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Total space in Resolution is at least width squared

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Abstract

Given an unsatisfiable k-CNF formula φ we consider two complexity measures in Resolution: width and total space. The width is the minimal W such that there exists a Resolution refutation of φ with clauses of at most W literals. The total space is the minimal size T of a memory used to write down a Resolution refutation of φ, where the size of the memory is measured as the total number of literals it can contain. We prove that $T = \Omega((W-k)^2)$.

1 Introduction

Resolution is a well known propositional proof system introduced by Blake in [16] and proposed by Robinson in [37] for automated theorem proving. Since then this proof system has become the most studied proof system in the sub-area of complexity theory that is *Proof Complexity*. Given a set of clauses φ, that is a set of disjunctions of literals or, equivalently, given a formula in Conjunctive Normal Form, *Resolution* is a method to infer new clauses according to the following inference rule:

$$\frac{C \vee D \quad D \vee \neg x}{C \vee D}, \tag{1}$$

where C, D are clauses and x is a variable. Resolution is sound and complete, that is it is possible to derive the empty clause $\perp$ if and only if φ is unsatisfiable. A Resolution *refutation* of φ is then just a sequence of clauses $C_1, \ldots, C_\ell$ with $C_\ell = \perp$ and each clause of the sequence is either a clause from φ or it is inferred by previous clauses in the sequence according to the inference rule in equation (1).

Nowadays the main reason for the interest in Resolution comes from a practical perspective: it is at the core of most of the state-of-the-art SAT solvers since the introduction of the DPLL algorithm [21, 22] and its improvements, the so called *Conflict Driven Clause Learning* (CDCL) algorithms [4, 31, 39]. The track of the running of such algorithms on unsatisfiable instances produces a (particular form of) Resolution proofs. Hence Resolution is a valuable tool to study their performances and limitations. In this work we are interested in more theoretical questions about the Resolution proof system and the reader interested in more details on the connections between Resolution and SAT solvers could look at the recent survey [34].

Given an unsatisfiable φ we are interested in measuring how complex a Resolution proof of φ must be. Certainly there are many ways of measuring the complexity of proofs and in this work we are interested in linking two of such measures. The main complexity measure we can associate to φ in Resolution, and by far the most important, is the minimal length of a Resolution refutation of φ. This measure is denoted with $\mathsf{size}(\varphi \vdash \perp)$ and since a long time now we know that there are certain

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φ requiring exponentially long proofs, e.g. the encodings of the Pigeonhole Principle [27] or Tseitin formulas [38, 41]. Another, easier to study, complexity measure is the *width*. Suppose that we focus on Resolution refutations of φ with clauses up to a certain length w. The minimal w such that we have a refutation of φ with clauses of length at most w is the *width*, $\mathsf{width}(\varphi \vdash \bot)$. We have a trivial upper bound connecting size and width, that is for every set of clauses φ in n variables

$$\mathsf{size}(\varphi \vdash \bot) \leqslant n^{O(\mathsf{width}(\varphi \vdash \bot))},$$

and indeed in [3] is proved that this trivial bound could be tight. Another, more useful, connection between width and size is the following result from the seminal paper by Ben-Sasson and Wigderson [12]:

$$\log_2 \mathsf{size}(\varphi \vdash \bot) \geqslant \frac{(\mathsf{width}(\varphi \vdash \bot) - k)^2}{16n}, \tag{2}$$

where φ is a collection of clauses over n variables and each of them has at most k literals. Hence if $\mathsf{width}(\varphi \vdash \bot) = \omega(\sqrt{n \log n})$ then immediately by the previous relationship $\mathsf{size}(\varphi \vdash \bot)$ is super-polynomial. Moreover this size-width relationship is essentially optimal [20].

Regarding the space complexity of proofs, its investigation was proposed in 1998 by Armin Haken as a natural analogue of the space complexity in the context of Touring machines and the first definitions of space measures in Resolution were given in [24, 1]. When talking about space, Resolution proofs are seen as a sequence of memory configurations $\mathfrak{M}_0, \dots, \mathfrak{M}_\ell$, where each $\mathfrak{M}_i$ is a set of clauses, $\bot \in \mathfrak{M}_\ell$ and each $\mathfrak{M}_{i+1}$ derive from $\mathfrak{M}_i$ in one of the two following ways:

Axiom download $\mathfrak{M}_{i+1} \subseteq \mathfrak{M}_i \cup \{C\}$, where $C \in \varphi$;

Inference $\mathfrak{M}_{i+1} = \mathfrak{M}_i \cup \{D \vee E\}$, where both $D \vee x$ and $E \vee \neg x$ belong to $\mathfrak{M}_i$, for some variable x.

We then have some notions of how "spacious" a memory configuration can be. The most natural space measure for a memory configuration is of course the number of bits needed to write down it. Unfortunately it turns out that this notion of space is quite hard to study and hence some alternative notions of space were introduced. For example the *clause space*, the number of distinct clauses a memory can contain; or the *total space*[1], the total number of literals it can contain. The minimal s such that we have a refutation of φ with memory configurations with total space at most s is the *Total Space* (needed to refute φ), $\mathsf{TSpace}(\varphi \vdash \bot)$. Similarly for the clause space we obtain $\mathsf{CSpace}(\varphi \vdash \bot)$. A more formal definition of $\mathsf{TSpace}(\varphi \vdash \bot)$, to avoid misunderstandings, is provided in Section 2.

We now recall some known results about the space complexity measures. Given a set of clauses φ in n variables, in [24] was proven that

$$\mathsf{CSpace}(\varphi \vdash \bot) \leqslant n + 1,$$

and, as a trivial consequence, we have that

$$\mathsf{TSpace}(\varphi \vdash \bot) \leqslant n(n + 1).$$

Both upper bounds are asymptotically tight, for example for random k-CNF formulas [8, 19]. Regarding lower bounds, in [2] it is proved that

$$\mathsf{CSpace}(\varphi \vdash \bot) \geqslant \mathsf{width}(\varphi \vdash \bot) - k + 1, \tag{3}$$

[1] In [1] this space complexity measure is called *variable space*, but we follow [10, 9, 32, 11, 33] in calling it *total space*. This is due to distinguish it from a different space complexity measure in which different occurrences of the same variable are not counted, the *variable space*, investigated for instance in [42].

where φ consists of clauses of at most k literals. Clearly $\mathsf{TSpace}(\varphi \vdash \bot) \geqslant \mathsf{CSpace}(\varphi \vdash \bot)$ and whenever $\mathsf{TSpace}(\varphi \vdash \bot) = \omega(\mathsf{CSpace}(\varphi \vdash \bot))$ we say to have a *non-trivial* total space lower bound.

The total space measure was introduced in [1] and there the first non-trivial lower bounds were proven, for two particular class of formulas the Complete Tree formulas and the Pigeonhole Principle formulas. After that, in [19] it was introduced a technique to prove total space lower bounds in Resolution. That technique was sufficiently strong to prove asymptotically optimal total space lower bounds for, for example, random k-CNFs [19, 13] but the proofs given there are quite long and involved. This paper, as a corollary, deeply simplify such proofs. Space complexity measures are also studied concerning trade-offs with other complexity measures, see for example [9, 35, 11, 33, 7, 5].

1.1 Contributions

This work is about proving an analogue of the inequality in (3) for the total space. This will add a nice bit to our knowledge of the lattice of relations between complexity measures in Resolution, it will simplify the proofs of existing total space lower bounds and it will imply new non-trivial total space lower bounds.

Theorem 1.1. *Let φ be a k-CNF formula, then*

$$\mathsf{TSpace}(\varphi \vdash \bot) \geqslant \frac{1}{16} \left(\mathsf{width}(\varphi \vdash \bot) - k - 4\right)^2 .$$

The general idea of the proof is the following: given a Resolution refutation, we identify a memory configuration where some small clause appear and then show that before that moment there must have been some memory configuration with a lot of clauses (and hence with large total space). This idea was originally used in [1] in some particular cases and in more generality in [19]. Indeed the proof we show has some close structural similarities with the total space lower bound from [19] and essentially it is a simplification of the proof of Theorem 2.5 from the author PhD. Thesis [17]. The proof we give is not direct since it involves another, less studied, complexity measure: the *asymmetric width*, $\mathsf{awidth}(\varphi \vdash \bot)$, and families of assignments closely related to it. The asymmetric width was introduced in [28, 29] and the definition is quite technical so we defer it to Section 2 where we collect all the preliminary definitions and notations. The proof of Therem 1.1 is purely combinatorial and implicitly uses some properties from a characterization of the asymmetric width from [15], cf. Section 3 for more details, together with a result tightly connecting the width and the asymmetric width, Theorem 2.1 (Lemma 8.5 from [28]).

Although defined quite differently, asymmetric width and width indeed share many properties. For instance, an analogue of the size-width inequality by Ben-Sasson and Wigderson [12]: given an unsatisfiable CNF formula φ in n variables

$$\ln\left(\mathsf{size}(\varphi \vdash \bot)\right) \geqslant \frac{\mathsf{awidth}(\varphi \vdash \bot)^2}{8n},$$

cf. Theorem 6.12 of [30]. For more information and history on the asymmetric width we refer to [15].

1.2 Examples of applications (and limitations) of Theorem 1.1

Since the seminal work of Ben-Sasson and Wigderson [12], the width measure has become one of the main tool to study Resolution proofs and their complexity. Hence we already have many relevant width lower bounds for many interesting class of formulas and then the range of applications of Theorem 1.1 is quite large. Below we recall some of the relevant examples.

Tseitin formulas. Given a d-regular graph G over n vertices, the Tseitin formula over G, $\mathsf{Tseitin}(G)$, is a CNF formula over $dn/2$ variables based on a propositional encoding of the fact that the total degree in any graph is even, see for example [12] for a formal definition. Such formulas were used by Tseitin to prove the first super-polynomial size lower bound for Resolution size [40]. Since then those formulas became one of the standard tools in proof complexity to prove lower bounds and trade-offs, see for example [38, 42, 12, 24, 7]. In particular given a connected 3-regular graph G over n vertices which is an expander, in [12] is proved that $\mathsf{width}(\mathsf{Tseitin}(G) \vdash \bot) \geqslant \Omega(n)$. Hence, by Theorem 1.1, we have an asymptotically optimal total space lower bound: $\mathsf{TSpace}(\mathsf{Tseitin}(G) \vdash \bot) = \Theta(n^2)$. This completely answer the open question 4 from [1].

Random k-CNFs. A random k-CNF with n variables and clause density Δ is a CNF formula picked as follows: choose independently uniformly at random Δn clauses from the set of all possible clauses in the variables $\{x_1, \ldots, x_n\}$ containing exactly k literals. If $\Delta = o(n^{1/4})$, Beame et al. [6] showed that random k-CNFs require exponential size Resolution proofs. Such result was simplified in [12] by showing a lower bound on width: if φ is a random k-CNF ($k \geqslant 3$) in n variables and Δn clauses, and Δ is a constant for simplicity, then with high probability $\mathsf{width}(\varphi \vdash \bot) \geqslant \Omega(n)$. Hence, by Theorem 1.1, with high probability $\mathsf{TSpace}(\varphi \vdash \bot) \geqslant \Omega(n^2)$. That is almost every k-CNF require asymptotically optimal total space to be refuted in Resolution. This result was proven in [19] for $k \geqslant 4$ and for $k = 3$ in [13] with some explicit but quite involved constructions. Instead, as we saw, an asymptotically optimal total space lower bound for such formulas follows immediately from Theorem 1.1.

Formulas with short proofs. Bonet and Galesi [20] showed that the size-width inequality by Ben-Sasson and Wigderson [12] is essentially optimal. That is they showed that there are arbitrarily large 3-CNF formulas φ_n with $\Theta(n^3)$ clauses, $\Theta(n^2)$ variables and such that

- $\mathsf{width}(\varphi_n \vdash \bot) = \Theta(n)$,
- $\mathsf{CSpace}(\varphi_n \vdash \bot) = \Theta(n)$,

but φ_n has some Resolution proof of size $O(n^3)$, width $O(n)$ and clause space $O(n)$. Theorem 1.1 in this case tells us that $\mathsf{TSpace}(\varphi_n \vdash \bot) = \Omega(n^2)$, which is just a linear lower bound in the number of variables of φ_n. On the other hand this is a non-trivial total space lower bound since $\mathsf{TSpace}(\varphi_n \vdash \bot) = \omega(\mathsf{CSpace}(\varphi_n \vdash \bot))$.

Regarding the limitations, Theorem 1.1 suffers from the same kind of limitations of size-width relation, equation (2), and the clause space-width relation, equation (3). That is it became trivially vacuous for CNF formulas in n variables with clauses with many literals. For example we see such phenomenon when considering encodings of the negation of the Pigeonhole Principle as CNFs having clauses of n literals, the PHP_n^{n+1} formulas. For such formulas $\mathsf{width}(\mathsf{PHP}_n^{n+1} \vdash \bot) = \Theta(n)$ and hence no size lower bound or clause space lower bound could be implied directly from equations (2) - (3). The same applies for Theorem 1.1. On the other hand, by different techniques, we still have size lower bounds [41, 36], $\mathsf{size}(\mathsf{PHP}_n^{n+1} \vdash \bot) \geqslant 2^{\Omega(n)}$, clause space lower bounds [1], $\mathsf{CSpace}(\mathsf{PHP}_n^{n+1} \vdash \bot) \geqslant n$, and total space lower bounds [1, 19], $\mathsf{TSpace}(\mathsf{PHP}_n^{n+1} \vdash \bot) \geqslant \frac{1}{4}n^2$.

1.3 Organization of the paper

Section 2 contains all the preliminary definitions and notations needed for the proof of Theorem 1.1. In Section 3 we prove Theorem 1.1 and we give some more detailed comments on the proof. Section 4 contains some open questions about total space.

2 Preliminaries

We consider fixed a set of variables X and, given a natural number n, we denote as $[n]$ the set $\{1, \ldots, n\}$. Given a set A, $\binom{A}{\leqslant 2}$ is the subset of the power set of A consisting of all the subsets of size at most 2.

Partial assignments. Given a set of variables X, a *partial (Boolean) assignment over* X is a function $\alpha : X \to \{0,1\} \cup \{\star\}$. The *domain* of α is $\mathrm{dom}(\alpha) = \alpha^{-1}(\{0,1\})$ and we say that α assigns a value to x if $x \in \mathrm{dom}(\alpha)$. With λ we denote the partial assignment with empty domain. Given two partial assignments over X, α and β we say that α extends β, $\beta \subseteq \alpha$, if for all $x \in X$ $\beta(x) \in \{\alpha(x), \star\}$. We denote a partial assignment with domain a single variable x set to a value $b \in \{0,1\}$ simply as $\{x \mapsto b\}$. Given two partial assignments α and β with disjoint domains, with $\alpha \cup \beta$ we denote the partial assignment with domain $\mathrm{dom}(\alpha) \cup \mathrm{dom}(\beta)$ such that for each $x \in \mathrm{dom}(\alpha) \cup \mathrm{dom}(\beta)$

$$\alpha \cup \beta(x) = \begin{cases} \alpha(x) & \text{if } x \in \mathrm{dom}(\alpha), \\ \beta(x) & \text{if } x \in \mathrm{dom}(\beta). \end{cases}$$

CNF formulas. A *literal* is a variable in X or the negation of a variable in X. A clause C is a formula of the form $\ell_1 \vee \cdots \vee \ell_k$, where the ℓ_i are literals and m is the *width* of the clause C, denoted as $|C|$. A formula in *Conjunctive Normal Form* (CNF) is a formula φ with variables in X of the form $C_1 \wedge \cdots \wedge C_m$, where the C_js are clauses. A k-CNF formula is a CNF formula where each clause has at most k distinct literals. With $\mathrm{var}(\varphi)$ we denote the set of variables occurring in the formula φ.

Given a CNF formula φ over a set of variables X and a partial assignment α over X, we can apply α to φ, obtaining a new CNF formula, denoted as $\varphi\restriction_\alpha$ or $\alpha(\varphi)$, in the following way: for each variable $x \in \mathrm{dom}(\alpha)$ substitute each occurrence of x in φ with $\alpha(x)$. Then simplify the resulting CNF according to the following rules: $\neg 0 \equiv 1$, $\neg 1 \equiv 0$, $0 \vee A \equiv A$, $1 \vee A \equiv 1$, $1 \wedge A \equiv A$, $0 \wedge A \equiv 0$. We say that α *satisfies* φ if $\alpha(\varphi) = 1$ and we say that α *falsifies* φ if $\alpha(\varphi) = 0$. Similarly, we can apply a partial assignment α to set of clauses $A = \{C_1, \ldots, C_\ell\}$ component-wise: $A\restriction_\alpha = \{C_1\restriction_\alpha, \ldots, C_1\restriction_\alpha\}$. Given a set of formulas F and a partial assignment α we say that α satisfies F, $\alpha \models F$, if and only if for every formula $\varphi \in F$, $\alpha(\varphi) = 1$.

Resolution proofs. A *Resolution derivation* of a clause C from a CNF formula φ is a sequence of clauses $\pi = (C_1, \ldots, C_\ell)$ such that $C_\ell = C$ and each C_i is either a clause from φ or it is inferred from C_j, C_k with $j, k < i$ and such that $\frac{C_j \quad C_k}{C_i}$ is a valid instance of the Resolution rule:

$$\frac{C \vee x \quad D \vee \neg x}{C \vee D}$$

where C, D are clauses and x is a variable; or, C_i is inferred from a C_j with $j < i$ and such that $\frac{C_j}{C_i}$ is a valid instance of the *weakening* inference rule[2]

$$\frac{C}{C \vee D}$$

where C, D are clauses. A *Resolution refutation* of a CNF formula φ is a Resolution derivation of the empty clause $\bot$ from φ. Resolution is sound and complete, that is it is possible to infer the empty clause $\bot$ from φ if and only if φ is unsatisfiable.

Similarly, we can apply a partial assignment α to a sequence of clauses $\pi = (C_1, \ldots, C_\ell)$ component-wise: $\pi\restriction_\alpha = (C_1\restriction_\alpha, \ldots, C_1\restriction_\alpha)$, moreover if π is a Resolution derivation of a clause C from φ then $\pi\restriction_\alpha$ is a Resolution derivation of $C\restriction_\alpha$ from $\varphi\restriction_\alpha$. The last notation we need is for the concatenation of Resolution derivations: given $\pi = (C_1, \ldots, C_\ell)$ and $\pi' = (C'_1, \ldots, C'_{\ell'})$ then $\pi \circ \pi' = (C_1, \ldots, C_\ell, C'_1, \ldots, C'_{\ell'})$.

[2]Notice that the weakening rule is not really needed but it will make simpler the exposition when dealing with restrictions of Resolution proofs.

Width. Given a sequence of clauses $\pi = (C_1, \ldots, C_\ell)$ we recall that

$$\mathsf{width}(\pi) = \max_{C_j \in \pi} |C_j|$$

and the minimal width needed to refute φ in Resolution is

$$\mathsf{width}(\varphi \vdash \bot) = \min_{\pi} \mathsf{width}(\pi),$$

where the min is taken over all refutations of φ in Resolution[3].

Asymmetric width. The notion of *asymmetric width* was introduced in [29, 30]. Let φ be a CNF formula and $\pi = (C_1, \ldots, C_\ell)$ be a Resolution derivation from φ. To define the asymmetric width of π, $\mathsf{awidth}(\pi)$ we preliminary need the notion of *witness function.* A *witness function for* $\pi = (C_1, \ldots, C_\ell)$ is a function $\sigma : [\ell] \to \binom{[\ell]}{\leqslant 2} \cup \{\star\}$ witnessing the fact that π is a derivation from φ, that is such that

- $\sigma(i) = \{j, k\}$ implies that $j, k < i$ and $\frac{C_j \quad C_k}{C_i}$ is a valid instance of the inference rule of Resolution and if $j = k$ we require $\frac{C_j}{C_i}$ to be a valid instance of the weakening rule; and
- $\sigma(i) = \star$ implies that C_i is a clause from φ.

Given $\pi = (C_1, \ldots, C_\ell)$ a Resolution derivation from φ and a witness function σ for π, the *asymmetric width of C_i with respect to π and σ, $aw_{\pi,\sigma}(C_i)$*, is defined as follows

$$aw_{\pi,\sigma}(C_i) = \begin{cases} 0 & \text{if } \sigma(i) = \star, \text{ that is } C_i \in \varphi, \\ \min_{j \in \sigma(i)} |C_j| & \text{otherwise.} \end{cases}$$

Then $\mathsf{awidth}(\pi)$ is the minimum over all the possible functions σ witnessing the validity of π of the maximum over i of $aw_{\pi,\sigma}(C_i)$, that is

$$\mathsf{awidth}(\pi) = \min_{\sigma} \max_{C_i \in \pi} aw_{\pi,\sigma}(C_i).$$

Finally, the asymmetric width needed to refute φ, $\mathsf{awidth}(\varphi \vdash \bot)$, is the minimum of $\mathsf{awidth}(\pi)$ over all possible sequence of clauses $\pi = (C_1, \ldots, C_\ell)$ that are Resolution refutations of φ.

Clearly holds that $\mathsf{awidth}(\varphi \vdash \bot) \leqslant \mathsf{width}(\varphi \vdash \bot)$. Interestingly the width cannot be much bigger than the asymmetric width.

Theorem 2.1 (Lemma 8.5 of [28]). *Let φ be an unsatisfiable k-CNF formula, then*

$$\mathsf{width}(\varphi \vdash \bot) \leqslant \mathsf{awidth}(\varphi \vdash \bot) + \max\{\mathsf{awidth}(\varphi \vdash \bot), k\}.$$

For completeness this result is proven in Appendix A, such proof is self-contained and essentially based on [14].

Total Space. As we saw in the introduction, a Resolution refutation of a CNF formula φ can be seen as a sequence of memory configurations $\pi = (\mathfrak{M}_0, \ldots, \mathfrak{M}_\ell)$, where each $\mathfrak{M}_i$ is a set of clauses, $\bot \in \mathfrak{M}_\ell$ and each $\mathfrak{M}_{i+1}$ derive from $\mathfrak{M}_i$ in one of the two following ways:

Axiom download $\mathfrak{M}_{i+1} \subseteq \mathfrak{M}_i \cup \{C\}$, where $C \in \varphi$;

Inference $\mathfrak{M}_{i+1} = \mathfrak{M}_i \cup \{D \vee E\}$, where both $D \vee x$ and $E \vee \neg x$ belong to $\mathfrak{M}_i$, for some variable x.

[3] If φ is a satisfiable CNF formula then is customary to define $\mathsf{width}(\varphi \vdash \bot) = \infty$.

Given π as above, the *total space* of π is

$$\mathsf{TSpace}(\pi) = \max_{i \in [\ell]} \sum_{C \in \mathfrak{M}_i} |C|$$

and given an unsatisfiable CNF formula φ, the total space needed to refute φ in Resolution is

$$\mathsf{TSpace}(\varphi \vdash \bot) = \min_{\pi} \mathsf{TSpace}(\pi),$$

where the min is taken over all the possible Resolution refutations of φ given as a sequence of memory configurations[4].

3 Proof of Theorem 1.1

First let's prove the main result of this work, Theorem 1.1, for convenience of the reader restated below. We postpone more detailed comments on the proof after the proof itself.

Restated Theorem 1.1. *Let φ be a k-CNF formula, then*

$$\mathsf{TSpace}(\varphi \vdash \bot) \geqslant \frac{1}{16} \left(\mathsf{width}(\varphi \vdash \bot) - k - 4\right)^2 .$$

Proof. Let $\mathsf{awidth}(\varphi \vdash \bot) = r + 1$. We prove that

$$\mathsf{TSpace}(\varphi \vdash \bot) \geqslant \frac{1}{4}(r-1)^2,$$

or, more precisely, we prove that every Resolution refutation of φ must pass trough a memory configuration of at least $(r-1)/2$ clauses each of width at least $(r-1)/2$. Once we prove this, the desired lower bound between total space and width follows:

$$\mathsf{TSpace}(\varphi \vdash \bot) \geqslant \frac{1}{4}(r-1)^2 \geqslant \frac{1}{16} \left(\mathsf{width}(\varphi \vdash \bot) - k - 4\right)^2 ,$$

where the last inequality uses that $\mathsf{width}(\varphi \vdash \bot) \leqslant 2(r+1) + k$, a consequence of Theorem 2.1.

Let Ξ and Ψ be two functions respectively mapping subsets of clauses into subsets of partial assignments and viceversa. Given a set of clauses A,

$$\Xi(A) = \{\alpha \text{ partial assignment} : \forall C \in A, \alpha(C) \neq 0\},$$

and given a set of partial assignments F,

$$\Psi(F) = \{C \text{ clause} : \exists \alpha \in F, \alpha(C) = 0\}.$$

Notice that, by construction, for every set of clauses A, $A \cap \Psi \circ \Xi(A) = \emptyset$ and $\bot \in \Psi(F)$ whenever F is non-empty. We consider the following special set:

$$W_r = \{C \text{ clause} : \mathsf{awidth}(\varphi \vdash C) \leqslant r\},$$

and its images $\Xi(W_r)$ and $\mathscr{S} = \Psi \circ \Xi(W_r)$. The main reason to consider the set $\Xi(W_r)$ is the following property:

> **Claim 3.1** (extension property of $\Xi(W_r)$)**.** *Let α be a $\subseteq$-maximal partial assignment in $\Xi(W_r)$ and x a variable not in $\mathrm{dom}(\alpha)$, then for every $\beta \subseteq \alpha$ such that $|\mathrm{dom}(\beta)| < r$ both $\beta \cup \{x \mapsto 0\}$ and $\beta \cup \{x \mapsto 1\}$ are in $\Xi(W_r)$.*

[4]If φ is a satisfiable CNF formula then is customary to define $\mathsf{TSpace}(\varphi \vdash \bot) = \infty$.

Proof. By contradiction let $\beta \subseteq \alpha$ such that $|\mathrm{dom}(\beta)| < r$ and $b \in \{0,1\}$ such that $\beta_b = \beta \cup \{x \mapsto b\} \notin \Xi(W_r)$. Without loss of generality we can restrict to $b = 0$. Since $\beta_0 \notin \Xi(W_r)$ it means that there exists a clause D in W_r such that $\beta_0(D) = 0$ but $\alpha(D) \neq 0$. This means that $D = D' \vee x$, $|D| \leqslant r$ and $\beta(D') = \alpha(D') = 0$. By maximality of α then both $\alpha_0 = \alpha \cup \{x \mapsto 0\} \notin \Xi(W_r)$ and $\alpha_1 = \alpha \cup \{x \mapsto 1\} \notin \Xi(W_r)$. In particular there exists a clause $E \in W_r$ such that $\alpha_1(E) = 0$, so, as before, we must have that $E = E' \vee \neg x$ and $\alpha(E') = 0$. But now

$$\frac{D' \vee x \qquad E' \vee \neg x}{D' \vee E'}$$

is a valid instance of the Resolution rule. Hence, by definition of asymmetric width,

$$\mathsf{awidth}(\varphi \vdash D' \vee E') \leqslant \max\{\mathsf{awidth}(\varphi \vdash D), \mathsf{awidth}(\varphi \vdash E), r\} \leqslant r,$$

since both D and E belong to W_r and $|D| \leqslant r$. So $D' \vee E' \in W_r$ and $\alpha(D' \vee E') = 0$ which is a contradiction. ☐

Let $\pi = (\mathfrak{M}_0, \ldots, \mathfrak{M}_\ell)$ be a Resolution refutation of φ given as a sequence of memory configurations. By definition of W_r, $\bot \notin W_r$ and hence the empty partial assignment is in $\Xi(W_r)$, so, in particular $\bot \in \mathscr{S}$. Hence the following set is non-empty:

$$\mathscr{A} = \{i \in [\ell] \ : \ \exists C \in \mathfrak{M}_i \cap \mathscr{S}, \ |C| < (r-1)/2\}.$$

Let $t = \min A$ and let $C \in \mathfrak{M}_t \cap \mathscr{S}$ be a clause of width less than $(r-1)/2$. Since $C \in \mathscr{S}$ there must exists a partial assignment $\alpha \in \Xi(W_r)$ that falsifies C and let α_C be the minimal partial assignment contained in α falsifying C. Notice that $|\mathrm{dom}(\alpha_C)| = |C| < (r-1)/2$. Our goal now is to show that there exists some $i < t$ such that $|\mathfrak{M}_i \cap \mathscr{S}| \geqslant (r-1)/2$. Since for every $i < t$ every clause in $\mathfrak{M}_i \cap \mathscr{S}$ has width at least $(r-1)/2$, this will give the desired result.

For sake of contradiction, suppose that for each $i < t$, $|\mathfrak{M}_i \cap \mathscr{S}| < (r-1)/2$. We inductively construct a sequence of assignments $\beta_0, \ldots, \beta_t$ in $\Xi(W_r)$ such that for each $i \leqslant t$ we have that $\alpha_C \subseteq \beta_i$ and that $\beta_i \models \mathfrak{M}_i \cap \mathscr{S}$. This immediately give a contradiction when we reach β_t, since α_C falsifies the clause $C \in \mathfrak{M}_t \cap \mathscr{S}$ and $\beta_t \supseteq \alpha_C$.

The first memory configuration $\mathfrak{M}_0$ is empty, so we can put $\beta_0 = \alpha$. Supposing that $0 \leq i < t$ and that we already have a suitable β_i, we construct β_{i+1} distinguishing between two cases.

Axiom download case. $\mathfrak{M}_{i+1} \subseteq \mathfrak{M}_i \cup \{D\}$, where D is a clause from φ. Since each clause D from φ belongs to W_r and we have that $W_r \cap \mathcal{S} = \emptyset$, then $\mathfrak{M}_i \cap \mathcal{S} = \mathfrak{M}_{i+1} \cap \mathscr{S}$ and hence we can simply put $\beta_{i+1} = \beta_i$.

Inference case. $\mathfrak{M}_{i+1} \subseteq \mathfrak{M}_i \cup \{D \vee E\}$ where $D \vee E$ follows by Resolution on some variable x from two clauses $D \vee x$ and $E \vee \neg x$ in $\mathfrak{M}_i$. Then, by the inductive hypothesis, there exists $\beta_i \in \Xi(W_r)$ such that $\beta_i \models \mathfrak{M}_i \cap \mathscr{S}$, let $\bar{\beta}_i \in \Xi(W_r)$ be a $\subseteq$-maximal partial assignment containing β_i and let β be an assignment contained in $\bar{\beta}_i$ $\subseteq$-minimal such that $\alpha_C \subseteq \beta$ and $\beta \models \mathfrak{M}_i \cap \mathscr{S}$. We have that

$$|\mathrm{dom}(\beta)| \leqslant |\alpha_C| + |\mathfrak{M}_i \cap \mathscr{S}| < (r-1)/2 + (r-1)/2 = r - 1,$$

where the first inequality follows easily from the fact that to satisfy a clause $F \in \mathfrak{M}_i \cap \mathscr{S}$ an assignment just have to satisfy a single literal in F. Notice that since $|\mathrm{dom}(\beta)| \leqslant r-2$ the extension property from Claim 3.1 can be applied twice and we will use this later. The main property of β that we now use is the following:

Claim 3.2. *Let $\gamma \in \Xi(W_r)$ and F be any clause in $\mathfrak{M}_i$, if $\mathrm{var}(F) \subseteq \mathrm{dom}(\gamma)$ and $\beta \subseteq \gamma$, then $\gamma \models F$.*

Proof. Since $\mathrm{var}(F) \subseteq \mathrm{dom}(\gamma)$, then $\gamma(F) \in \{0,1\}$. If by contradiction $\gamma(F) = 0$, then, by construction $F \in \mathscr{S}$ and, again by construction, $\beta \models \mathfrak{M}_i \cap \mathscr{S}$. So $\beta \models F$, which is is a contradiction since $\beta \subseteq \gamma$. $\square$

The remaining part of the proof is just case analysis. If there is some variable y in $D \vee E$ unassigned by $\bar{\beta}_i$ then we can use the extension property (Claim 3.1) extending β to some $\beta' \in \Xi(W_r)$ setting y and satisfying $D \vee E$.

If $\mathrm{var}(D \vee E) \subseteq \mathrm{dom}(\beta)$ then we can extend β to some assignment $\beta' \in \Xi(W_r)$ setting x to some value (either by choosing $\bar{\beta}_i$ if $x \in \mathrm{dom}(\bar{\beta}_i)$ or otherwise by the extension property). Then $\mathrm{var}(D \vee x) \subseteq \mathrm{dom}(\beta')$, and, by the previous claim, $\beta' \models D \vee x$. The same happens for $E \vee \neg x$ and hence $\beta' \models D \vee E$ by the soundness of the Resolution rule.

The only remaining possibility is that $\mathrm{var}(D \vee E) \not\subseteq \mathrm{dom}(\beta)$ but $\mathrm{var}(D \vee E) \subseteq \mathrm{dom}(\bar{\beta}_i)$, and without loss of generality suppose that $\mathrm{var}(D) \not\subseteq \mathrm{dom}(\beta)$. If $x \in \mathrm{dom}(\bar{\beta}_i)$ then, by the previous claim $\bar{\beta}_i \models (D \vee x) \wedge (E \vee \neg x)$ so $\bar{\beta}_i \models D \vee E$. Suppose then that $x \notin \mathrm{dom}(\bar{\beta}_i)$. By Claim 3.1 we have that $\beta' = \beta \cup \{x \mapsto 0\} \in \Xi(W_r)$. Take a $\subseteq$-maximal assignment in $\Xi(W_r)$ containing β', let $\bar{\beta}'$ be such assignment. If $\mathrm{var}(D \vee x) \subseteq \mathrm{dom}(\bar{\beta}')$ then, by the previous claim, $\bar{\beta}_i \models D \vee x$, but $\beta'(x) = 0$ so $\bar{\beta}' \models D$ and hence $\bar{\beta}' \models D \vee E$. If $\mathrm{var}(D \vee x) \not\subseteq \mathrm{dom}(\bar{\beta}')$ then there is some variable y in $D \vee x$ not assigned by $\bar{\beta}'$ and since $|\mathrm{dom}(\beta')| = |\mathrm{dom}(\beta)| + 1 < r$ we can apply the extension property to β' extending it setting y and satisfying D. $\square$

First of all notice that we proved something actually stronger, that is we proved that given an unsatisfiable CNF formula φ, every Resolution refutation of φ must pass trough a memory configuration of at least $\frac{1}{2}(\mathsf{awidth}(\varphi \vdash \bot) - 1)$ clauses each of width at least $\frac{1}{2}(\mathsf{awidth}(\varphi \vdash \bot) - 1)$.

A crucial point in the proof of Theorem 1.1 is Claim 3.1. It is related with the following characterization of asymmetric width in Resolution.

Theorem 3.3 (Theorem 22 from [15])**.** *Let φ be an unsatisfiable CNF formula, then the following are equivalent:*

1. $\mathsf{awidth}(\varphi \vdash \bot) > r$,
2. *there exists a non-empty set $\mathscr{F}$ of partial assignments such that:*

   Consistency *for every $\alpha \in \mathscr{F}$ and every clause C of φ, $\alpha(C) \neq 0$;*

   Extension *If $\alpha \in \mathscr{F}$ and $\beta \subseteq \alpha$ is such that $|\mathrm{dom}(\beta)| < r$, then for every variable $x \notin \mathrm{dom}(\alpha)$ and for every $\epsilon \in \{0,1\}$ there exist $\beta_\epsilon \in \mathscr{F}$ with $\beta \subseteq \beta_\epsilon$ such that $\beta_\epsilon(x) = \epsilon$.*

Claim 3.1 is based on the proof of the implication from 1. to 2. in the previous theorem. The other implication (easier to prove) is not needed for Theorem 1.1. Indeed it is easy to see that given 1. the set of $\subseteq$-maximal partial assignments in $\Xi(W_r)$ satisfies the properties claimed in 2., and the crucial extension property is essentially Claim 3.1.

4 Open questions

We conclude this work with some open questions about the behaviour of the total space measure. Most of the questions are motivated by some analogy with the behaviour of the clause space measure.

On super-linear lower bounds. Is there any family of k-CNF formulas φ_n in n variables and $\mathsf{poly}(n)$ clauses such that $\mathsf{size}(\varphi_n \vdash \bot) = \mathsf{poly}(n)$ and $\mathsf{TSpace}(\varphi_n \vdash \bot) = \Theta(n^2)$?

For the formulas from [20] we saw in Section 1.2 we just have a linear total space lower bound. If we could find some formulas ψ_n with polynomial size Resolution proofs[5] but such that

[5] Recall that by equation 2 this implies that $\mathsf{width}(\psi_n \vdash \bot) = O(\sqrt{n \log n})$.

$\mathsf{width}(\psi_n \vdash \bot) = \omega(\sqrt{n})$ then, by Theorem 1.1, we would have that $\mathsf{TSpace}(\psi_n \vdash \bot) = \omega(n)$. This is anyway quite far from the question we are asking here and it seems that a positive answer should need some new techniques.

On simpler proofs for total space lower bounds. Is there a simpler more direct proof of a total space-width lower bound?

The clause space the relationship $\mathsf{CSpace}(\varphi \vdash \bot) \geqslant \mathsf{width}(\varphi \vdash \bot) - k + 1$, where φ is a k-CNF, can be proven using some families of assignments and a characterization of Resolution width [2] or it can be proven (with some small loss in an additive constant) via some operation on Resolution proofs [25]. The proof we have of Theorem 1.1 is in a sense similar to the proof of the clause-width lower bound in [2] although not quite simple as that since we pass trough the asymmetric width and more complicated families of assignments.

Beyond Resolution. Space measures are defined in [1] also for proof systems stronger than Resolution. For example for Polynomial Calculus, a proof system where instead of clauses we infer polynomials, or $\mathsf{Res}(k)$, a (stronger) version of Resolution where instead of clauses k-DNF can be inferred, or Frege systems. In all such systems very little is known about space especially when it comes to total space. Regarding other space measures something is known for example for $\mathsf{Res}(k)$ [10, 23] and for Polynomial Calculus [1, 18, 26]. In [1] it is proven that in Frege system the total space is always at most linear and regarding Polynomial Calculus, the only lower bounds known for total space are from [1] and those are for the PHP_n^{n+1} formulas and the Complete Tree formulas. Is there any family of k-CNF formulas in n variables φ_n with polynomially many clauses requiring super-linear total space to be refuted say in Polynomial Calculus?

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A Proof of Theorem 2.1

The proof of Theorem 2.1 we give is essentially a self-contained exposition of the analogous one in the underlying report of [15]. Given a set of clauses A we call *A-input Resolution derivation* of a clause C a Resolution derivation of C from A such that each application of the Resolution rule has at least a premise from A. We use the following property of A-input Resolution derivations:

Lemma A.1. *Let A be a set of clauses and C a clause, if there exists an A-input derivation of a clause C then*

$$\mathsf{width}(A \vdash C) \leqslant \mathsf{width}(A) + |C|.$$

Proof. Without loss of generality we can restrict to consider A-input *refutations*, that is A-input derivations of the empty clause $\bot$. Indeed, suppose we have π an A-input derivation of a clause C, and let ρ the minimal partial assignment mapping C to 0. Clearly $|\mathrm{dom}(\rho)| \leqslant |C|$ and $\pi|_\rho$ is a $A|_\rho$-input refutation, hence, if the property we want to prove holds for input refutations, then $\mathsf{width}(A|_\rho \vdash \bot) \leqslant \mathsf{width}(A|_\rho)$. So, by the fact that ρ is removing at most $|C|$ literals from each clause, then $\mathsf{width}(A \vdash C) \leqslant \mathsf{width}(A) + |C|$.

Hence suppose there exists an A-input Resolution refutation, we prove that

$$\mathsf{width}(A \vdash \bot) \leqslant \mathsf{width}(A).$$

Let $\mathcal{A}$ be the set of all set of clauses A that have an A-input Resolution refutation but $\mathsf{width}(A \vdash \bot) > \mathsf{width}(A)$. By contradiction suppose that $\mathcal{A}$ is non-empty, so there will be some $\bar{A} \in \mathcal{A}$ with the minimum number of variables. Since $\bar{A} \in \mathcal{A}$ then it must be that $\bar{A}$ is non-trivial, that is $\bot$ is not in $\bar{A}$.

By hypothesis there exists some $\bar{A}$-input refutation π and let ℓ be the last literal resolved in π. Since π is an $\bar{A}$-input refutation it must be that either $\ell \in \bar{A}$ or $\neg\ell \in \bar{A}$. Without loss of generality suppose that $\neg\ell \in \bar{A}$. Now consider $\pi|_{\ell\mapsto 0}$, this is an $\bar{A}|_{\ell\mapsto 0}$-input Resolution refutation and $\bar{A}|_{\ell\mapsto 0}$ has strictly less variables than $\bar{A}$, hence, by the minimality of $\bar{A}$, $\bar{A}|_{\ell\mapsto 0}$ cannot be in $\mathcal{A}$. So $\mathsf{width}(\bar{A}|_{\ell\mapsto 0} \vdash \bot) \leqslant \mathsf{width}(\bar{A}|_{\ell\mapsto 0})$ and there must exist some π' which is a refutation of $\bar{A}|_{\ell\mapsto 0}$, but not necessarily an $\bar{A}|_{\ell\mapsto 0}$-input refutation, and such that $\mathsf{width}(\pi') \leqslant \mathsf{width}(\bar{A}|_{\ell\mapsto 0})$.

Now we just construct π'' as follows: $\pi'' = \bar{A} \circ \pi'$, that is we just write down before π' all the clauses in $\bar{A}$: π'' is a valid Resolution refutation of $\bar{A}$. This is because $\neg\ell \in \bar{A}$ and hence each clause in $\bar{A}|_{\ell\mapsto 0}$ can be seen as the result of an inference step between some clause in $\bar{A}$ and $\neg\ell$. Since $\mathsf{width}(\pi') \leqslant \mathsf{width}(\bar{A}|_{\ell\mapsto 0})$, we clearly have that $\mathsf{width}(\pi'') \leqslant \mathsf{width}(\bar{A})$, which implies that

$$\mathsf{width}(\bar{A} \vdash \bot) \leqslant \mathsf{width}(\pi'') \leqslant \mathsf{width}(\bar{A}).$$

On the other hand $\bar{A} \in \mathcal{A}$ implies that $\mathsf{width}(\bar{A} \vdash \bot) > \mathsf{width}(\bar{A})$ and this clearly contradicts the previous equation. $\square$

Proof of Theorem 2.1. Let $w = \mathsf{awidth}(\varphi \vdash \bot)$ and let $\mathscr{S}$ be the closure of φ under input derivations, that is: $\mathscr{S} = \bigcup_i S_i$, where

$$\begin{cases} S_0 & = \varphi, \\ S_{i+1} & = S_i \cup \{C \text{ clause} : |C| \leqslant w \wedge C \text{ has an } S_i\text{-input Resolution derivation}\}. \end{cases}$$

Notice that each clause in $\mathscr{S}$ has width at most $\max\{w, k\}$ and hence $\mathscr{S}$ is just a finite union, as S_{i+1} can be strictly bigger than S_i at most $O(n^{\max\{w,k\}})$ many times, since this is the number of clauses in n variables of width at most $\max\{w, k\}$. Now we claim to have the two following properties:

1. $\bot$ has an $\mathscr{S}$-input Resolution derivation;

2. if C has an $\mathscr{S}$-input Resolution derivation then $\mathsf{width}(\varphi \vdash C) \leqslant w + \max\{w, k\}$.

From **1.** and **2.** we immediately have the desired inequality between $\mathsf{width}(\varphi \vdash \bot)$ and $\mathsf{awidth}(\varphi \vdash \bot)$. To prove **1.**, consider a refutation π of φ such that $\mathsf{awidth}(\pi) = w$: we show that π is an $\mathscr{S}$-input Resolution derivation of $\bot$. Let, by contradiction, C be the first clause in π inferred from previous C', C'' in π[6] with both $C', C'' \notin S$. Since $\mathsf{awidth}(\pi) = w$ we have that without loss of generality $|C'| \leqslant w$, hence it must be that for each i, C' does not have an S_i-input Resolution derivation, otherwise $C' \in S_{i+1}$ but we are supposing that $C' \notin \mathscr{S}$. Hence, C' doesn't have a $\mathscr{S}$-input Resolution derivation either, contradicting the minimality of C in π.

We prove **2.** by induction on S_i. That is, we prove that if C has an S_i-input Resolution derivation then $\mathsf{width}(\varphi \vdash C) \leqslant w + \max\{w, k\}$. For S_0 this is clearly true. For the inductive step let C be a clause in $S_{i+1} \setminus S_i$, let $S_i = \{C_1, \dots, C_m\}$ and let π be an S_i-input Resolution derivation of C. By what observed before, there exists some π' which is a Resolution derivation of C from S_i such that

$$\mathsf{width}(\pi') \leqslant |C| + \mathsf{width}(S_i) \leqslant w + \max\{w, k\}.$$

Finally, by induction, for each $j = 1, \dots, m$, C_j has a Resolution derivation π_j from φ of width at most $w + \max\{w, k\}$, hence $\tilde{\pi} = \pi_1 \circ \cdots \circ \pi_m \circ \pi'$ is a Resolution derivation of C from φ and

$$\begin{aligned} \mathsf{width}(\varphi \vdash C) &\leqslant \mathsf{width}(\tilde{\pi}) \\ &= \max\{\mathsf{width}(\pi_1), \dots, \mathsf{width}(\pi_m), \mathsf{width}(\pi')\} \\ &\leqslant w + \max\{w, k\}. \square \end{aligned}$$

[6] If C is the result of an application of the weakening rule we just assume $C' = C''$.

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