A $\tilde{O}(n)$ Non-Adaptive Tester for Unateness

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Abstract

Khot and Shinkar (RANDOM, 2016) recently describe an adaptive, $O(n \log(n)/\varepsilon)$-query tester for unateness of Boolean functions $f : \{0,1\}^n \to \{0,1\}$. In this note, we describe a simple non-adaptive, $O(n \log(n/\varepsilon)/\varepsilon)$ -query tester for unateness for real-valued functions over the hypercube.

1 Introduction

Let $f$ be a function $f : \{0,1\}^n \to R$ defined over the Boolean hypercube where $R$ is some ordered range. We use $e_i$ to denote the unit vectors in $\{0,1\}^n$ that has 1 in the $i$th coordinate, and 0s at other coordinates. The $i$-th partial derivate at $x$ is $f(x \oplus e_i) - f(x)$ if $x_i = 0$ and $f(x) - f(x \oplus e_i)$ is $x_i = 1$. This is denoted by the function $\partial_i f$. Note that a function is monotonically increasing (or simply monotone), if $\partial_i f(x) \geq 0$ for all $x \in \{0,1\}^n$.

Unateness is a generalization of monotonicity. A function is unate if in every coordinate it is either monotone or anti-monotone. More precisely, a function $f$ is unate if for all $i \in [n]$, either $\partial_i f(x) \geq 0$ for all $x$, or $\partial_i f(x) \leq 0$ for all $x$. The problem of unateness testing was introduced by Goldreich et al. [GGL+00] in their seminal paper on testing monotonicity. For Boolean functions $f : \{0,1\}^n \to \{0,1\}$, [GGL+00] described a non-adaptive $O(n^{3/2}/\varepsilon)$-query tester (while for monotonicity of Boolean functions they described a non-adaptive $O(n/\varepsilon)$-query tester). To our knowledge, there was no further progress on this problem, until the recent result of Khot and Shinkar [KS16] that gives an adaptive $O(n \log(n)/\varepsilon)$-query tester for Boolean functions. Our main theorem is the following.

**Theorem 1.1.** Consider functions $f : \{0,1\}^n \to R$, where $R$ is an arbitrary ordered set. There exists an one-sided error, non-adaptive, $O((n/\varepsilon) \log(n/\varepsilon))$-time tester for unateness.

Monotonicity testing has been extensively studied in the past two decades [GGL+00, DGL+99, EKK+00, HK03, HK08, BCG10, BCGS12, CS14, CS13, BRY14, CST14, CDST15, CDJS15, BB16]. We employ a previous result on testing derivative bounded properties by Chakrabarty et al. [CDJS15]. That paper provides general theorems about the testability of properties that are specified in terms of the partial derivative being bounded.

**Definition 1.2.** Given an $n$-dimensional bit vector $b$, call a function $f : \{0,1\}^n \to R$ $b$-monotone if for all $i$ with $b_i = 0$ we have $\partial_i f(x) \geq 0$ for all $x$, and for all $i$ with $b_i = 1$ we have $\partial_i f(x) \leq 0$ for all $x$. 
Note that $0$-monotonicity is simply the standard notion of monotonicity. Also note that a function is unate iff it is $b$-monotone for some $b$.

The property of $b$-monotonicity is a derivative-bounded property, in the language of [CDJS15]. A dimension reduction theorem for derivative properties (Theorem 8 in Arxiv version of [CDJS15]), when instantiated for $b$-monotonicity, implies the following theorem.

**Theorem 1.3.** Fix bit vector $b$ and function $f : \{0,1\}^n \to R$. Let $\varepsilon$ denote the distance of $f$ to $b$-monotonicity. Let $\mu_i$ be the fraction of points where $\partial_i f$ violates $b$-monotonicity, that is, the number of hypercube edges across dimension $i$ which violate $b$-monotonicity is $\mu_i 2^n$. Then $\sum_{i=1}^n \mu_i \ge \varepsilon/4$.

The above theorem (in fact a stronger version without the 4 in the denominator) can also be obtained by observing that if $f$ is $b$-monotone (resp, $\varepsilon$-far from being $b$-monotone) iff the function $g(x) := f(x \oplus b)$ is monotone (resp, $\varepsilon$-far from being monotone). Every hypercube edge that violates monotonicity for $g$ violates $b$-monotonicity for $f$. A previous result of the authors shows that if a function $g$ is $\varepsilon$-far from being monotone, then it has $\varepsilon 2^{n-1}$ hypercube edges violating monotonicity [CS13]. (Such results were previously known for the case of Boolean range [GGL+00], and weaker results for general range [DGL+99]. Refer to [CS13] for more details.)

One can show that it suffices to query $\partial_i f$ at $O(1/\mu_i)$ points to detect a violation to unateness. We need to “interpolate” between two opposite scenarios: exactly one $\mu_i = \Omega(\varepsilon)$ and all others are 0 versus all $\mu_i = \Theta(\varepsilon/n)$. An efficient strategy for achieving this is Levin’s investment strategy (refer to Section 8.2.4 of Goldreich’s book [Gol15]). A tighter analysis of this method is given by Berman et al [BRY14], which is effectively what we use. For the sake of completeness, we repeat the calculations of [BRY14] for a complete proof.

## 2 The Tester

### Unate-test($f, \varepsilon$)

1. For $r = 1, 2, \ldots, L := \lceil \log(8n/\varepsilon) \rceil$:
   1. Repeat $s_r = \lceil \frac{20n}{2^r} \rceil$ times
      1. Sample u.a.r. dimension $i$.
      2. Sample a set $R_i$ of $3 \cdot 2^r$ u.a.r. points in the hypercube and evaluate $\partial_i f$ at all these points.
      3. If there is some $x \in R_i$ such that $\partial_i f(x) > 0$ and $y \in R_i$ such that $\partial_i f(y) < 0$, reject and abort.
2. Accept (since tester has not rejected so far)

It is evident that this is a non-adaptive, one-sided tester. Furthermore, the running time is $O((n/\varepsilon) \log(n/\varepsilon))$. It suffices to prove the following.

**Theorem 2.1.** If $f$ is $\varepsilon$-far from being unate, Unate-test rejects with probability at least $1 - 1/e$.

**Proof.** For dimension $i$, let $U_i$ be the set of points in $\{0,1\}^n$ where $\partial_i f(x) > 0$. Analogous, let $D_i$ be the set of points where $\partial_i f(x) < 0$. The tester rejects iff it finds a triple $i, x, y$ such that $x \in U_i$ and $y \in D_i$. Let $\mu_i := \min(|U_i|, |D_i|)/2^n$.

Define the $n$-dimensional bit vector $b$ as follows: $b_i = 0$ if $|U_i| > |D_i|$ and $b_i = 1$ otherwise. Observe that the fraction of points where $\partial_i f$ violates $b$-monotonicity is precisely $\mu_i$. Since $f$ is $\varepsilon$-far from being unate, $f$ is $\varepsilon$-far from being $b$-monotone. By Theorem 1.3, $\sum_i \mu_i \ge \varepsilon/4$.

For any integer $r \ge 1$, define $S_r := \{i \in [n] : \mu_i \in (1/2^r, 1/2^{r-1})\}$.
Claim 2.2. $\sum_{r=1}^{L} |S_r|/2^r \geq \varepsilon/16$.

Proof. Observe that $\sum_{r>L} |S_r|/2^r \leq (\varepsilon/8n) \sum_r |S_r| = \varepsilon/8$ since $\sum_r |S_r| = n$. Since for any $i \in S_r$ we have $\frac{1}{2^{r-1}} \geq \mu_i$, we get that $\sum_r |S_r|/2^r \geq \sum_i \mu_i/2 \geq \varepsilon/8$ from Theorem 1.3. We subtract these bounds to prove the claim. \hfill \Box

Fix $r$. Let $p_r$ be the probability that Step 1a, Step 1b, and Step 1c reject for this $r$. Then the probability that the tester rejects is

$$1 - \prod_{r=1}^{L} (1 - p_r)^{s_r} \geq 1 - e^{-\sum_{r=1}^{L} p_r s_r} \quad (1)$$

We now lower bound $p_r$. The tester rejects iff the set $R_i$ in Step 1b contains a point in $U_i$ and in $D_i$. The probability that $R_i$ does not contain a point in $U_i$ or $D_i$ is at most $(1 - |U_i|/2^n)^{|R_i|} + (1 - |D_i|/2^n)^{|R_i|} \leq 2(1 - \mu_i)^{|R_i|}$. Note that if the sampled dimension $i$ lies in $S_r$, then this probability is at most $2(1 - 1/2^r)^{3 \cdot 2^r} < 1/6$. Therefore, we get $p_r > \frac{5}{6} \cdot \frac{|S_r|}{n}$. Since $s_r \geq \frac{20n}{\varepsilon \cdot 2^r}$, we get

$$\sum_{r=1}^{L} p_r s_r \geq \sum_{r=1}^{L} \left( \frac{5}{6} \cdot \frac{|S_r|}{n} \cdot \frac{20n}{\varepsilon \cdot 2^r} \right) \geq \frac{100}{6\varepsilon} \sum_{r=1}^{L} \frac{|S_r|}{2^r} > 1$$

where the second inequality follows from Claim 2.2. Substituting in (1), we get the theorem. \hfill \Box

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References


