A Simple $\tilde{O}(n)$ Non-Adaptive Tester for Unateness

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Abstract

Khot and Shinkar (RANDOM, 2016) recently describe an adaptive, $O(n \log(n)/\varepsilon)$-query tester for unateness of Boolean functions $f : \{0, 1\}^n \rightarrow \{0, 1\}$. In this note we describe a simple non-adaptive, $O(n \log(n/\varepsilon)/\varepsilon)$-query tester for unateness for functions over the hypercube with any ordered range.

1 Introduction

Let $f$ be a function $f : \{0, 1\}^n \rightarrow R$ defined over the Boolean hypercube where $R$ is some ordered range. We use $e_i$ to denote the unit vectors in $\{0, 1\}^n$ that has 1 in the $i$th coordinate, and 0s at other coordinates. The $i$-th partial derivative at $x$ is $f(x \oplus e_i) - f(x)$ if $x_i = 0$ and $f(x) - f(x \oplus e_i)$ is $x_i = 1$. This is denoted by the function $\partial_i f$. Note that a function is monotonically increasing (or simply monotone), if $\partial_i f(x) \geq 0$ for all $x \in \{0, 1\}^n$.

Unateness is a generalization of monotonicity. A function is unate if in every coordinate it is either monotone or anti-monotone. More precisely, a function $f$ is unate if for all $i \in [n]$, either $\partial_i f(x) \geq 0$ for all $x$, or $\partial_i f(x) \leq 0$ for all $x$. The problem of unateness testing was introduced by Goldreich et al. [GGL+00] in their seminal paper on testing monotonicity. For Boolean functions $f : \{0, 1\}^n \rightarrow \{0, 1\}$, [GGL+00] described a non-adaptive $O(n^{3/2}/\varepsilon)$-query tester (while for monotonicity of Boolean functions they described a non-adaptive $O(n/\varepsilon)$-query tester). To our knowledge this problem was not studied after this, till very recently Khot and Shinkar [KS16] gave the first improvement for Boolean functions by designing an adaptive $O(n \log(n)/\varepsilon)$-query tester. Our main theorem is the following.

Theorem 1.1. Consider functions $f : \{0, 1\}^n \rightarrow R$, where $R$ is an arbitrary ordered set. There exists an one-sided error, non-adaptive, $O((n/\varepsilon) \log(n/\varepsilon))$-time tester for unateness.

Monotonicity testing has been extensively studied in the past two decades [GGL+00, DGL+99, EKK+00, HK03, HK08, BGJ+09, BCGSM12, CS14, CS13, CST14, CDST15, CDJS15, BB16]. We employ a previous result on testing derivative bounded properties by Chakrabarty et al. [CDJS15]. That paper provides general theorems about the testability of properties that are specified in terms of the partial derivative being bounded.

Definition 1.2. Given an $n$-dimensional bit vector $b$, call a function $f : \{0, 1\}^n \rightarrow R$ $b$-monotone if for all $i$ with $b_i = 0$ we have $\partial_i f(x) \geq 0$ for all $x$, and for all $i$ with $b_i = 1$ we have $\partial_i f(x) \leq 0$ for all $x$. 

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Note that 0-monotonicity is simply the standard notion of monotonicity and 1-monotonicity is the notion of anti-monotonicity. Also note that a function is unate if it is \( b\)-monotone for some \( b \).

The property of \( b\)-monotonicity is a *derivative-bounded property*, in the language of [CDJS15]. A dimension reduction theorem for derivative properties (Theorem 8 in Arxiv version of [CDJS15]), when instantiated for \( b\)-monotonicity, immediately implies the following theorem.

**Theorem 1.3.** Fix bit vector \( b \) and function \( f : \{0,1\}^n \rightarrow \mathbb{R} \). Let \( \varepsilon \) denote the distance of \( f \) to \( b\)-monotonicity. Let \( \mu_i \) be the fraction of points where \( \partial_i f \) violates \( b\)-monotonicity, that is, the number of hypercube edges across dimension \( i \) which violate \( b\)-monotonicity is \( \mu_i 2^n \). Then \( \sum_{i=1}^n \mu_i \geq \varepsilon/4 \).

The above theorem can also be obtained by observing that \( f \) is \( b\)-monotone (resp, \( \varepsilon\)-far from being \( b\)-monotone) iff the function \( g(x) := f(x \oplus b) \) is monotone (resp, \( \varepsilon\)-far from being monotone). Furthermore, every hypercube edge that violates monotonicity for \( f \) also violates \( b\)-monotonicity for \( f \). A previous result of the authors shows that if a function \( g \) is \( \varepsilon\)-far from being monotone, then it has \( \varepsilon 2^{n-1} \) hypercube edges violating monotonicity [CS13].

## 2 The Tester

<table>
<thead>
<tr>
<th>Unate-test ((f, \varepsilon))</th>
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</thead>
<tbody>
<tr>
<td>1. For ( r = 1, 2, \ldots, L := \lceil \log(8n/\varepsilon) \rceil ):</td>
</tr>
<tr>
<td>( \text{Repeat } s_r = \lceil 20n/\varepsilon^2 \rceil \text{ times} )</td>
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<tr>
<td>( a ) Sample u.a.r. dimension ( i ).</td>
</tr>
<tr>
<td>( b ) Sample a set ( R_i ) of ( 3 \cdot 2^r ) u.a.r. points in the hypercube and evaluate ( \partial_i f ) at all these points.</td>
</tr>
<tr>
<td>( c ) If there is some ( x \in R_i ) such that ( \partial_i f(x) &gt; 0 ) and ( y \in R_i ) such that ( \partial_i f(y) &lt; 0 ), reject and abort.</td>
</tr>
<tr>
<td>2. Accept (since tester has not rejected so far)</td>
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</table>

It is evident that this is a non-adaptive, one-sided tester. Furthermore, the running time is \( O((n/\varepsilon) \log(n/\varepsilon)) \). It suffices to prove the following.

**Theorem 2.1.** If \( f \) is \( \varepsilon\)-far from being unate, \text{Unate-test} rejects with probability at least 2/3.

**Proof.** For dimension \( i \), let \( U_i \) be the set of points in \( \{0,1\}^n \) where \( \partial_i f(x) > 0 \). Analogous, let \( D_i \) be the set of points where \( \partial_i f(x) < 0 \). The tester rejects iff it finds a triple \( i, x, y \) such that \( x \in U_i \) and \( y \in D_i \). Let \( \mu_i := \min(|U_i|, |D_i|)/2^n \).

Define the \( n\)-dimensional bit vector \( b \) as follows: \( b_i = 0 \) if \( |U_i| > |D_i| \) and \( b_i = 1 \) otherwise. Observe that the fraction of points where \( \partial_i f \) violates \( b\)-monotonicity is precisely \( \mu_i \). Since \( f \) is \( \varepsilon\)-far from being unate, \( f \) is \( \varepsilon\)-far from being \( b\)-monotone. By Theorem 1.3, \( \sum_i \mu_i \geq \varepsilon/4 \).

For any integer \( r \geq 1 \), define \( S_r := \{ i \in [n] : \mu_i \in (1/2^r, 1/2^{r-1}) \} \).

**Claim 2.2.** \( \sum_{r=1}^L |S_r|/2^r \geq \varepsilon/8 \).

**Proof.** Observe that \( \sum_{r>L} |S_r|/2^r \leq (\varepsilon/8n) \sum_r |S_r| = \varepsilon/8 \) since \( \sum_r |S_r| = n \). Also, \( \sum_r |S_r|/2^r \geq \sum_i \mu_i \geq \varepsilon/4 \) from Theorem 1.3. We subtract these bounds to prove the claim. \( \square \)
Fix \( r \). Let \( p_r \) be the probability that Step 1a, Step 1b, and Step 1c reject for this \( r \). Then the probability that the tester rejects is

\[
1 - \prod_{r=1}^{L} (1 - p_r)^{s_r} \geq 1 - e^{-\sum_{r=1}^{L} p_r s_r}
\]

(1)

We now lower bound \( p_r \). The tester rejects iff the set \( R_i \) in Step 1b contains a point in \( U_i \) and in \( D_i \). The probability that \( R_i \) does not contain a point in \( U_i \) or \( D_i \) is at most \((1 - |U_i|/2^n)^{|R_i|} + (1 - |D_i|/2^n)^{|R_i|} \leq 2(1 - \mu_i)^{|R_i|}\). Note that if the sampled dimension \( i \) lies in \( S_r \), then this probability is at most \( 2(1 - 1/2^r)^{3 \cdot 2^r} < 1/6 \). Therefore, we get \( p_r > \frac{5}{6} \cdot \frac{|S_r|}{n} \). Since \( s_r \geq \frac{20n}{\varepsilon 2^r} \), we get

\[
\sum_{r=1}^{L} p_r s_r \geq \sum_{r=1}^{L} \frac{5}{6} \cdot \frac{|S_r|}{n} \cdot \frac{20n}{\varepsilon 2^r} \geq \frac{100}{6\varepsilon} \sum_{r=1}^{L} \frac{|S_r|}{2^r} \geq 2
\]

where the second inequality follows from Claim 2.2. Substituting in (1), we get the theorem. \( \square \)

References


