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A $\widetilde{O}(n)$ Non-Adaptive Tester for Unateness

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Abstract

Khot and Shinkar (RANDOM, 2016) recently describe an adaptive, $O(n\log(n)/\varepsilon)$-query tester for unateness of Boolean functions $f : \{0,1\}^n \mapsto \{0,1\}$. In this note, we describe a simple non-adaptive, $O(n\log(n/\varepsilon)/\varepsilon)$ -query tester for unateness for real-valued functions over the hypercube.

1 Introduction

Let f be a function $f : \{0,1\}^n \mapsto R$ defined over the Boolean hypercube where R is some ordered range. We use e_i to denote the unit vectors in $\{0,1\}^n$ that has 1 in the ith coordinate, and 0s at other coordinates. The *i-th partial derivate* at x is $f(x \oplus e_i) - f(x)$ if $x_i = 0$ and $f(x) - f(x \oplus e_i)$ is $x_i = 1$. This is denoted by the function $\partial_i f$. Note that a function is monotonically increasing (or simply monotone), if $\partial_i f(x) \geq 0$ for all $x \in \{0,1\}^n$.

Unateness is a generalization of monotonicity. A function is unate if in every coordinate it is either monotone or anti-monotone. More precisely, a function f is unate if for all $i \in [n]$, either $\partial_i f(x) \geq 0$ for all x, or $\partial_i f(x) \leq 0$ for all x. The problem of unateness testing was introduced by Goldreich et al. [GGL$^+$00] in their seminal paper on testing monotonicity. For Boolean functions $f : \{0,1\}^n \to \{0,1\}$, [GGL$^+$00] described a non-adaptive $O(n^{3/2}/\varepsilon)$-query tester (while for monotonicity of Boolean functions they described a non-adaptive $O(n/\varepsilon)$-query tester). To our knowledge, there was no further progress on this problem, until the recent result of Khot and Shinkar [KS16] that gives an adaptive $O(n\log(n)/\varepsilon)$-query tester for Boolean functions. Our main theorem is the following.

Theorem 1.1. *Consider functions $f : \{0,1\}^n \mapsto R$, where R is an arbitrary ordered set. There exists an one-sided error, non-adaptive, $O((n/\varepsilon)\log(n/\varepsilon))$-time tester for unateness.*

Monotonicity testing has been extensively studied in the past two decades [GGL$^+$00, DGL$^+$99, EKK$^+$00, HK03, HK08, BGJ$^+$09, BCGSM12, CS14, CS13, BRY14, CST14, CDST15, CDJS15, BB16]. We employ a previous result on testing derivative bounded properties by Chakrabarty et al. [CDJS15]. That paper provides general theorems about the testability of properties that are specified in terms of the partial derivative being bounded.

Definition 1.2. *Given an n-dimensional bit vector $\mathbf{b}$, call a function $f : \{0,1\}^n \to R$ $\mathbf{b}$-monotone if for all i with $\mathbf{b}_i = 0$ we have $\partial_i f(x) \geq 0$ for all x, and for all i with $\mathbf{b}_i = 1$ we have $\partial_i f(x) \leq 0$ for all x.*

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Note that $\mathbf{0}$-monotonicity is simply the standard notion of monotonicity. Also note that a function is unate iff it is $\mathbf{b}$-monotone for some $\mathbf{b}$.

The property of $\mathbf{b}$-monotonicity is a *derivative-bounded property*, in the language of [CDJS15]. A dimension reduction theorem for derivative properties (Theorem 8 in Arxiv version of [CDJS15]), when instantiated for $\mathbf{b}$-monotonicity, implies the following theorem.

Theorem 1.3. *Fix bit vector* $\mathbf{b}$ *and function* $f : \{0,1\}^n \to R$. *Let* ε *denote the distance of* f *to* $\mathbf{b}$-*monotonicity. Let* μ_i *be the fraction of points where* $\partial_i f$ *violates* $\mathbf{b}$-*monotonicity, that is, the number of hypercube edges across dimension* i *which violate* $\mathbf{b}$-*montonicity is* $\mu_i 2^n$. *Then* $\sum_{i=1}^n \mu_i \geq \varepsilon/4$.

The above theorem (in fact a stronger version without the 4 in the denominator) can also be obtained by observing that f is $\mathbf{b}$-monotone (resp, ε-far from being $\mathbf{b}$-monotone) iff the function $g(x) := f(x \oplus b)$ is monotone (resp, ε-far from being monotone). Every hypercube edge that violates monotonicity for g violates $\mathbf{b}$-monotonicity for f. A previous result of the authors shows that if a function g is ε-far from being monotone, then it has $\varepsilon 2^{n-1}$ hypercube edges violating monotonicity [CS13]. (Such results were previously known for the case of Boolean range [GGL$^+$00], and weaker results for general range [DGL$^+$99]. Refer to [CS13] for more details.)

One can show that it suffices to query $\partial_i f$ at $O(1/\mu_i)$ points to detect a violation to unateness. We need to "interpolate" between two opposite scenarios: exactly one $\mu_i = \Omega(\varepsilon)$ and all others are 0 versus all $\mu_i = \Theta(\varepsilon/n)$. An efficient strategy for achieving this is Levin's investment strategy (refer to Section 8.2.4 of Goldreich's book [Gol15]). A tighter analysis of this method is given by Berman et al [BRY14], which is effectively what we use. For the sake of completeness, we repeat the calculations of [BRY14] for a complete proof.

2 The Tester

`Unate-test`(f, ε)

1. For $r = 1, 2, \ldots, L := \lceil \log(8n/\varepsilon) \rceil$:
   Repeat $s_r = \lceil \frac{20n}{\varepsilon \cdot 2^r} \rceil$ times
   (a) Sample u.a.r. dimension i.
   (b) Sample a set R_i of $3 \cdot 2^r$ u.a.r. points in the hypercube and evaluate $\partial_i f$ at all these points.
   (c) If there is some $x \in R_i$ such that $\partial_i f(x) > 0$ **and** $y \in R_i$ such that $\partial_i f(y) < 0$, reject and abort.
2. Accept (since tester has not rejected so far)

It is evident that this is a non-adaptive, one-sided tester. Furthermore, the running time is $O((n/\varepsilon)\log(n/\varepsilon))$. It suffices to prove the following.

Theorem 2.1. *If* f *is* ε-*far from being unate,* `Unate-test` *rejects with probability at least* $1 - 1/e$.

Proof. For dimension i, let U_i be the set of points in $\{0,1\}^n$ where $\partial_i f(x) > 0$. Analogous, let D_i be the set of points where $\partial_i f(x) < 0$. The tester rejects iff it finds a triple i, x, y such that $x \in U_i$ and $y \in D_i$. Let $\mu_i := \min(|U_i|, |D_i|)/2^n$.

Define the n-dimensional bit vector $\mathbf{b}$ as follows: $\mathbf{b}_i = 0$ if $|U_i| > |D_i|$ and $\mathbf{b}_i = 1$ otherwise. Observe that the fraction of points where $\partial_i f$ violates $\mathbf{b}$-monotonicity is precisely μ_i. Since f is ε-far from being unate, f is ε-far from being $\mathbf{b}$-monotone. By Theorem 1.3, $\sum_i \mu_i \geq \varepsilon/4$.
For any integer $r \geq 1$, define $S_r := \{i \in [n] \ : \ \mu_i \in (1/2^r, 1/2^{r-1}]\}$.

Claim 2.2. $\sum_{r=1}^{L} |S_r|/2^r \geq \varepsilon/16$.

Proof. Observe that $\sum_{r>L} |S_r|/2^r \leq (\varepsilon/8n) \sum_r |S_r| = \varepsilon/8$ since $\sum_r |S_r| = n$. Since for any $i \in S_r$ we have $\frac{1}{2^{r-1}} \geq \mu_i$, we get that $\sum_r |S_r|/2^r \geq \sum_i \mu_i/2 \geq \varepsilon/8$ from Theorem 1.3. We subtract these bounds to prove the claim. $\square$

Fix r. Let p_r be the probability that Step 1a, Step 1b, and Step 1c reject for this r. Then the probability that the tester rejects is

$$1 - \prod_{r=1}^{L} (1 - p_r)^{s_r} \geq 1 - e^{-\sum_{r=1}^{L} p_r s_r} \tag{1}$$

We now lower bound p_r. The tester rejects iff the set R_i in Step 1b contains a point in U_i and in D_i. The probability that R_i *does not* contain a point in U_i or D_i is at most $(1 - |U_i|/2^n)^{|R_i|} + (1 - |D_i|/2^n)^{|R_i|} \leq 2(1-\mu_i)^{|R_i|}$. Note that if the sampled dimension i lies in S_r, then this probability is at most $2\left(1 - 1/2^r\right)^{3 \cdot 2^r} < 1/6$. Therefore, we get $p_r > \frac{5}{6} \cdot \frac{|S_r|}{n}$. Since $s_r \geq \frac{20n}{\varepsilon \cdot 2^r}$, we get

$$\sum_{r=1}^{L} p_r s_r \geq \sum_{r=1}^{L} \frac{5}{6} \cdot \frac{|S_r|}{n} \cdot \frac{20n}{\varepsilon 2^r} \geq \frac{100}{6\varepsilon} \sum_{r=1}^{L} \frac{|S_r|}{2^r} > 1$$

where the second inequality follows from Claim 2.2. Substituting in (1), we get the theorem. $\square$

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