

Almost Cubic Bound for Depth Three Circuits in VP

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Abstract

In "An almost Cubic Lower Bound for Depth Three Arithmetic Circuits", [KST16] present an infinite family of polynomials in VNP, $\{P_n\}_{n \in \mathbb{Z}^+}$ on n variables with degree n such that every $\Sigma \Pi \Sigma$ circuit computing P_n is of size $\tilde{\Omega}(n^3)$. A similar result was proven in [BLS16] for polynomials in VP with lower bound $\Omega\left(\frac{n^3}{2^{\sqrt{\log n}}}\right)$. We present a modified polynomial and perform a tighter analysis to obtain an $\tilde{\Omega}(n^3)$ lower bound for a family of polynomials in VP effectively bridging the VP and VNP results up to a $\log^5 n$ factor. More generally, we show that for every N and D satisfying $\text{poly}(N) > D > \log^2 N$, there exist polynomials $P_{N,D}$ on N variable of degree D in VP that can not be computed by circuits of size $\tilde{\Omega}(N^2 D)$.

1 Introduction

A depth three $\Sigma \Pi \Sigma$ circuit consists of a layer of sum gates, followed by a layer of multiplication gates, followed by a single sum gate that outputs the computation of the circuit. The fan-in is unbounded, and the circuit size is measured in terms of the number of edges. As such, depth three circuits capture "sums of products of linear polynomials". A recent line of work on depth reduction [AV08, Koi10, GKKS16, Tav15] has shown that moderately strong lower bounds for circuits of depth three imply a super-polynomial lower bound for general circuits. In addition, [Raz13] shows that a strong enough lower bound for set-multilinear depth three circuits implies a super-polynomial lower bound for general arithmetic formulas. These depth reduction results pave an avenue towards proving super-polynomial lower bounds for general circuits/formulas by leveraging the apparent simple structure of depth three circuits. Unfortunately, it is still an open problem to prove super-polynomial lower bounds for depth three circuits over fields of characteristic zero. Below we present some of the seminal results in depth three lower bounds.

In [SW02], Shpilka and Wigderson proved a $\Omega(n^2)$ depth three circuit computing the elementary symmetric polynomials $ES_{Y M^d}(x_1, x_2, \dots, x_n) = \sum_{S \subseteq [n], |S|=d} \prod_{i \in S} x_i$

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on n variables and degree $d = \Theta(n)$. In the same paper, the authors prove a near quadratic lower bound for the determinant polynomial [SW02]. Restricting the circuit model (homogeneity, multilinearity) and restricting the field characteristic yields better results. Over fixed finite fields, [GR00] prove an exponential lower bound for the determinant and in [NW96] it was shown that any homogeneous depth three circuit computing $ESYM_n^{2d}$ has size $\Omega((\frac{n}{4d})^d)$. More recently, in [KS15] a $n^{\Omega(\sqrt{d})}$ lower bound was proven for depth three circuits, with bottom fan-in bounded by n^ϵ for any fixed $\epsilon < 1$, computing an explicit n -variate polynomial of degree d .

Despite success in many restricted settings (homogenous, degree bounded product gates) the lower bounds in general cases remain relatively weak. Recently [KST16] gave near cubic $\tilde{\Omega}(n^3)$ lower bounds for a polynomial family in VNP, which was followed by [BLS16] who gave a $\Omega(\frac{n^3}{2^{\sqrt{\log n}}})$ lower bounds for a polynomial family in VP.

In this work we strengthen the latter lower bound to get a polynomial in VP on N variables and degree D satisfying $\text{poly}(N) > D > \log^2 N$, with size lower bound $\tilde{\Omega}(N^2 D)$. Setting $D = N$, this recovers the VNP result up to a $\log^5(N)$ factor. Along the way we present a simplified polynomial and a tighter analysis of its multiplicative complexity. We also expand on the trade off between circuit size as a joint function of the degree of the polynomial and the number of variables — something that does not seem to have been explicitly clarified before.

Our main result is as follows.

Theorem 1.1. *There exists an explicit polynomial family $P_{N,D}$ computable in VP on N variables of degree D satisfying $\text{poly}(N) > D > \log^2 N$ such that any depth 3 circuit computing it has size $\tilde{\Omega}(N^2 D)$. Setting $D = N$ as in previous works recovers, up to a $\log^4(N)$ factor the $\tilde{\Omega}(N^3)$ bound for polynomials in VNP [KST16]*

2 Preliminaries

We discuss some of the language and common techniques relating to arithmetic circuits. An extended treatment can be found in the survey [SY10] of Shpilka and Yehudayoff.

Our general organization is as follows. Section (3) constructs a "hard" polynomial and bounds its size for bounded fan-in circuits. Section (4) presents the embedding procedure producing a polynomial that can be analyzed for unbounded fan-in.

2.1 Basic Notation

The ideal generated by a set of polynomials of the ring P will be denoted $\langle P \rangle$. We use $\text{poly}(N)$ to denote polynomial in N with an arbitrary constant exponent. A $\Sigma \Pi^Y \Sigma$ circuit computes polynomials that are the sum of the product of at most Y affine linear forms. Similarly, a $\Sigma \Pi^Q \Pi^R \Sigma$ circuit consists of a layer of sum gates, followed by two layers of product gates with fan-in bounded by R and Q respectively, followed by a final sum gate. We

observe that each $\Sigma \Pi^Q \Pi^R \Sigma$ circuit can be converted to a $\Sigma \Pi^{QR} \Sigma$ circuit with constant factor overhead in size.

2.2 Shifted Partial Derivative Measure

As in previous works, we use a measure $\mu : \mathbb{F}[\mathbf{x}] \rightarrow \mathbb{N}$ to capture weakness of a circuit model in opposition to a "hard" family of polynomials giving us a lower bound for the circuit family. Our choice of measure is the "dimension of the shifted partials" introduced in [Kay12]. For polynomial $P \in \mathbb{F}[x_1, x_2, \dots, x_N]$, let $\langle P \rangle_{\leq k}$ be the set of k 'th order partials of P . Furthermore, let

$$\langle P \rangle_{\leq \ell}^k := \{f \cdot p \mid \forall \text{ monomials } f \text{ s.t. } \deg(f) \leq \ell, \forall p \in \langle P \rangle_{\leq k}\} \quad (1)$$

Then for $k, \ell \in \mathbb{N}$, the shifted derivative measure is defined to be

$$\mu_{k, \ell} P = \dim(\langle P \rangle_{\leq \ell}^k) \quad (2)$$

Adding the parameter ℓ produces this shifted derivative measure that introduces "leeway" into the measure of the "dimension of the partial derivatives" introduced in [NW96]

2.3 Circuits under Affine Projections

Given polynomial $P \in \mathbb{F}[x_1, x_2, \dots, x_N]$ as above, let $A : \mathbb{F}^N \rightarrow \mathbb{F}^N$ be an affine linear transform, then it is easy to show that $\mu_{k, \ell} P \circ A \leq \mu_{k, \ell} P$. In which case if A is invertible, then $\mu_{k, \ell} P \circ A = \mu_{k, \ell} P$. The takeaway is that the shifted derivative measure is invariant under invertible affine transforms.

Now let V be a subspace of \mathbb{F}^N and V^\perp be its complement. Then if A is an affine projection onto the space V , then we say $P \circ A$ is a subspace restriction $P|_V$. If we let U_V be the orthogonal projection of \mathbb{F}^N to V , by the above discussion we observe that $\mu_{k, \ell} P \circ U_V \circ A = \mu_{k, \ell} P \circ U_V$. This is useful for the following reason.

The central barrier to proving lower bounds for bounded depth circuits is the unbounded fan-in. The key idea is then to restrict the polynomial with an affine transform A to an affine subspace V so that the product gates with large fan-in can be pruned. We are then left with a bounded fan-in circuit which we can analyze. However, we must now compute the measure of the polynomial $P \circ A$. We do this precisely by noting that $\mu_{k, \ell} P \circ U_V \circ A = \mu_{k, \ell} P \circ U_V$ and construct P so that its shifted derivative measure is easy to compute under orthogonal affine restrictions. In some sense we are "embedding" a polynomial for which we can analyze its shifted derivative measure within P . Section 3 constructs the embedded polynomial and section 4 details how the subspace restrictions are performed in practice.

3 Embedded Polynomial

First we construct a polynomial in VP for which we can analyze its shifted derivative measure and bound its circuit size for constant depth circuits with bounded fan-in.

3.1 Polynomial Construction

Let X be a b -by- n matrix of formal variables as shown below.

$$X = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{b1} & x_{b2} & \dots & x_{bn} \end{bmatrix} \quad (3)$$

Let $J = (j_1, j_2, \dots, j_b)$ for $J \in [n]^b$. Then define the function $Permute(X)$ to be

$$Permute(X) = \prod_{i=1}^b \sum_{j=1}^n x_{ij}^{\frac{D}{b}} = \sum_{J \in [n]^b} x_{1j_1}^{\frac{D}{b}} x_{2j_2}^{\frac{D}{b}} \dots x_{bj_b}^{\frac{D}{b}} \quad (4)$$

Notice that $Permute(X)$ has $N = nb$ variables and has degree D . For $b = \log n$, $Permute(X)$ is in VP by inspecting the sum and product in the definition.

3.2 Bounding Measure for Target Polynomial

The first lemma is presented as Proposition 9 in [AG13]. If polynomial $f \in \mathbb{F}[x_1, \dots, x_N]$ is of the form $f = \sum_{i=1}^s \prod_{j=1}^Q G_{ij}(x_1, x_2, \dots, x_N)$ where each G_{ij} is a polynomial of degree no greater than R , then the following inequality bounds the size of s .

Lemma 3.1. *For all $k, \ell \in \mathbb{N}$ let the shifted partial derivative measure $\mu_{k, \ell} f = \dim(\langle f \rangle_{\leq \ell}^k)$. Then for $k < Q$ the following lower bounds the size of s*

$$\frac{\mu_{k, \ell} f}{\binom{Q+k}{k} \binom{N+\ell+k(R-1)}{\ell+k(R-1)}} \leq s \quad (5)$$

With respect to circuits, s is the size of the top fan-in which is what we'll be using as a lower bound for circuit size. Q can then be interpreted as the top layer product gate fan-in. So long as each product gate has a fan-in consisting of polynomials of degree no greater than R , the above lemma holds. Summarizing these remarks, we find that the left hand side of the inequality is dependent only on the circuit model, and that k and ℓ are chosen for analytical convenience.

The next lemma has several formulations. We will present the formulation in Lemma 3 of [CM13].

First, we define a distance metric between any pair of monomials g and g' of identical degree. Let h be the monomial of minimum degree divisible by both g and g' . Then let $|g \Delta g'| = \deg(h) - \deg(g)$ which is well defined because $\deg(g) = \deg(g')$.

Lemma 3.2. *Let $f \in \mathbb{F}[x_1, x_2, \dots, x_N]$ be a polynomial, then the following inequality lower bounds the shifted partial derivative measure $\mu_{k, \ell} f$ for all $k, \ell \in \mathbb{N}$. If $S \subseteq \partial_k \langle f \rangle$ is a set of monomials satisfying for distinct $g, g' \in S$, $|g \Delta g'| \geq \tau$ then*

$$|S| \binom{N+\ell}{\ell} - |S|^2 \binom{N+\ell-\tau}{\ell-\tau} \leq \mu_{k, \ell} f \quad (6)$$

Putting Lemma 0.1 and 0.2 together we obtain

$$\frac{|S|^{\binom{N+\ell}{\ell}} - |S|^{2\binom{N+\ell-\tau}{\ell-\tau}}}{\binom{Q+k}{k}\binom{N+\ell+k(R-1)}{\ell+k(R-1)}} \leq s \quad (7)$$

Now we must determine the size of a set S satisfying the properties of Lemma 0.2 with a corresponding minimum distance τ for our polynomial $Permute(X)$. Consider the following, we set $k = b = \log n$, and define $\partial_J Permute(X)$ for $J = (j_1, j_2, \dots, j_k) \in [n]^k$ to be the k 'th order derivative obtained by differentiating $Permute(X)$ by $x_{1j_1} x_{2j_2} \dots x_{kj_k}$. Then

$$\partial_J Permute(X) = x_{1j_1}^{\frac{D}{\log n} - 1} x_{2j_2}^{\frac{D}{\log n} - 1} \dots x_{kj_k}^{\frac{D}{\log n} - 1} \quad (8)$$

Then we define $S := \{\partial_J Permute(X) \mid \forall J \in [n]^k\}$ which gives us $|S| = n^k$. Furthermore, for any distinct $J, J' \in [n]^k$, J and J' differ in some coordinate j_i implying $\tau = \frac{D}{\log n} - 1$. Armed with our values of $|S|$ and τ , we can set the circuit parameters Q, R and the shifted derivative parameters k, ℓ and compute a lower bound on bounded fan-in depth four circuits.

3.3 Calculation

Lemma 3.3. *For any $\Sigma \Pi^Q \Pi^R \Sigma$ circuit computing $Permute(X)$, if we set the values for the circuit parameters $Q = n^{1 - \frac{5}{\log n}}$, $R = \frac{\tau}{\log^2 n}$ and the shifted derivative parameters $k = \log n$, $\ell = \frac{n \log n}{2^{\frac{\log^2 n + 1}{\tau} - 1} - 1}$, then the top fan-in s is greater than N^4 . Adjusting the constant in the definition of Q gives us an $\text{poly}(N)$ bound of arbitrary constant degree.*

Proof. Plugging these parameters into (5) we find

$$\frac{n^k \binom{N+\ell}{\ell} - n^{2k} \binom{N+\ell-\tau}{\ell-\tau}}{\binom{Q+k}{k} \binom{N+\ell+k(R-1)}{\ell+k(R-1)}} \leq s \quad (9)$$

We apply standard binomial inequalities to obtain

$$\frac{n^k \binom{N+\ell}{\ell} - n^{2k} \binom{N+\ell}{\ell} \left(\frac{N+\ell}{\ell}\right)^{-\tau}}{\binom{Q+k}{k} \binom{N+\ell}{\ell} \left(\frac{N+\ell}{\ell}\right)^{k(R-1)}} \leq s \quad (10)$$

And remove the $\binom{N+\ell}{\ell}$ term to obtain

$$\frac{n^k - n^{2k} \left(\frac{N+\ell}{\ell}\right)^{-\tau}}{\binom{Q+k}{k} \left(\frac{N+\ell}{\ell}\right)^{k(R-1)}} \leq s \quad (11)$$

Now our setting of ℓ gives us $\left(\frac{N+\ell}{\ell}\right)^{-\tau} = \frac{1}{2} n^{-k}$ so that the numerator reduces to

$$n^k - n^{2k} \left(\frac{N+\ell}{\ell}\right)^{-\tau} = \frac{1}{2} n^k \quad (12)$$

The denominator reduces to

$$\binom{Q+k}{k} \left(\frac{N+\ell}{\ell}\right)^{kR} = \binom{Q+k}{k} n^{\frac{k^2 R}{\tau}} 2^{\frac{kR}{\tau}} \quad (13)$$

Now combining numerator and denominator we obtain

$$s \geq \frac{\frac{1}{2}n^k}{\binom{Q+k}{k} n^{\frac{k^2 R}{\tau}} 2^{\frac{kR}{\tau}}} \geq \frac{\frac{1}{2}n^k}{\binom{Q+k}{k} n} \geq \frac{\frac{1}{2}n^k}{Q^k n} \geq \frac{\frac{1}{2}n^k}{n^{(1-\frac{5}{\log n})k}} = \frac{1}{2}n^4 \quad (14)$$

This concludes our analysis of $Permute(X)$. We can obtain any polynomial lower bound by adjusting the constant parameter 5 in the setting of Q which is all we need for the subspace restrictions detailed next. \square

4 Putting it Together

We present the technique of subspace restrictions following the general presentation in [BLS16, KST16]. The proof idea is to construct an explicit polynomial $F_{N',D'}$ in VP with $N' = \Theta(N \log N)$ variables and degree $D' = \Theta(D \log N)$ where any circuit computing $F_{N',D'}$ satisfies the property that restricting any N product gates yields a circuit computing $Permute(X)$. So long as $Permute(X)$ must be computed by a $poly(N)$ sized circuit with some large constant degree, then $F_{N',D'}$ must be computed by a $\Omega(NQR) = \tilde{\Omega}(N^2 D) = \tilde{\Omega}(N'^2 D')$ sized circuit. Note, it is for $F_{N',D'}$, not $Permute(X)$, for which we produce our almost cubic lower bound. First we present the construction of $F_{N',D'}$, then we present the subspace restriction procedure, and finally we prove Theorem 0.1.

4.1 Polynomial Embedding

$Permute(X)$ takes $N = n \log n$ variables. We now introduce the formal variables $W = \{w_1, w_2, \dots, w_{2N}\}$ and $U = \{U_1, U_2, \dots, U_N\}$. Where each $U_i \in U$ is a collection of q variables $U_i = \{u_{i1}, u_{i2}, \dots, u_{iq}\}$ for $q = C \log n$ for constant factor C . Now let $M = \{m_1, m_2, \dots, m_{2N}\}$ be $2N$ pairwise distinct subsets of $[C \log n]$ where each $m_i \in M$ is of size $|m_i| = C' \log n$. Then for $i \in [2N]$ and $j \in N$, we define $\phi_i(U_j) = \prod_{y \in m_i} u_{jy}$. Now we are ready to define $F_{N',D'}(U, W)$. Let V be a set of N formal variables, defined as follows

$$V = \begin{bmatrix} \phi_1(U_1) & \phi_2(U_1) & \dots & \phi_{2N}(U_1) \\ \phi_1(U_2) & \phi_2(U_2) & \dots & \phi_{2N}(U_2) \\ \dots & \dots & \dots & \dots \\ \phi_1(U_N) & \phi_2(U_N) & \dots & \phi_{2N}(U_N) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{2N} \end{bmatrix} \quad (15)$$

Then we define

$$F_{N',D'}(U, W) = Permute(V) \quad (16)$$

There is slight notational abuse since we initially defined $Permute$ to be a function taking a matrix of N variables but V is a vector. It is to be understood that in writing $Permute(V)$ we implicitly arrange V into a matrix.

First we observe that $F_{N',D'}(U, W)$ has $N' = CN \log N + 2N = \Theta(N \log N)$ variables. Furthermore, the degree $D' = C'D \log N = \Theta(D \log N)$. Since the sets $m_i \in M$ are pairwise distinct, for each subset $A \in [2N]$ satisfying $|A| = N$ there exists a setting of the variables in U such that $F_{N',D'}(U, W) = \text{Permute}(\chi_A(W))$ where $\chi_A(W)$ selects N variables from W corresponding to A . Therefore, we call the W 's "relevant" variables and the U 's "indicator" variables that we eventually set to be $\{0, 1\}$. We restate this critical property in the following lemma.

Lemma 4.1. *For each subset $A \in [2N]$ satisfying $|A| = N$, there exists a setting of the variables in U such that $F_{N',D'}(U, W) = \text{Permute}(\chi_A(W))$ where $\chi_A(W)$ selects N variables from W corresponding to A .*

4.2 Affine Subspace Restriction

Here we finish proving Theorem 0.1. For any $\Sigma \Pi \Sigma$ circuit computing $F_{N',D'}(U, W)$ we say a product gate is "heavy" if its fan-in consists of more than QR sum gates that have a relevant variable $w_i \in W$ in their fan-in. Then there are two cases.

case 1: If there are more than $N = \Theta(n \log n)$ product gates with fan-in greater than $QR = n^{1-\frac{5}{\log n}} \frac{\tau}{\log^2 n} = \frac{nD}{32 \log^3 n}$, then we have an $N \frac{nD}{32 \log^3 n} = \Omega(\frac{N^2 D}{\text{polylog}(N)})$ lower bound on the number of wires in the circuit and we're done.

case 2: Consider a $\Sigma \Pi \Sigma$ circuit with top fan-in s computing $F_{N',D'}(U, W)$. If there are fewer than N heavy product gates then we remove them in the following manner. Let $P(U, W)$ be a heavy product gate, then choose any sum gate $L(U, W)$ in the fan-in of $P(U, W)$ that is the affine sum of variables including a relevant $w_i \in W$. Therefore we can write $L(U, W) = \alpha w_i + L'(U, W)$ where $L'(U, W)$ is an affine linear form not involving w_i . Then rewiring the circuit so that $w_i = \frac{-1}{\alpha} L'(U, W)$ removes the sum gate $L(U, W)$ and the product gate $P(U, W)$. Repeating this process at most N times for all heavy product gates we are eventually left with a $\Sigma \Pi^Q R \Sigma$ circuit which we then pull apart to a $\Sigma \Pi^Q \Pi^R \Sigma$ circuit (Note: pulling the product apart does not change the size of the top fan-in). Now let $Y \in [2N]$ be the set of indices corresponding to the unrestricted variables in W , and let $A \subseteq Y$ be a subset of the unrestricted variables of size $|A| = N$. Then by lemma 0.5 we can set the U 's so that $F_{N',D'}(U, W) = \text{Permute}(\chi_A(W))$. Taken together, we have a $\Sigma \Pi^Q \Pi^R \Sigma$ circuit with some top fan-in s' computing our hard polynomial $\text{Permute}(\chi_A(W))$. In the process of converting from $\Sigma \Pi \Sigma$ to $\Sigma \Pi^Q \Pi^R \Sigma$ we have performed affine restrictions and set the variables in U , operations that can only decrease the size of the top fan-in. Therefore $s > s'$, and by lemma 0.4 we know $s > s' > N^4$.

Taking the minimum of case 1 and case 2 we obtain the size of any $\Sigma \Pi \Sigma$ circuit computing $F_{N',D'}(U, W)$ is greater than $\min(\frac{N^2 D}{\text{polylog}(N)}, N^4) = \tilde{\Omega}(N'^2 D')$ where we understand that N^4 can be any $\text{poly}(N)$. As a final comment, the $\tilde{\Omega}$ hides a $\log^7 N$ factor, whereas the VNP result in [KST16] is almost cubic by a $\log^2 N$ factor. One avenue towards removal is avoiding the overhead in both variables and degree in the polynomial embedding.

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