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New hardness results for graph and hypergraph colorings

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Abstract

Finding a proper coloring of a t-colorable graph G with t colors is a classic NP-hard problem when $t \geq 3$. In this work, we investigate the approximate coloring problem in which the objective is to find a proper c-coloring of G where $c \geq t$. We show that for all $t \geq 3$, it is NP-hard to find a c-coloring when $c \leq 2t-2$. In the regime where t is small, this improves, via a unified approach, the previously best known hardness result of $c \leq \max\{2t-5, t+2\lfloor t/3 \rfloor -1\}$ [GJ76, KLS00, GK04]. For example, we show that 6-coloring a 4-colorable graph is NP-hard, improving on the NP-hardness of 5-coloring a 4-colorable graph.

We also generalize this to related problems on the strong coloring of hypergraphs. A k-uniform hypergraph H is t-strong colorable (where $t \geq k$) if there is a t-coloring of the vertices such that no two vertices in each hyperedge of H have the same color. We show that if $t = \lceil 3k/2 \rceil$, then it is NP-hard to find a 2-coloring of the vertices of H such that no hyperedge is monochromatic. We conjecture that a similar hardness holds for $t = k+1$.

We establish the NP-hardness of these problems by reducing from the hardness of the Label Cover problem, via a "dictatorship test" gadget graph. By combinatorially classifying all possible colorings of this graph, we can infer labels to provide to the label cover problem. This approach generalizes the "weak polymorphism" framework of [AGH14], though interestingly our results are "PCP-free" in that they do not require any approximation gap in the starting Label Cover instance.

1 Introduction

A t-coloring of a graph $G = (V,E)$ is a coloring of its vertices with t colors such that the endpoints of every edge receive distinct colors, i.e., a map $c : V \to \{1,2,\ldots,t\}$ such that or every $(u,v) \in E$, $c(u) \neq c(v)$. The chromatic number of a graph G, denoted $\chi(G)$, is the minimum t for which G admits a t-coloring. A graph G is said to be t-colorable if $\chi(G) \leq t$. For $t \geq 3$, finding a t-coloring of a t-colorable graph is one of the classic NP-hard problems. The problem remains difficult even when one is allowed to use many more colors. In fact, the best known efficient algorithms to color a 3-colorable graph require $n^{\Omega(1)}$ colors. However, the known NP-hardness results only rule out coloring a 3-colorable graph with a mere 4 colors [KLS00, GK04]. By an easy reduction on this implies the NP-hardness of coloring a t-colorable graph with $t+2\lfloor t/3 \rfloor -1$ colors. The work of Garey and Johnson [GJ76] gave an elegant reduction from 3-colorability using Kneser graphs to show NP-hardness of $(2t-5)$-coloring a t-colorable graph, for all $t \geq 6$.[1] Much stronger hardness results are known for larger t, and as well as

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[1]The applicability of this result to graphs with small chromatic number seems to have been somewhat overlooked in the literature.

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conditional hardness results for $t = 3,4$ under variants of the Unique Games conjecture; we review some of the literature on inapproximability of graph/hypergraph coloring in Section 1.1.

In this work, we prove the following NP-hardness result for coloring t-colorable graphs which improves the previous best known result in the challenging regime where t is small. The result holds for graphs whose degree is bounded by a function of t, which can be taken to be $5t^6$.

Theorem 1.1. *For every $t \geq 3$, it is NP-hard to distinguish, given an input graph G, whether $\chi(G) \leq t$ or $\chi(G) \geq 2t-1$. In particular, $(2t-2)$-coloring a t-colorable graph is NP-hard.*

While the above does not improve the state of affairs for $t = 3$, it does yield new results for other small t, such as the NP-hardness of 6-coloring a 4-colorable graph.[2] We also note that by plugging in the NP-hardness of telling if $\chi(G) \leq 3$ or $\chi(G) \geq 5$ from [KLS00, GK04] as the starting point in the reduction of Garey and Johnson [GJ76], together with bounds for multicoloring Kneser graphs [Sta98], one can show that it is NP-hard to $2t-3$-color a t-colorable graph for $t \geq 6$ (improving the $2t-5$ bound in [GJ76]).

The improvement in Theorem 1.1 is quantitatively modest, but we feel our proof methodology reveals insights into the source of the hardness, and also gives results stronger than previous works for small t in a *unified* manner. Our reduction is inspired by techniques used to show hardness of constraint satisfaction problems and employs dictatorship gadgets in a modular fashion, and the analysis hinges on combinatorial arguments to classify colorings of the gadget (more about our techniques in Section 1.2). It is worth pointing out that Theorem 1.1, as well as our results for hypergraph coloring below, are "PCP-free" in that they reduce from standard NP-hardness results for decision problems (as opposed to promise problems with an approximation gap in the optimal value). This is also true of the hardness results in [GJ76, GK04].

Hypergraph coloring. We also prove new hardness results for coloring hypergraphs. We will be interested in k-uniform hypergraphs for small k where each hyperedge has exactly k vertices. A hypergraph is *t-colorable* if its vertices can be colored with t colors so that there is no monochromatic hyperedge. We say that a hypergraph is *t-strong colorable* (or *t-partite*) if its vertices can be t-colored so that every hyperedge has no two vertices of the same color; in other words, it is "t-partite" with vertices partitioned into t parts so that every hyperedge has at most one vertex from each part. Note that t-strong coloring is equivalent to t-coloring the graph obtained by converting each hyperedge into a clique.

Compared to graph coloring, the situation for hardness results for hypergraph coloring is much better. We know that it is NP-hard to color a 2-colorable 3-uniform hypergraph with any constant number of colors [DRS05], and a recent line of work has led to quasi-NP-hardness of coloring 2-colorable n-vertex hypergraphs of $O(1)$-uniformity with $\exp(\Omega(\log^{0.1-o(1)} n))$ colors [DG13, GHH+14, KS14, Var14, Hua15], which is approaching the ballpark of polynomially many colors needed by current algorithms.

The 2-coloring problem is easy on hypergraphs $H = (V,E)$ which admit balanced partial colorings. Namely, if there are subsets $A, B \subset V$ such that for each $e \in E$, $|e \cap A| = |e \cap B|$, then one can efficiently find a 2-coloring of H that leaves no hyperedge monochromatic [McD93]. In particular, a t-uniform t-partite hypergraph, is easy to 2-color. However, even a slight relaxation of the perfect balance condition seems to render 2-coloring intractable. For example, with the promise that there is a near-balanced 2-coloring, finding a 2-coloring without monochromatic edges is still NP-hard [AGH14], and further even c-coloring is NP-hard for any constant c [GL15].

[2]Note that the NP-hardness of 6-coloring 4-colorable graphs would immediately follow from the (as yet unknown) NP-hardness of 5-coloring a 3-colorable graph by adding a new vertex adjacent to all nodes in the graph.

It will be really interesting to establish further powerful hardness results that show in some formal sense that 2-coloring is hard unless the perfect balance promise is met. Towards this end, the following ultra-strong conjecture postulates that the generally believed hardness of $O(1)$-coloring 3-colorable graphs extends to all strongly colorable hypergraphs with one more color than uniformity (i.e., just beyond the case of a perfectly balanced strong coloring).

Conjecture 1.2. *For all $k,c \geq 2$, $(k,c) \neq (2,2)$, given a $k+1$-strongly colorable k-uniform hypergraph, it is NP-hard to find a c-coloring of its vertices that leaves no hyperedge monochromatic.*

Note that a k-uniform hypergraph that is strongly colorable with $k+1$ colors is also 2-colorable, so the problem in the above conjecture makes sense for any $c \geq 2$. Note the conjecture would immediately yield as a corollary the NP-hardness of telling if a graph G has $\chi(G) \leq t$ or $\chi(G) > c$ for all $c,t \geq 3$, so in this form the conjecture might be well beyond current techniques. However, proving it for $c = 2$ would already be very interesting and this challenge might be within reach by developing more sophisticated analysis tools in the broader framework employed in this paper.

In this work, we prove the following hardness result for 2-coloring strongly colorable hypergraphs, which is the first such result for any promise of strong coloring that implies 2-colorability. Note that a k-uniform hypergraph that is t-strongly colorable for $t \leq 2k-2$ is also 2-colorable (as one can partition the $2k-2$ colors into two groups of $k-1$ and each hyperedge must have colors from both groups).

Theorem 1.3. *For $k \geq 3$, given a k-uniform hypergraph, it is NP-hard to tell if it is $\lceil \frac{3k}{2} \rceil$-strongly colorable or if it is not 2-colorable. Further, for $k = 3,4$, it is NP-hard to 2-color a $k+1$-strongly colorable k-uniform hypergraph.*

The proofs of this theorem can be found in Sections 3.1.1, 3.3, 4, and A.2. In addition, in Appendix A.3, using a simple Fourier-analytic argument, we note the hardness of a variant of $[k,k+1,2]$, k-odd, in which the sought after two-coloring must be *balanced* (have discrepancy 1) — note that such a balanced 2-coloring *exists* if the hypergraph is $k+1$-partite.

1.1 Prior related work

Towards describing the previous related results in a compact and easy to reference manner, we introduce the following expressive notation, which we will also use in the body of the paper. A k-uniform hypergraph is said to be *t-rainbow colorable* (for some $t \leq k$) if its vertices can be t-colored so that every hyperedge has vertices of every color (note that 2-rainbow colorability is the same as 2-colorability, and for larger t the notion gives a more structured coloring).

Definition 1.1. Let $t,k,c \geq 2$ be positive integers. Define *$[k,t,c]$-coloring* to be the following decision problem: Given a be a k-uniform hypergraph H, distinguish between the following two cases.

- YES: If $t < k$, G is t-rainbow colorable; or if $t \geq k$, G is t-strongly colorable. (Note that when $t = k$, t-rainbow and t-strong colorability are the same notion.)
- NO: H is not c-colorable.

Note that when $k = 2$, this is the well-known problem of deciding whether a graph can be colored with at most t colors or requires more than c colors. The algorithm to 2-color a hypergraph in the presence of a balanced partial coloring [McD93] shows that $[t,t,2]$-coloring is polynomial time solvable for all $t \geq 2$. The known results on the complexity of $[k,t,c]$-coloring are tabulated below. We will not discuss rainbow coloring further in the paper, but include it in the table below, which also includes two conjectures that 2-coloring is hard if the t-uniform t-strong/rainbow colorability is relaxed for either strong/rainbow coloring. The table does not include algorithmic results for graph/hypergraph coloring where the number of colors used is a function of the number of vertices, or recent hardness results which show hardness of hypergraph coloring with super-polylogarithmically many colors.

Problem	Parameters	Known Hardness	References
Graph coloring	$[2,t,2t-5]$	NP-hard	[GJ76]
	$[2,t,t+2\lfloor \frac{t}{3} \rfloor -1]$	NP-hard	[KLS00, GK04]
	$[2,t,2^{\Omega(t^{1/3})}]$, large t	NP-hard	[Hua13b]
	$[2,t,c]$, $c \geq t \geq 3$	UG-variant-hard	[DMR09, DS10]
	$[2,t,2t-2]$	NP-hard	this paper
k-uniform hypergraph coloring	$[k,k,2]$	in P	[McD93, Alo14]
	$[k,2,c]$, $k \geq 4$, $c \geq 2$	NP-hard	[GHS02]
	$[3,2,c]$, $c \geq 2$	NP-hard	[DRS05]
t-strong hypergraph coloring	$[3,4,2],[4,5,2],[\lfloor \frac{2t}{3} \rfloor,t,2]$, $t \geq 6$	NP-hard	this paper
	$[t-1,t,2]$, $t \geq 6$	NP-hard (conjectured)	this paper
t-rainbow coloring	$[2t,t,c]$, $t \geq 2$, $c \geq 2$	NP-hard	[GL15]
	$[t+1,t,2]$, $t \geq 3$	NP-hard (conjectured)	this paper

Note that we prove the hardness of $[3,4,2]$ and $[4,5,2]$ coloring separately, for $t \geq 6$, the challenge of proving hardness of $[t-1,t,2]$ coloring remains open.

We believe that our techniques can also be used to show that $[4,3,2]$-coloring and some other problems in the setting of rainbow coloring are NP-hard, but for simplicity and a focused presentation we decided to restrict our study to strong coloring in this version of the paper.

1.2 Techniques

Previous hardness results for approximate coloring of graphs with chromatic number bounded by a constant t fall into three categories:

- NP-hardness for small t, e.g. the $2t-5$-coloring hardness in [GJ76], or the hardness of 4-coloring for $t=3$: these are based on clever ad hoc reductions from some NP-hard coloring/independent set exact optimization problem (in [KLS00] an approximation version was needed, but the later proof in [GK04] required only hardness of exact independent set).
- NP-hardness of $f(t)$-coloring for large constant t, such as $f(t) = t^{\Omega(\log t)}$ [Kho01] or the current record $f(t) = \exp(\Omega(t^{1/3}))$ [Hua13b]: these are based on designing a PCP with very good query vs. soundness error trade-off and reducing to graph coloring via the FGLSS graph. These results also show that finding an independent set of density $1/f(t)$ is NP-hard, but they don't kick in until t is reasonably large.
- Hardness of $O(1)$-coloring for $t=3,4$ based on variants of the Unique Games Conjecture [DMR09]: these design a 2-query verifier checking the Not-Equal predicate that directly corresponds to graph coloring, and the soundness analysis, which shows that there is no large independent set, relies on appropriate invariance principles. The results showing hardness of $O(1)$-coloring hypergraphs [GHS02, Hol02, Kho02, DRS05] also proceed along this route, but since the Not-All-Equal predicate makes more than two queries, the PCP can be analyzed unconditionally using Fourier analytic tools of the sort pioneered by Håstad [Hås01].

The primary method used to obtain the hardness results in this work departs from the above approaches. We treat coloring as a constraint satisfaction problem (CSP), and our approach is inspired by techniques in the CSP dichotomy literature, where NP-hardness emerges due to the lack of non-dictator "polymorphisms" for the predicate. A polymorphism gives a way to combine several assignments satisfying the predicate into another satisfying assignment. Formally, for an arity k predicate $P \subseteq D^k$ over domain D, a polymorphism for P is a function $f : D^L \to D$ (for some arity L) such that for all $a_1, a_2, \ldots, a_L \in P$, applying f coordinate-wise to the i'th coordinates of $a_1, \ldots, a_L$ for $1 \le i \le k$, yields $b \in D^k$ that also belongs P. The dictator functions $f(z) = z_j$ for $j = 1, \ldots, L$ are trivially polymorphisms. If there are no other polymorphisms, then the associated CSP is NP-hard (this connection is folklore, and is mentioned in [BJK05] in a more algebraic language). For instance, if $P \subseteq \mathbb{Z}_3^2$ (for domain $\mathbb{Z}_3 = \{0,1,2\}$) is the predicate $\{(x,y) \mid x \neq y\}$, then the only polymorphisms are dictators (this is a nice exercise, and we will prove stronger forms of this for our results). As a result, the associated CSP, which is simply graph 3-colorability, is NP-hard.

Since we seek hardness even when one is allowed more colors, we work in the framework of "weak polymorphisms" from the recent work [AGH14] on hardness of satisfiability even when a near-balanced satisfying assignment exists. Here, the objects of study are relaxations of polymorphisms that map assignments satisfying a predicate P into those that satisfy a *weaker* predicate Q. For instance, to show hardness of 4-coloring 3-colorable graphs, we study functions $f : \mathbb{Z}_3^L \to \mathbb{Z}_4$ satisfying $f(x) \neq f(y)$ whenever $x_i \neq y_i \ \forall i \in \{1,2,\ldots,L\}$ (in other words, we study 4-colorings of a *dictatorship gadget graph* with vertex set $\mathbb{Z}_3^L$ where two nodes are adjacent precisely when they differ in every coordinate). With 4 colors we can no longer say that f must depend on only coordinate — indeed, we can start with a dictator 3-coloring and corrupt it by recoloring any independent set with the 4'th color. We prove that in fact this is the only thing that can happen — for some $c \in \mathbb{Z}_4$, f restricted to $\mathbb{Z}_4^L \setminus f^{-1}(c)$ is a dictator. For t-colorable graphs, we prove a similar statement classifying functions $f : \mathbb{Z}_t^L \to \mathbb{Z}_{2t-2}$ as comprising of a dictator function for t colors corrupted with $t-2$ independent sets.

Our proof of Theorem 1.1 follows the common paradigm of reducing from Label Cover, with dictatorship gadgets at each node and cross edges testing the projection constraints. However, our analysis ensures that one can decode, based on a $(2t-2)$-coloring of the resulting graph, a *unique* label to each vertex that satisfies *all* the label cover constraints. Therefore, as a starting point, we only need the NP-hardness of deciding if a Label Cover instance is satisfiable, and do *not* need a gap version based on PCPs. For the results in [AGH14], the functions which satisfy the dictatorship test are juntas which depend on few variables. This requires starting the reduction from a gap version of Label Cover, as the decoding of labels is not unique. On the other hand, the functions which pass the dictatorship test in [AGH14] are either exact juntas or very close to one, which interfaces nicely with the Label Cover reduction. The challenge in our setting is that the characterization reveals a dictator function corrupted with a *large* amount of "noise." This is because we have to test functions $f : \mathbb{Z}_t^L \to \mathbb{Z}_c$ with a larger range (with $c > t$, for graph coloring), and for hypergraph coloring, the weak 2-coloring predicate is much weaker than the strong t-coloring promise. For example, for our hardness result for $(2t-2)$-coloring t-colorable graphs, the dictator could be corrupted on almost a $1 - 2/t$ fraction of the hypercube by $t-2$ independent sets. However, the non-noisy portion has a nice structure which helps ensure that the decoded dictatorial coordinate is unique, and further satisfies the projection constraints in the Label Cover instance.

We also abstract a notion of robust decoding of a dictatorship test, which makes the interface with Label Cover more modular, and might help with future reductions based on dictatorship tests.

1.3 Discussion and Limitations

Although the techniques developed produce some new hardness results, there are technical limitations which prevent us from proving better hardness results. For graph coloring, there seem to be fundamental barriers preventing our robust decoding framework from being extended to $[2,t,2t]$. The primary challenge is that many colorings of the $[2,t,2t]$ dictatorship test involve nontrivial dependence in multiple coordinates. For example, consider $f_1 : \mathbb{Z}_t^L \to \mathbb{Z}_{2t}$ and $f_2 : \mathbb{Z}_t^L \to \mathbb{Z}_{2t}$ defined by $f_1(x) = x_1$ and $f_2(x) = x_2 + t$. Since the colorings of f_1 and f_2 use separate color sets, any 'interleaving' $f : \mathbb{Z}_t^L \to \mathbb{Z}_{2t}$, for which we choose $f(x) = f_1(x)$ or $f(x) = f_2(x)$ arbitrarily for each x, is a valid coloring of this dictatorship test, too. Furthermore, in Appendix B, we formalize this intuition, we show that there exists no robust decoder for the $[2,3,6]$-coloring gadget, which implies that our current methods cannot be used directly to show the NP-hardness of $[2,3,6]$-coloring. For similar but more subtle reasons, it also seems likely that no robust decoder exists for the $[2,3,5]$-coloring gadget either.

On possible remedy to this technical challenge would be to use a stronger variant of Label Cover known as *smooth* Label Cover. In smooth Label Cover, the edges and projection maps are guaranteed to have pseudorandom properties, allowing for weaker inner verifiers to obtain NP-hardness results. This variant of label cover has been able to prove the NP-hardness of approximation of problems for which the basic variant does not appear to suffice, (e.g., [Kho02, GRSW12, Hua13a, GL15]). Currently though, smooth Label Cover does not seem to be sufficient in itself to overcome these technical challenges.

On the other hand, to generalize hypergraph coloring, the primary challenge appears to be the opposite problem. For certain instances, such as $[5,6,2]$-coloring, we have a conjectured robust decoder which interfaces well with multipartite Label Cover, but at this time we are unable to determine a combinatorial proof that the robust decoder captures all colorings of the $[5,6,2]$ dictatorship test. We conjecture, albeit less confidently, that the situation is similar for $[t-1,t,2]$-coloring for $t \geq 6$

1.4 Paper Organization

Section 2 constructs the dictatorship gadgets and formally defines the notion of a *robust decoder* of a gadget. Section 3 combinatorially proves the existence of robust decoders for a variety of gadgets. Section 4 uses label cover reductions similar to that of [AGH14] to prove the main theorems. Appendix A contains proofs omitted from Section 3, including a combinatorial classification of the $[4,5,2]$-dictatorship test. Appendix B shows that our techniques cannot directly obtain the NP-hardness of 6-coloring a 3-colorable graph.

2 Preliminaries

2.1 The $[k,t,c]$-coloring Gadget

Adapting the techniques of [AGH14], to prove hardness results we use a label cover reduction with a combinatorial gadget as an inner-verifier long code test. Generalizing the Boolean hypercube, we construct our long code with the *tensor product* of hypergraphs (e.g., [ADFS04]).

Definition 2.1. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be a k-uniform hypergraphs. The *tensor product* $G \otimes H$ of G and H is the k-uniform hypergraph on vertex set $V_G \times V_H$ such that for all $(g_1, \ldots, g_k) \in E_G$ and $(h_1, \ldots, h_k) \in E_H$ and for all permutations $\sigma : [k] \to [k]$, $((g_1, h_{\sigma(1)}), \ldots, (g_k, h_{\sigma(k)}))$ is an edge of $G \otimes H$.

We let $\otimes^n G$ to denote the tensor product of n copies of G. The most common graph we will be taking the tensor product of is the complete k-uniform hypergraph on t vertices (when $k \le t$), which we denote K_t^k. We identify the vertices of K_t^k with $\mathbb{Z}_t$ and the edges with k-element subsets of $\mathbb{Z}_t$.

Definition 2.2 (Dictatorship test gadget)**.** Let $k,t,c \ge 2$ be positive integers such that $k \le t$ $c \ge t/(k-1)$ and let $L \ge 1$ be an integer. The *dictatorship gadget for* $[k,t,c]$ *on* L *labels* is the k-uniform hypergraph $\otimes^L K_t^k$, the vertices of which we identify with $\mathbb{Z}_t^L$. A *valid coloring* of the dictatorship gadget is a function $f : \mathbb{Z}_t^L \to \mathbb{Z}_c$ such that for all k element subset $S \subseteq \mathbb{Z}_t^L$ which corresponds to an edge of $\otimes^L K_t^k$, $|\{f(x) : x \in S\}| \ge 2$. If f is a valid coloring, then we say that f *satisfies the* $[k,t,c]$*-coloring gadget*.

The constraint $c \ge t/(k-1)$ guarantees that a k-uniform, t-strongly colorable hypergraph has a c-coloring. Note that if $c < t/(k-1)$, then K_t^k is not c-colorable.

We can identify the hyperedges of $\otimes^L K_t^k$ with the k-tuples $(x^{(1)},\dots,x^{(k)})$ of $\mathbb{Z}_t^L$ such that for all $i \in [L] = \{1,2,\dots,L\}$, $x_i^{(1)},\dots,x_i^{(k)}$ are all distinct. Since it is tedious to refer to the underlying graph of a gadget coloring, we formulate a simple syntactic way to check if f satisfies the $[k,t,c]$-coloring gadget. First, we define the notion of being *disjoint* which captures the idea of strong coloring.

Definition 2.3. Let $L \ge 1$. A subset $S \subseteq \mathbb{Z}_t^L$ is *disjoint* if $|S| \le t$ and for all $i \in [L], |\{x_i : x \in S\}| = |S|$. Similarly, we say that $x,y \in \mathbb{Z}_t^L$ are *disjoint* if $\{x,y\}$ is a disjoint subset.

It is easy to verify that the following definition of a coloring of the dictatorship gadget is equivalent.

Proposition 2.1. *Let $k,t,c \ge 2$ and $L \ge 1$ be positive integers such that $k \le t$, $c \ge t/(k-1)$. Let $f : \mathbb{Z}_t^L \to \mathbb{Z}_c$ be a function. We have that f satisfies the $[k,t,c]$-coloring gadget if and only if for all $S \subset \mathbb{Z}_t^L$ such that $|S| = k$ and S is disjoint, we have that $|\{f(x) : x \in S\}| \ge 2$.*

In our label cover reduction (see Lemma 4.3), a $[k,t,c]$-coloring gadget will be a long code test for a specific vertex of the label cover instance. To represent the edges, we need to construct a $[k,t,c]$-coloring *co-gadget*. This co-gadget is analogous to the edge constraints of [AGH14].

Definition 2.4. Let $k,t,c,L1$ be positive integers such that $k \le t$, $c \ge t/(k-1)$. Let $f_1, f_2 : \mathbb{Z}_t^L \to \mathbb{Z}_c$ be functions. We say that $\{f_1, f_2\}$ satisfy the $[k,t,c]$ *co-gadget* if for all disjoint $A \subseteq \mathbb{Z}_t^L$ such that $|A| = k$, and for all partitions[3] $A = A_1 \cup A_2$,

$$|\{f_1(x) : x \in A_1\} \cup \{f_2(x) : x \in A_2\}| \ge 2.$$

Notice that if f and g satisfy the $[k,t,c]$ co-gadget, then f and g must both satisfy the $[k,t,c]$ gadget.

2.2 Decoding of Gadgets

With the dictatorship gadgets formulated, we move on to define what it means to *decode* the coloring of a gadget. Toward this, we need to formally define what *dictators* and *juntas* are.

Definition 2.5 (Dictators)**.** A function $f : \mathbb{Z}_t^L \to \mathbb{Z}_c$ is a *dictator* if there exists $i \in [L]$ and $g : \mathbb{Z}_t \to \mathbb{Z}_c$ such that $f(x) = g(x_i)$ for all $x \in \mathbb{Z}_t^L$.

Definition 2.6 (Juntas)**.** A function $f : \mathbb{Z}_t^L \to \mathbb{Z}_c$ is a ℓ*-junta* if there exists a $S \subseteq [L]$ with $|S| = \ell$ and $g : \mathbb{Z}_t^S \to \mathbb{Z}_c$ such that $f(x) = g(x|^S)$ for all $x \in \mathbb{Z}_t^L$, where $x|^S$ is the restriction of x to entries indexed by S.

[3]Some sets in the partition may be empty.

If f satisfies the $[k,t,c]$ gadget and is a dictator we would like to *decode* f into some small subset $S \subseteq [L]$ of coordinates which dictate f's behavior the most. In general though, f is not necessarily a dictator or an ℓ-junta for small ℓ, but it will often be quite close to one. This motivates the following definition.

Definition 2.7. For a fixed choice of $k,t,c \geq 2$, A *decoder* is a family[4] of functions $\mathsf{Dec} = \{\mathsf{Dec}^L_{[k,t,c]} : (\mathbb{Z}_t^L \to \mathbb{Z}_c) \to \mathscr{P}([L]) : L \in \mathbb{N}\}$ satisfying the following properties.

- (nontrivial) For all $f : \mathbb{Z}_t^L \to \mathbb{Z}_c$ satisfying the $[k,t,c]$ gadget, $\mathsf{Dec}(f) \neq \emptyset$.
- (sensible) If f depends on the coordinates $S \subseteq [L]$, then $\mathsf{Dec}(f) \subseteq S$. In particular, if f is dictated by the ith coordinate, then $\mathsf{Dec}(f) = \{i\}$.
- (compatible) For all pairs $f, g : \mathbb{Z}_t^L \to \mathbb{Z}_c$, which satisfy the $[k,t,c]$ co-gadget, $\mathsf{Dec}(f) \cap \mathsf{Dec}(g) \neq \emptyset$.
- (bounded) There exists a constant $C = C(k,t,c)$ independent of L, such that $|\mathsf{Dec}(f)| \leq C$ for all choices of f.

We say that $\mathsf{Dec}(f)$ is the *decoding* of f.

Due to the technical details of our label cover reduction, to obtain NP-hardness results we need our decoder to have one additionally property, that the decoding of f needs to compose nicely with *projections*.

Definition 2.8. Let $f : \mathbb{Z}_t^L \to \mathbb{Z}_c$ be a function. Let $\pi : [L] \to [L]$ be a projection. Define the *restriction of f with respect to π*, denoted $f \square \pi : \mathbb{Z}_t^L \to \mathbb{Z}_c$ to be the unique map satisfying

$$(f \square \pi)(x) = f(y) \text{ where } y_i = x_{\pi(i)} \text{ for all } i \in [L].$$

In other words, $f \square \pi$ applies f after 'copying' coordinates in the image of π to coordinates with that projection.

Note that if f satisfies the $[k,t,c]$ gadget, then $f \square \pi$ also satisfies the $[k,t,c]$ gadget for all π.

Definition 2.9. We say that a decoder $\mathsf{Dec} = \mathsf{Dec}_{[k,t,c]}$ is *robust* if for all $f : \mathbb{Z}_t^L \to \mathbb{Z}_c$ satisfying the $[k,t,c]$ gadget and all projections $\pi : [L] \to [L]$, $\mathsf{Dec}(f \square \pi) \subseteq \pi[\mathsf{Dec}(f)]$.

3 Colorings of Dictatorship Gadgets

Now that have constructed our dictatorship gadgets/co-gadgets and defined the notion of a robust decoder, we proceed to demonstrate to prove that a number of $[k,t,c]$-coloring gadgets indeed have robust decoders.

3.1 Small Examples

To better understand gadget colorings, we often examine subsets $S \subseteq \mathbb{Z}_t^L$ such that no pair of elements of S are disjoint. We call such an S an *independent set*. For any $S \subseteq \mathbb{Z}_t^L$ we let the *clique number* $\omega(S)$ denote the size of the largest disjoint subset of S. The following fact relates the clique number of S to the density of S in $\mathbb{Z}_t^L$.

[4]For $f : \mathbb{Z}_t^L \to \mathbb{Z}_c$, we use $\mathsf{Dec}(f)$ or $\mathsf{Dec}_{[k,t,c]}(f)$ as a shorthand for $\mathsf{Dec}^L_{[k,t,c]}(f)$.

Claim 3.1. *For any $S \subseteq \mathbb{Z}_t^L$,*

$$\frac{|S|}{|\mathbb{Z}_t^L|} \leq \frac{\omega(S)}{t} \tag{1}$$

with equality if and only if the indicator function for S is a dictator.

Remark. For independent sets ($\omega(S) = 1$), this claim is well-known (e.g., [GL74]). See [ADFS04] for a particularly elegant proof involving Fourier analysis.

Proof. See Appendix A. □

3.1.1 Strong coloring: $[3,4,2]$ gadget

Using Claim 3.1, we may easily classify the colorings of $[3,4,2]$ gadget. Recall that this gadget corresponds to the 2-coloring of 3-uniform 4-strong colorable graphs.

Claim 3.2. *Let $f : \mathbb{Z}_4^L \to \mathbb{Z}_2$ satisfy the $[3,4,2]$-coloring gadget. Then, there exists $i \in [n]$ and $S \subset \mathbb{Z}_4$ such that $|S| = 2$ and for all $x \in \mathbb{Z}_4^L$, $f(x) = 0$ iff $x_i \in S$.*

Proof. From the definition of the gadget $\omega(f^{-1}(0)), \omega(f^{-1}(1)) \leq 2$. Thus, $|f^{-1}(0)|, |f^{-1}(1)| \leq \frac{1}{2}|\mathbb{Z}_4^L|$. Since $\mathbb{Z}_4^L = f^{-1}(0) \cup f^{-1}(1)$, we must have that $|f^{-1}(0)| = |f^{-1}(1)| = \frac{1}{2}|\mathbb{Z}_4^L|$. Thus, the indicator function for $|f^{-1}(1)|$ must be a dictator in the ith coordinate, implying that f is a dictator in the ith coordinate. This implies the conclusion of the claim. □

Since any f which satisfies this gadget must be a dictator, the natural choice for a decoder $\mathsf{Dec}_{[3,4,2]}$ is to decode the index of the dictatorial coordinate.

Claim 3.3. *There exists a robust decoder $\mathsf{Dec} = \mathsf{Dec}_{[3,4,2]}$ for $[3,4,2]$-coloring.*

Proof. Let $\mathsf{Dec}(f) = \{i\}$, where i is the coordinate that f is a dictator. Clearly Dec is nontrivial, sensible, and bounded. To establish that Dec is compatible, assume for sake of contradiction f_1 and f_2 satisfy the $[3,4,2]$-coloring co-gadget, but f_1 and f_2 are dictated by different coordinates. Without loss of generality, we may assume that f_1 is dictated by the first coordinate, and f_2 is dictated by the second coordinate. Furthermore, we may assume without loss of generality that $f_1(x) = 0$ if and only if $x_1 \in \{0,1\}$ and that $f_2(x) = 0$ if and only if $x_2 \in \{0,1\}$. But then, we may select $A_1 = \{(0,2,0,0,\ldots,0),(1,3,1,1,\ldots,1)\}$ and $A_2 = \{(2,0,2,2,\ldots,2)\}$ such that $A_1 \cup A_2$ is disjoint, but $f_1(A_1) \cup f_2(A_2) = \{0\}$, violating the $[3,4,2]$-coloring co-gadget, a contradiction. Thus, Dec is indeed compatible. Therefore, Dec is a decoder.

Note that if f is a dictator in coordinate i, then for any projection $\pi : [L] \to [L]$, $f \square \pi$ is a dictator in coordinate $\pi(i)$. Thus, $\mathsf{Dec}(f \square \pi) \subseteq \pi(\mathsf{Dec}(f))$, establishing that Dec is robust, as desired. □

If we combine Claim 3.3 with Lemma 4.3, we have that $[3,4,2]$-coloring is NP-hard.

In Appendix A.2, we give a combinatorial classification of the $[4,5,2]$ dictatorship gadget.

3.1.2 Graph coloring: $[2,3,4]$ gadget

Next, we classify the $[2,3,4]$ gadget which corresponds to problem of coloring a 3-colorable graph with 4 colors. This will form the base case for our more general $[2,t,2t-2]$-hardness result in Section 3.2. The following lemma is a key ingredient in the proof of our main result.

Lemma 3.4. *Let $f : \mathbb{Z}_3^L \to \mathbb{Z}_4$ satisfy the $[2,3,4]$ gadget. Then, there exists $a \in \mathbb{Z}_4$ such that f restricted to $\mathbb{Z}_3^L \setminus f^{-1}(a)$ is a dictator.*

Proof. We say that three points $x, y, z \in \mathbb{Z}_3^L$ form an *axis-parallel line* if x, y, z differ in exactly one coordinate.

Claim 3.5. *There does not exist $x \in \mathbb{Z}_3^L$ and lines $\{x, y, z\}$ and $\{x, y', z'\}$ such that both lines are axis-parallel to the coordinate axes and each line takes on 3 distinct values with respect to f.*

For ease of notation, when referring to a subset of $\mathbb{Z}_3^L$ we may concatenate digits to indicate an ordered tuple. For example,

- 012 is the ordered tuple $(0,1,2)$.
- $12\{0,1\}^2$ is the set $\{(1,2,0,0),(1,2,0,1),(1,2,1,0),(1,2,1,1)\}$
- $\{12,21\} \times \{0\}$ is $\{(1,2,0),(2,1,0)\}$.

Proof. If $L = 1$, the claim is trivial. Assume for sake of contradiction that such an x exists. Without loss of generality, such x is $0\dots0$ the two axis-parallel lines differ in the first and second coordinates. Thus, we may assume without loss of generality that

$$f(0\dots0) = 0 \qquad f(10\dots0) = 1 \qquad f(20\dots0) = 2\ .$$

We may also assume without loss of generality that

$$f(010\dots0) = 1.$$

Since 01 and 10 are disjoint, $L = 2$ is impossible. Now we have two cases.
Case 1, $f(020\dots0) = 2$. Notice that we may then deduce that

$$f(12\{1,2\}^{L-2}) = 3 \qquad f(21\{1,2\}^{L-2}) = 3\ .$$

Since there are disjoint elements in $\{12,21\} \times \{1,2\}^{L-2}$, we have a contradiction.
Case 2, $f(020\dots0) = 3$. Now, we may deduce that

$$f(11\{1,2\}^{L-2}) = 1 \tag{2}$$
$$f(12\{1,2\}^{L-2}) = 3 \tag{3}$$
$$f(21\{1,2\}^{L-2}) = 2. \tag{4}$$

Notice that this implies that

$$f(00\mathbb{Z}_3^{L-2}) = 0. \tag{5}$$

From this, we may deduce that

$$f(\{22\} \times \mathbb{Z}_3^{L-2}) \subseteq \{2,3\}. \qquad \text{[using (2) and (5)]}$$

If there exists, $x, y \in \{0,1\}^2 \times \mathbb{Z}_3^{L-2}$ such that $f(x) = 2$ and $f(y) = 3$, then there is some $z \in \{22\} \times \mathbb{Z}_3^{L-2}$ that is disjoint from both x and y, a contradiction. Notice then, due to symmetry, we may assume without loss of generality that

$$f(x) \neq 2, x \in \{0,1\}^2 \times \mathbb{Z}_3^{L-2}. \tag{6}$$

Therefore, we have that

$$f(01\mathbb{Z}_3^{L-2}) \subseteq \{0,1\}. \qquad \text{[using (3) and (6)]}$$

Thus, similar logic, there cannot be $x, y \in \{10\} \times \mathbb{Z}_3^{L-2}$ such that $f(x) = 0$ and $f(y) = 1$. Since $f(10\ldots0) = 1$, we have that

$$f(x) \neq 0, x \in \{10\} \times \mathbb{Z}_3^{L-2}. \tag{7}$$

Therefore,

$$f(\{10\} \times \mathbb{Z}_3^{L-2}) \subseteq \{1,3\}. \qquad \text{[using (4) and (7)]}$$

But, this is at odds with $f(020\ldots0) = 3$ and $f(010\ldots0) = 1$, leading to a contradiction. $\square$

Claim 3.6. *If there exist $x_0, x_1, x_2 \in \mathbb{Z}_3^L$ which each differ in exactly one coordinate (that is, form a line parallel to an axis) and $f(x_0) \neq f(x_1) = f(x_2)$, then f satisfies the conclusion of Lemma 3.4.*

Proof. Assume without loss of generality that $x_0 = 0\ldots0$, $x_1 = 10\ldots0$, and $x_2 = 20\ldots0$, and

$$f(x_0) = f(0\ldots0) = 0 \qquad f(x_1) = f(10\ldots0) = 1 \qquad f(x_2) = f(20\ldots0) = 1\ .$$

From this, we may deduce that

$$f(\{1,2\}^L) \subseteq \{2,3\}. \tag{8}$$

Additionally, any two disjoint points of $\{1,2\}^L$ must take on different values with f. Consider any $y_0, y_1 \in \{1,2\}^L$ which differ in exactly one coordinate but also differ with respect to f:

$$f(y_0) = 2, f(y_1) = 3.$$

Let y_2 be the third point on the axis-parallel line between y_0 and y_1, and let y_0' and y_1' be the points of $\{1,2\}^L$ disjoint from y_0 and y_1, respectively. Thus,

$$f(y_0') = 3, f(y_1') = 2.$$

Since y_2 is disjoint from both y_0' and y_1', we have that

$$f(y_2) \in \{0,1\}.$$

Hence, we have deduced that for any $y_0, y_1 \in \{1,2\}^L$ which differ in exactly one coordinate and also differ with respect to f, the third point on the line between them must also differ with respect to f. Therefore, by Claim 3.5, we necessarily have that for all $y \in \{1,2\}^L$, there is at most one $z \in \{1,2\}^L$ differing with f in exactly one coordinate with $f(y) \neq f(z)$.

From this fact, we can assign a function $g : \{1,2\}^L \to [L] \cup \{\text{None}\}$ such that if $y, z \in \{1,2\}^L$ are neighbors but $f(y) \neq f(z)$ then $g(y) = g(z)$ is the coordinate they differ in. Furthermore, if $g(y) \neq \text{None}$, we claim that for all $w \in \{1,2\}^L$ which differ by y in exactly one coordinate, $g(y) = g(w)$. If it were the case $g(y) \neq g(w)$, let $y_0, w_0 \in \{1,2\}^L$ which differ from y and w in the $g(y)$th coordinate. Then $f(y) = f(w) = f(w_0) \neq f(y_0)$, implying that $g(y_0)$ should be both $g(y)$ (due to y_0)and $g(w)$ (due to w_0), a contradiction.

Since the Hamming graph on $\{1,2\}^L$ connected and f takes on multiple values in $\{1,2\}^L$, we have that g takes on a non-None value at at least one point and since the Hamming graph is connected, we must have that g is a constant function. Thus, f restricted to $\{1,2\}^L$ is a dictator. Let i be the coordinate in which f restricted to $\{1,2\}^L$ is a dictator. And assume without loss of generality due to (8) that

$$x \in \{1,2\}^L, f(x) = 2 \text{ if and only if } x_i = 1.$$

From this and (8), we may deduce that for a general $x \in \mathbb{Z}_3^L$,

$$\begin{aligned} x_i = 0 &\text{ then } f(x) \in \{0,1\} \\ x_i = 1 &\text{ then } f(x) \in \{0,1,2\} \\ x_i = 2 &\text{ then } f(x) \in \{0,1,3\}. \end{aligned}$$

Notice that if there are $x, y \in \mathbb{Z}_3^L$ (not necessarily disjoint) with $x_i, y_i \neq 0$ such that $f(x) = 0$ and $f(y) = 1$, then we can identify at least one $z \in \mathbb{Z}_3^L$ with $z_i = 0$ such that z is disjoint from x and y, implying that $f(z) \notin \{0,1\}$, a contradiction. Thus, without loss of generality, we can say that $f(x) \neq 1$ if $x_i \neq 0$. Therefore, f restricted to $\mathbb{Z}_3^L \setminus f^{-1}(0)$ is a dictator in the ith coordinate ($f(x_i) = x_i + 1$), as desired. $\square$

(Back to proof of Lemma 3.4) Assume for sake of contradiction that there is a counterexample. From Claim 3.6, we know that no three of $y_0, y_1, y_2 \in \mathbb{Z}_3^L$ differing in exactly one coordinate which take on two distinct values with respect to f. But, we also know from Claim 3.5 that no $x \in \mathbb{Z}_3^L$ has two axis-parallel lines through it that take on 3 different values. This implies that for each $x \in \mathbb{Z}_3^L$, there is at most coordinate for which changing x changes the value of f. As in the proof of Claim 3.6, we may construct $g : \mathbb{Z}_3^L \to [L] \cup \{\text{None}\}$ such that if x and x' are neighbors which differ then $g(x) = g(x')$ is the coordinate they differ in. Again, for all $x \in \mathbb{Z}_3^L$ such that $g(x) \neq \text{None}$, we have that all the neighbors (at Hamming distance 1) of x must take on the same value for g. Since f is not constant, g takes on at least one non-None value. Thus by a flood-fill argument, g must be a constant function. Hence, f must be a dictator, implying that there is no counterexample, as desired. $\square$

We wait to show that the $[2,3,4]$ gadget has a robust decoder until we establish the generalization for the $[2,t,2t-2]$ gadget.

3.2 Graph coloring: The general case

Lemma 3.7. *Let $f : \mathbb{Z}_t^L \to \mathbb{Z}_{2t-2}$ satisfy the $[2,t,2t-2]$-colorability gadget where $t \geq 3$. Then there exists $S \subset \mathbb{Z}_{2t-2}$ such that $|S| = t-2$ and f restricted to $\mathbb{Z}_t^L \setminus f^{-1}(S)$ is a dictator.*

Since by Claim 3.1, we have that for all $c \in \mathbb{Z}_{2t-2}$, $|f^{-1}(c)| \leq t^{L-1}$. Thus, by 'discarding' only $t-2$ of the colors, we have an understanding of the structure of at least $2/t$ fraction of the coloring. This is enough structure to obtain NP-hardness in our label cover reduction in Section 4.

Proof. We proceed by induction on t. The base case $t = 3$ follows from Lemma 3.4.

Claim 3.8. *If $t \geq 4$ and f restricted to $\mathbb{Z}_{t-1}^L$ satisfies the $[2,t-1,2t-4]$-colorability gadget and we assume the result is true for $t-1$, then f satisfies the conclusion of the lemma.*

Proof. From the inductive hypothesis, we have that there is $S' \subseteq \mathbb{Z}_{2t-4}$ with $|S'| = (t-1) - 2 = t-3$ such that f restricted to $\mathbb{Z}_{t-1}^L \setminus f^{-1}(S')$ is a dictator. Since $(t-1) + (t-3) = 2t-4$, there is a subset $S = \mathbb{Z}_{2t-4} \setminus S' \subseteq \mathbb{Z}_{2t-4}$ of size $t-1$ such that f restricted to $f^{-1}(S) \cap \mathbb{Z}_{t-1}^L$ is a dictator in some coordinate. Assume without loss of generality that $S = \{0, 1, \ldots, t-2\}$. Additionally, we may assume that $f(x) = x_1$ when $x \in f^{-1}(S) \cap \mathbb{Z}_{t-1}^L$. Thus, f in this restricted domain is a dictator in the first coordinate.

Still working in $\mathbb{Z}_{t-1}^L$, we let T_i be the set of colors in the image of f with respect to the set of points where the first coordinate is i. More formally, $T_i = \{f(x) : x \in \mathbb{Z}_{t-1}^L, x_1 = i\}$. Since $T_i \subset \mathbb{Z}_{2t-4}$ and by our assumption $T_i \cap S = \{i\}$, we have that $|T_i| \leq t-2$ for all $i \in \mathbb{Z}_{t-1}$.

As a key part of our inductive step, for each $i \in \mathbb{Z}_{t-1}$, we seek to select a color $c_i \in T_i$ such that for *all* $x \in \mathbb{Z}_t^L$, $f(x) = c_i$ implies that $x_1 = i$. Note that it might be the case that $c_i \neq i$. Assume for sake of contradiction that there exists $i \in \mathbb{Z}_t$ such that for all $c \in T_i$, there is $x^{(c)} \in \mathbb{Z}_t^L$ with $x_1^{(c)} \neq i$ but $f(x^{(c)}) = c$. Since $|T_i| \leq t-2$, there must $z \in \mathbb{Z}_{t-1}^L$ which is disjoint from every element of the set $\{x^{(c)} : c \in T_i\}$. Since we stipulated that $x_1^{(c)} \neq i$ for all $c \in T_i$, we may select that $z_1 = i$. Since $z \in \mathbb{Z}_{t-1}^L$ and $x_1 = i$, by definition of T_i, $f(z) \in T_i$. But, by definition of z, z is disjoint from $x^{(f(z))}$, so $f(z) \neq f(x^{f(z)}) = f(z)$, a contradiction. Thus, for all $i \in \mathbb{Z}_{t-1}$, we can find an *exclusive* color c_i; that is, $f(x) = c_i$ implies $x_1 = i$ for all $x \in \mathbb{Z}_t^L$.

To complete the claim, it suffices to find a color c_{t-1} such that $f(x) = c_{t-1}$ implies $x_1 = t-1$. Let $T_{t-1} = \mathbb{Z}_{2t-2} \setminus \{c_i : i \in \mathbb{Z}_{t-1}\}$. That is, T_{t-1} is set of colors that are not already exclusive. Thus, if $x \in \mathbb{Z}_t^L$ and $x_1 = t-1$, then we must have that $f(x) \in T_{t-1}$. It is easy to see that $|T_{t-1}| = 2t-2-(t-1) = t-1$. Assume for sake of contradiction that an exclusive color c_{t-1} does not exist. Thus, for all $c \in T_{t-1}$, there is $y^{(c)}$ such that $f(y^{(c)}) = c$ but $y_1^{(c)} \neq t-1$. Thus, we may select $z \in \mathbb{Z}_t^L$ disjoint from every element of $\{y^{(c)} : c \in T_{t-1}\}$. Furthermore, since $y_1^{(c)} \neq t-1$ for all $c \in T_{t-1}$, we can let $z_1 = t-1$. By choice of z, we have that $f(z) \notin T_{t-1}$. Thus, $f(z) \in \mathbb{Z}_{2t-2} \setminus T_{t-1} = \{c_i : i \in \mathbb{Z}_{t-1}\}$. Thus, $f(z) = c_i$ for some $i \in \mathbb{Z}_{t-1}$, implying that $z_1 = i$, a contradiction. Therefore, there is $c_{t-1} \in T_{t-1}$ such that $f(x) = c_{t-1}$ implies that $x_1 = t-1$, as desired.

Hence, f restricted to $f^{-1}(\{c_i : i \in \mathbb{Z}_t\})$ is a dictator, as desired. $\square$

Consider any axis-parallel line ℓ of $\mathbb{Z}_t^L$. If there exists $x, y, z \in \ell$ such that $f(x) \neq f(y) = f(z)$, then the $(t-1)^L$ subgrid disjoint from x cannot have either $f(x)$ or $f(y)$ in the image of f. Thus this subgrid satisfies the $[2, t-1, 2t-4]$ gadget and then we are done by Claim 3.8.

Thus, every axis-parallel line must be entirely distinct or entirely the same. Next, we seek to show that any counterexample cannot contain two perpendicular axis-parallel lines which take on entirely distinct values. Without loss of generality, we may assume that for all $i \in \mathbb{Z}_t$, $f(i0\ldots0) = i$. We may also assume that $f(010\ldots0) = c \in \{1, \ldots t-1\}$ since there are only $t-2$ values in $\{t, \ldots, 2t-3\}$ and $f(0\ldots0) = 0$. Now notice that $f(c2\{1, \ldots, t-1\}^{L-2}) \subseteq \{t, t+1, \ldots, 2t-3\}$ Thus, this $(t-1)^{L-2}$ grid can only take on $t-2$ values. In order for every axis-parallel line to be completely the same or completely distinct, we must have that $f(c2\mathbb{Z}_t^{L-2}) = c_2$ for some $c_2 \notin \mathbb{Z}_t$. Similarly, $f(c\ell\mathbb{Z}_t^{L-2}) = c_\ell$ for some $c_\ell \notin \mathbb{Z}_t$. Since $f(c0\ldots0) = c$, we cannot have any two c_i be equal. Thus, we may assume without loss of generality that $c_k = t+k-2$ for all $k \geq 2$. Since $t \geq 4$, we must have that $f(c1\{1, \ldots, t-1\}^{L-2})$ cannot take on any element in $\{t, \ldots, 2t-3\}$ without forcing a non-distinct, non-homogeneous axis-parallel line. Thus, $f(c1\{1, \ldots, t-1\}^{L-2})$ can only take on the value c. Since every axis-parallel line through at least two points in $c1\{1, \ldots, t-1\}^{L-2}$ must take on the value c, by an inductive argument we can deduce that $f(c1\mathbb{Z}_t^{L-2}) = c$. Hence, $f(c0\ldots0) = f(c10\ldots0) = c$ but $f(c20\ldots0) = c_2 \neq c$, a contradiction. Thus, any counterexample cannot contain two perpendicular axis-parallel lines taking on distinct values.

Clearly f cannot be constant. Thus, there is an least one distinct line. Using an argument quite similar to the one in the proof of Lemma 3.4, for any x in this line, the neighbors of x (those at Hamming distance at most 1 away) must also be on an axis-parallel-line in the same direction. Thus, f is forced to be a dictator, as desired. $\square$

Now that we understand the $[2, t, 2t-2]$ gadget well, we can now establish the existence of robust decoders.

Lemma 3.9. *For all $t \geq 3$, the $[2, t, 2t-2]$ gadget has a robust decoder* Dec.

Proof. For any $f : \mathbb{Z}_t^L \to \mathbb{Z}_{2t-2}$, let $\mathsf{Dec}(f) \subseteq [L]$ be the set of coordinates i such that there is a t-element subset $S \subset \mathbb{Z}_{2t-2}$ such that f restricted to $f^{-1}(S)$ is a dictator. We call S a *witness* for i. We now show that our decoder meets all of the conditions of Definitions 2.7 and 2.9.

- nontrivial: From Lemma 3.7, we know that $\mathsf{Dec}(f) \neq \emptyset$.
- sensible: If $i \in \mathsf{Dec}(f)$, let S be a witness for i. For every $x \in f^{-1}(S)$ and all x' such that x' and x only differ in the ith coordinate, $f(x) \neq f(x')$. Thus, f has dependence in the ith coordinate.
- bounded: We claim that $|\mathsf{Dec}(f)| = 1$ always, implying boundedness. Assume that there exist $i \neq j$ such that $i, j \in \mathsf{Dec}(f)$. Let S_i and S_j be the subsets of t colors such that f restricted to $f^{-1}(S_i)$ and $f^{-1}(S_j)$ are dictators in the ith and jth coordinate respectively. Let $A = S_i \cap S_j$. Clearly $|A| \geq |S_i| + |S_j| - |\mathbb{Z}_{2t-2}| = 2$. Let $B_i = \{x_i : f(x) \in A\}$ and $B_j = \{x_j : f(x) \in A\}$. It is easy to see that $|B_i| = |B_j| = |A|$. Now consider $K = \{x \in \mathbb{Z}_t^L : x_i \in B_i, x_j \in B_j\}$. It is easy to see that $|K|/|Z_t^L| = |B_i||B_j|/t^2 = |A|^2/t^2$. By definition of i and j, we can see that K is disjoint from $f^{-1}((S_i \cup S_j) \setminus A)$.

  We claim that for any color $c \in \mathbb{Z}_{2t-2}$, $|f^{-1}(c) \cap K| \leq |K|/|A|$. The proof of this is similar to the proof of Claim 3.1. Without loss of generality, assume that $B_i = B_j = \mathbb{Z}_{|A|}$. We then can see the following is a partition of K into disjoint cliques of size K

  $$\bigcup_{x \in K, x_i = 0} \{(x, x + (1, \dots, 1), \dots, x + (|A| - 1, \dots, |A| - 1))\},$$

  where addition in the ith and jth coordinates is modulo $|A|$.

  Also, for any $c \in A$, it is apparent that $|f^{-1}(c) \cap K| \leq |K|/|A|^2$. Combining these two facts, $|f(K)| \geq 2|A| - 1$ (the $|A|$ colors in A and the additional $|A| - 1$ colors needed by the bound above). Thus, $2t - 2 \geq |f(K)| + |(S_i \cup S_j) \setminus A| \geq 2|A| - 1 + 2t - 2|A| = 2t - 1$, a contradiction. Thus, $|\mathsf{Dec}(f)| = 1$, as desired.
- compatible: Consider any $f, g : \mathbb{Z}_t^L \to \mathbb{Z}_{2t-2}$ which satisfy the $[2, t, 2t-2]$ co-gadget. We claim that $\mathsf{Dec}(f) = \mathsf{Dec}(g)$. Consider $h : \mathbb{Z}_t^{L+1} \to \mathbb{Z}_{2t-2}$ such that $h(x, 0) = f(x)$ and $h(x, i) = g(x)$ for all $i \in \mathbb{Z}_t \setminus \{0\}$. Since f and g satisfy the $[2, t, 2t-2]$ co-gadget, h satisfies the $[2, t, 2t-2]$ gadget. Thus, we can decode h in a unique coordinate i. Let S be the witness for i of h. If $i = L + 1$, then $h(\mathbb{Z}_t^L \times \{0\}) = f(\mathbb{Z}_t^L)$ can only have $2t - 2 - (t - 1) = t - 1$ colors, a contradiction. Thus, $i \in [L]$. It is easy then to see that S is also a witness for f and g. Thus, due to unique decoding, $\mathsf{Dec}(f) = \mathsf{Dec}(g) = \{i\}$, as desired.
- robust: Consider any projection $\pi : [L] \to [L]$. For $\{i\} = \mathsf{Dec}(f)$, let S our witness. Since f is a dictator with respect to $f^{-1}(S)$, $f \Box \pi$ is a dictator in coordinate $\pi(i)$ respect to $(f \Box \pi)^{-1}(S)$. Thus, S is a witness of $\pi(i)$ for $f \Box \pi$, $\mathsf{Dec}(f \Box \pi) = \{\pi(i)\}$. Thus, $\mathsf{Dec}(f \Box \pi) \subseteq \pi(\mathsf{Dec}(f))$.

Hence, the $[2, t, 2t-2]$ gadget has a robust decoder. $\square$

3.3 $[k, \lceil 3k/2 \rceil, 2]$ combinatorial characterization

To obtain hardness results for our hypergraph coloring problem, we first prove a characterization of the two-colorings of the strong hypergraph coloring dictatorship gadget.

Lemma 3.10. *Let $f : \mathbb{Z}_t^L \to \mathbb{Z}_2$ satisfy the $[k, t, 2]$ gadget, where $\lceil t/2 \rceil + 1 \leq k \leq \lfloor 2t/3 \rfloor$. Then, there exists $i \in [L]$ and dictator-bounding $D_j \subseteq \mathbb{Z}_t$ for all $j \in [L]$ such that*

$$\begin{pmatrix} 0 & 1 & & & & \\ & & 0 & & & \\ & & 1 & & & \\ & & & \blacksquare & \blacksquare & \blacksquare \\ & & & \blacksquare & \blacksquare & \blacksquare \\ & & & \blacksquare & \blacksquare & \blacksquare \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & & & & \\ \blacksquare & \blacksquare & 0 & 0 & 1 & 1 \\ \blacksquare & \blacksquare & 0 & 0 & 1 & 1 \\ \blacksquare & \blacksquare & 0 & 0 & 1 & 1 \\ \blacksquare & \blacksquare & 0 & 0 & 1 & 1 \\ \blacksquare & \blacksquare & 0 & 0 & 1 & 1 \end{pmatrix}$$

Figure 1: Illustration of the proof of Lemma 3.10 in the case $L=2, k=4, t=6$. The grids represent values deduced of $f : \mathbb{Z}_6^2 \to \{0,1\}$. Left: if $f(00) = f(21) = 0$ and $f(10) = f(22) = 1$, it is impossible to fill in the values of $\{3,4,5\}^2$ without forcing a monochromatic 4-uniform hyperedge. Right: if $f(00) = 0$, $f(10) = 1$, and $f : \{2,3,4,5\} \times \{1,2,3,4,5\} \to \{0,1\}$ is a 'balanced' dictator in the first coordinate, then in order to avoid a monochromatic 4-uniform hyperedge, $f(0x) = 0$ and $f(1x) = 1$ for all $x \in \{1,2,3,4,5\}$.

- $|D_j| = 2t-2k+2$ *for all* $j \in [L]$
- *there is a function* $g : D_i \to \mathbb{Z}_2$ *such that for all* $x \in \mathbb{Z}_t^L$ *such that* $x_j \in D_j$ *for all* j, $f(x) = g(x_i)$. *Furthermore* $|g^{-1}(0)| = |g^{-1}(1)| = t-k+1$.

To motivate the structure of the proof, we first handle the special case $t = 2k-2$.

Proof of $t = 2k-2$ case. We seek to show that f must be a dictator with an even split of 0s and 1s. Assume without loss of generality that

$$f(00\ldots0) = 0, f(10\ldots0) = 1$$

Now, consider f restricted to $S = \{2,\ldots,t-1\} \times \{1,\ldots,t-1\}^{L-1}$. If this is not a dictator in the first coordinate, then we may select axis parallel $x,y \in S$ which are the same in the first coordinate such that $f(x) = 0, f(y) = 1$. It is easy to see that there is a disjoint set of size $t-3 = 2k-5$ which is disjoint from all of $\{0\ldots0, 10\ldots0, x, y\}$. Thus are at least $\lceil (t-3)/2 \rceil = k-2$ points in this disjoint set which are of the same value. Adding to this set either $\{0\ldots0, x\}$ or $\{10\ldots0, y\}$, we have that there is a disjoint set of size k all of which take on the same value with respect to f, a contradiction. Thus, f restricted to S must be a dictator in the first coordinate. In order to avoid any disjoint sets of size k, we must have that f restricted to S has an equal number of 1s and 0s.

Now, take any axis-parallel pair $x,y \in S$ differing in the first coordinate such that $f(x) = 0$ and $f(y) = 1$. Using the same argument (where we replace $00\ldots0$ with x and $10\ldots0$ with y), we have that the set of points disjoint from x and y must be a dictator in the first coordinate. Applying this fact to every such x and y in S, we can see that f restricted to $\{0,1\} \times \mathbb{Z}_t^{L-1}$ is a dictator in the first coordinate with $f(0\mathbb{Z}_t^{L-1}) = 0$ and $f(1\mathbb{Z}_t^{L-1}) = 1$. Next, if we consider all axis-parallel pairs $x \in \{0\} \times \mathbb{Z}_t^{L-1}, y \in \{1\} \times \mathbb{Z}_t^{L-1}$, we may deduce that f restricted to $\{2,\ldots,t-1\} \times \mathbb{Z}_t^{L-1}$ is a dictator in the first coordinate. Thus, f is a dictator in the first coordinate with an equal number of 0 and 1s (in order to avoid a $(t/2+1)$-sized monochromatic hyperedge), as desired. □

Full argument. We proceed by strong induction on t. Our base case $k=4, t=6$ is handled above.

Let $x^1 = 0\ldots0, y^1 = 10\ldots0$. Assume without loss of generality that $f(x^1) = 0, f(y^1) = 1$. Now, consider the subgrid $T = \{2,\ldots,t-1\} \times \{1,\ldots,t-1\}^{L-1}$ which is disjoint from x^1 and y^1.

Claim 3.11. *There exists $D'_j \subseteq \mathbb{Z}_t$ $(j \in [L])$ such that $|D'_j| \geq 2t-2k$ for all j, f restricted to the Cartesian product $\bigtimes_{j \in L} D'_j \subseteq T$ is a dictator in some coordinate ℓ. Furthermore, exactly $t-k$ values of D'_ℓ set f to 0.*

Proof. First, assume we can find axis-parallel $x^2, y^2 \in T$ such that $f(x^2) = 0, f(y^2) = 1$, but the coordinate x^2 and y^2 differ in is not the first coordinate. Without loss of generality assume that $x^2 = 21\dots1$ and $y^2 = 221\dots1$. Now consider $T' = \{3, \dots t-1\}^L$ which is disjoint from x^1, x^2, y^1, and y^2. Clearly f restricted to T' satisfies the $[k-2, t-3, 2]$ gadget since if f were to have a disjoint set of size $k-2$ in T', either $\{x^1, x^2\}$ or $\{y^1, y^2\}$ could be augmented to yield a disjoint set of size k which is constant with respect to f. Clearly if $\lfloor 2t/3 \rfloor \geq k$ then $\lfloor 2(t-3)/3 \rfloor \geq k-2$. The last thing we need to check to apply the induction hypothesis is that $t-3 \geq 6$. We can find a disjoint set of size $\lceil (t-3)/2 \rceil$ which is constant with respect to f, so $k-2 > \lceil (t-3)/2 \rceil$. It is easy to check this is false if $t < 9$ since $k \leq \lfloor 2t/3 \rfloor$, yielding a contradiction. Thus, $t \geq 9$, so $t-3 \geq 6$ so we may use the induction hypothesis to find D'_j which satisfy the claim.

Now, assume that no such $x^2, y^2 \in T$ exist. Then, f restricted to T is a dictator in the first coordinate. All we need to check is that there are at least $t-k$ choices for the first coordinate so that f restricted to T is equal to 0 (respectively 1) in the first coordinate. If we cannot find $t-k$ choices for, without loss of generality 0, then there are at least $(t-2)-(t-k-1) = k-1$ choices for 1. Since there are at least $t-2$ (which is at least $k-1$) choices in each of the other coordinates, we can select $k-1$ disjoint points which take on the value 1 within T. If we add in y^1, we have a set of k disjoint points which take on the value 1 within T. □

Thus, we have that f restricted to $T' = \bigtimes_{j \in [L]} D'_j$ is a dictator in the ℓth coordinate for ℓ. We would like to let $D'_j = \mathbb{Z}_t$ in every coordinate except ℓ. If this is not the case, then there exists $x \in \mathbb{Z}_t^L$ such that $x_\ell \in D'_\ell$, but $f(x)$ is not equal to the the common value of $f(y)$ where $y \in T'$ and $x_\ell = y_\ell$. Let T_x be the subset of $\mathbb{Z}_t^L$ disjoint from x. We claim that f restricted to T_x satisfies the $[k-1, t-1, 2]$ gadget. If there were a disjoint set of size $k-1$ within T_x which all take on the value $f(x)$, then adding in x, we get a disjoint set of size k which all take on the value $f(x)$, a contradiction. Now, consider the case when there is a disjoint set of size $k-1$ within T_x which all take on the value $1-f(x)$, Since $\lfloor 2t/3 \rfloor \geq k$, $2(t-k) > k-1$, and no point in this set take on the value x_ℓ in the ℓth coordinate, there exists $y \in T'$ such that $y_\ell = x_\ell$ which is disjoint from every element of this set. Since $f(y) = 1-f(x)$, we have that there then is a disjoint of size k such that f takes on a constant value, yielding a contradiction. Thus, f restricted to T_x satisfies the $[k-1, t-1, 2]$ gadget. Since $k-1 > \lceil (t-1)/2 \rceil$ and $k-1 \leq \lfloor 2(t-1)/3 \rfloor$, we have by the induction hypothesis there is a grid of width at least $2((t-1)-(k-1))+2 = 2(t-k)+2$ which has the desired properties. Thus, we are done.

Hence, we may now assume that $D'_j = \mathbb{Z}_t$ for all $j \neq \ell$. Since $2(t-k) \geq 2$, we may select x^ℓ, y^ℓ such that $f(x^\ell) = 0, f(y^\ell) = 1$ and $x^\ell_\ell, y^\ell_\ell \in D'_\ell$. Repeating the same argument where we replace x^1, y^1 with x^ℓ, y^ℓ we may deduce that either we have the desired conclusion or there exists a set $D''_m \subset \mathbb{Z}_t$ of size $2(t-k)$ such that f is a dictator in the mth coordinate when the mth coordinate is in D''_m If $m \neq \ell$, then $f(x)$ must be a function of only x_m (respectively x_ℓ) when $(x_m, x_\ell) \in D''_m \times D'_\ell$. Since both dictators take on both the value 0 and 1, this is impossible. Thus, $\ell = m$. Let $D_\ell = D'_\ell \cup D''_\ell$. Since $x^\ell_\ell, y^\ell_\ell \notin D''_\ell$ we have that $|D_\ell| \geq 2(t-k)+2$, and there are at least $t-k+1$ values of D_ℓ which take on 0 with respect to f (respectively 1). For the other D_j's we can select an arbitrary $2(t-k)+2$ element subset of $\mathbb{Z}_t$. □

Corollary 3.12. *If $k \leq \lceil 2t/3 \rceil$, then the choice of i is unique.*

Proof. For sake of contradiction, imagine that there are $i \neq i'$ and and families D_j and D'_j satisfying the properties of Lemma 3.10. For all $j \in [L]$, we have that $|D_j \cap D'_j| \geq |D_j| + |D'_j| - t = 3t - 4k + 4$. Note that f restricted to $\bigtimes_j (D_j \cap D'_j)$ is a dictator in i and a dictator in i'. Thus, f must be a constant function. Since the dictator is evenly split between 0s and 1s, we have that $|D_i \cap D'_i| \leq t - k + 1$. Thus, $3t - 4k + 4 \leq t - k + 1$ or $k \geq 2t/3 + 1$, which contradicts the assumed bound on k. □

Thus, if $k \leq \lfloor 2t/3 \rfloor$, a natural choice for $\mathsf{Dec}(f)$ is this unique i for which there is a large "sub-dictator." We now show that this is indeed a dictator.

Claim 3.13. *Let $f : \mathbb{Z}_t^L \to \mathbb{Z}_2$ satisfy the $[k,t,2]$ gadget for $k \leq \lfloor 2t/3 \rfloor$. Let $\mathsf{Dec}(f)$ be the unique $i \in [L]$ from Corollary 3.12. Then,* Dec *is a decoder.*

Proof. From Lemma 3.10 and Corollary 3.12, we have that Dec is nontrivial, sensible, and bounded. It remains to prove that Dec is compatible. Assume for sake of contradiction that there exists $f, g : \mathbb{Z}_t^L \to \mathbb{Z}_2$ which satisfy the $[k,t,2]$ co-gadget, but $\{i\} = \mathsf{Dec}(f) \neq \{i'\} = \mathsf{Dec}(g)$. Let $D_j \subseteq \mathbb{Z}_t$ and $D'_j \subseteq \mathbb{Z}_t$ be the sets guaranteed by Lemma 3.10 for f and g, respectively. Additionally, let $T_0 \subset D_i$ for which f takes on the value 0. Note that $|T_0| = t - k + 1$, so $|D'_i \setminus T_0| \geq (2t - 2k + 2) - (t - k + 1) = t - k + 1$. Thus, there exists disjoint $A_g \subset \mathbb{Z}_t^L$ such that $|A_g| = t - k + 1$, and $g[A_g] = \{0\}$. Since $|A_g| = t - k + 1$ and $a_i \notin T_0$ for all $a_i \in A_g$, we have that we may select $A_f \subset \mathbb{Z}_t^L$ such that $|A_f| = t - k + 1$, $f[A_f] = \{0\}$, and $A_f \cup A_g$ is disjoint. Since f and g satisfy the $[k,t,2]$ co-gadget, we have that $k - 1 \geq |A_f \cup A_g| = 2t - 2k + 2$. Thus, $k \geq 2t/3 + 1$, a contradiction. □

For this decoder to work well with our label cover reduction in Section 4, we need to show that this decoder is robust.

Lemma 3.14. *Let $f : \mathbb{Z}_t^L \to \mathbb{Z}_2$ satisfy the $[k,t,2]$ gadget for $k \leq \lfloor 2t/3 \rfloor$. Then for all projections $\pi : [L] \to [L]$,*

$$\pi[\mathsf{Dec}(f)] = \mathsf{Dec}(f \square \pi).$$

That is, Dec *is robust.*

Proof. Assume for sake of contradiction that there exists π and $i \neq i'$ such that $\mathsf{Dec}(f) = \{i\}$, $\mathsf{Dec}(f \square \pi) = \{i'\}$ and $\pi(i) \neq i'$. Let $D_j \subseteq \mathbb{Z}_t$ be the family of 'dictator-bounding' sets of f guaranteed by Lemma 3.10, and likewise let $D'_j \subseteq \mathbb{Z}_t$ be the corresponding family of sets for $f \square \pi$. Let $T_0 \subset D_i$ be the subsets of values for which f takes on the value 0 in the Cartesian product of the D_ℓ's. Similarly, let $T'_0 \subset D'_j$ be the subset of values for which $f \square \pi$ takes on the value 0 in the Cartesian product of the D'_ℓ's.

Note that $|T_0| = |T'_0| = t - k + 1$. Thus, $|D'_{\pi(i)} \setminus T_0| \geq 2t - 2k + 2 - (t - k + 1) = t - k + 1$. Therefore, we may select a disjoint set $S' \subseteq \mathbb{Z}_t^L$ such that $|S'| = t - k + 1$, $f_\pi(s) = 0$, and $s_i \in D'_{\pi(i)} \setminus T_0$ for all $s \in S'$. Let $S_0 = \{s \in \mathbb{Z}_t^L : \text{exists } t \in S' \text{ such that } s_j = t_{\pi(j)} \text{ for all } j \in [L]\}$. Note that S_0 is also disjoint and $|S_0| = |S'|$. Since for all $s \in S_0$, $s_i \notin T_0$, we may select a disjoint set S_1 of size $t - k + 1$ disjoint from S_0 such that $f(s) = 0$ for all $s \in S_1$. Thus, $S_0 \cup S_1$ is a disjoint set of size $2t - 2k + 2$ which takes on only 0s as values. Since f satisfies the $[k,t,2]$ gadget, $2t - 2k + 2 \leq k - 1$. Thus, $k \geq 2t/3 + 1$, a contradiction. □

4 Hardness of Gadget Decoders

4.1 The projection co-gadget

In our label cover reduction in this section, we need to be able to integrate projections in our co-gadget constraints. To do that, we generalize the co-gadget to deal with arbitrary projections. The co-gadget constraints are similar to the edge constraints in [AGH14].

Definition 4.1. Let $k,t,c,\ell \geq 2, L \geq 1$ be positive integers such that $k \leq t$, $c \geq t/(k-1)$. Let $f,g : \mathbb{Z}_t^L \to \mathbb{Z}_c$ be a functions and $\pi : [L] \to [L]$ be a projection. We say that (f,g) satisfy the $[k,t,c]$ *π-co-gadget* if for all $A \subset \mathbb{Z}_t^L$ disjoint with $|A| = k$ and all parititions $A_1 \cup A_2 = A$ such that

- (Strong constraint) for all $x \in A_1$, $y \in A_2$, $a \in [L]$, then $x_a \neq y_{\pi(a)}$

we have that

$$|\{f(x) : x \in A_1\} \cup \{g(y) : y \in A_2\}| \geq 2.$$

From the definition, it is clear that the π-co-gadget constraint can be implemented as a k-uniform t-strong hypergraph. The following claim is the main motivation for the previous definition.

Lemma 4.1. *Let k,t,c,L be positive integers such that $k \leq t$, $c \geq t/(k-1)$. Let $f,g : \mathbb{Z}_t^L \to \mathbb{Z}_c$ be functions and $\pi : [L] \to [L]$ a projection. If (f,g) satisfy the $[k,t,c]$ π-co-gadget, then $(f\Box\pi, g)$ satisfy the $[k,t,c]$ co-gadget.*

Remark. It turns out the converse is false: the π-co-gadget between f and g is strictly stronger than the co-gadget on the projections. In spite of this, the reduction in power is offset by the modularity achieved by having the robust decoder act as a liaison between the coloring gadget and the label cover reduction.

Proof. It suffices to show that every $[k,t,c]$ co-gadget constraint on $(f\Box\pi, g)$ is reflected in a constraint in π-co-gadget for (f,g).

Consider every disjoint $A \subseteq \mathbb{Z}_t^L$ such that $|A| = k$. Let $A_1 \cup A_2 = A$ be a partition of A. We seek to show that

$$|\{(f\Box\pi)(x) : x \in A_1\} \cup \{g(y) : y \in A_2\}| \geq 2.$$

Define $B_1 \subseteq \mathbb{Z}_t^L$ to be

$$B_1 = \{z \in \mathbb{Z}_t^L : \text{there exists } x \in A_1 \text{ such that } z_j = x_{\pi(j)} \text{ for all } j \in [L]\}.$$

For each $x \in A_1$, there is a unique corresponding $z \in B_1$, and vice versa, so $|A_1| = |B_1|$ We claim that (B_1, A_2) is a hyperedge constraint in Definition 4.1. Clearly $|B_1| + |A_2| = k$ since $|A_1| = |B_1|$. For any $z \in B_1, y \in A_2$, we have that there is a $x \in A_1$ such that $z_j = x_{\pi(j)}$ for all $j \in [L]$. Since (A_1, A_2) is a valid hyperedge for the co-gadget, $x_{\pi(j)} \neq y_{\pi(j)}$ for all $j \in [L]$. Thus, $z_j \neq y_{\pi(j)}$ for all $j \in [L]$. Therefore, (B_1, A_2) is a valid hyperedge in the π-co-gadget so

$$|\{f(z) : z \in B_1\} \cup \{g(y) : y \in A_2\}| \geq 2.$$

Now, by definition of B_1, we have that $\{f(z) : z \in B_1\} = \{(f\Box\pi)(x) : x \in A_1\}$. Thus,

$$|\{(f\Box\pi)(x) : x \in A_1\} \cup \{g(y) : y \in A_2\}| \geq 2.$$

Thus, $(f\Box\pi, g)$ satisfy this constraint in the co-gadget. Since the choice of this constraint was arbitrary, we have that $(f\Box\pi, g)$ satisfy the $[k,t,c]$ co-gadget, as desired. $\square$

4.2 The main reduction

Like [AGH14], to obtain NP-hardness results, we reduce from *Label Cover*.

Definition 4.2. An instance of *Label Cover* consists of $\Psi = (U,V,E,[L],\{\pi_e : [L] \to [L]\}_{e\in E})$ a bipartite graph for which each edge has been assigned a projection constraint. The constraint satisfaction problem is to find labelings $\sigma_1 : U \to [L], \sigma_2 : V \to [L]$ of the vertices such that for all $(u,v) \in E$, $\pi_{(u,v)}(\sigma_1(u)) = \sigma_2(v)$.

Although label cover is well-known to be hard with a large approximation gap, we only need that the problem of finding a fully correct labeling is NP-hard.

Lemma 4.2. *It is* NP*-hard to determine if a Label Cover instance* Ψ *is satisfiable (whether a correct labeling exists), where* L *is a constant.*

Proof for completeness. We show that we can take $L = 6$ by reducing from 3-coloring. Let $G = (V,E)$ be a graph we seek to three color. We let the U of our Ψ be the set of edges E and we let V of our Ψ be the vertices V. If our color set is $\{0,1,2\}$, we identify our label set with $\{(0,1),(0,2),(1,0),(1,2),(2,0),(2,1)\}$, the six possible colorings of an edge $(u,v) \in E$. We also identify our label set with $\{0,1,2,3,4,5\}$. For $(u,v) \in E$, we have edge from (u,v) to u and one from (u,v) to v. The projection constraints are

$$\pi_{(u,v)\to u}((c_1,c_2)) = c_1, \pi_{(u,v)\to v}((c_1,c_2)) = c_2$$

Note that we do not use labels 3, 4, and 5 on the right side of Ψ. Now, it is easy to check given these constraints that if G has a three coloring $\gamma : V \to \{0,1,2\}$, then the labelings for all $(u,v) \in E$ and $u \in V$

$$\sigma_1((u,v)) = (\gamma(u),\gamma(v)), \sigma_2(u) = \gamma(u)$$

will satisfy Ψ. Conversely, any correct labeling for Ψ will correspond to a proper coloring from G. Thus, since 3-coloring is NP-complete, Label Cover is NP-hard, as desired. □

Lemma 4.3. *If the* $[k,t,c]$*-coloring gadget (*$k \le t$*) has a robust decoder* $\mathsf{Dec} = \mathsf{Dec}_{[k,t,c]}$ *such that* Dec *always decodes into a unique coordinate (*$C(k,t,c) = 1$*), then* $[k,t,c]$*-coloring is* NP*-hard.*

Proof. We reduce from label cover. Let $\Psi = (U,V,E,[L],\{\pi_e : [L] \to [L]\}_{e\in E})$ be our instance of label cover. Replace each vertex $u \in U$ and $v \in V$ with $[k,t,c]$-gadgets whose colorings are represented by f_u and f_v, respectively. Replace each $(u,v) \in E$ with projection $\pi_{(u,v)}$ with the $\pi_{(u,v)}$-co-gadget for f_u and f_v. Call the resulting (hyper)graph G_Ψ. Since L is a constant G_Ψ is polynomial (in fact, linear) in the size of Ψ. To complete the reduction it suffices to prove the following

Claim 4.4 (Completeness)**.** *If* Ψ *is satisfiable, then* G_Ψ *is* t*-strong colorable.*

Proof. Let $\sigma_1 : U \to [L]$ and $\sigma_2 : V \to [L]$ be the labelings. For all $u \in U$ and $v \in V$ and $x \in \mathbb{Z}_t^L$ let

$$f_u(x) = x_{\sigma_1(u)}, f_v(x) = x_{\sigma_2(v)}.$$

Clearly this is a t-coloring of G_Ψ. Now, we show that every hyperedge is t-strong colored. In every $[k,t,c]$-gadget every two vertices in each hyperedge differ in every coordinate, so their colors must be different. For any $(u,v) \in E$, if not every $[k,t,c]$ $\pi_{(u,v)}$-co-gadget constraint is t-strongly colored, then there are $x,y \in \mathbb{Z}_t^L$ such that $f_u(x) = f_v(y)$ but $x_j \ne y_{\pi_{(u,v)}(j)}$ for all j (we cannot possible have two vertices of the same color in the same hyperedge on the same side of the bipartite graph). In particular, this implies that $x_{\sigma_1(u)} = f_u(x) = f_v(y) = y_{\sigma_2(v)} = y_{\pi_{(u,v)}(\sigma_1(u))}$, contradicting our assumption about the labeling. Thus, G_Ψ is indeed t-strong colorable. □

Claim 4.5 (Soundness). *If G_Ψ is c-colorable, then Ψ is satisfiable.*

Proof. From the assumption we know there exist $\{f_u : u \in U\}$ and $\{f_v : v \in V\}$ which satisfy the $[k,t,c]$ gadget. Thus, since Dec does unique decoding, we may set

$$\sigma_1(u) = \mathsf{Dec}(f_u), \sigma_2(v) = \mathsf{Dec}(f_v).$$

It suffices to check for all $(u,v) \in E$ that $\pi_{(u,v)}(\sigma_1(u)) = \sigma_2(v)$. Since f_u and f_v satisfy the $[k,t,c]$ $\pi_{(u,v)}$-co-gadget (by construction), we have that $f_u \Box \pi_{(u,v)}$ and f_v satisfy the $[k,t,c]$-co-gadget. Thus, since Dec is a decoder with unique decoding, $\mathsf{Dec}(f_u \Box \pi_{(u,v)}) = \mathsf{Dec}(f_v)$. Because Dec is robust, $\pi_{(u,v)}(\mathsf{Dec}(f_u)) = \mathsf{Dec}(f_u \Box \pi_{(u,v)})$. Thus, $\pi_{(u,v)}(\sigma_1(u)) = \sigma_2(v)$ for all $(u,v) \in E$, so Ψ is satisfiable, as desired. □

Thus, we have reduced from Label Cover to $[k,t,c]$-coloring, so $[k,t,c]$-coloring is NP-hard. □

From this, Theorem 1.1 follows from Lemma 3.9. Theorem 1.3 follows from Claim 3.3, Lemma 3.14, and Lemma A.6.

Remark. Using the techniques of [AGH14], we can drop the constraint that $C(k,t,c) = 1$ by reducing from Label Cover with an approximation gap.

Remark. Label Cover is also NP-hard if Ψ has bounded degree. This follows from the hardness of 3-coloring on bounded-degree graphs and applying the reduction in Lemma 4.2. Our reduction then shows that *bounded degree* $[k,t,c]$-coloring is NP-hard, obtaining a result similar to that of [GK04].

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A Combinatorial Gadget Classifications

A.1 Claim 3.1

Proof of Claim 3.1. The proof given is similar to the combinatorial proof in [ADFS04] (Claim 4.1). Let $I = \{(i,\dots,i) \in \mathbb{Z}_t^L : i \in \mathbb{Z}_t\}$. Clearly I and all of its translates are disjoint, so for all $x \in \mathbb{Z}_t^L$, $|S \cap (x+I)| \le \omega(S)$. Since $|I| = t$,

$$t|S| = \sum_{x \in \mathbb{Z}_t^L} |S \cap (x+I)| \le |\mathbb{Z}_t^L|\omega(S)$$

which implies (1). Note that equality holds if and only if $|S \cap (x+I)| = \omega(S)$. In fact, I can be replaced by any set of t disjoint points in $\mathbb{Z}_t^L$. It is easy to then see that if S is a dictator, equality always holds.

Now, we show if equality holds, then S is a dictator. We present a proof using the Fourier Analysis techniques of [ADFS04]. Let $f : \mathbb{Z}_t^L \to \{1,-1\}$ be the indicator function for S in the sense that $f(x) = -1$ if and only if $x \in S$. Using the notation of [ADFS04], let $\hat{f} : \mathbb{Z}_t^L \to \mathbb{C}$ be f's Fourier transform, and consider the following function:

$$A(f)(x) = \frac{1}{(t-1)^L} \sum_{y \in (\mathbb{Z}_t \setminus \{0\})^L} f(x+y)$$

Due to the structure proven above, it is easy to see that for all $x \in \mathbb{Z}_t^L$,

$$\begin{aligned}
A(f)(x) &= \frac{1}{t-1}\sum_{i=1}^{t-1} f(x+(i,\ldots,i)) \\
&= -\frac{f(x)}{(t-1)} + \frac{1}{t-1}\sum_{i=0}^{t-1} f(x+(i,\ldots,i)) \\
&= -\frac{f(x)}{(t-1)} + \frac{1}{t-1}(t-2\omega(S))
\end{aligned}$$

Thus,

$$\widehat{A(f)}(0,\ldots,0) = \frac{t-2\omega(S)}{t-1} - \frac{\hat{f}(0,\ldots,0)}{t-1}$$

and if $x \neq (0,\ldots,0)$,

$$\widehat{A(f)}(x) = -\hat{f}(x)/(t-1).$$

Let $|x|$ be the number of nonzero coordinates of x. Claim 4.3 of [ADFS04] shows though that

$$\widehat{A(f)}(x) = \hat{f}(x)\left(\frac{-\omega(S)}{t-1}\right)^{|x|},$$

Combining these two, we must have that $\hat{f}(x) = 0$ unless $|x| \leq 1$. Thus, f is nonzero on only its first two levels which Lemma 2.3 of [ADFS04] implies that f is a dictator, as desired.

A.2 Hardness of [4,5,2]-coloring

Claim A.1. *Consider $f : \mathbb{Z}_3 \times \mathbb{Z}_4^{L-1} \to \{0,1\}$ such that for all $x,y,z \in \mathbb{Z}_3 \times \mathbb{Z}_4^{L-1}$ disjoint, $\{f(x), f(y), f(z)\} = \{0,1\}$. Then, either there exists $a \in \mathbb{Z}_3$ such that $f(x)$ is constant for all $x \in \{a\} \times \mathbb{Z}_4^{L-1}$ or f is a dictator in one of the coordinates $\{2,\ldots,L\}$.*

Proof. If $L = 1$, the claim is obvious. This proof proceeds by casework. If f is constant on all of coordinates $\{2,\ldots,L\}$, then we are done. Otherwise, without loss of generality, we are in one of the following two cases

1. $f(0\ldots00) = f(0\ldots01) = f(0\ldots02) = 0, f(0\ldots03) = 1$
2. $f(0\ldots00) = f(0\ldots01) = 0, f(0\ldots02) = f(0\ldots03) = 1$

Case 1. If $f(1\ldots1) = 1$, then we must have that $f(2\ldots2) = 0$ which is "equivalent" to $f(1\ldots1) = 0$. Thus, we only need to consider the case $f(1\ldots1) = 0$. Now we have that following series of implications

$$f(2\{2,3\}^{L-2}\{0,2,3\}) = 1. \tag{9}$$

$$f(1\{1,2,3\}^{L-2}\{0,1,2\}) = 0. \tag{10}$$

$$f(2\{1,2,3\}^{L-2}\{0,1,2,3\}) = 1. \tag{11}$$

If $f(1x) = 0$ for all $x \in \mathbb{Z}_4^{L-1}$, then we are done. Otherwise, there exists $x_0 \in \mathbb{Z}_4^{L-1}$ such that $f(1x_0) = 1$. Using (11), this implies that $f(0y) = 0$ for all $y \in \mathbb{Z}_4^{L-1}$ disjoint from x_0. Furthermore, by (10) we can conclude that $f(2z) = 1$ for all $z \in \mathbb{Z}_4^{L-1}$ where z is disjoint from y. Since for all $z \in \mathbb{Z}_4^{L-1}$, we can find

$y_0 \in \mathbb{Z}_4^{L-1}$ such that y_0 is disjoint from both x_0 and z, we must have that $f(2z) = 1$ for all $z \in \mathbb{Z}_4^{L-1}$, as desired.

Case 2. First assume that $f(1\dots1) = 1$, then we get that

$$f(2\{2,3\}^{L-2}\{0,2,3\}) = 0. \tag{12}$$

$$f(1\{1,2,3\}^{L-2}\{0,1,2,3\}) = 1. \tag{13}$$

$$f(2\{1,2,3\}^{L-2}\{0,1,2,3\}) = 0. \tag{14}$$

if $f(1x) = 1$ for all $x \in \mathbb{Z}_4^{l-1}$, we are done, else $f(1x_0) = 0$ for some $x_0 \in \mathbb{Z}_4^{L-1}$. Applying the same reasoning as in Case 1, we reach the same conclusion.

Now, we may assume that for all $x \in \{1,2\} \times \{1,2,3\}^{L-2} \times \{0,1,2,3\}$, $f(x) = 0$ if and only if $x_L \in \{0,1\}$. Otherwise, we fall into a case already covered by permuting the coordinates or output labels. If $f(x)$ is a dictator in the last coordinate, we are done. Else, we may assume without loss of generality that there is $x \in \mathbb{Z}_3 \times \mathbb{Z}_4^{L-2}$ such that $f(x0) = 1$. Thus, $f(y\{2,3\}) = 0$ for all $y \in \mathbb{Z}_3 \times \mathbb{Z}_4^{L-2}$ disjoint from x. But we also know that there is some $y' \in \{1,2\} \times \{1,2,3\}^{L-2}$ disjoint from x such that $f(y'\{2,3\}) = 1$, a contradiction. Thus, we have exhausted all cases. □

Corollary A.2. *Consider $f : \mathbb{Z}_3^k \times \mathbb{Z}_4^{L-k} \to \{0,1\}$ such that for all $x,y,z \in \mathbb{Z}_3^k \times \mathbb{Z}_4^{L-k}$ disjoint, $\{f(x), f(y), f(z)\} = \{0,1\}$. Then, either there exists $a \in \mathbb{Z}_3^k$ such that $f(x)$ is constant for all $x \in \{a\} \times \mathbb{Z}_4^{L-k}$ or f is a dictator in one of the coordinates $\{k+1,\dots,L\}$.*

Proof. For any three $x_0, y_0, z_0 \in \mathbb{Z}_3^k$ disjoint, construct the map $f'_{x_0,y_0,z_0} : \mathbb{Z}_3 \times \mathbb{Z}_4^{L-k}$, such that for all $w \in \mathbb{Z}_4^{L-k}$,

$$f'_{x_0,y_0,z_0}(\{0\}w) = f(x_0w)$$
$$f'_{x_0,y_0,z_0}(\{1\}w) = f(y_0w)$$
$$f'_{x_0,y_0,z_0}(\{2\}w) = f(z_0w).$$

It is clear that f'_{x_0,y_0,z_0} meets the hypothesis of Claim A.1. Thus, either f'_{x_0,y_0,z_0} is a dictator in of coordinates $\{2,\dots,L-k+1\}$, or for one of $w_0 \in \{x_0,y_0,z_0\}$, $f(w_0 \times \mathbb{Z}_4^{L-K})$ is constant. Since the latter is sufficient to establish the claim, we assume the former in all cases. That is, for all disjoint $\{x_0,y_0,z_0\} \subseteq \mathbb{Z}_3^k$, f'_{x_0,y_0,z_0} is a dictator. Notice that this implies for all $x_0 \in \mathbb{Z}_3^k$ there is $i_{x_0} \in \{k+1,\dots,L\}$ such that f restricted to $\{x_0\} \times \mathbb{Z}_4^{L-k}$. Additionally, note that for all $x_0, y_0 \in \mathbb{Z}_3^k$ disjoint we must have that $i_{x_0} = i_{y_0}$ since there exists $z_0 \in \mathbb{Z}_3^k$ disjoint from both x_0 and y_0. Since the "disjoint" property induces a connected graph on $\mathbb{Z}_3^k$, we have that i_{x_0} is constant for all $x_0 \in \mathbb{Z}_3^k$. Thus, f is a dictator on one of $\{k+1,\dots L\}$, as desired. □

Lemma A.3. *For all $f : \mathbb{Z}_5^L \to \mathbb{Z}_2$ satisfying the $[4,5,2]$ gadget, there exists $i \in [L]$ and $a,b \in \mathbb{Z}_5$ distinct such that for all $x \in \{x \in \mathbb{Z}_5^L : x_i = a \text{ or } x_i = b\}$, $f(x)$ is constant.*

Proof. First, we show (up to symmetry) a wide class of f have this property.

Claim A.4. *Let $f : \mathbb{Z}_5^L \to \mathbb{Z}_2$ satisfy the $[4,5,2]$ gadget and assume that f restricted to $\mathbb{Z}_4^L$ satisfies the $[3,4,2]$ gadget. Then f satisfies the conclusion of Lemma A.3.*

Proof. Clearly by Claim 3.2 f restricted to $\mathbb{Z}_4^L$ is a dictator in one of the coordinates. Assume without loss of generality that $f(\{0,1\} \times \mathbb{Z}_4^{L-1}) = 0$ and $f(\{2,3\} \times \mathbb{Z}_4^{L-1}) = 1$. In order for the claim to not be immediately true, we must have without loss of generality that there exist $x, y \in \mathbb{Z}_4^{L-1}$ such that $f(0x) = 1$ and $f(2y) = 0$. Thus, $f(4z) = 0$ for all $z \in \mathbb{Z}_5^{L-1}$ disjoint from x and $f(4z) = 1$ for all $z \in \mathbb{Z}_5^{L-1}$ disjoint from y. Since there are $z \in \mathbb{Z}_5^{L-1}$ disjoint from both x and y, we have a contradiction. Thus, the claim is true. $\square$

Now, we show an even wider class of f satisfy the lemma.

Claim A.5. *Let $f : \mathbb{Z}_5^L \to \mathbb{Z}_2$ satisfy the $[4,5,2]$ gadget and assume that f restricted to $\{0\} \times \mathbb{Z}_4^{L-1}$ is always* 0. *Then either f restricted to $\{0\} \times \mathbb{Z}_5^{L-1}$ or f satisfies the conclusion of Lemma A.3.*

Proof. Assume the first conclusion is false. Thus, there is $x \in \mathbb{Z}_5^{L-1}$ such that $f(0x) = 1$. Consider the hypercube H_{0x} of points disjoint from $0x$. If any three disjoint points in H_{0x} have the same value, we could find a fourth value in $\mathbb{Z}_5^{L-1}$ which is disjoint from all three but has the same value, a contradiction. Thus, f restricted to H_{0x} satisfies the $[3,4,2]$ gadget. Thus, by Claim A.4 we have that f satisfies the conclusion of Lemma A.3. $\square$

Now, we may finish the proof. Clearly f must depend in at least one coordinate. Assume that $f(0\ldots0) = 0$ and $f(10\ldots0) = 1$. Thus, f restricted to $S = \{2,3,4\} \times \{1,2,3,4\}^{L-1}$ meets the hypothesis of Claim A.1. Therefore, either f restricted to this set is a dictator or f has be constant on $\{a\} \times \{1,2,3,4\}^{L-1}$ for some $a \in \{2,3,4\}$. First, assume that the former case occurs and that f is dictated by coordinate $i \in \{2,\ldots,L\}$. Now, assume without loss of generality that $f(20\ldots0) = 1$. Thus, f restricted to $T = \{1,3,4\} \times \{1,2,3,4\}^{L-1}$ also meets the hypothesis of A.1. Because S and T have a large overlap, it is not possible for f restricted to T to T to be dictated by any coordinate other than i. But it is possible for $f(\{1\} \times \{1,2,3,4\}^{L-1})$ to be constant. In the first case, f restricted to $\{1,2,3,4\}^L$ is also a dictator. Thus, f restricted to this set satisfies the $[3,4,2]$ gadget. Thus, by Claim A.4, f satisfies the conclusion. In the second case, by Claim A.5 f $f(\{1\} \times \mathbb{Z}_5^{L-1}) = 1$ or else we are done. This yields a contradiction since we can pick $x \in \{1\} \times \mathbb{Z}_5^{L-1}$ and $y, z \in \{3,4\} \times \{1,2,3,4\}^{L-1}$ such that x, y, z, and $20\ldots0$ are all disjoint but $f(x) = f(y) = f(z) = f(20\ldots0) = 1$ since f restricted to $\{3,4\} \times \{1,2,3,4\}^{L-1}$ is a dictator in a coordinate other than than the first.

Thus, we may now assume without loss of generality that f restricted to $\{2\} \times \{1,2,3,4\}^{L-1}$ is always 1. By A.5, we may assume that $f(\{2\} \times \mathbb{Z}_5^{L-1}$ is always 1 (or else we are immediately done). Since $f(0\ldots0) = 0 \neq f(20\ldots0) = 1$, we have that f restricted to $\{1,3,4\} \times \{1,2,3,4\}^{L-1}$ also satisfies the $[3,4,2]$ gadget. Thus, $f(\{a\} \times \{1,2,3,4\}^{L-1}$ is constant and so we may assume that $f(\{a\} \times \mathbb{Z}_5^{L-1})$ is constant. Clearly if this constant value is 1 we are done, otherwise Thus, assume that $f(\{a\} \times \mathbb{Z}_5^{L-1}) = 0$. Therefore, $f(a0\ldots0) = 0$. Thus, f restricted to $(\mathbb{Z}_5 \setminus \{2,a\}) \times \mathbb{Z}_4^{L-1}$ also satisfies the $[3,4,2]$ gadget. Thus, there is $b \in \mathbb{Z}_5 \setminus \{2,a\}$ such that $f(b \times \{1,\ldots,4\}^{L-1})$ is constant, so $f(b \times \mathbb{Z}_5^{L-1})$ is constant (or else we are done). Thus, $i = 1$ and either $\{a,b\}$ or $\{1,b\}$ is the desired pair, as desired. $\square$

Lemma A.6. *The $[4,5,2]$-coloring gadget has a robust decoder.*

Proof. We omit the proof. The proof is similar to that of Claim 3.13 and Lemma 3.14. $\square$

A.3 Classification of $\langle t-1, t, 2\rangle$

In this subsection, we examine a balanced variant of the strong hypergraph coloring problem.

Definition A.1. Let $k, t, c \geq 2$ be positive integers with $t \geq k$. Define $\langle k,t,c\rangle$*-coloring* to be the following promise problem. Let G be a k-uniform hypergraph which is promised to be t-strong colorable. Can G be efficiently colored with c colors such that the discrepancy is minimal?

Definition A.2. Let L,k,t,c be positive integers with $t \geq k$, and let $f : \mathbb{Z}_t^L \to \mathbb{Z}_c$ be a function. We say that f *satisfies the* $\langle k,t,c \rangle$ *gadget* if for all $S \subset \mathbb{Z}_t^L$ such that $|S| = k$ and S is disjoint otherwise, we have that the multiset $\{f(x) : x \in S\}$ is as equi-distributed as possible.

Note the $\langle 3,4,2 \rangle$ is equivalent to $[3,4,2]$ (both gadget and problem) and that $\langle 4,5,2 \rangle$ is equivalent to $[4,5,2]$. We now prove a result that holds for $\langle t-1,t,2 \rangle$ for all odd t.

Lemma A.7. *If* $f : \mathbb{Z}_t^L \to \mathbb{Z}_2$ *satisfies the* $\langle t-1,t,2 \rangle$ *gadget, where* t *is even, then* f *is a dictator.*

Proof. We present a proof using the Fourier Analysis techniques of [ADFS04]. Using their notation, remap f so that its output if $\{-1,1\}$ instead of $\{0,1\}$, let $\hat{f} : \mathbb{Z}_t^L \to \mathbb{C}$ be f's Fourier transform, and consider the following function:

$$A(f)(x) = \frac{1}{(t-1)^L} \sum_{y \in (\mathbb{Z}_t \setminus \{0\})^L} f(x+y)$$

For combinatorial reasons, it is easy to see in our context that $A(f)(x) = -f(x)/(t-1)$ for all $x \in \mathbb{Z}_t^L$. Thus, $\widehat{A(f)}(x) = -\hat{f}(x)/(t-1)$. Claim 4.3 of their paper shows though that

$$\widehat{A(f)}(x) = \hat{f}(x) \left(\frac{-1}{t-1} \right)^{|x|}.$$

Combining these two, we must have that $\hat{f}(x) = 0$ unless $|x| = 1$. That is, x has only one nonzero coordinate. Thus, f is nonzero on only its first two levels which Lemma 2.3 of their paper implies that f is a dictator, as desired. $\square$

We omit the proof that there exists a robust decoder and the subsequent Label Cover argument.

Definition A.3. A function $f : \mathbb{Z}_t^L \to \mathbb{Z}_2$ is an almost dictator if there exists an independent set I of $\mathbb{Z}_t^L$ (i.e., a subset no two of whose elements are disjoint) such that f restricted to $\mathbb{Z}_t^L \setminus I$ is a dictator.

Conjecture A.8. *If* $f : \mathbb{Z}_t^L \to \mathbb{Z}_2$ *satisfies the* $\langle t-1,t,2 \rangle$ *gadget, where* t *is odd, then* f *is an almost dictator.*

This conjecture, with a suitable application of Label Cover, would imply that finding a discrepancy two 2-coloring of a t-colorable graph is NP-hard.

$\square$

B Nonexistence of Robust Decoding of $[2,3,6]$

Claim B.1. *There does not exist a robust decoding of the* $[2,3,6]$*-coloring gadget. Even if the projections considered are* $p(L)$*-to-1 for any* $p(L) = \omega(1)$.

Proof. Assume for sake of contradiction that there exist a robust decoding Dec. Let $p(L) : \mathbb{N} \to \mathbb{N}$ be any function in $\omega(1)$. For all $L \geq 1$, consider $f : \mathbb{Z}_3^L \to \mathbb{Z}_6$ with the following properties.

- For any $x \in \mathbb{Z}_3^L$ such that there is $s \in \mathbb{Z}_3$ such that $|\{i \in [p(L)] : x_i = s\}| > p(L)/2$, then $f(x) = s+3$.
- Otherwise, $f(x) = x_1$.

As a sanity check, note that for each $s \in \mathbb{Z}_6$, $f^{-1}(s)$ is an independent set. For each $S \subseteq [p(L)]$ such that $|S| > p(L)/2$, let π_S be the projection such that $\pi_S(i) = \min S$ if $i \in S$ and $\pi_S(i) = i$ otherwise. Since $|S| > p(L)/2$, $f \square \pi_S$ has a range of $\{3,4,5\}$. Furthermore, $f \square \pi_S$ is a dictator in coordinate $\min S$, so $\mathsf{Dec}(f \square \pi_S) = \{\min S\}$. Since Dec is robust, $(\pi_S)^{-1}(\min S) = S$ must have nontrivial intersection with $\mathsf{Dec}(f)$. Thus, $\mathsf{Dec}(f)$ must have nontrivial intersection with every S such that $|S| > p(L)/2$. Thus, $|\mathsf{Dec}(f)| \geq p(L)/2 = \omega(1)$ (since otherwise we could exhibit a non-intersecting S), but $|\mathsf{Dec}(f)| \leq C$ for some constant C independent of L, a contradiction. $\square$

Note that this arguments suggests that the 'robust decoder' techniques could not work, unless we use a d-to-1 variant of label cover, of which hardness is only conjectured. A similar argument shows that there does not exist a robust decoding of the $[2,t,2t]$-coloring gadget.

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