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Relating two width measures for resolution proofs

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1 Introduction

In this short note, we revisit two hardness measures for resolution proofs: width and asymmetric width. It is known that for every unsatisfiable CNF F,

$$\mathsf{width}(F \vdash \Box) \leq \mathsf{awidth}(F \vdash \Box) + \max\{\mathsf{awidth}(F \vdash \Box), \mathsf{width}(F)\}.$$

We give a simple direct proof of the upper bound, also shaving off a +1.

2 The Background

A clause is a disjunction of literals. We use set notation; a set of literals denotes the clause that is their disjunction. The empty clause is denoted by $\Box$. The width of a clause C, denoted $|C|$, is the number of literals in it. The width of a CNF formula F, denoted $\mathsf{width}(F)$, is the maximum width of any clause in F. For a partial assignment ρ, $C|_\rho$ denotes the restriction of the clause C by applying the partial assignment to all literals in C in the domain of ρ, and $F|_\rho$ denotes the conjunction of the clauses $C|_\rho$ for each $C \in F$.

The resolution rule infers a clause $C \cup D$ from clauses $C \cup \{x\}$, $D \cup \{\neg x\}$. A resolution proof π deriving a clause C from a set of initial clauses $C_1, C_2, \ldots, C_m$ (called the axioms) is a sequence of clauses $D_1, D_2, \ldots, D_t$ where D_t is the clause C, and each D_i is either an axiom, or is obtained from some D_j, D_k for $j, k < i$ by resolution. The width of π is $\max_i\{|D_i|\}$, and its size is t. By $\mathsf{width}(F \vdash C)$ we denote the minimum width of a resolution proof π deriving C from F. The width measure has proven to be extremely useful in obtaining lower bounds on the total size of resolution proofs; see [BW01].

A clause C' is a weakening of a clause C if $C \subseteq C'$, that is, every literal in C also appears in C'. We may also allow weakening steps in a resolution proof; in the sequence above, D_i may be obtained from D_j for some $j < i$ by weakening. Weakening steps are redundant and can be eliminated. The measure $\mathsf{width}(F \vdash \Box)$ remains unchanged whether or not weakening is allowed.

If π is restricted so that each derived clause D_i is used at most once subsequently (that is, the underlying graph structure is a tree), then we have a tree-like resolution proof. Making a proof tree-like by duplicating sub-derivations as needed can increase its size significantly, but does not affect width. So without loss of generality we consider only tree-like proofs.

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(In the standard definition of a tree, each non-root node has a unique parent. While talking of resolution trees, it is customary to invert the parent-child relation. So each non-root node has a unique child, but a resolution node has two parents and a weakening node has one parent.)

Another width measure for tree-like proofs is the asymmetric width awidth, introduced in [Kul00, Kul04]. At each internal node u of the resolution tree, let $f(u)$ denote the minimum, over all parents v of u, of the width of the clause at node v. Then $\mathsf{awidth}(\pi)$ is the maximum value of $f(u)$ seen at any internal node. Thus if $\mathsf{awidth}(\pi) \leq k$, then every internal node has at least one parent node with a clause of width no more than k; the other parent could have an arbitrarily large clause. By $\mathsf{awidth}(F \vdash C)$ we denote the minimum asymmetric width of a tree-like resolution proof π deriving C from F. Asymmetric width can also be defined for non-tree-like proofs, and requires a more complicated definition (see for instance [Bon16a]), but in fact the value of the measure remains the same even if restricted to tree-like proofs.

The definitions imply that for all F, C, $\mathsf{awidth}(F \vdash C) \leq \mathsf{width}(F \vdash C)$. The gap can be arbitrarily large. For instance, for a minimally unsatisfiable Horn formula F with width k, $\mathsf{width}(F \vdash \Box) \geq k$. However, there is a proof where every step is obtained by "unit-resolution" (resolution where one of the two clauses used has width 1); hence this proof has asymmetric width 1. The interesting relationship between these measures is the following upper bound of width in terms of awidth:

Theorem 1 ([Kul04, BK14b, BK14a]) *For every unsatisfiable F other than $F = \Box$,*

$$\mathsf{width}(F \vdash \Box) \leq \mathsf{awidth}(F \vdash \Box) + \max\{\mathsf{awidth}(F \vdash \Box), \mathsf{width}(F)\}.$$

In [BK14a] it is conjectured that

$$\mathsf{width}(F \vdash \Box) \leq \mathsf{awidth}(F \vdash \Box) + \mathsf{width}(F) - 1.$$

This is certainly true for 2SAT, because if F is a non-trivial unsatisfiable 2CNF then the resolution proof has width 2, and asymmetric width is at least 1. However there has been no progress towards this conjecture in general.

This result was recently used in [Bon16b] to obtain a lower bound on the total space of resolution proofs. For completeness, in [Bon16a] the author includes in the appendix a self-contained proof of the above theorem.

We give here another self-contained proof of the above theorem. It is essentially the same proof as in [Kul04, BK14a, Bon16a], but presented in a direct constructive way rather than by contradiction. In the process we reduce the upper bound by 1; we show:

Theorem 2 *For every unsatisfiable F other than $F = \Box$,*

$$\mathsf{width}(F \vdash \Box) \leq \mathsf{awidth}(F \vdash \Box) + \max\{\mathsf{awidth}(F \vdash \Box), \mathsf{width}(F)\} - 1.$$

3 The proof of Theorem 2

We start with a tree-like resolution proof (without weakening) of small asymmetric width, and produce a resolution proof with weakening of the required width.

Let $q = \mathsf{width}(F)$, and let $k = \mathsf{awidth}(F \vdash \Box)$ as witnessed by a tree-like resolution proof π. Let r denote $\max\{q, k\}$; note $r \geq 1$. Order the parents of each binary node in π so that the parent with

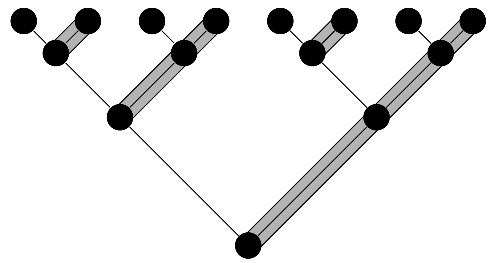

Figure 1: Cutting the proof tree into sub-trees

the wider clause appears on the right. Now cut up π into maximal sub-trees with no edge from a node to its left parent. See Figure 1. Each sub-tree is simply a path all the way up to a pair of axioms. The root of each such sub-tree is either the left parent of its child, or the root of π; hence the root of each sub-tree is labeled by a clause of width at most k. We restructure each sub-tree and piece together the restructured versions to obtain a small-width proof for F.

Let T be one of these sub-trees. Let G denote the set of clauses labeling nodes that feed into T. Each clause in G is either an axiom from F or is the left parent of a node in π; hence $\mathsf{width}(G) \leq r$. Let C be the clause at the root of T. Then T is an "input-resolution" proof showing $G \vdash C$; every step is a resolution step, and at least one parent is an axiom. Lemma 1 below shows that any such proof can be restructured so that all internal nodes have width at most $|C| + \mathsf{width}(G) - 1$. The restructured proof may also have weakening steps. Piecing together the restructured sub-trees gives a proof π', with resolution and weakening steps, for $F \vdash \Box$. Observe that each node in π' is either an internal node of a restructured sub-tree (and hence has width at most $k+r-1$), or is the root of a sub-tree (and hence has width at most k), or is an axiom (and hence has width atmost q). Thus $\mathsf{width}(\pi') \leq k + r - 1$.

The weakening steps in π' can be removed in a standard way without increasing the width. Hence $\mathsf{width}(F \vdash \Box) \leq k + r - 1$.

It remains to prove the lemma.

Lemma 1 *If C can be derived from G by input-resolution, then C can be derived from G by a resolution proof where all internal nodes have width at most $|C| + \mathsf{width}(G) - 1$.*

Proof: Let τ be the proof deriving C from G using input-resolution.

Let ρ be the minimal partial assignment that makes C false. Restricting all the clauses of G by ρ gives a proof-like structure. This structure may have redundant steps; specifically, it may have, apart from resolution steps, some weakening steps. Pruning out weakening steps in the standard way gives an input-resolution proof (without weakening) τ' for $G|_\rho \vdash \Box$. We will restructure τ' into a proof τ'' for $G|_\rho \vdash \Box$ that uses only unit-resolution; every step is a resolution step where at least one parent is a unit clause. Since unit-resolution necessarily shortens clauses, every internal

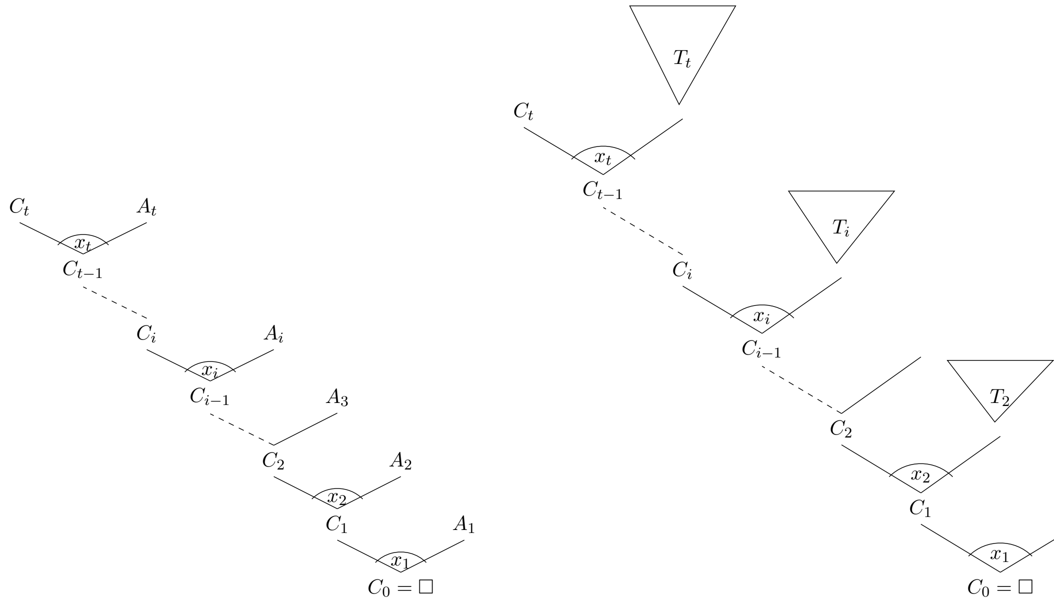

Figure 2: From input-resolution to unit-resolution

node of τ'' has width at most $\mathsf{width}(G|_\rho) - 1$. We then put back the literals of C into the clauses of G wherever they appeared, to get a resolution proof τ''' for $G \vdash C'$, where C is either C' or is a weakening of C'. This increases the width of internal nodes by at most $|C|$. Applying the weakening rule to the root if necessary gives the final restructured proof.

We now describe the transformation τ' to τ''. See Figure 2 for an illustration of the transformation.

Let G' denote the clauses in $G|_\rho$, so τ proves $G' \vdash \Box$ using input-resolution. Let the clauses at the resolution nodes be $\Box = C_0, C_1, C_2, \ldots, C_{t-1}$. Each C_{i-1} is obtained by resolving C_i and an axiom $A_i \in G'$ on the variable x_{j_i}. (At the end, C_t is also in G'.) Of the literals $x_{j_i}, \neg x_{j_i}$, exactly one appears in C_i and one in A_i; let ℓ_{j_i} denote the literal in C_i. Define $C'_i = C_i \setminus \{\ell_{j_i}\}$ and $A'_i = A_i \setminus \{\neg \ell_{j_i}\}$. From the definition of resolution, it follows that for each $i \in [t]$, $A'_i \cup C'_i = C_{i-1}$. Further, since $C_0 = \Box$, all literals in C_{i-1} are "resolved away", hence $A'_i \cup C'_i = C_{i-1} \subseteq \{\ell_{j_1}, \ell_{j_2}, \ldots, \ell_{j_{i-1}}\}$.

We inductively establish the following: For $i \in [t]$, the unit clause $\neg \ell_{j_i}$ can be generated by a unit-resolution proof from the axioms $A_1, \ldots, A_i$.

The basis, when $i = 1$, is immediate. Consider the inductive step at i. By the induction hypothesis, for each $s \leq i - 1$, there is a proof T_s that derives the unit clause $\neg \ell_{j_s}$ using unit-resolution and the axioms $A_1, \ldots, A_s$. Use these proofs sequentially to resolve away the literals of A'_i from A_i, leaving just $\neg \ell_{j_i}$. This is the proof T_i, deriving $\neg \ell_{j_i}$ using unit-resolution and the axioms $A_1, \ldots, A_i$. This completes the induction.

To obtain the desired proof τ'', replace each axiom A_i in τ' by the coresponding tree T_i. This produces a sequence of clauses $C''_0, C''_1, \ldots, C''_t$, where each C_i is (possibly a weakening of) C''_i. Hence $C''_0 = \Box$, and τ'' derives $\Box$ from G' using unit-resolution alone.

$\Box$

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