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One-way Communication and Linear Sketch for Uniform Distribution

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Abstract

This note is prepared based on the article titled "Linear Sketching over $\mathbb{F}_2$"(ECCC TR16-174) by Sampath Kannan, Elchanan Mossel and Grigory Yaroslavtsev. We quantitatively improve the parameters of Theorem 1.4 of the above work. In particular, our result implies that the one-way communication complexity of any function $f^+(x,y) := f(x \oplus y)$ corresponding to the uniform distribution over the input domain $\{+1,-1\}^n \times \{+1,-1\}^n$ and error $\frac{1}{18}$ is asymptotically lower bounded by the linear sketch complexity of $f(x)$ corresponding to the uniform distribution over the input domain $\{+1,-1\}^n$ and error $\frac{1}{3}$. Our proof is information theoretic; our improvement is obtained by studying the mutual information between Alice's message and the evaluation of certain parities in the Fourier support of f on her input.

We recall the definition of approximate Fourier dimension by Kannan et al. (TR16-174).

Definition 1 (δ-approximate Fourier dimension, Kannan et al. 2016) *The δ-approximate Fourier dimension of a Boolean function $f(x) = \sum_S \widehat{f}(S)\chi_S(x)$ is defined to be the smallest dimension of any linear subspace $\mathcal{A} \in \mathbb{F}_2^n$ such that $\sum_{S\in\mathcal{A}} \widehat{f}^2(S) \geq \delta$.*

We will need the following basic fact about the Shannon entropy of ± 1 valued random variables, that can be easily proved by Taylor expanding the binary entropy function $H(p)$ about $p = \frac{1}{2}$.

Fact 2 *There is a universal constant $k > 0$ such that for any random variable X supported on $\{+1,-1\}$, $H(X) \leq 1 - k(\mathbb{E}X)^2$.*

For the rest of the note, fix an arbitrary $f : \{+1,-1\}^n \to \{+1,-1\}$, and let $f^+(x,y) = f(x \oplus y)$. We denote the δ-approximate Fourier dimension of f

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by $d_\delta(f)$. The following theorem is the main technical contribution of this note. The improvement on Theorem 1.4 in TR16-174 that is indicated in the abstract is presented in Corollary 5.

Theorem 3 *For every $\delta > 0$ the following holds. Let Π be any deterministic one-way protocol for the function $f^+(x,y)$ of cost c_Π that makes error $\epsilon_\Pi := \mathbb{P}_{x,y\sim U_n}[\Pi(x,y) \neq f^+(x,y)] \leq \frac{1}{4}(1 - \widehat{f}^2(\emptyset) - 2\delta)$. Then $c_\Pi \geq k\delta d_\delta(f)$, where k is the constant from Fact 2.*

Proof: Towards a contradiction assume that $c_\Pi < k\delta d_\delta(f)$. Let M be the random message sent by Alice to Bob. We will abuse notation and also denote the distribution of messages by M. Let $\mathcal{D}_m$ be the distribution of Alice's input x conditioned on the event that $M = m$. For any fixed input y of Bob, define $\epsilon_m^{(y)} := \mathbb{P}_{x\sim\mathcal{D}_m}[\Pi(x,y) \neq f^+(x,y)]$. Thus,

$$\epsilon_\Pi = \mathbb{E}_{m\sim M}\mathbb{E}_{y\sim U_n}\epsilon_m^{(y)}. \tag{1}$$

Observe that

$$\epsilon_m^{(y)} \geq \min_{b\in\{0,1\}} \mathbb{P}_{x\sim\mathcal{D}_m}[f^+(x,y) = b] \geq \frac{\mathrm{Var}_{x\sim\mathcal{D}_m} f^+(x,y)}{4}. \tag{2}$$

Now,

$$\begin{aligned}
&\mathrm{Var}_{x\sim\mathcal{D}_m} f^+(x,y) = 1 - \left(\mathbb{E}_{x\sim\mathcal{D}_m} f^+(x,y)\right)^2 \\
&= 1 - \left(\sum_S \widehat{f}(S)\chi_S(y)\mathbb{E}_{x\sim\mathcal{D}_m}\chi_S(x)\right)^2 \\
&= 1 - \Bigg(\sum_S \widehat{f}^2(S)\left(\mathbb{E}_{x\sim\mathcal{D}_m}\chi_S(x)\right)^2 \\
&\qquad + \sum_{\{S_1,S_2\}:S_1\neq S_2} 2\widehat{f}(S_1)\widehat{f}(S_2)\chi_{S_1\triangle S_2}(y)(\mathbb{E}_{x\sim\mathcal{D}_m}\chi_{S_1}(x))(\mathbb{E}_{x\sim\mathcal{D}_m}\chi_{S_2}(x))\Bigg).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbb{E}_{y\sim U_n}\mathrm{Var}_{x\sim\mathcal{D}_m} f^+(x,y) \\
&= 1 - \sum_S \widehat{f}^2(S)\left(\mathbb{E}_{x\sim\mathcal{D}_m}\chi_S(x)\right)^2 \\
&= 1 - \widehat{f}^2(\emptyset) - \sum_{S\neq\emptyset} \widehat{f}^2(S)\left(\mathbb{E}_{x\sim\mathcal{D}_m}\chi_S(x)\right)^2.
\end{aligned}$$

Taking expectation over messages it follows from (1) and (2) that,

$$\epsilon_\Pi \geq \frac{1}{4}\left(1 - \widehat{f}^2(\emptyset) - \sum_{S \neq \emptyset} \widehat{f}^2(S) \cdot \mathbb{E}_{m\sim M}\left(\mathbb{E}_{x\sim\mathcal{D}_m}\chi_S(x)\right)^2\right) \tag{3}$$

Define $\mathcal{T} := \{S \neq \emptyset \mid \mathbb{E}_{m\sim M}(\mathbb{E}_{x\sim\mathcal{D}_m}\chi_S(x))^2 \geq \delta\}$. For each $S \in \mathcal{T}$,

$$\begin{aligned} H(\chi_S(x) \mid M) &= \mathbb{E}_{m\sim M} H(\chi_S(x) \mid M = m) \\ &\leq \mathbb{E}_{m\sim M}(1 - k \cdot (\mathbb{E}_{x\sim\mathcal{D}_m}\chi_S(x))^2) \qquad \text{(Fact 2)} \\ &\leq 1 - k\delta. \end{aligned}$$

Let $\{T_1, \ldots, T_d\} \subseteq \mathcal{T}$ be a basis of $\mathcal{T}$. Then,

$$\begin{aligned} &c_\Pi \geq I(\chi_{T_1}(x), \ldots, \chi_{T_d}(x); M) = H(\chi_{T_1}(x), \ldots, \chi_{T_d}(x)) - H(\chi_{T_1}(x), \ldots, \chi_{T_d}(x) \mid M) \\ &\geq d - (\sum_{i=1}^{d} H(\chi_{T_i}(x) \mid M)) \\ &\geq d - d(1 - k\delta) = dk\delta. \end{aligned}$$

which implies that $d \leq c_\Pi / k\delta < d_\delta(f)$. We conclude that $\sum_{S\in\mathcal{T}} \widehat{f}^2(S) < \delta$. Thus we have,

$$\begin{aligned} &\sum_{S\neq\emptyset} \widehat{f}^2(S) \cdot \mathbb{E}_{m\sim M}\left(\mathbb{E}_{x\sim\mathcal{D}_m}\chi_S(x)\right)^2 \\ &= \sum_{S\in\mathcal{T}} \widehat{f}^2(S) \cdot \mathbb{E}_{m\sim M}\left(\mathbb{E}_{x\sim\mathcal{D}_m}\chi_S(x)\right)^2 + \sum_{S\notin\{\emptyset\}\cup\mathcal{T}} \widehat{f}^2(S) \cdot \mathbb{E}_{m\sim M}\left(\mathbb{E}_{x\sim\mathcal{D}_m}\chi_S(x)\right)^2 \\ &< \delta + \delta = 2\delta. \end{aligned}$$

From (3) we have

$$\epsilon_\Pi > \frac{1}{4}(1 - \widehat{f}^2(\emptyset) - 2\delta).$$

which is a contradiction. This concludes the proof.

Theorem 4 *Let* $\epsilon \in (0, \frac{1}{2})$. *Let* $\Delta := 1 - \sqrt{\left(\frac{1}{2} - \epsilon\right)}$. *Let* k *be the constant from Fact 2. Then,*

$$D_\epsilon^{\rightarrow,U}(f^+) \geq \min\left\{1, \left(2\sqrt{\left(\frac{1}{2} - \epsilon\right)} - 1\right) \cdot k\right\} \cdot D_\Delta^{lin,U}(f)$$

Proof: We split the proof into two cases:

Case 1: $\min_{b\in\{0,1\}} \mathbb{P}_{x\sim U_n}[f(x) = b] \leq \Delta$.
In this case $D^{lin,U}_{\Delta}(f) = 0$, as the algorithm that just outputs the more popular value of f errs with probability at most Δ. Thus we have,

$$D^{\rightarrow,U}_{\epsilon}(f^+) \geq D^{lin,U}_{\Delta}(f).$$

Case 2: $\min_{b\in\{0,1\}} \mathbb{P}_{x\sim U_n}[f(x) = b] > \Delta$.
In this case, $1 - \widehat{f}^2(\emptyset) > 1 - (1 - 2\Delta)^2 = 4\Delta - 4\Delta^2$. Applying Theoem 3 with $\delta = 2\Delta - 2\Delta^2 - 2\epsilon$, we have that $D^{\rightarrow,U}_{\epsilon}(f^+) \geq k(2\Delta - 2\Delta^2 - 2\epsilon) \cdot d_{2\Delta-2\Delta^2-2\epsilon}(f)$. Now, from Theorem 3.4 (Part 1) in the work of Kannan et al. (TR16-174), we have that $d_{2\Delta-2\Delta^2-2\epsilon}(f) \geq D^{lin,U}_{(1-2\Delta+2\Delta^2+2\epsilon)/2}(f)$. Thus,

$$D^{\rightarrow,U}_{\epsilon}(f^+) \geq k(2\Delta - 2\Delta^2 - 2\epsilon) \cdot D^{lin,U}_{(1-2\Delta+2\Delta^2+2\epsilon)/2}(f).$$

The theorem follows by substituting the value of Δ and verifying that $(1 - 2\Delta + 2\Delta^2 + 2\epsilon)/2 = \Delta$, and $2\Delta - 2\Delta^2 - 2\epsilon = 2\sqrt{\left(\frac{1}{2} - \epsilon\right)} - 1$.

The following corollary of Theorem 4 is obtained by setting $\epsilon = \frac{1}{18}$.

Corollary 5

$$D^{\rightarrow,U}_{\frac{1}{18}}(f^+) = \Omega\left(D^{lin,U}_{\frac{1}{3}}(f)\right).$$

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