1 = \ell$ as desired.

Now we describe the color of the edges of the previously defined set of edges. For $\alpha \in \{0,1\}$, $i = 0,\ldots,\ell-3$ and $j = 1,\ldots,\ell$, the edges $\{\alpha 2^{\ell-2}+j, \alpha 2^{\ell-1}+j-1+2i\}$ are in color class $C_{j+2i+1(\mod \ell)}$. For each vertex, each of the $\ell-2$ edges have different color so, for each chain, we have $2^{\ell-2}$ (the number of vertices in each chain) $\times(\ell-2)$ (the extra edges per vertex we add in this second step) $=(\ell-2)2^{\ell-2}$ and since we have $2^{\ell-2}$ color classes, each appearing uniquely in each vertex, we conclude that the number of extra edges of color j we add on the second step is $\ell-2$, for a total of $2\ell-4$ edges (because we repeat the process on the two consecutive chains) plus 2 edges from the first step $= 2\ell$ edges of color C_j.

Observe that such graphs are actually bipartite.

Now define the vector $y \in F_\psi$ in $[0,1]^\eta$, $\eta = \sum_{q\in[\psi]}\binom{n}{q}$ as follows:

$$\boldsymbol{y} = \begin{cases} y_{\{\emptyset\}} = & 1 \\ y_{\{e\}} = & \frac{1-\epsilon}{\ell+(\psi-1)(1-\epsilon)}, \qquad \forall e \in E(G) \\ y_I \;\;= & 0, \qquad \forall I \subseteq [n], |I| \geq 2 \end{cases}$$

Lemma 1. *The vector y as defined above is feasible for the ψ-th level of the Sherali-Adams hierarchy applied on $\mathcal{M}_c$.*

Proof. In order to prove the claim of the lemma, it suffices to prove that all constraints defined by the ψ-th level of the Sherali-Adams hierarchy are satisfied by such a $\boldsymbol{y}$. We have the following sets of constraints:

Degree constraints: These corresponds to all the constraints

$$\Big(1 - \sum_{e\in\delta(v)} y_e\Big)\prod_{\gamma\in\Gamma} y_\gamma \prod_{\delta\in\Delta}(1-y_\delta) \equiv \Big(1 - \sum_{e\in\delta(v)} y_e\Big)\sum_{H\subseteq\Delta}(-1)^{|H|}y_{\Gamma\cup H} \geq 0$$

where $\Gamma, \Delta \subseteq [n]$: $\Gamma \cap \Delta = \emptyset$ and $|\Gamma|, |\Delta| \leq \psi$. This is still not a linear constraint. If we insist to fully linearize them, then they will take the form

$$\sum_{H\subseteq\Delta}(-1)^{|H|}y_{\Gamma\cup H} - \sum_{H\subseteq\Delta, e\in\delta(v)}(-1)^{|H|}y_{\Gamma\cup H\cup\{e\}} \geq 0.$$

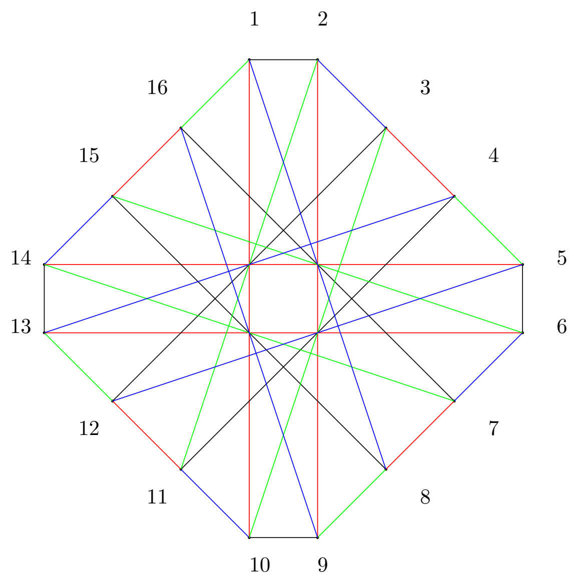

Fig. 1. An example of the graph constructed for $\ell = 4$. This graph has $2^4 = 16$ vertices, $2^{4-2} = 4$ color classes, and $2\ell = 8$ edges per color for a total of 32 edges. The edges are defined as follows: $E_{black} = \{\{1,2\},\{5,6\},\{9,10\}\{13,14\}, \{3,12\},\{4,11\},\{7,16\},\{8,15\}\}$, $E_{blue} = \{\{2,3\},\{6,7\}, \{10,11\},\{14,15\}, \{4,13\},\{5,12\},\{8,1\},\{9,16\}\}$, $E_{red} = \{\{3,4\},\{7,8\}, \{11,12\}\{15,16\}, \{1,10\},\{2,9\},\{5,14\},\{6,13\}\}$ and $E_{green} = \{\{4,5\},\{8,9\}, \{12,13\}\{16,1\}, \{2,11\},\{3,10\},\{6,15\},\{7,14\}\}$. By setting $w_j = 2(1-\epsilon)$ for some $\epsilon > 0$, we see that the optimal integral solution will select one edge per color for a total of 4. On the other hand, the LP can set the values on the variables (corresponding to edge) to $\frac{1}{\ell+\epsilon}$ for a total LP value $2^{\ell-2}(2\ell)\frac{1}{\ell+\epsilon} \approx 2^{\ell-1} = 8 - \epsilon'$ in our example.

Color bound constraints: Similarly, for all the color constraints we add all the constraints of the form

$$w_j \cdot \sum_{H \subseteq \Delta} (-1)^{|H|} y_{\Gamma \cup H} - \sum_{e \in C_j} \sum_{H \subseteq \Delta} (-1)^{|H|} y_{\Gamma \cup H \cup \{e\}} \geq 0$$

Non-negativity constraints: These are the constraints $1 - y_e \geq 0$ and $y_e \geq 0$, $\forall e \in E$. Identically with the previous cases, these constraints will become

$$\sum_{H \subseteq \Delta \cup \{e\}} (-1)^{|H|} y_{\Gamma \cup H} \geq 0 \quad \text{and} \quad \sum_{H \subseteq \Delta} (-1)^{|H|} y_{\Gamma \cup H \cup \{e\}} \geq 0$$

We will show that the vector $\boldsymbol{y}$ previously defined satisfies all the above constraints. First of all, it is immediate from the definition that $\boldsymbol{y}$ satisfies all the initial constraints. We will prove that it satisfies all the color bound constraints raising after t rounds. The rest can be shown to be true using identical arguments.

So, we have to show that for the defined $\boldsymbol{y}$ we have that

$$2(1-\epsilon)\sum_{H\subseteq \Delta}(-1)^{|H|}y_{\Gamma\cup H}-\sum_{H\subseteq \Delta, e\in C_j}(-1)^{|H|}y_{\Gamma\cup H\cup\{e\}}\geq 0.$$

We begin with the following technical but very helpful claim.

Claim. Given subsets Γ and Δ as above, we have that

$$\sum_{e\in E_j}\sum_{H\subseteq \Delta}(-1)^{|H|}y_{\Gamma\cup H\cup\{e\}}\leq \frac{1-\epsilon}{\ell+(t-1)(1-\epsilon)}|E_j|\sum_{H\subseteq \Delta}(-1)^{|H|}y_{\Gamma\cup H}$$

Proof of Claim: Indeed, if $|\Gamma|>2$ then, by definition, $y_{\Gamma\cup H\cup\{e\}}=0$ i.e., both terms are equal to zero and the inequality holds.

If $|\Gamma|=1$, we have to consider two cases:

(1) $|\Gamma|=\{e'\}$, for some $e'\in E_j$. In this case $|H|$ should be zero (otherwise the term is zero) and so

$$\sum_{H\subseteq \Delta, e\in C_j}(-1)^{|H|}y_{\Gamma\cup H\cup\{e\}}=y_{\{e'\}\cup\{e'\}}=y_{\{e'\}}=\frac{1-\epsilon}{\ell+(t-1)(1-\epsilon)}.$$

But in that case, the sum in the second term, $\sum_{H\subseteq\Delta}(-1)^{|H|}y_{\Gamma\cup H}=y_{\{e'\}}$ and so the second term becomes $\frac{|E_j|(1-\epsilon)^2}{(\ell+(t-1)(1-\epsilon))^2}$ which is greater than $\frac{1-\epsilon}{\ell+(t-1)(1-\epsilon)}$ (the first term) when $t\leq\ell$.

Now the second case: (2) $\Gamma=\{e'\}\notin E_j$. In this case the first some equals to zero, since we would have terms of the form $y_{e'\cup e}$ for $e'\neq e$ and, by definition, such terms are all zero. The right hand side of the above inequality will again become $\frac{|E_j|(\ell-\epsilon)^2}{(1+(t-1)(1-\epsilon))^2}\geq 0$, so again in this case the inequality is true.

Finally, if $\Gamma=\emptyset$ we have that $\Gamma\cup H\cup\{e\}=(\Gamma\cup\{e\})\cup H$. So, $\{e\}\notin J$ and again the first sum $\sum_{e\in E_j}\sum_{H\subseteq\Delta}(-1)^{|H|}y_{\Gamma\cup H\cup\{e\}}$ equals to $\frac{1-\epsilon}{1+(t-1)(1-\epsilon)}$. Since $\Gamma=\emptyset$, the sum in the second term is equal to the sum of the terms corresponding to $H=\emptyset$ and $|H\cap\Delta|=1$. In the first case the term becomes $(-1)^0y_\emptyset=1$ and in the second case $(-1)^1y_H$, where H is a single element set $\{h\}\in\Delta$ in this case. Summing these terms we see that

$$\sum_{H\subseteq\Delta}(-1)^{|H|}y_{\Gamma\cup H}=1-|\Delta|\underbrace{\frac{1-\epsilon}{\ell+(t-1)(1-\epsilon)}}_{\rho}=1-|\Delta|\rho$$

In order the inequality to be true, we require that $\rho\leq\rho|E_j|(1-|\Delta|\rho)\Rightarrow |\Delta|\rho\leq\frac{|E_j|-1}{|E_j|}$. Since $|\Delta|\leq t$ (because, by definition $|\Gamma\cup\Delta|\leq t$ and $\Gamma=\emptyset$), we have that

$$|\Delta|\rho\leq\frac{t(1-\epsilon)}{\ell+(t-1)(1-\epsilon)}=\frac{t}{\frac{\ell}{1-\epsilon}+t-1}$$

and since $|E_j| = 2\ell, \forall C_j$ we require that

$$\frac{t}{\frac{\ell}{1-\epsilon} + t - 1} \leq \frac{2\ell - 1}{2\ell} \Rightarrow t \leq \frac{2\ell}{1-\epsilon}(2\ell - 1).$$

So, in any case, we see that the inequality is true for any $\epsilon > 0$ and any $t \leq \ell$. $\square$

Let $A = w_j \sum_{H \subseteq \Delta} (-1)^{|H|} y_{\Gamma \cup H} - \sum_{H \subseteq \Delta, e \in C_j} (-1)^{|H|} y_{\Gamma \cup H \cup \{e\}}$ (the linearized version of the initial color bound constraint for color class C_j after multiplying it with $\prod_{\gamma \in \Gamma} x_\gamma \prod_{\delta \in \Delta}(1 - x_\delta)$). By using the previous claim we can very easily verify that

$$A \geq (w_j - |E_j|\rho)\Big(\sum_{H \subseteq \Delta} (-1)^{|H|} y_{\Gamma \cup H} \Big)$$

Claim. We have that $(w_j - |E_j|\rho) \sum_{H \subseteq \Delta} (-1)^{|H|} y_{\Gamma \cup H} \geq 0$.

This would immediately give us that also $A \geq 0$ and so we will be done showing that the defined $\boldsymbol{y}$ satisfied the corresponding constraint.

Proof of Claim: First, we immediately see that $(w_j - |E_j|\rho) \geq 0, \forall t \geq 1$. Indeed we have that

$$(w_j - |E_j|\rho) = 2(1-\epsilon) - 2\ell \frac{1-\epsilon}{\ell + (t-1)(1-\epsilon)} \geq 0$$

since $\ell + (t-1)(1-\epsilon) \geq \ell$, $\forall t \geq 1$. In other words, $\boldsymbol{y}$ satisfies the *initial* constraints corresponding to color bounds of the LP (trivially this is true for all the other initial constraints).

So, everything boils down in proving that $\sum_{H \subseteq \Delta} (-1)^{|H|} y_{\Gamma \cup H} \geq 0$. To prove that, we will again distinguish cases regarding the cardinality of Γ.

$|\Gamma| \geq 2$: In this case we have that $|\Gamma \cup H| > 2, \forall H \subseteq \Delta$ and so, by the definition of vector $\boldsymbol{y}$, we have that $y_{\Gamma \cup H} = 0$ i.e., the entire sum is zero and we are done.

$|\Gamma| = 1$: Then $\Gamma = \{e\}$ for some edge $e \in E(G)$. In that case, $\{e\} \notin \Delta$. Assume not, then $\Gamma \cap \Delta \neq \emptyset$, as required by the definition of the Sherali-Adams hierarchy. The reason of the disjoint requirement is that if we let the two sets to intersect, then simple calculations show that the whole sum is zero. Indeed, let $H \subseteq \Delta$ such that $\{e\} \notin H$. Then, the corresponding term in the sum becomes $(-1)^{|H|} y_{\Gamma \cup H}$. Consider now the term $H \cup \{e\}$. The corresponding term in the sum is now

$$(-1)^{|H|+1} y_{\Gamma \cup H \cup \{e\}} = (-1)^{|H|+1} y_{\Gamma \cup H \cup} = -y_{\Gamma \cup H}$$

(since $\{e\} \in \Gamma$, we have that $y_{\Gamma \cup H \ cup\{e\}} = y_{\Gamma \cup H}$).
So, the two terms cancel each other, and the whole sum is zero and we have that $\{e\} \notin \Delta$, so in fact $\Delta = \emptyset$ (otherwise $|\Gamma \cup \Delta| > 1 \Rightarrow y_{\Gamma \cup \Delta = 0}$). In that case, all the terms in the sum are of the form $(-1)^0 y_{\Gamma \cup \emptyset} \Rightarrow$ the sum is equal to $y_{\{e\}} = \rho \geq 0$.

$|\varGamma| = 0$**:** In that case, the only terms of the sum that survive are those for which $H = \emptyset$ and those for which $H = \{h\}$. In the first case, there is only one term to consider, namely $(-1)^0 y_{\varGamma\cup H} = y_{\{\emptyset\}} = 1$. In the second case, the terms are of the form $(-1)^1 y_{\varGamma\cup H} = -y_h = -\rho$, and we have $|\varDelta|$ many such terms. All in all, the sum becomes $1 - |\varDelta|\rho$ which means that

$$\sum_{H\subseteq\varDelta} (-1)^{|H|} y_{\varGamma\cup H} \geq 0 \Leftrightarrow 1 - \rho|\varDelta| \geq 0$$

.

Indeed, as before, we have that

$$\frac{|\varDelta|(1-\epsilon)}{\ell + (t-1)(1-\epsilon)} \leq \frac{t(1-\epsilon)}{\ell + (t-1)(1-\epsilon)} = \frac{t}{\frac{\ell}{(1-\epsilon)} + (t-1)} \leq 1$$

so that $1 - \rho|\varDelta| \geq 0, \forall t$, and so we are done.

We have proved that the the vector $\boldsymbol{y}$ defined previously, satisfies all the color bound constraints arising after at most ℓ rounds of the Sherali-Adams hierarchy. The rest of the constraints can be proved to be true for the same bounds of t in identical manner and the calculations are omitted in the current reading.

Since we have a feasible vector for the t-th level of the Sherali-Adams Hierarchy, the next task is to bound the value the objective function takes for this $\boldsymbol{y}$. We have that for t less than ℓ

$$\mathsf{value}(\boldsymbol{y}) = \sup_{\ell\to\infty} \lim_{\epsilon\to 0} \left(\frac{\ell 2^{\ell-1}(1-\epsilon)}{\ell + (t-1)(1-\epsilon)} \right) \geq \frac{2^{\ell-1}}{\frac{1}{1-\epsilon} + \frac{t-1}{\ell}} = 2^{\ell-1}$$

On the other hand, any integer solution can select at most one edge per color class and so the integer optimum is exactly $2^{\ell-2}$. So, we have that the integrality gap of at least 2 remains even after t rounds of the Sherali-Adams hierarchy applied on $\mathcal{M}_c$, for any $t \leq \ell$. Since there exists a $\frac{1}{2}$-approximation algorithm, this value is tight.

Theorem 1. *For any $\epsilon > 0$, we have that the integrality gap of the t-th level of the Sherali-Adams hierarchy for $\mathcal{M}_c$ is at least $\frac{2}{\frac{1}{1+\beta}}$, for $t = \beta\lambda$, for $\beta = o(1)$.*

3 The BCC Operator

Here we will show that the integrality gap of 2 of the natural LP formulation for BCM resists even for a *linear* number of rounds for the BCC operator. This is fact true even for the general LP with the blossom inequalities and even for bipartite graphs with the normal (degree constraints only) LP. In particular, we will show that the BCC [1] *rank* of the $\mathcal{M}_c$ polyhedron is large (linear). Recall the BCC operator applied to a polyhedron $P_0 = \{x \in [0,1]^n : \ Ax \leq \boldsymbol{\beta}\}$, $A \in \mathbb{R}^{m\times n}, \beta \in \mathbb{R}^m$:

BCC-1 Fix an index $i \in [n]$.
BCC-2 Multiply each constraint $\boldsymbol{a_j}^T x \leq \boldsymbol{\beta}_j$ with x_i and $(1-x_i)$ and obtain the quadratic system $x_i(Ax - \boldsymbol{\beta}) \leq 0, (1-x_i)(Ax-\boldsymbol{\beta}) \leq 0$.
BCC-3 Linearize the quadratic system: replace x_i^2 with x_i and $x_i x_{i'}$ with $y_{i'}$. Let $P_i(P_0) \equiv P_i$ the resulting (lifted) polyhedron.
BCC-4 Project P_i back to the original n-dimensional space.

The polyhedron P_i that the above process returns, for some index $i \in [n]$ has a remarkable property:

Theorem 2 ([1]). $P_i = conv\{P_0 \cap \{x \in [0,1]^n \text{ such that } x_i \in \{0,1\}\}\}$.

We can apply the previous procedure iteratively on indexes $(i_1, i_2, \ldots, i_k)$ for some $k \leq n$ thus defining a lifted polyhedron $P_{(i_1,\ldots,i_k)} = P_{i_k}(\cdots(P_{i_1}(P_0)))$ that has the property, similarly as before, that

$$P_{(i_1,\ldots,i_k)} = \mathsf{conv}(P_0 \cap x \in [0,1]^n : x_{i_1}, \ldots, x_{i_k} \in \{0,1\})$$

If $B \subset [n]$ we define

$$P_B = \mathsf{conv}\{x \in P_0 \wedge x_i \in \{0,1\},\ \forall i \in B\}$$

This implies that if we apply the BCC operator iteratively to all the n variables then we obtain a tight linear characterization of the convex hull of the integer solutions of P_0. Observe that the resulting polyhedron does not depend on the order of the indexes that we apply the operator. This allows us to write

$$P_{(i_1,\ldots,i_k)} = P_{\{i_1,\ldots,i_k\}} \Rightarrow P_{[n]} = \mathsf{conv}(P_0 \cap \{0,1\}^n)$$

The integrality gap example: Now, consider the following family of instances, $\mathcal{F}$, for Bounded Color Matching problem: let k copies of the C_4 graph, $k \in \mathbb{N}$, where C_4 is the usual 4-cycle. Let the i-th copy of C_4, C_4^i, $1 \leq i \leq k$, have vertices $\alpha_i^1, \alpha_i^2, \alpha_i^3, \alpha_i^4$. Let $E_i^r = \{\{\alpha_i^1, \alpha_i^2\}, \{\alpha_i^3, \alpha_i^4\}\}$ and $E_i^b = \{\{\alpha_i^1, \alpha_i^4\}, \{\alpha_i^2, \alpha_i^3\}\}$. Now, connect the i-th copy of C_4, C_4^i, $1 \leq i \leq k-1$ with the $(i+1)$-th as follows: add the edge $\{\alpha_i^2, \alpha_{i+1}^4\}$ and assign this edge with a new, color let's say c_{w1} and add the edge $\{\alpha_i^3, \alpha_{i+1}^1\}$ with color c_{w2}. All in all our graph has $4k$ vertices and $4k + 2(k-1)$ edges such that $E = E_{w1} \cup E_{w2} \cup (\bigcup_{i=1}^k (E_i^r \cup E_i^b))$ where $|E_{w1}| = |E_{w2}| = k-1$ and $|E_i^j| = 2$, $\forall i \in [k], j \in \{r, b\}$. We put bounds equal to 1 for every color class. Observe that the constructed graph is *bipartite*.

The optimal *integral* solution selects one edge per C_4 plus one edge from E_{w1} and one from E_{w2} for a total value of $k+2$. We will show that even after a large (linear) number of rounds, the BCC operator has a very high fractional value. Indeed, define $B = \bigcup_{i=1}^k (E_i^r \cup E_i^b) \subset E$ and consider the following solution vector:

$$\boldsymbol{x} = \begin{cases} x_e = \frac{1}{k-1}, & \forall e \in E_{w1} \\ x_e = 0, & \forall e \in E_{w2} \\ x_e = \frac{k-3}{2(k-1)} (< \frac{1}{2}), & \forall e \in B \end{cases}$$

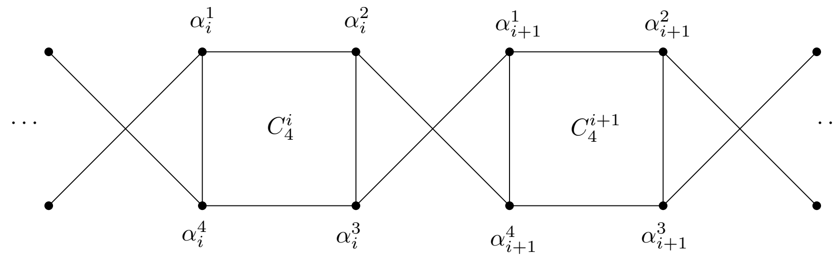

Fig. 2. A graph from $\mathcal{F}$.

It is immediate, by definition, that this solution vector belongs in $P_{E\setminus(B\cup E_{w1})} = P_{E_{w2}} = \mathsf{conv}\{x \in \mathcal{M}_c : \ x_e \in \{0,1\} \ \forall e \in E \setminus (B \cup E_{w1})\}$ and so it belongs in the $|E \setminus B|$-level of the BCC hierarchy. Moreover, its value is

$$k \cdot 4 \cdot \frac{k-3}{2(k-1)} + (k-1)\frac{1}{k-1} = 2k \cdot \frac{k-3}{k-1} + 1 \equiv 2k \text{ for large } k$$

and so we obtain the following:

Theorem 3. *If $G = (V, E)$ is a graph as constructed above, then the integrality gap of 2 for the $\mathcal{M}_c$ polyhedron resists a linear (in fact at least $\frac{m}{5} - 1$, $m = |E|$) number of rounds of the BCC operator applied to it, even for bipartite graphs, i.e. there are graphs where the integrality gap is at least $\frac{2k\cdot\frac{k-3}{k-1}+1}{k+2} \approx 2$ for large enough k.*

By experimenting slightly with the instance, we can construct graphs such that the *rank* of the BCC operator on $\mathcal{M}_c$ is at least $m-3$. (We remind that the rank is the maximum number of rounds in order to fully close the integrality gap).

4 The Effect of SA on $\mathcal{F}$

In this section we will show that the bad integrality gap instances, as captured by the family of graphs $\mathcal{F}$, cannot "fool" the SA hierarchy for more but a very small number of rounds.

For convenience, let us re-define $\mathcal{F}$ in an even simpler setting (instead of E_{w_1}, E_{w_2} we have only E_w): we have k copies of the following graph of 4 vertices, let's call it Σ. The i-th copy of Σ is denoted as Σ^i and has four vertices $V(\Sigma^i) = \{\gamma_i^1, \gamma_i^2, \gamma_i^3, \gamma_i^4\}$ and the following set of edges: $E_i^r = \{\{\gamma_i^1, \gamma_i^4\}, \{\gamma_i^2, \gamma_i^3\}\}$, $E_i^b = \{\{\gamma_i^1, \gamma_i^3\}, \{\gamma_i^2, \gamma_i^4\}\}$ and $E_i^g = \{\{\gamma_i^1, \gamma_i^2\}, \{\gamma_i^3, \gamma_i^4\}\}$. Connect Σ^i with Σ^{i+1} for $1 \le i \le k-1$, by connecting (for example) γ_i^1 with γ_{i+1}^1. Let $E_w = \bigcup_{i=1}^{k-1}\{\gamma_i^1, \gamma_{i+1}^1\}$. Set each color bound equal to 1 for every edge class. The optimal integral value is $k+1$ and the optimal LP value for the stronger $\mathcal{M}_c$ is $2k$.

Define $\Xi = E \setminus \bigcup_{i=1}^{k-1}\{\gamma_i^1, \gamma_{i+1}^1\}$ and observe that $|\Xi| = 6k$. Let $y \in F_\psi$ be a vector in $[0,1]^\eta$, $\eta = \sum_{q\in[\psi]}\binom{n}{q}$:

$$
\boldsymbol{y} = \begin{cases} y_{\{\emptyset\}} = & 1 \\ y_{\{e\}} = \frac{1}{3+\epsilon(\psi-1)}, & \forall e \in \Xi \\ y_{\{e\}} = \frac{\epsilon(\psi-1)}{3+\epsilon(\psi-1)}, & \forall e \in E_w \\ y_I \;\; = \;\; 0, & \forall I \subseteq [n], |I| \geq 2 \end{cases}
$$

In order to calculate the value of the vector $\boldsymbol{y}$, for appropriate values of ψ, we see that

$$
\begin{aligned}
\mathsf{value}(\boldsymbol{y}) &= \sup_k \lim_{\epsilon \to 0} \Big(\frac{6k}{3+\epsilon(\psi-1)} + (k-1)\frac{\epsilon(\psi-1)}{3+\epsilon(\psi-1)} \Big) \\
&= \sup_k \lim_{\epsilon \to 0} \Big(k\frac{6+\epsilon(\psi-1)}{3+\epsilon(\psi-1)} - \frac{\epsilon(\psi-1)}{3+\epsilon(\psi-1)} \Big) \\
&= 2k
\end{aligned}
$$

and as before the optimal integral solution is $k+1$.

Lemma 2. *The vector y as defined above is feasible for the ψ-th level of the Sherali-Adams hierarchy applied on $\mathcal{M}_c$.*

Proof. As usual, we need to show that the proposed vector satisfies all the corresponding constraints after their linearization. We begin with the non-negativity constrains. We will show that $\sum_{H \subseteq \Delta} (-1)^{|H|} y_{\Gamma \cup H} \geq 0$ which is equivalent to prove it for $|\Gamma| \leq 1$ since otherwise by definition $\sum_{H \subseteq \Delta} (-1)^{|H|} y_{\Gamma \cup H} = 0$ (the case of the other non-negativity constraint, namely $\sum_{H \subseteq \Delta} (-1)^{|H|} y_{\Gamma \cup H \cup \{e\}} \geq 0$ is much simpler because this constraint boils down to show that $y_e \geq 0$ which is trivially true).

So we assume that $|\Gamma| \leq 1$. If $|\Gamma| = 1$ and $\Delta \supseteq \Gamma$, it is not hard to show that $\sum_{H \subseteq \Delta} (-1)^{|H|} y_{\Gamma \cup H} = 0$. Indeed each term $y_{\Gamma \cup H'}$ will appear twice in the sum, each time with inverse sign: one time for $H' \subseteq \Delta \setminus \Gamma$ (i.e. $\Gamma \notin H'$) and one time for the term $H' \cup \Gamma$. Since in these two cases the cardinality of $H' \cup \Gamma$ differs by exactly 1, the two identical terms $y_{H' \cup \Gamma}$ will appear with inverse sign and hence will cancel each other forcing the total sum to be zero. This suggests that $\Gamma \notin \Delta$. But in this case observe that for $|H| > 0$, the corresponding term $y_{\Gamma \cup H}$ will be zero by definition. So the only surviving term is the term that corresponds to $H = \emptyset$ and in that case $y_{\Gamma \cup H} = y_\Gamma = \frac{1}{3+\epsilon(\psi-1)} \geq 0$ or $y_\Gamma = \frac{\epsilon(\psi-1)}{3+\epsilon(\psi-1)}$ which is again ≥ 0. Similarly, if $|\Gamma| = 0$, then the only surviving terms in the sum are those for which $|H| \leq 1$. The interesting case is when $|H| = 1$, since otherwise $y_{\Gamma \cup H} = y_\emptyset = 1$ by definition and the coefficient is 1. For $|H| = 1$, the sum is equal to

$$
\begin{aligned}
\sum_{H\subseteq\Delta}(-1)^{|H|}y_{\Gamma\cup H} &= \sum_{H\subseteq\Delta}(-1)^{|H|}y_H = y_\emptyset + \sum_{\delta\in\Delta} -y_\delta \\
&= y_\emptyset - \sum_{\delta\in\Delta\cap\Xi} y_\delta - \sum_{\ell\in\Delta\cap E_w} y_\ell \\
&= 1 - \frac{|\Phi|}{3+\epsilon(\psi-1)} - (|\Delta\setminus\Phi|)\frac{(\psi-1)\epsilon}{3+\epsilon(\psi-1)} \\
&\geq 1 - |\Delta|\frac{\max\{1,\epsilon(\psi-1)\}}{3+\epsilon(\psi-1)} \quad \geq \quad 1-\psi\cdot\frac{\max\{1,\epsilon(\psi-1)\}}{3+\epsilon(\psi-1)} \\
&= \frac{3+\epsilon(\psi-1)-\psi\max\{1,\epsilon(\psi-1)\}}{3+\epsilon(\psi-1)} = \Theta_1
\end{aligned}
$$

Now, we move forward to the task of proving that the degree and color bound constraints are satisfied by the proposed vector y and delivering a (weaker) bound on the number of rounds that this is true. In fact we will prove only the degree constrains, since the color bound constrains have identical structure (and they are even easier and they will give us an even better bound on ψ). At the end we will use the worst bound on ψ delivered by the degree constraints to give a global bound on the number of rounds ψ that satisfies the given constraints. Given the linearized versions of the degree constraints, and by following similar arguments as before, we have that

$$
\begin{aligned}
\sum_{H\subseteq\Delta}(-1)^{|H|}y_{\Gamma\cup\Delta} - \sum_{H\subseteq\Delta, e\in\delta(v)}(-1)^{|H|}y_{\Gamma\cup\Delta\cup\{e\}} &\equiv \\
y_\emptyset - \sum_{t\in T} y_t - y_{\{e\}} &\geq \\
1-\psi\cdot\frac{\max\{1,\epsilon(\psi-1)\}}{3+\epsilon(\psi-1)} - \frac{\epsilon(y-1)}{3+\epsilon(y-1)} &= \Theta_2
\end{aligned}
$$

Both expressions Θ_1 and Θ_2 involve the term $\max\{1,\epsilon(\psi-1)\}$. We will consider the two cases for this term:

$\max\{1,\epsilon(\psi-1)\} = 1:$ This implies that $\psi \leq \frac{\epsilon+1}{\epsilon}$. Since we require $\Theta_2 \geq 0$, we have that $\Theta_2 = 1-\psi\frac{1}{3+\epsilon(\psi-1)} - \frac{\epsilon(\psi-1)}{3+\epsilon(\psi-1)} \geq 0 \Rightarrow 3+\epsilon(\psi-1)-\psi-\epsilon(\psi-1) \geq 0 \Rightarrow \psi \leq 3$. This means that after 3 only rounds the SA hierarchy declares the proposed vector (that fools the starting LP) as infeasible.

$\max\{1,\epsilon(\psi-1)\} = \epsilon(\psi-1):$ This implies that $\epsilon(\psi-1)\geq 1 \Rightarrow \psi \geq \frac{\epsilon+1}{\epsilon}$ as lower bound for the number of rounds ψ. On the other hand, since we require that the quantity Θ_1 is ≥ 0, we have that its numerator should be ≥ 0 (since the denominator clearly is) and so

$$
3+\epsilon(\psi-1)-\psi(\epsilon(\psi-1)) \geq 0 \Rightarrow -\epsilon\psi^2+2\epsilon\psi+3 \geq 0 \Rightarrow \psi \leq \frac{2\epsilon+\sqrt{12\epsilon}}{2\epsilon}.
$$

So we have both a lower and an upper bound on the number of rounds ψ as function of ϵ:

$$\frac{\epsilon+1}{\epsilon} \leq \psi \leq \frac{2\epsilon+\sqrt{12\epsilon}}{2\epsilon}$$

The above is true only for $\epsilon \geq 7-4\sqrt{3} \approx 0.0718$. Since the function $f(\epsilon) = \frac{2\epsilon+\sqrt{12\epsilon}}{2\epsilon}$ is decreasing with ϵ, it attains its maximum value for $\epsilon = 7-4\sqrt{3}$ and has value ≈ 6.46. This means that again SA realizes after at most 7 rounds that the vector is not feasible for the integrality gap example, unlike the standard LP.

□

Given the symmetric properties of any graph $G \in \mathcal{F}$, the above can be immediately generalized for any possible fractional vector and any target integrality gap: after a very short number of rounds (linear function of the target integrality gap), the SA hierarchy applied to the family of graphs $\mathcal{F}$ would correctly output that the proposed symmetric fractional vector (that fools the initial LP) is not feasible and would output the correct integral solution, in extreme contrast to the BCC operator that can be "fooled" even for a linear number of rounds.

5 Conclusions

In this paper we provided strong integrality gap results for the Bounded Color Matching which is a case of the multi-budgeted matching problem. In particular, we proved that even a logarithmic (linear) number of rounds of the Sherali-Adams (BCC) hierarchy is not enough to improve the integrality gap of the natural LP relaxation of the problem to the slightest. The integrality gap instances instances for the SA hierarchy have the property that have uniform lengths and fractional bounds, similar to the Knapsack instances from [16]. Such simply structured instances provide a great challenge even for SA hierarchy to realize the core of their bad behaviour. Such intsances can be treated by simple "cutting" operations but SA is unable to realize that even after a very large number of rounds. On the other hand, it is a very interesting open question to provide examples with integral bounds with same behaviour for the SA hierarchy. So far, we are able to come up with instances that can fool the SA hierarchy for a constant number of rounds. The graph is the n-dimensional hypergraph with careful assignments of colours to the edges. Following a very similar pattern with the one employed, we have that SA can be fooled for $1+\sqrt{1+\frac{3+\epsilon}{\epsilon}}$ rounds, i.e., such many rounds are not enough to improve the integrality gap to the slightest.

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