An average-case depth hierarchy theorem for higher depths

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Abstract

We extend the recent hierarchy results of Rossman, Servedio and Tan [2] to any \( d \leq \frac{c \log n}{\log \log n} \) for an explicit constant \( c \).

To be more precise, we prove that for any such \( d \) there is a function \( F_d \) that is computable by a read-once formula of depth \( d \) but such that any circuit of depth \( d - 1 \) and size at most \( 2^{O(n^{1/8d})} \) agrees with \( F_d \) on a fraction at most \( \frac{1}{2} + O(n^{-1/8d}) \) of inputs.

1 Introduction

This technical note is an extension of the work of Rossman, Servedio and Tan [2]. Their result is that there is a constant \( c \) such that for \( d \leq \frac{c \sqrt{\log n}}{\log \log n} \) there is a function \( F_d \) that is computed by a read-once formula of depth \( d \) but such that any circuit of size at most \( \exp(n^{1/(4d)}) \) and depth \( d - 1 \) agrees with \( F_d \) on a fraction of inputs that is at most \( \frac{1}{2} + n^{-\Omega(1/d)} \). We extend their result by allowing \( d \) to be as large as \( \frac{\log n}{\log \log n} \) and in particular we prove the following theorem.

**Theorem 1.1** For sufficiently large \( n \) and \( d \leq \frac{\log n}{\log \log n} \), then there is a Boolean function \( F_d \), depending on \( n \) Boolean inputs which is computable by a read-once circuit of depth \( d \) such that for any circuit \( C \) of depth \( d - 1 \) and size at most \( 2^{n^{1/5(d-1)}} \) we have

\[
Pr[F_d(x) = C(x)] \leq \frac{1}{2} + O(n^{-\Omega(1/d)}).
\]

The theorem is only really interesting for \( d \leq \frac{\log n}{\log \log n} \) as the bound on the size of \( C \) is sublinear for larger values of \( d \). For the record let us state that we

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have not tried to optimize the value of the constants 5 and 8, but even being more careful, we do not believe they can be pushed below 2.

In fact, as was also done in [2], we prove a lower bound for a slightly larger class of circuits including those of depth $d$ and small bottom fanin and even depth $d + 1$ circuits with small bottom fanin if the output gate is restricted to be the other type compared to the output gate of $F_d$.

As the objective is to keep this as a short technical note we refer to the very nice introduction of [2] for a discussion of the background of the problem, its connection to results in various branches of circuit complexity and structural complexity theory, as well as the historical developments leading up the current results.

Let us, however, stress the fact that the current work builds very much on the work of [2] and we use the same framework focused on restrictions and their generalization in the form of projections.

2 An outline of the paper

The function $F_d$ we prove is difficult to compute on average is more or less the first function that comes to mind and slight variants of this function are used in earlier paper proving hierarchy results. It is computed by a read-once formula which is a tree of depth $d$ and governed by a parameter $m \approx \log n / 2d$. We have alternating levels of and-gates and or-gates, the top fanin is $\Theta(2^{2m})$, the bottom fanin is $2m$ while the fanin at other levels is $\Theta(m2^{2m})$. A little care is needed to make it balanced but this is straightforward.

We are given a depth $d - 1$ circuit $C$ that we wish to prove does not even approximate $F_d$. The early papers [3, 4, 1] picked a random restriction (giving values to most variables) keeping a few variables non-assigned and made sure that $F_d$ turned into $F_{d-1}$ while the depth of the circuit $C$ could be decreased. A new element was introduced by [2] by allowing projections (identifying several different old variables with one new variable) but this paper also relied on induction.

We follow almost the same approach but there is a subtle difference. After we have done a random restriction and projection, $F_d$ does not turn exactly into $F_{d-1}$ but to a similar looking function and previous papers uses a clean-up stage to turn it exactly into $F_{d-1}$ in order to apply the induction hypothesis. This clean-up is fairly costly and we omit it. Instead, starting with $F_d$ we apply a sequence of $d - 1$ restrictions and projections and after the $i$'th stage $F_d$ is reduced to something similar to, but not exactly equal to $F_{d-i}$, while $C$ has, with high probability, lost $i$ levels. Apart from this little change the approach of the paper is what can be expected and an outline of the paper is as follows.

Defining the function $F_d$ is done in Section 3. We define our set of restrictions in Section 4. Their basic properties, that they generate uniformly random inputs and that they, with high probability and to a significant extent, preserve $F_d$ is established in Section 5.
One way of thinking about picking the restriction $\rho^i$ is by first picking independent restrictions $\rho^{i-1}$ to each sub-formula of depth $i-1$ and then doing some additional fixing. This is also the way we reason about $\rho^i$ when performing the simplifications, via a switching lemma, of the competing circuit $C$. The needed switching lemma is found in Section 6. We put all the pieces together proving our main theorem in Section 7.

3 Defining the functions $F_d$

We have a parameter $m$ guiding our construction. We define $F_d$ to be a tree with alternating layers of and-gates and or-gates. The layer next to the inputs is defined to contain and-gates and thus the output gate is an and-gate if $d$ is odd and otherwise an or-gate.

The intuitive version of the definition is that we want internal and-gates to be 1 with probability $2^{-2m}$ for random inputs and or-gates to be 0 with the same probability. Furthermore we want the output to be unbiased. This cannot be achieved exactly but by an inductive definition we can come very close. Let us turn to the formal details and give the parameters.

**Definition 3.1** Let $c_0 = \frac{1}{2}$ and for $1 \leq i \leq d-1$ let $f_i$ be the smallest integer such that

$$(1 - c_{i-1})^{f_i} \leq 2^{-2m},$$

and set $c_i = (1 - c_{i-1})^{f_i}$. Finally set $f_d$ to be the smallest integer such that

$$(1 - c_{d-1})^{f_d} \leq \frac{1}{2}.$$  

The function $F_d$ is defined by a read-once formula of alternating levels of and-gates and or-gates. The fan-out at distance $i$ from the inputs is $f_i$.

It is not difficult to see that $2^{-2m} - 2^{-4m} \leq c_i \leq 2^{-2m}$ for $1 \leq i \leq d-1$ and $f_1 = 2m$, $f_i = 2m \ln 2 \cdot 2^{2m(1 + O(2^{-m}))}$ for $2 \leq i \leq d-1$, while $f_d = \ln 2 \cdot 2^{2m(1 + O(2^{-m}))}$. It follows that the number of inputs of $F_d$ is

$$\prod_{i=1}^d f_i = 2^{2(d-1)m} m^{d-1} 2^{O(d)}$$

(1)

and we denote this number by $n$. We note that if $d \leq \frac{\log n}{2\log \log n}$ then the first factor of (1) is the dominating factor and $m = \frac{\log n}{2d-2}(1 + o(1))$.

It follows by construction that if we feed random independent uniform bits as inputs into the formula defining $F_d$ then an and-gate on level $i$ (which assumes that $i$ is odd) is one with probability $c_i$ while an or-gate is zero with the same probability for even $i$. It follows that output of $F_d$ is, within an error $2^{-2m}$, unbiased. We turn to defining the space of restrictions.
4 The space of restrictions $R^i$

In this paper we, as was first done in [2], complement restrictions with projections. A classical restriction maps each variable, $x_i$, to one of the three values 0, 1 and *. The two first values indicate that the corresponding constant should be substituted for this variables while the third value says that that the value of $x_i$ remains undetermined.

We combine classical restriction with projections under which groups of variables are identified with the same new variable. This makes further simplifications possible. The mapping of old variables to new variables could be completely arbitrary but to avoid a generality that we do not utilize we define only a special class of projections.

The universe of variables used by our restrictions is given by $x_v$ where $v$ ranges over all nodes in the tree defining $F_d$. Let $V_i$ be the set of variables $x_v$ where $v$ is at height $i$, i.e. at distance $i$ from the inputs. Note that the set of original inputs to $F_d$ is exactly given by $V_0$.

**Definition 4.1** A level $i$ restriction, $\rho^i$, is a mapping of $V_0$ into the set $\{0, 1\} \cup V_i$. The possible values of $\rho^i(x_w)$ are 0, 1 and $x_v$ where $v$ is the height $i$ ancestor of $w$.

The only way we construct a level $i$ restriction in this paper is to first do a level $i - 1$ restriction, then apply a classical restriction to the variables in $V_{i-1}$ and finally identify any live variables with its parent. Thus when going from $\rho^{i-1}$ to $\rho^i$ we define a mapping from $V_{i-1}$ to $\{0, 1\} \cup V_i$ and $\rho^i$ is the composition of these two mapping. Any input mapped to a constant under $\rho^{i-1}$ is still mapped to the same constant under $\rho^i$.

A central role in our proof is played by a probability distribution of level $i$ restrictions, $R^i$, and let us give its basic properties. The restrictions operate independently on each height $i$ sub-formula and let us assume that $i$ is odd and hence the top gate of such a sub-formula is an and-gate. Let $F_v$ the be sub-formula rooted at $v$, a gate at level $i$ in the formula defining $F_d$. We have four basic properties of our space of restrictions.

1. With probability $2^{-5m/2}$ all variables of $F_v$ are fixed to constants and $F_v \downarrow_{\rho^i} \equiv 1$.
2. With probability $1 - 2^{-m}$ all variables of $F_v$ are fixed to constants and $F_v \downarrow_{\rho^i} \equiv 0$.
3. With probability $2^{-m} - 2^{-5m/2}$ we have $F_v \downarrow_{\rho^i} \equiv x_v$.
4. If $x_v$ is set to 1 with probability $b_i$ defined as
   \[ b_i = \frac{c_i - 2^{-5m/2}}{2^{-m} - 2^{-5m/2}} \] (2)
   then $\rho^i$ combined with this setting gives a uniformly random input to all variables in $F_v$. 

4
For future reference we note that \( b_v = 2^{-m}(1 + O(2^{-m/2})) \). If the gate is an or-gate we reverse the roles of 0 and 1. The spaces of restrictions are constructed recursively for increasing values of \( i \) and let us start by formally defining \( R^1 \). Let \( v \) be a gate at level one and let \( B_v \) be the inputs that appear in \( F_v \), which is thus a set of size \( 2m \).

**Definition 4.2** A random restriction \( \rho \in R^1 \) is constructed independently for each gate, \( v \), on level 1 as follows.

1. Pick a uniformly random assignment \( \alpha \in \{0,1\}^{2m} \) to all inputs \( x_w \in B_v \).

2. If \( \alpha = 1^{2m} \) then with probability \( 2^{-m/2} \) set \( \rho(x_w) = 1 \) for all \( x_w \in B_v \), and otherwise proceed as follows. Pick a uniformly random non-empty subset \( S \) of \( B_v \) and set \( \rho(x_w) = * \) for \( x_w \in S \) and otherwise set \( \rho(x_w) = 1 \).

3. If \( \alpha \neq 1^{2m} \), then with probability \( (1 - 2^{-m})/(1 - 2^{-2m}) \) set \( \rho(x_w) = \alpha_w \) for all \( x_w \in B_v \) and otherwise set \( \rho(x_w) = * \) for all \( w \) such that \( \alpha_w = 0 \) while \( \rho(x_w) = 1 \) when \( \alpha_w = 1 \).

4. Identify all variables \( x_w \in B_v \) such that \( \rho(x_w) = * \) with \( x_v \).

We let \( \rho^1 \) denote a typical level 1 restriction after also the projection in the last step has been applied. If some variable \( x_w \) is mapped to \( x_v \) then we say that \( x_v \) is alive and also write this as \( \rho^1(x_v) = x_v \). Keeping with this convention we also write \( \rho^1(x_v) = c \) when \( F_v \) is fixed to the constant \( c \). In general we sometimes use \( \rho^1(x_v) \) for \( F_v[\rho^1] \) and remember that this takes values 0, 1 or \( x_v \).

We think of the assignment \( \alpha \) as a “tentative” assignment to all variables. We forget some of the values but if we later assign \( x_v \), with the correct bias we get the same probability distribution as if we had kept the original \( \alpha \). This assures that such a substitution creates a uniformly random input.

It is not difficult to see that \( R^1 \) has the four basic properties we described above but let us still check them in detail.

The probability that \( F_v \) is fixed to 1 is \( 2^{-5m/2} \) as we need to pick \( \alpha = 1^{2m} \) and then decide to use this assignment fully under step 2. Similarly the probability that \( F_v \) is fixed to 0 is

\[
(1 - 2^{-2m}) \cdot (1 - 2^{-m})/(1 - 2^{-2m}) = 1 - 2^{-m}
\]

as this must happen under step 3. This ensures the two first properties. Note that in all other cases we have a non-empty set \( S \) such that \( \rho(x_w) = x_v \) for all \( x_w \in S \) while all other variables of \( B_v \) are mapped to 1. This implies that the value at \( F_v[\rho^1] = x_v \). We now turn to the property that if any live \( x_v \) is set to 1 with probability \( b_1 = \frac{2^{-2m} - 2^{-5m/2}}{2^{-m} - 2^{-5m/2}} \) then we have the distribution of a random input.

The probability that a set \( S \) is chosen under step 2 is

\[
\rho_S^2 = \frac{2^{-2m}(1 - 2^{-m/2})}{2^{2m} - 1}
\]
while the probability that the same set is chosen under step 3 is

\[ p^3_S = \frac{2^{-2m}(2^{-m} - 2^{-2m})}{(1 - 2^{-2m})} = \frac{(2^{-m} - 2^{-2m})}{(2^{2m} - 1)}. \]

This implies that, conditioned that \( S \) is the set of inputs such that \( \rho^1(x_w) = x_v \), the probability the probability that it was chosen under step 2 is

\[ \frac{p^S_2}{p^S_2 + p^S_3} = \frac{2^{-2m} - 2^{-5m/2}}{2^{-m} - 2^{-5m/2}} \tag{3} \]

and this is exactly \( b_1 \). This implies that if \( x_v \) is set to 1 with probability \( b_1 \) then we get the same probability distribution on \( x \) as if we had set \( x_w = \alpha_w \) immediately. We conclude that we get a uniformly random input to the gate \( v \).

We conclude that \( R^1 \) has all the desired properties and proceed to the general case.

When picking a restriction in the space \( R^i \) we first pick a restriction \( \rho^i-1 \) from \( R^{i-1} \) which reduces each sub-formula, \( F_w \) of depth \( i - 1 \) to a constant or a variable \( x_w \). This implies that going from \( \rho^{i-1} \) to \( \rho^i \) is essentially picking the inputs to a single gate and thus quite similar to a restriction from \( R^1 \). We again center the construction around a Boolean vector \( \alpha \) which plays the role of a set of independent but suitably biased set of values at level \( i - 1 \). The bits of \( \alpha \) come in two flavors. Those that are “hard” which should be thought of as already fixed by \( \rho^{i-1} \) and hence cannot be changed to \( x_v \) and those that are “soft” and can be changed.

Let us assume that \( i \) is odd and hence that each gate \( v \) on level \( i \) is an \( \text{and}-\text{gate} \). In the case of even \( i \) the role of 0 and 1 are reversed in the definition below. As before we let \( B_v \) be the set of input gates to \( v \) which is now of size \( f_i \).

We first pick an input \( \alpha \in \{0, 1\}^{f_i} \) where some values are hard while other are soft. For each \( \alpha_w \) independently.

1. Make it a hard zero with probability \( 2^{-5m/2} \).
2. Make it a hard one with probability \( 1 - 2^{-m} \).
3. Make it a soft zero with probability \( c_{i-1} - 2^{-5m/2} \).
4. Make it a soft one with probability \( 2^{-m} - c_{i-1} \).

We note that a coordinate that is not given a hard value is set to a soft zero with probability exactly \( b_{i-1} \).

Let \( T \) be the set of inputs that are given soft values. Thus typically \( T \) is of size roughly \( 2^{-m}f_i \) and let \( f_v \) be the actual number. Let \( S \) be a potential set of soft zeroes. For a non-empty set \( T \), by a “uniformly non-empty subset of \( T \) of bias \( b_{i-1} \)” we mean that we include each element of \( T \) with probability \( b_{i-1} \) in \( S \) and if \( S \) turns out to be empty we try again. We denote this probability distribution by \( q_{S,T} \) and it is not difficult to see that

\[ q_{S,T} = (1 - (1 - b_{i-1})^{f_v})^{-1} b_{i-1}^{\vert S \vert} (1 - b_{i-1})^{f_v - \vert S \vert}. \tag{4} \]
Note that if \( T \) is about its typical size, then the probability (conditioned on picking \( T \)) that \( S \) is empty is about \( 2^{-2m} \) so this conditioning is in general mild. Now we proceed to the determine the value of \( \rho' \) by determining the values of the gates in \( B_v \) to be 0, 1 or \( x_v \). As indicated above we never change a hard value while soft values are either made permanent or turned into \( x_v \).

1. If there is at least one hard zero in \( B_v \) set \( \rho(x_w) = \alpha_w \) for all \( w \).
2. If \( |f_v - f(\overline{\alpha})| \geq 2^{3m/4} \) then set \( \rho(x_w) = \alpha_w \) for all \( w \).

Let us turn to the more interesting part of \( \rho \). Let \( q_3 \) and \( q_4(f_v) \) be constants, which in rough terms satisfy \( q_3 \approx 2^{-m/2} \) and \( q_4(f_v) \approx 2^{-m} \), but whose exact values are given during the analysis.

3. If \( \alpha = 1 \) then with probability \( q_3 \) set \( \rho(x_w) = 1 \) for all \( w \) and otherwise proceed as follows. Choose a non-empty subset \( S \) of \( T \) with bias \( b_{i-1} \) and set \( \rho(x_w) = \star \) for \( x_w \in S \) while \( \rho(x_w) = 1 \) otherwise.
4. Suppose we get a non-empty set \( S \) of soft zeroes. Then with probability \( 1 - q_4(f_v) \) set \( \rho(x_w) = \alpha_w \) for all \( x_w \in B_v \) and otherwise we set \( \rho(x_w) = \star \) for \( w \in S \) and \( \rho(x_w) = 1 = \alpha_w \) otherwise.
5. Identify all live variables in \( B_v \) with \( x_v \).

Before proceeding let us observe that if any coordinate \( \alpha_w \) is set to a hard value, \( x_w \) is always set to this value. This follows as \( S \) is a subset of \( T \) and hence not given hard ones as values and if a hard zero is assigned, all values of \( \alpha \) are used.

The above description tells us how to go from \( \rho^{i-1} \) to \( \rho^i \) and there are a couple of equivalent ways to view the combination into a full assignment. One way is the following.

1. Pick an assignment \( \alpha \).
2. For hard coordinates \( w \) of \( \alpha \) pick a restriction \( \rho^{i-1} \in R^{i-1} \) conditioned on \( \rho^{i-1}(x_w) \) being this constant.
3. For soft coordinates \( w \) of \( \alpha \) pick a restriction \( \rho^{i-1} \in R^{i-1} \) conditioned on \( \rho^{i-1}(x_w) = x_w \) and then set \( \rho'(x_w) \) as in the above procedure.

Of course an equivalent way to describe the procedure is to first pick random independent \( \rho^{i-1} \in R^{i-1} \) for each depth \( i - 1 \) circuit and then a value of \( \alpha \) conditioned on getting hard coordinates with the correct value whenever \( \rho^{i-1}(x_w) \) was chosen to be a constant. In more detail this is the following procedure.

1. Pick a random \( \rho^{i-1} \in R^{i-1} \). Whenever \( \rho^{i-1}(x_w) \) is a constant we fix \( \alpha_w \) to be that constant in hard way.
2. For any \( w \) such that \( \alpha_w \) is not set in step 1 pick it to be a soft value with bias \( b_{i-1} \).
3. Fix the values of some $x_w$ to constants based on cases 1 and 2. This is done by a traditional restriction taking values 0, 1 and * and we denote this restriction by $\rho_1$.

4. Fix the values of some $x_w$ for $w \in V_{i-1}$ based on cases 3 and 4. This as a traditional restriction that we denote by $\rho$.

5. For all $x_w \in B_v$ with $\rho(x_w) = *$ set $\rho^i(x_w) = x_v$. We call this projection $\pi$.

We say that $\rho^{i-1}, \rho_1, \rho$ and $\pi$ are the components of $\rho^i$. The most interesting part when going from $\rho^{i-1}$ to $\rho^i$ turns out to be the third step $\rho$. For a function $f$ we let $f|_\rho$ denote the function after this step. We let $f|_{\rho+\pi}$ denote the function also after the projection has been made. We have that $f|_\rho$ is a function of $x_w$ where $w \in V_{i-1}$ while $f|_{\rho+\pi}$ is a function of $x_v$ where $v \in V_i$.

As an example suppose that $f = x_{w_1} \lor \overline{x_{w_2}}$ where $w_1$ and $w_2$ are two nodes in the same depth i sub-formula. Suppose furthermore that $\rho$ does not fix either of these variables. In this situation $f|_\rho$ is the same function as $f$ while $f|_{\rho+\pi}$ is identically true.

5 Simple properties of $R^i$

The construction of the space of restrictions, $R^i$ has been carefully crafted to, more or less by definition, satisfy the four basic properties and we establish these as a sequence of lemmas.

**Lemma 5.1** For any node $v$ on level $i$ in $F_d$ such that $x_v$ is alive after $\rho^i$ we have $F_v|_{\rho^i} = x_v$. Furthermore if $F_v|_{\rho^i}$ is a constant then $\rho^i$ assigns constants to all variables in $F_v$.

**Proof:** Going over the construction line by line it is not difficult to see that this is true.

The second lemma says that a sub-formula is reduced to 0 with the correct probability.

**Lemma 5.2** We can determine a value of $q_3 = 2^{-m/2}(1 + o(m))$ such that $Pr[F_v|_{\rho^i} = 1] = 2^{-5m/2}$.

**Proof:** Let $p_2$ be the probability that that case 2 happens. By standard Chernoff bounds we have $p_2 = \exp(-\Omega(2^{m/2}/m))$. Let $p_{2,1}$ be the probability that the value of $F_v$ is fixed to 1 under case 2.

Now let $p_3$ be the event that we are in case 3. As the probability that $\alpha = 1_f$ is $c_i$ we have that $p_3 = c_i - p_{2,1}$ and thus $p_3 = 2^{-2m}(1 + o(m))$. Fix the value of $q_3$ to be $(2^{-5m/2} - p_{2,1})/p_3$ and note that $q_3 = 2^{-m/2}(1 + o(m))$ as promised. The probability of fixing $F_v$ to the value 1 is $p_{2,1} + p_3q_3$ and this, by the choice of $q_3$, equals $2^{-5m/2}$.
We next determine a suitable value for \( q_4(f_v) \).

**Lemma 5.3** We can determine a value of \( q_4(f_v) = 2^{-m}(1 + o(m)) \) such that setting \( x_v = 1 \) with probability \( b_i \) gives the same distribution as setting \( x_w = \alpha_w \) for all \( w \).

**Proof:** We prove that for any \( S \) and \( T \), conditioned on \( T \) being the non-hard ones and \( S \) being the set of variables set to \( x_v \), the probability that this happened in case 3 is \( b_i \) while the probability that this happened in case 4 is \( 1 - b_i \). As we in case 3 change the values of the variables in \( S \) from 1 to \( x_v \) and in case 4 from 0 to \( x_v \), this is sufficient to prove the lemma. From now on we condition on a particular set \( T \) of size \( f_v \) being chosen and no hard zero being picked.

The conditional probability of ending up with a specific set \( S \) and being in case 3 is
\[
p_{S,T}^{4,3} = (1 - b_i) q_{S,T}^{4,3} f_v,
\]
where \( q_{S,T} \) is the conditional probability as in (4). The probability of getting the same sets in case 4 is
\[
p_{S,T}^{4,4} = (1 - (1 - b_i) f_v) q_{S,T} q_4(f_v).
\]

We now set
\[
q_4(f_v) = \frac{(1 - b_i) f_v(1 - b_i)}{(1 - (1 - b_i) f_v)b_i}
\]
with the result that
\[
\frac{p_{S,T}^{4,4}}{p_{S,T}^{4,3}} = \frac{1 - b_i}{b_i}.
\]

Since we did not fix the values of all variables in case 2 we know that \( f_v = f_i 2^{-m}(1 + o(m)) \) and hence \((1 - b_i) f_v = 2^{-2m}(1 + o(m))\). Furthermore as \( b_i = 2^{-m}(1 + o(m)) \) it is possible to satisfy (5) with \( q_4(f_v) = 2^{-m}(1 + o(m)) \).

From now on we assume that we use the values of \( q_3 \) and \( q_4(f_v) \) determined by Lemma 5.2 and Lemma 5.3. The next lemma is, more or less, an immediate consequence of Lemma 5.3.

**Lemma 5.4** Let \( v \) be a node on level \( i \) of the formula for \( F_d \). Then picking a random \( \rho^i \in R^i \) and then setting \( x_v \) with bias \( b_i \) gives the uniform distribution on inputs to the formula \( F_v \).

For completeness let us point out that by “bias \( b_i \)” we here mean that \( x_v \) is more likely to be 0 if \( i \) is odd and more likely to be 1 if \( i \) is even.

**Proof:** We proceed by induction over \( i \) and for \( i = 1 \) we already established the lemma in the discussion after the definition of \( R^1 \).

Lemma 5.3 tells us that picking a \( x_v \) with bias \( b_i \) is the same as setting the soft values according to \( \alpha \). This, in its turn, is the same as picking independent restrictions from \( \rho^{i-1} \) for each sub-formula \( F_w \) and setting any live \( x_w \) with bias \( b_{i-1} \). By induction this results in the uniform distribution. ■
Let us also verify the last basic property of $R^i$.

**Lemma 5.5** We have $\Pr[F_v[\rho] = 0] = 1 - 2^{-m}$.

**Proof:** This could be done by a tedious calculation, but in fact it can be seen by a high level argument. The restriction $\rho'$ can reduce $F_v$ to 0, 1 or $x_v$. Lemma 5.2 says that second value is taken with the correct probability and Lemma 5.4 says that if $x_v$ is set to 1 with bias $b_i$ then we get a uniformly random input and hence the output of $F_v$ is one with probability $c_i$. This implies that

$$2^{-5m/2} + b_i \Pr[F_v[\rho] \equiv x_v] = c_i$$

and hence, by the definition of $b_i$, we conclude that $\Pr[F_v[\rho] \equiv x_v] = 2^{-m} - 2^{-5m/2}$ and as the probabilities of obtaining the three possible values for $F_v[\rho']$ sum to one, the lemma follows. 

The most interesting property of our restrictions is that we can prove a switching lemma and we proceed with this step.

## 6 The switching lemmas

To establish a general hierarchy result (in particular distinguishing depth $d$ and $d - 2$) it is sufficient to prove a switching lemma for $R^i$ for $i \geq 2$ and in view of this we prove this lemma first. To get a tight result we later prove a modified lemma for $R^1$.

As discussed in Section 4, a restriction $\rho^i \in R^i$ is chosen by first picking $\rho^{i-1} \in R^{i-1}$, followed by $\rho_1$, $\rho$ and finally making a projection $\pi$. In this section we assume any fixed values of $\rho^{i-1}$ and $\rho_1$ and consider the effect of $\rho$. The fact that the distribution of $\rho$ is dependent on the actual values of $\rho^{i-1}$ and $\rho_1$ is left implicit.

A set $\mathcal{F}$ of restrictions is said to be “downward closed” if changing the value of $\rho$ on some input from the value $*$ to a constant cannot make it leave the set. Let us write this formally.

**Definition 6.1** A set $\mathcal{F}$ of restrictions is downward closed if when $\rho \in \mathcal{F}$ and $\rho'(x_w) = \rho(x_w)$ for $w \neq w_0$ and $\rho(x_{w_0}) = *$ then $\rho' \in \mathcal{F}$.

We can now formulate the main lemma.

**Lemma 6.2** Let $\rho^i \in R^i$ be a random restriction with components $\rho^{i-1}$, $\rho_1$, $\rho$ and $\pi$ and $f$ be an arbitrary function. Suppose $g = f[\rho^{i-1}]$ is computed by a depth-2 circuit of bottom fanin $t \leq 2^m/8$. Let $\mathcal{F}$ be a downward closed set of restrictions and let $\text{depth}(g[\rho])$ be the minimal depth of the decision tree computing $g[\rho]$. Then, for sufficiently large $m$,

$$\Pr[\text{depth}(g[\rho]) \geq s \mid \rho \in \mathcal{F}] \leq D^s,$$

where $D = t2^{3m/2}$. 

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Proof: By symmetry we may assume that $i$ is odd. By possibly looking at the negation of $g$ (which has the same depth decision tree as $g$) we can assume that $g$ is a CNF, and after $\rho_1$ has been applied it can be written as

$$g|_{\rho_1} = \bigwedge_{i=1}^t C_i,$$

where each $C_i$ is a disjunction of at most $t$ literals. The proof proceeds by induction over $\ell$ and the base case is when $\ell = 0$ in which case $g|_{\rho_1}$ is always computable by a decision tree of depth 0.

We divide the analysis into two cases depending on whether $C_1$ is forced to one or not. We can bound the probability of the lemma as the maximum of

$$\Pr[\text{depth}(g|_{\rho_1}) \geq s \mid \rho \in \mathcal{F} \land C_1|_{\rho} \equiv 1],$$

and

$$\Pr[\text{depth}(g|_{\rho_1}) \geq s \mid \rho \in \mathcal{F} \land C_1|_{\rho} \not\equiv 1]. \tag{7}$$

The first term is taken care of by induction applied to $g$ without its first conjunction (and thus having size at most $\ell - 1$) and using that the conditioning in this case is a new downward closed set. We need to consider the second term (7).

To avoid that $g|_{\rho_1} \equiv 0$ there must be some non-empty set, $Y$ of variables appearing in $C_1$ which are given the value $*$ by $\rho$. Let a set $B_v$ of variables be called a “block” and suppose the variables in $C_1$ come from $t_1$ different blocks. Say that a block is “undetermined” if it contains a variable given the value $*$ by $\rho$. Let $Z$ be the set of undetermined blocks and let us assume it is of size $r$.

Let us introduce the notation $\text{undet}(Z)$ to denote the event that all blocks in $Z$ are undetermined and $\text{det}(C_1/Z)$ to say that all variables in $C_1$ outside $Z$ are fixed to non-$*$ values by $\rho$.

We start constructing a decision tree for $g|_{\rho_1}$ by querying the new variables corresponding to $Z$. Let $\tau$ be an assignment to these variables. We can now bound (7) as

$$\sum_{\tau, Z} \Pr[\text{depth}(g|_{\tau \rho_1}) \geq s - r \land \text{undet}(Z) \land \text{det}(C_1/Z) \mid \rho \in \mathcal{F} \land C_1|_{\rho} \not\equiv 1], \tag{8}$$

where $r$ is the size of $Z$ which is non-empty. We will use the estimate

$$\Pr[\text{depth}(g|_{\tau \rho_1}) \geq s - r \mid \text{undet}(Z) \land \text{det}(C_1/Z) \land \rho \in \mathcal{F} \land C_1|_{\rho} \not\equiv 1] \times \Pr[\text{undet}(Z) \mid \rho \in \mathcal{F} \land C_1|_{\rho} \not\equiv 1] \tag{9}$$

for each term in (8) and hence a key lemma is the following.

Lemma 6.3 If $Z$ is a set of set $r$ blocks appearing in $C_1$ and $\rho$ a random restriction appearing in the construction of $R^i$ for $i \geq 2$, then, for sufficiently large $m$,

$$\Pr[\text{undet}(Z) \mid \rho \in \mathcal{F} \land C_1|_{\rho} \not\equiv 1] \leq 2^{r(1-m/2)}.$$
Proof: The crux of the proof is to, given a restriction \( \rho \) that contributes to the probability in question, create a restriction \( \rho' \) that also satisfies the conditioning but fixes all variables in the blocks of \( Z \). We describe how to do this for \( r = 1 \) but the general case follows immediately as we can do the changes independently on each block. Thus let us assume that \( Z \) is the single block \( B_v \) and fix a restriction \( \rho \) that contributes to the event of the lemma. Let \( P \) be the set of variables of \( B_v \) that appears positively in \( C_1 \) and \( N \) the set of variables that appear negatively.

We can assume that we have no hard zero in \( B_v \) and the number of non-hard ones in \( B_v \) is close to \( f_1 2^{-m} \) as otherwise already \( \rho_1 \) would have fixed all variables in \( B_v \) to constants.

Clearly for \( \rho \) we must have \( \rho(x_v) = * \). For variables \( x_w \in P \) we must have \( \rho(x_w) = * \) while for variables in \( N \) we have either \( \rho(x_w) = * \) or \( \rho(x_w) = 1 \).

We now define a companion restriction \( \rho' = H(\rho) \). If \( \rho \) maps some variable outside \( N \) to * (and in particular if \( P \) is non-empty) we set \( \rho'(x_v) = 0 \) and otherwise \( \rho'(x_v) = 1 \). For a \( x_w \in P \) we set \( \rho'(x_w) = 0 \) while for \( x_w \in N \) we set \( \rho'(x_w) = 1 \), independently of the value of \( \rho(x_w) \). Outside \( C_1 \) but in \( B_v \) we set \( \rho'(x_w) = 1 \) if \( \rho(x_w) = 1 \) and \( \rho'(x_w) = \rho(x_w) \) otherwise. Outside \( B_v \), \( \rho \) and \( \rho' \) agree. First observe that \( \rho' \) satisfies the conditioning. We only have \( \rho(x_w) \neq \rho'(x_w) \) when \( \rho(x_w) = * \) and by the definition of \( P \) and \( N \) we are careful not to satisfy \( C_1 \).

The mapping \( H \) is many-to-one as given \( \rho' \) we do not know the values of \( \rho(x_w) \) when \( x_w \in N \) (but we do for all other variables in \( B_v \)).

First note that

\[
\frac{\Pr(\rho)}{\Pr(\rho')} = \frac{\Pr(\rho_v)}{\Pr(\rho'_v)}
\]

where \( \rho_v \) is only the behavior of \( \rho \) on \( B_v \) and similarly for \( \rho' \). This is true as \( \rho \) and \( \rho' \) take the same values outside \( B_v \) and the restrictions are picked independently on each \( B_v \).

Assume first that \( \rho'(x_v) = 0 \). In this situation \( \rho \) could have been picked under case 3 or case 4 while \( \rho' \) can only have been produced under case 4. We know, by (6), that each \( \rho \) is about a factor \( 2^m \) more likely to have been produced under case 4 than under case 3 so let us ignore case 3, introducing a small error factor \((1 + O(2^{-m}))\) that we temporarily suppress.

Let \( N_1 \) be subset of \( N \) that was actually given the value * by \( \rho \). If \( N_1 \) is empty then \( \Pr(\rho_v) = \frac{q_4(f_v)}{1 - q_4(f_v)} \Pr(\rho'_v) \) and in general we pick up an extra factor \( b_{i-1} [N_1] (1 - b_{i-1})^{-|N_1|} \). As

\[
\sum_{N_1 \subseteq N} b_{i-1} [N_1] (1 - b_{i-1})^{-|N_1|} = (1 + \frac{b_{i-1}}{1 - b_{i-1}})^{|N|}
\]

we get

\[
\sum_{H(\rho) = \rho'} \Pr(\rho) \leq \left(1 + \frac{b_{i-1}}{1 - b_{i-1}}\right)^{|N|} \frac{q_4(f_v)}{1 - q_4(f_v)} \Pr[\rho'] \leq 2^{1-m} \Pr[\rho'], \quad (10)
\]
for sufficiently large \( m \). This follows as \(|N| \leq 2^m/8\), \( b_i = (1 + o(m))2^{-m} \) and \( q_4 = (1 + o(m))2^{-m} \).

If, \( \rho'(x_v) = 1 \) the situation is similar except that \( \rho' \) is produced under case 3 and thus we pick up a factor \( q_3 \) instead of \( 1 - q_4(f_v) \). We get in this case

\[
\sum_{H(\rho) = \rho'} \Pr(\rho) \leq \left( 1 + \frac{b_{i-1}}{1 - b_{i-1}} \right) \frac{|N|}{q_4(f_v)} \Pr[\rho'] \leq 2^{1-m/2} \Pr[\rho'],
\]

again for sufficiently large \( m \). The fact that we ignored restrictions \( \rho \) produced under case 3 gives an additional factor \( (1 + O(2^{-m})) \) in the above estimates and thus the calculations remain valid, possibly by making \( m \) slightly larger, to make sure that the “sufficiently large \( m \)” statements are true.

The case of general \( r \) follows from the fact that we do the modifications on all blocks of \( Z \) independently.

\textbf{Remark 1} The careful reader might have noticed that in the case with the \( \rho'(x_v) = 1 \) then we can conclude that \( N_1 \) is non-empty giving a slightly better estimate especially in the case when \( t \) is small. This observation can probably be used to get a slightly better constant in the main theorem, but to keep the argument simple we ignore this point. We return to main argument.

We now estimate

\[
\Pr[\min depth(g_{\tau^r}) \geq s - r \mid \text{undet}(Z) \wedge \det(C_1/Z) = 0 \wedge \rho \in \mathcal{F} \wedge C_1[\rho \neq 1],
\]

by induction. We need to check that the conditioning defines a downward closed set. This is not complicated but let us spell out some details. Fix any behavior of \( \rho \) inside the blocks of \( Z \) and satisfying the conditioning. As \( g|_{\rho^r} \) does not depend on the variables corresponding to \( Z \) the event in (12) depends only the values of \( \rho \) outside \( Z \). Changing \( \rho \) from \( * \) to a constant value for any variable outside \( Z \) cannot violate any of the conditions in the conditioning and hence we have a downward closed set when considering \( \rho \) as a restriction outside \( Z \). We conclude that the probability of the event in (12) is, by induction, bounded by \( D^{s-r} \).

Our goal is to estimate the sum (8), using the bound (9) for each term, Lemma 6.3 and the inductive case. If \( C_1 \) intersects \( t_1 \) different blocks (where of course \( t_1 \leq t \)) then, using the fact that we have at most \( 2^t \) (remember that \( r \) is the number of blocks of \( Z \) different \( \tau \), we get the total estimate

\[
\sum_{Z \neq \emptyset} 2^r 2^{(1-m/2)} D^{s-r} = D^s \left( (1 + D^{-1}2^{-m/2})^{t_0} - 1 \right) \leq D^s \left( 1 + \frac{1}{2t} \right)^{t_0} - 1 \leq D^s
\]

and we are done.

\textbf{Lemma 6.2} is sufficient to prove a fairly tight hierarchy theorem. To prove a tight variant we need also to see how \( R^3 \) simplifies circuits.
Lemma 6.4 Let $g$ be computed by a depth-2 circuit of bottom fanin $t \leq m/4$. Let $\mathcal{F}$ be a downward closed set of restrictions and $\rho^1$ a random restriction with the distribution $R^1$. Let depth$(g[\rho^1])$ be the minimal depth of the decision tree computing $g[\rho^1]$. Then, for sufficiently large $m$,

$$Pr[\text{depth}(g[\rho^1]) \geq s \mid \rho \in \mathcal{F}] \leq D^s,$$

where $D = t2^{3+t-m/2}$.

Proof: The proof of this lemma is almost identical to the proof of Lemma 6.2 and let us only discuss the differences. Lemma 6.3 is replaced by the following.

Lemma 6.5 If $Z$ is a set of set $r$ blocks appearing in $C_1$ and $\rho$ a random restriction appearing in the construction of $R^1$, then, for sufficiently large $m$,

$$Pr[\text{undet}(Z) \mid \rho \in \mathcal{F} \land C_1[\rho] \neq 1] \leq 2^t(t+1-m/2).$$

Proof: The proof is almost the same as the proof of Lemma 6.3. The reason for the loss in parameters is that the factor

$$\left(1 + \frac{b_{i-1}}{1-b_{i-1}}\right)^{|N|}$$

that used to be bounded by a constant strictly less than two can now be as large as $2^t$.

The rest of the proof of how Lemma 6.4 follows from Lemma 6.5 is identical with how Lemma 6.2 followed from Lemma 6.3 with the obvious change in the final calculation.

7 The proof of the main theorem

We now proceed to prove Theorem 1.1. In fact we are going to prove the following, slightly stronger, theorem.

Theorem 7.1 Let $C$ be a circuit depth $d$ with bottom fanin at most $m/4$ and which is of size $S$ then, for sufficiently large $m$,

$$Pr[F_d(x) = C(x)] \leq \frac{1}{2} + O(2^{-m/4}) + S2^{-2m/2-4}.$$

It is not difficult to see that this theorem implies Theorem 1.1 as a depth $d-1$ circuit can be seen as a depth $d$ circuit with bottom fanin one and that $m = \frac{\log n}{2d-2}(1 + o(1))$. We turn to proving Theorem 7.1.
Proof: Let us apply a random restriction $\rho^{d-1} \in \mathbb{R}^{d-1}$ to both $F_d$ and $C$. Let us assume that $i$ is odd and hence the output gate of $F_d$ is an and-gate. The case of even $i$ is completely analogous. By Lemma 5.4 we have

$$Pr[F_d(x) = C(x)] = Pr[F_{\rho^{d-1}}(x) = C[\rho^{d-1}(x)]$$

where the latter probability is over a random $\rho^{d-1}$ and random assignment to the live variables in $V_{d-1}$ where each variable is given the value 1 with probability $1 - b_{d-1}$. Let us first see how $\rho^{d-1}$ affects $F_d$.

We have the output gate, $v$, of fanin $f_d$. With probability $O(2^{-m/2})$ some input gate is forced to 0 by $\rho^{d-1}$. Suppose that this does not happen and let $b_1$ be the number of input gates to $v$ that are not fixed to one. With probability $1 - \exp(-\Omega(2^{m/2}))$ we have $|b_1 - f_d 2^{-m}| \leq 2^{3m/4}$. Thus we conclude that, with probability $1 - O(2^{-m/2})$, $F_d$ has been reduced to an and-gate of fanin $\ln 2 \cdot 2^m(1 + O(2^{-m/4}))$.

Now let us see how $\rho^{d-1}$ affects $C$. We prove by induction that $\rho^i$, with high probability, reduces the depth of $C$ by $i$. Let us assume that $C$ has $S_i$ gates at distance $i$ from the inputs.

Consider any gate in $C$ at distance two from the inputs and suppose it is an or or of and-gates, the other case being similar. By Lemma 6.4, for sufficiently large $m$, after $\rho^i$ has been applied, except with probability $2^{-2^m/4}$ this subcircuit can be computed by a decision tree of depth of depth at most $2^{m/2-4}$. This implies that it can we written as an and of or-gates of fan-out at most $2^{m/2-4}$. We conclude that except with probability $S_i 2^{2^m/2-4}$, by collapsing two adjacent levels of and-gates, $C[\rho^i]$ can be computed by a depth $d-1$ circuit with bottom fanin at most $2^{m/2-4}$ where each gate at distance at least two from the inputs corresponds to a gate at distance at least three in the original circuit.

Applying Lemma 6.2 for $i = 2, 3 \ldots d-2$ in a similar way we conclude that except with probability $\sum_{i=3}^{d-2} S_i 2^{-2^m/2-4}$, $C[\rho^i]$ can be computed by a depth 2 circuit of bottom fanin $2^{m/2-4}$. A final application of Lemma 6.2 says that except with an additional failure probability $2^{-2^m/2-4}$, $C[\rho^{d-1}]$ can be computed by a decision tree of depth $2^{m/2-4}$.

By the above reasoning we know that except with probability $O(2^{-m/2}) + S 2^{-2^m/2-4}$, it is true that $F_d[\rho^{d-1}]$ is an and of size $\ln 2 \cdot 2^m (1 + O(2^{-m/4}))$ and $C[\rho^{d-1}]$ is computed by a decision tree of depth $2^{m/2-4}$. As the former is equal to 1 with probability $\frac{1}{2}(1 + O(2^{-m/4}))$ and the output of any decision tree of depth $s$ of inputs that are $b_{d-1}$ biased has a $sb_{d-1}$ biased output, we conclude that

$$Pr[F_d[\rho^{d-1}](x) = C[\rho^{d-1}(x)] = \frac{1}{2} + O(2^{-m/4}) + S 2^{-2^m/2-4}$$

and the proof is complete.

Looking more closely at the proof we can derive an even stronger theorem.

Theorem 7.2 Suppose $d$ is odd and let $C$ be a circuit depth $d + 1$ with output gate that is an or-gate, with bottom fanin at most $m/4$ and of size at most $S_i$,
then, for sufficiently large $m$, 
\[ \Pr[F_d(x) = C(x)] \leq \frac{1}{2} + O(2^{-m/4}) + S2^{-2m/2-4}. \]

The same is true for even $d$ if the output gate of $C$ is an and-gate.

**Proof:** Let us assume that $d$ is odd, the even case being completely analogous. We follow exactly the proof of Theorem 7.1 until the very last step. We can conclude that $C_{\lceil \rho d - 1 \rceil}$, with high probability, is reduced to the disjunction of a set of functions each computable by a decision tree of depth $2^{m/2-4}$. We can convert this to a DNF formula of bottom fanin $2^{m/2-4}$ and we must analyze the probability that such a formula equals an and of size $\ln 2 \cdot 2^m (1 + O(2^{-m/4}))$. We have two cases.

Suppose first that each term in the DNF-formula contains a negated variable. Then $C_{\lceil \rho d - 1 \rceil}$ rejects the all-one input which is chosen with probability $\frac{1}{2} + O(2^{-m/4})$ and as this input is accepted by $F_d\lceil \rho d - 1 \rceil$ we have
\[ \Pr[F_d\lceil \rho d - 1 \rceil(x) = C\lceil \rho d - 1 \rceil(x)] \leq \frac{1}{2} + O(2^{-m/4}) \] (13)

in this situation (where the probability is only over a random input) and this case follows.

On the other hand if there is a term in $C_{\lceil \rho d - 1 \rceil}$ that only contains positive variables then it (and hence $C_{\lceil \rho d - 1 \rceil}$) is true with probability $1 - O(2^{-m/2})$. As $F_d\lceil \rho d - 1 \rceil$ is close to unbiased, (13) is true also in this case and the theorem follows.

As stated previously we have not done a serious effort to get the best constants in our main theorems. They are, however, not too far from the truth as we may take $C$ to be one input to the output gate of $F_d$. This is a depth $d - 1$ circuit of sub-linear size that agrees with $F_d$ for a fraction $\frac{1}{2} + \Omega(2^{-2m})$ of the inputs.

8 Some final words

The main difference between the current paper and the early proof of the hierarchy theorem in [1] is the use of projections. The projections serve two purposes. The first is to make sure that once a single $\ast$ is found in $\rho$ we do not bias any other value of $\rho'$ to be $\ast$. This was achieved in [1] by fixing the values of neighboring variables to constants while here we identify all the neighboring variables with the same new variable and hence we only query one variable in the decision tree. We feel that this difference is minor.

The more important difference is that projections enables us to choose a uniformly random input where this seemed difficult to achieve. It is amazing how seemingly simple ideas can take care of problems that, at least initially, looks like fundamental obstacles.
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References


