Electronic Colloquium on Computational Complexity, Report No. 3 (2017)

Magic Adversaries Versus Individual Reduction: Science Wins Either Way*

Yi Deng[1,2]

[1] SKLOIS, Institute of Information Engineering, CAS, P.R.China
[2] State Key Laboratory of Cryptology, P. O. Box 5159, Beijing ,100878,China
deng@iie.ac.cn

Abstract. We prove that *at least* one of the following statements is true:

– (Infinitely-often) Public-key encryption and key agreement can be based on injective one-way functions;
– For every inverse polynomial ϵ, the 4-round protocol from [Feige and Shamir, STOC 90] is distributional concurrent zero knowledge for any efficiently samplable distribution over any OR NP-relations with distinguishability gap bounded by ϵ.

Both these statements have been shown to be unprovable [Impagliazzo and Rudich, STOC 89; Canetti et. al., STOC 01] via black-box reduction. Our win-win result also establishes an unexpected connection between the complexity of public-key encryption and the round-complexity of concurrent zero knowledge.

As the main technical contribution, we introduce a dissection procedure for concurrent adversaries, which enables us to show that, if there is a magic concurrent adversary that breaks the ϵ-distributional concurrent zero knowledge of the Feige-Shamir protocol for some OR NP-relations, then we transform it to an (infinitely-often) public-key encryption and key agreement based on injective one-way functions. If it could be proved that the reduction from injective one-way functions to public-key encryption does not exist, then our dissection reveals that all possible concurrent verifiers for the Feige-Shamir protocol share a common structure in their computation.

This dissection of adversary algorithms also gives insight into the fundamental gap between the known *universal* security reduction that works for *any* adversaries, and the security definition (of almost all cryptographic primitives/protocols), which switches the order of qualifiers and only requires that for every adversary there *exists* an *individual* reduction.

1 Introduction

The seminal work of Impagliazzo and Rudich [IR89] provides a methodology for studying the limitations of black-box reduction. Following this methodology, a plenty of black-box barriers, towards building cryptographic systems on simpler primitives/assumptions and achieving more efficient constructions, have been found in the last three decades. These findings have long challenged us to develop new reduction methods and get around the limitations of black-box reduction, however, the progress towards this goal is quite slow, and for most of the known black-box barriers, it is still unclear whether they even hold for arbitrary reductions.

We revisit two seemingly unrelated fundamental problems, for both of which the black-box impossibility results are well known. We show that these impossibility results cannot *coexist* unconditionally, and there must be a new reduction technique that can help us bypass at least one of them.

* "Science wins either way" is credited to Silvio Micali. This work was supported by the National Natural Science Foundation of China (Grant No. 61379141), and the Open Project Program of the State Key Laboratory of Cryptology.

ISSN 1433-8092

The first problem is whether we can base public-key cryptography on general one-way functions. Ever since the invention of public key cryptography by Diffie and Hellman [DH76], the complexity of public-key cryptography, i.e., lowering the underlying complexity assumptions for cryptographic primitives/protocols, is one of the most basic problems. In the past four decades, for some primitives, including pseudorandom generator, signature and statistically-hiding commitment, we witnessed huge success on this line of research and can now base them on the existence of one-way functions [Rom90, HILL99, HR07], which is the minimum assumption in the sense that, as showed by [IL89], almost all cryptographic primitives/protocols imply the existence of one-way functions.

But for public-key encryption and key agreement– the concepts that were conceived in the original paper of Diffie and Hellman, we did not make that successful progress yet. On the positive side, there are numerous efficient constructions ([RSA78, Rab79, GM82, CS99, Reg09, HKS03], to name a few) for public-key encryption with various security notions based on specific assumptions with various algebraic structures, and some less efficient constructions [NY90, BHSV98, Sah99, Lin03a] based on more abstract assumptions– enhanced trapdoor permutations or trapdoor functions with polynomial pre-image size. Since public-key encryption implies key agreement (secure against eavesdropping adversaries), the same assumption is sufficient for the latter. On the negative side, Impagliazzo and Rudich proved in their seminal work [IR89] that there is no black-box reduction of one-way permutations to key agreement, and since public-key encryption implies key agreement, their result also separates one-way permutations from public-key encryption with respect to black-box reduction. The recent work of [BHSV98, GMR01, DS16] strengthens the black-box separation of [IR89] by allowing the reduction to take the code of the underlying primitive as input.

Though these black-box separations provide some strong negative evidences, they do not rule out the possibility of constructing public-key encryption from one-way functions, i.e., rule out the "minicrypt" of Impagliazzo's five possible worlds [Imp95].

The other fundamental problem we consider is the round-complexity of concurrent zero knowledge. The notion of concurrent zero-knowledge, put forward by Dwork, Naor and Sahai [DNS98], extends the standard-alone zero-knowledge security notion [GMR89] to the case where multiple concurrent executions of the same protocol take place and a malicious adversarial verifier may control the scheduling of the messages and corrupt multiple verifiers. In the last two decades, concurrent zero knowledge attracted considerable attention, and actually lies at the core of advanced compositions of general cryptographic protocols [CLOS02, PR03, Lin03b, PR05, Pas04, Lin08, GGJ13, GGJS12, GGS15, GLP+15].

As observed in [DNS98], the traditional black-box simulator does not work for the classic constant-round protocols (including Feige-Shamir type protocol[FS89] and Goldreich-Kahan type protocol [GK96]) in the concurrent setting. Indeed, Canetti et al. [CKPR01] proved that concurrent zero-knowledge with black-box simulation requires a logarithmic number of rounds for languages outside BPP. Prabhakaran et al. [PRS02] later refined the analysis of the Kilian and Petrank's [KP01] recursive simulator and gave an (almost) logarithmic round concurrent zero knowledge protocol for NP.

In his breakthrough work, Barak [Bar01] introduced a non-black-box simulation technique based on PCP mechanism and constructed a constant-round public-coin zero knowledge protocol for NP, which breaks several known lower bounds for black-box zero knowledge. The original construction of Barak satisfies only *bounded-concurrent* zero knowledge. Goyal [Goy13] extended Barak's idea to achieve fully concurrent zero knowledge in polynomial rounds. In the globe hash model, Canetti et al. [CLP13a] showed that public-coin concurrent zero knowledge can be obtained with logarithmic round-complexity. Recently, Chung et al. [CLP15a] (based on [CLP13b]) presented the first public-coin constant-round concurrent zero knowledge protocol based on indistinguishability obfuscation with super-polynomial security.

The problem of whether we can achieve constant-round concurrent zero knowledge based on standard assumptions is still left open. Note also that the known constructions that beat the lower bound on the black-box round-complexity are rather complicated and therefore impractical. Given the current state of the art, a more ambitious question is whether we can prove the concurrent zero knowledge property of the classic 4-round protocols (such as Feige-Shamir protocol), although it is known to be impossible to give such a proof for these simple and elegant constructions via black-box simulation.

1.1 Universal Reduction "$\exists R \forall A$" Versus Individual Reduction "$\forall A \exists R$"

We observe that almost all known reduction/simulation techniques (including the known black-box reduction and the non-black-box reduction) are *universal* in the sense that, in the security proof of a protocol/premitive, the reduction R works for all possible efficient adversaries and turn the power of a give adversary A into the power of breaking the underlying assumption (i.e., "$\exists R \forall A$"). However, for almost all security definitions, it is only required that for a given specific adversary A there *exists* an *individual* reduction R that works for A (i.e., "$\forall A \exists R$").

This motivates us to step back and look at the concurrent security of the simplest Feige-Shamir protocol. We will show that, though Canetti et al. [CKPR01] constructed an adversarial verifier for which the known black-box simulator fails, we are still able to show an *individual* simulator for this specific verifier (and thus it is not a concrete "attacker"). Sure, showing the existence of a simulator for a specific verifier does not mean that the Feige-Shamir protocol is concurrent zero knowledge, but this example does reveal a gap between the universal reduction/simulation "$\exists R \forall A$" and the individual reduction/simulation "$\forall A \exists R$" .

The Feige-Shamir protocol for proving $x \in L$ proceeds as follows. In the first phase, the verifier picks two random strings α_1 and α_2, computes two images, $\beta_1 = f(\alpha_1)$, $\beta_2 = f(\alpha_2)$, of a one-way function f, and then proves to the prover via a constant-round witness indistinguishability protocol that he knows either α_1 or α_2; in the second phase, the prover proves that either $x \in L$ or he knows one of α_1, α_2. The adversary V^* constructed in [CKPR01] adopts a delicate scheduling strategy, and when computing a verifier message, it applies a hash function h with high dependence to the history hist sofar and generates the randomness $r = h(\mathsf{hist})$ for computing the current message. In our case, the randomness for the first verifier step of a session includes the two pre-images α_1 and α_2.

Canetti et al. showed that it is impossible for an efficient simulator to simulate V^*'s view when treating it as a black-box[1]. However, as mentioned before, the concurrent zero knowledge condition does not require a universal (or black-box) simulator that works for all adversarial verifiers, but just requires that for every specific V^* there *exists* an *individual* simulator.

Note that the individual simulator may depends on the specific verifier, and more importantly, since we are only required to show the *mere existence* of such a simulator, we can assume that the individual simulator knows (or equivalently, takes as input) the verifier's *functionality*, *randomness*, etc.

Indeed, for the adversary V^* of [CKPR01], there *exists*, *albeit* probably not efficiently constructible from a given (possibly obfuscated) code of V^*, a simple simulator for the above specific V^*: Note that there *exists* an adversary V' that acts exactly in the same way as V^* except that at each step V' outputs $r = h(\mathsf{hist})$ together with the current message, and thus a trivial simulator $\mathsf{Sim}(V')$, incorporating V' and using the fake witness (one of α_1 and α_2[2]) output by V' at the first verifier step of each session, can easily generate a transcript that is indistinguishable from the real interaction between V^* and honest provers .

The above example shows, even for simple construction, it is really hard to construct a concrete attack that would rule out *all* security reductions/simulations. As we will show, any concrete

[1] I.e., the simulator is given only oracle access to V^*, and does not have knowledge about its code, running time, etc.

[2] Note that α_1 and α_2 are part of the randomness r used in the first verifier message of a session.

concurrent attack for the Feige-Shamir protocol will yield a surprising consequence: We can base public-key encryption on *injective* one-way function.

1.2 Our Contribution

We prove that, at least one of these two problems (with respect to infinitely-often version and distributional version respectively) mentioned above has a *positive* answer. That is, there must *exist* a new reduction method that can break one of the known black-box lower bounds for them. We now state our theorem more formally.

Let f be an arbitrary *injective* one-way function, L and R_L be an arbitrary NP language and its associated NP relation respectively. We define $R_L^n := \{(x, w) : (x, w) \in R_L \wedge |x| = n\}$, and define the OR language $L \vee L$[3] and the corresponding relation $R_{L_{OR}}$ in a natural way.

Given an arbitrary efficiently samplable distribution ensemble $D = \{D_n\}_{n \in N}$ over R_L (each D_n defined over R_L^n), and an arbitrary efficiently samplable distribution Z_n over $\{0, 1\}^*$[4], we define the joint distribution $\{(X_n, W_n, Z_n)\}_{n \in N}$ over $R_{L_{OR}} \times \{0, 1\}^*$ in the following way: Sample $(x_1, w_1) \leftarrow D_n$,$(x_2, w_2) \leftarrow D_n$, $z \leftarrow Z_n$, $b \leftarrow \{1, 2\}$, and output $((x_1, x_2), w_b)$. We prove the following theorem.

Theorem 1. *Let the joint distribution $\{(X_n, W_n, Z_n)\}_{n \in N}$ and f be as above. Then,* at least *one of the following statements is true:*

- *(Infinitely-often) Public-key encryption and key agreement can be constructed from the injective one-way function f;*
- *For every inverse polynomial ϵ, the Feige-Shamir protocol based on f is distributional concurrent zero knowledge for $\{(X_n, W_n, Z_n)\}_{n \in N}$ with distinguishability gap bounded by ϵ.*

In the infinitely-often version of a primitive, the correctness and security of a construction are required to hold only for infinitely many security parameter n. The notion of *ϵ-distributional* concurrent zero knowledge (defined also in [Gol93, DNRS03, CLP15b]) differs from the traditional zero knowledge in that its zero knowledge property holds on average (i.e., holds for distributions over the statements), and that the indistinguishability gap for any PPT distinguisher is bounded by an arbitrary inverse polynomial (instead of a negligibly function).

We note that the black-box lower bounds [IR89, CKPR01] also hold for the infinitely-often version of public-key encryption and the *ϵ-distributional* concurrent zero knowledge[5].

The basic strategy for proving this theorem is to transform a magic adversary V^* that breaks the distributional concurrent zero knowledge condition of the Feige-Shamir protocol into constructions for (infinitely-often) public-key encryption and key agreement.

On the very high level, if a concurrent adversary verifier V^* that can break concurrent zero knowledge of the Feige-Shamir protocol, then in the real interaction there must exist a step i (verifier steps are ordered according to their appearance in the concurrent setting) such that:

- With high probability, V^* will output a pair of images β_1 and β_2, i.e., the first verifier message of some session j at this step i, and at a later time it will reach its second step of session j, i.e., completes its 3-round proof that it knows one pre-image of β_1 and β_2 under f.
- But for any PPT algorithm T, even taking the history prefix up to the step i of V^*, the probability that T inverts any one of these two images β_1 and β_2 is bounded away from 1.

[3] For simplicity, we consider only the OR composition of the same NP language L, but our result holds with respect to the OR composition of any two NP languages.

[4] The element z from Z_n will be given as auxiliary input to the verifier of Feige-Shamir protocol.

[5] By applying the lower-bound proof strategy of [CKPR01], we conclude that the Feige-Shamir protocol cannot be ϵ-distributional concurrent *black-box* zero knowledge for any non-trivial language outside heurBPP, where heurBPP refers to the distributional version of BPP (see [BT08] for a formal definition).

The intuition behind this observation is as follows. If the above two items does not hold simultaneously, then at each verifier step, either V^* does not output a pair of images of a session, or it outputs a pair of images of session j but will never reach its second message of session j, or there is an efficient algorithm that can find one of the corresponding pre-images. In each case we will have a simple simulator that can simulate the view of the V^*, which leads to a contradiction.

Thus, for a given successful adversary V^* the above two items must hold simultaneously. This means V^* magically endow the above two images β_1 and β_2 with a trapdoor (i.e., the witness w to the common input x): With the trapdoor w, one can play the role of honest prover until V^* completes his 3-round proof, then using standard rewinding technique to obtain one of the pre-images, while, without the knowledge of w, no PPT algorithm can invert any one of β_1 and β_2 with overwhelming probability. This is the key observation that enables us to construct PKE from the injective one-way f.

Our proof proceeds as follows.

STEP I: We introduce a dissection procedure and prove that there must be infinitely many n, for each of which there exists a step i of V^*, such that the above two items hold simultaneously. This illustrates the power of V^* that magically endows f with a sort of trapdoor.
This step is presented in section 3.

STEP II: With the code of V^*, we then construct a pair of (non-interactive) algorithms C and E such that for each (n, i) obtained in the above step:
- C (with knowledge of a witness w to x) outputs a *single* image β with high probability;
- E (with knowledge of a witness w to x) will extract the pre-image of β output by C;
- No PPT algorithm can compute the pre-image of β except with negligibly close to 1.

This step is presented in section 4.

STEP III: Using standard techniques, we amplify the gap between the success probability of E and the success probability of any PPT algorithms without knowing a witness to x, and obtain two algorithms M and Find, where M takes a sequence of (x, w) as input and outputs a sequence of images β, and Find takes the same sequence of (x, w) and outputs all pre-images corresponding to the sequence of images β, both with probability negligibly close to 1; further, there is no PPT algorithm that can invert all the images output by M simultaneously with non-negligible probability.
This step is presented in section 5.

STEP IV: Note that the Feige-Shamir protocol is concurrent witness indistinguishable, and thus the above holds when M and Find use different witnesses. Starting with a magic adversary V^* that breaks the distributional concurrent zero knowledge of the Feige-Shamir protocol for distribution over OR NP-statements of the form $(x_1 \vee x_2)$, we construct the public-key encryption scheme (and key-exchange scheme) in a natural way: The receiver generates a sequence of (x_1, w_1) as the public/secret key pair; to encrypt a bit, the sender generates a sequence of (x_2, w_2) and runs M on input the sequence of OR statements $(x_1 \vee x_2)$ and the sequence of witnesses w_2 to generate a sequence of images, computes the hard-core of the corresponding pre-images and XOR the plaintext bit with the hardcore; to decrypt, the receiver runs Find on input the ciphertext and the sequence of witnesses w_1, obtains the corresponding pre-images, and then computes the hardcore and gets the plaintext.
This step is presented in section 6.

1.3 A Wide Perspective on Reductions

As mentioned, the mostly common used security proof methods– black-box reduction (see [RTV04, BBF13] for refined treatments) and the known non-black-box reductions [Bar01, DGS09, BP15]– are universal reduction, where a single universal reduction algorithm works for all possible adversaries. Note that the description of an adversary that the reduction has access to probably is an obfuscated code. This causes a trouble for the reduction algorithm in cases where the *functionality* of the adversary is crucial for the reduction to go through (as showed in the example of simulation for the adversary in [CKPR01], and see also [DGL$^+$16]), since we cannot expect the efficient reduction algorithm to figure out the functionality from a given obfuscated code of an arbitrary adversary.

However, in almost all cases, the security proof only requires *arbitrary* reductions, which are allowed to depend not only on the code of the adversary, but also on any arbitrary "nice" properties of the adversary (if exist), such as functionality, good random tapes, etc. Furthermore, to show the mere existence of such an arbitrary reduction, we do not need to care about whether such properties can be efficiently extracted from the code of the adversary, but just assume that the reduction takes these properties as input. We refer to an arbitrary reduction as *individual* reduction, which is also called non-constructive reduction or non-uniform reduction in some previous work [BU08, CLMP13]. We stress that it is not always possible to turn an individual reductions into a universal reduction with a non-uniform advice because, in many cases, even if we can prove all possible adversaries share a certain property, this property may not have a short description. (This will be clear in the following example.)

Recall that, to complete a security proof, we have to show for *every* adversary there is an individual reduction. This would be impossible unless we can prove that all possible adversaries have certain properties *in common*. Indeed, we observe that a few exceptional individual reductions in complexity (e.g., [Adl78]) and hardness amplification (e.g., [GNW95, CHS05, HS11]) literature are based a property– the existence of "good" random tapes– shared by all possible adversaries. Let's take the reduction for BPP $\subseteq$ P/poly [Adl78] as an example. The first step of the proof of [Adl78] is to show a common property that every machine deciding a language $L \in$ BPP must have at least one good random tape on which this machine will make correct decisions on all instances of a given size. Using the mere existence of a good random tape, we can then simply hardwire this good random tape into the circuit family that decide the language L deterministically. This circuit family can be thought of as a reduction, which varies depending on the specific BPP machine since different machines may have different good random taps.

Besides the structure (success/failure) of the random tapes, there seems to be a more important structure of the adversaries, i.e., the structure of the adversary's computation, that would empower the individual reduction greatly. In cryptography, we actually already exploited structures of this type, such as the knowledge of exponent assumption and extractable one-way functions [Dam91, BCPR14], but most of them are viewed as just non-standard assumption. Our work seems to raise some hope that we may be able to prove highly non-trivial structures of the adversary's computation in some settings under standard assumptions in the future.

2 Preliminaries

A function $negl(n)$ is called negligible if it vanishes faster than any inverse polynomial.

If D is a distribution (or random variable), we denote by $x \leftarrow D$ the process of sampling x according to D, and by $\{x_i\}_{i=1}^k \leftarrow D^{\bigotimes k}$ the process of sampling k times x from D independently. Similarly, for a function $f : \{0,1\}^n \rightarrow \{0,1\}^{\ell(n)}$, $f^{\bigotimes k}$ denotes the function that maps $(x_1, x_2, ..., x_k)$ to $(f(x_1), f(x_2), ..., f(x_k))$. We abbreviate probabilistic polynomial-time with PPT.

Given a two-party protocol $\Pi = (P_1, P_2)$, for $i \in \{1,2\}$, we denote by $\mathsf{Trans}_{P_i}(P_1(x), P_2(y))$ the transcript of an execution of Π including the input to P_i (i.e., the view of P_i) when P_1's input is x and P_2's input is y. For a joint distribution (X, Y) over the two parties' inputs, $\mathsf{Trans}_{P_i}(P_1(X), P_2(Y))$ naturally defines the distribution over all possible view of P_i.

We refer readers to [Gol01, KL07] for formal definitions of basic notions and primitives such as computational indistinguishability, one-way function, pseudorandom generator and commitment.

Throughout the paper, we let n be the security parameter. We write $\{X_n\}_{n\in\mathbb{N}} \overset{c}{\approx} \{Y_n\}_{n\in\mathbb{N}}$ to indicate that the two distribution ensembles $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ are computationally distinguishable.

Arguments, WI and Distributional CZK

Fix an NP language L and its associated relation R_L. An interactive argument system (P, V) for L is a pair of interactive Turing machines, in which the prover P wants to convince the verifier V of some statement $x \in L$.

Definition 1 (Interactive Argument [BCC88]). *A pair of interactive Turing machines (P, V) is called an* interactive argument system for language L *if the machine V is a PPT machine and the following conditions hold:*

- *Completeness: For every $x \in L$, $w \in R_L(x)$, V accepts the transcripts at the end of interaction with $P(x, w)$ with probability negligibly close to 1.*
- *Soundness: For every $x \notin L$, and every PPT prover P^*, V rejects at the end of interaction with P^* with probability negligibly close to 1.*

Definition 2 (Witness Indistinguishability). *An interactive argument (P, V) for language L is said to be witness indistinguishable (WI) if for every PPT V^*, every auxiliary input $z \in \{0,1\}^*$ to V, every $\{(x, w_0, w_1)\}_{x\in L}$ such that both (x, w_0) and $(x, w_1) \in R_L$, it holds that*

$$\{\mathsf{Trans}_{V^*}(P(x, w_0), V^*(z))\}_{x\in L, z\in\{0,1\}^*} \overset{c}{\approx} \{\mathsf{Trans}_{V^*}(P(x, w_1), V^*(z))\}_{x\in L, z\in\{0,1\}^*}$$

where both distributions are over the random tapes of P and V^*.

A zero knowledge argument system is an interactive argument for which the view of the (even malicious) verifier in an interaction can be efficiently reconstructed. In this paper, we consider *distributional* zero knowledge, defined by Goldreich [Gol93], for which the indistinguishability between the real interaction and the simulation is only required to hold for any distribution over the inputs to each party, rather than to hold for every individual inputs. We follow the definition of [CLP15b], which departs from the one of [Gol93] in that it only requires that for each distribution over the inputs there exists an efficient simulator[6], and furthermore, we only consider the case (following [DNRS03, CLP15b]) where the indistinguishability gap between the simulation and the real interaction is less than any inverse polynomial ϵ (instead of a negligible function). As we will show, the size of encryption algorithm of our encryption scheme is polynomial in the value $\frac{1}{\epsilon}$, which needs to be upper-bounded by a fixed polynomial.

Steps of the concurrent verifier and steps of a session. We also allow the adversary V^* to launch a *concurrent* attack [DNS98, PRS02] in which it interacts with a polynomial number of independent provers over an asynchronous network, and fully controls over the scheduling of all messages in these interactions.

We refer to the action of sending a message by V^* as a step (of V^*). In a real concurrent interaction, we order the steps of V^* according to their appearance. Note that in the concurrent setting, sessions of the Feige-Shamir protocol are executed in interleaving way, and thus, "the second verifier step of a session" refers to the second verifier step that appears in this specific session, not to the second step of V^* in the real concurrent interaction.

Definition 3 (ϵ-Distributional Concurrent zero knowledge). *We say that an interactive argument (P, V) for language L is ϵ-distributional concurrent zero knowledge if for every concurrent adversary V^*, and every distribution ensemble $\{(X_n, W_n, Z_n)\}_{n\in\mathbb{N}}$ over $R_L^n \times \{0,1\}^*$ (where*

[6] Instead, the definition of [Gol93] requires an efficient simulator for all distributions over the inputs.

$R_L^n = \{(x,w) \in R_L : |x| = n\}$), there exists a PPT Sim *such that for all PPT* D *and sufficient large n it holds that*

$$\Pr[\mathsf{D}(\mathsf{Trans}_{V^*}(P(X_n, W_n), V^*(Z_n))) = 1] - \Pr[\mathsf{D}(\mathsf{Sim}(V^*, X_n, Z_n)) = 1] < \epsilon$$

where both distributions are over (X_n, W_n, Z_n) and the random tapes of P and V^*.

Parallelized Blum's WI Proofs for NP Based on Injective One-Way Functions

The basic building block of the Feige-Shamir protocols is witness indistinguishable proofs. For our purpose, we will use the parallelized 3-round Blum's proof system based on injective one-way functions [Blu86][7].

Denote by (a, e, t) the three messages exchanged by the prover and the verifier in a execution of the n-parallel-repetition of the 3-round Blum's protocol. Our results rely on the following nice properties of this protocol:

- Witness indistinguishability when the common input x has two different witnesses;
- Special soundness: the soundness error is $\frac{1}{2^n}$, and from any common input x and any pair of accepting transcripts (a, e, t) and (a, e', t') with the same first message a but different challenges $e \neq e'$, one can efficiently compute w such that $(x, w) \in R_L$.

The Feige-Shamir ZK Argument for NP Based on Injective One-Way Functions

We here describe the Feige-Shamir constant-round[8] zero knowledge argument for NP based on an injective one-way function $f : \{0,1\}^n \to \{0,1\}^{\ell(n)}$.

PROTOCOL FEIGE-SHAMIR

Common input: $x \in L$.
The prover P's input: w such that $(x, w) \in R_L$.
The verifier V's (auxiliary) input: z

First phase:
Execute the n-parallel-repetition of the 3-round Blum's protocol in which V plays the role of the prover:

$V \longrightarrow P$**:** Choose $\alpha_1, \alpha_2 \leftarrow \{0,1\}^n$ independently and at random, compute $\beta_1 = f(\alpha_1)$, $\beta_2 = f(\alpha_2)$, and compute the first prover message a of the 3-round n-parallel-repetition of the Blum's protocol in which V proves to P that he knows one of α_1, α_2.
Send β_1, β_2 and a.
$P \longrightarrow V$**:** Send a random challenge $e \leftarrow \{0,1\}^n$.
$V \longrightarrow P$**:** Send t.

Second phase:
P and V execute the n-parallel-repetition of the 3-round Blum's protocol in which P proves to V that either $x \in L$ or he knows one of α_1, α_2.

[7] Note that perfect binding commitment scheme can be constructed from injective one-way function.
[8] By merging the first and the second prover messages, one can obtain a 4-round Feige-Shamir protocol.

3 The Dissection of a Concurrent Verifier

In this section we develop a technique to dissect a concurrent verifier, which enables us to prove a lemma on the consequence of a supposed verifier V^* that magically breaks ϵ-distributional concurrent zero knowledge of the Feige-Shamir protocol. This is the key step towards proving our main result.

As mentioned in the introduction, this lemma asserts that a magic adversary V^* will endow the one-way function f with a trapdoor in the following sense: there are infinitely many n, for each of which there exists a step index i_n, such that the images (β_1, β_2) output by V^* at its step i_n can *only* be inverted by PPT algorithms with the trapdoor knowledge of a witness to the common input x with overwhelming probability.

We need the following notations to give a formal statement of our lemma:

- Trans^{i_n} and $h \leftarrow \mathsf{Trans}^{i_n}$: The former denotes the distribution of the history prefix in the view of V^* up to its i_n-th step in the real concurrent interaction $\mathsf{Trans}_{V^*}(P(X_n, W_n), V^*(Z_n))$; the latter denotes the event of drawing a history prefix h from Trans^{i_n}, i.e., the event of generating h in the real concurrent interaction between honest prover(s) and V^*, where h consists of the statement x, the auxiliary input z to V^* and the interaction history prefix upto the step i_n of the verifier.
- $V^* |_h \rightsquigarrow (j, 2)$ denotes the event that, conditioned on the given history prefix h, V^* reaches the second verifier step of session j in the real concurrent interaction, i.e., V^* completes its proof of knowledge of one pre-image in session j.
- PartR_h consists of the randomness used by V^* and the *partial* randomness used by honest provers in those *incomplete* sessions in h (i.e., sessions in which the last prover message does not appear in h) in a real concurrent interaction.
  Observe that in a session of the Feige-Shamir protocol, the honest prover uses the knowledge of corresponding witness w *only* in its last step, and the transcript of a session before the prover last step is independent of w. Thus, the transcript of an *incomplete* session together with the prover's randomness used do not help reveal the witness w, but this is not the case for a *complete* session.

In the real concurrent interaction, given a history prefix h up to the i_n-th step of V^*, we denote by $h = h'||(\beta_1^j, \beta_2^j, a^j)$ the event that V^* outputs the first verifier message $(\beta_1^j, \beta_2^j, a^j)$ of some session j at its i_n-th step, where "$||$" denotes concatenation of messages.

Let ϵ be an arbitrary inverse polynomial, and $\mathrm{poly}(\cdot)$ be an arbitrary polynomial. Define

$$p(\cdot) := \frac{\epsilon(\cdot)}{2\mathrm{poly}^2(\cdot)}.$$

Lemma 1. *Let ϵ, p,* poly *be as above, and f be the one-way function used in the Feige-Shamir protocol. Assume that there is a PPT verifier V^*, running in at most* $\mathrm{poly}(n)$ *steps, that breaks ϵ-distributional concurrent zero knowledge of the Feige-Shamir protocol on a joint distribution ensemble $\{(X_n, W_n, Z_n)\}_{n \in N}$ with respect to a NP language L. Then, there exists an infinite set $I = \{(n, i_n)\}$ for which the following two conditions simultaneously hold:*

1. *For a random history prefix generated in the real concurrent interaction,*

$$\Pr\left[h \leftarrow \mathsf{Trans}^{i_n} : \begin{array}{c} h = h'||(\beta_1^j, \beta_2^j, a^j) \wedge \\ \Pr[V^* |_h \rightsquigarrow (j, 2)] \geq p(n) \end{array}\right] \geq p(n).$$

2. *For every (non-uniform) PPT T, there is N_0 such that for every $n > N_0$ (s.t. $(n, \cdot) \in I$) it holds that,*

$$\Pr\left[T(h, \mathrm{PartR}_h) \in \{f^{-1}(\beta_1^j), f^{-1}(\beta_2^j)\} \middle| \begin{array}{c} h'||(\beta_1^j, \beta_2^j, a^j) = h \leftarrow \mathsf{Trans}^{i_n} \\ \wedge \ \Pr[V^* |_h \rightsquigarrow (j, 2)] \geq p(n) \end{array}\right] \leq 1 - p(n).$$

Remark 1. Note that if, conditioned on outputting the first verifier message $(\beta_1^j, \beta_2^j, a^j)$ of session j at its i_n-th step, V^* reaches the second verifier step of session j (i.e., completes the proof of knowledge of one pre-image) in the real concurrent interaction with probability greater than an inverse polynomial, we can construct an efficient algorithm, taking the corresponding witness w as input and playing the role of the honest prover, that extracts one of pre-images of (β_1^j, β_2^j) from V^* by rewinding it with probability negligibly close to 1. The first condition of our lemma asserts that it is relatively easy to obtain images of f for which there is an efficient algorithm with knowledge of w can invert one of them with overwhelming probability, while the second condition of the above lemma guarantees that for any efficient algorithm without knowledge of w the success probability of inversion is bounded away from 1. This illustrates the magic power that the supposed adversary V^* endows f with a sort of trapdoor.

As we shall see later, in the final construction of public key encryption, the partial randomness PartR_h together with some images of f will be part of cipher-text, and to ensure the CPA security it is naturally required that for any efficient algorithm with PartR_h as input the success probability of inversion the images of f in the challenge cipher-text is small. This is guaranteed by the second condition of the above lemma.

Remark 2. (On the role of the value ϵ) The main reason we deal only with ϵ-distributional concurrent zero knowledge, rather than the standard one, is that, as we will see later, our approach will yield encryption algorithm of the size $poly(\frac{1}{\epsilon})$, and thus the value $\frac{1}{\epsilon}$ has to be upper-bounded by a fixed but arbitrarily large polynomial, since otherwise the size of our encryption algorithm cannot be bounded by any polynomial.

3.1 The Dissection Procedure Leading to a Proof of Lemma 1

Formally, if for an arbitrary inverse polynomial ϵ, V^* breaks ϵ-distributional concurrent zero knowledge of Feige-Shamir protocol over distribution $\{(X_n, W_n, Z_n)\}_{n\in\mathbb{N}}$, then $\forall$ PPT Sim $\exists$ PPT D and infinitely many n, such that

$$\Pr[\mathsf{D}(\mathsf{Trans}_{V^*}(P(X_n, W_n), V^*(Z_n))) = 1] - \Pr[\mathsf{D}(\mathsf{Sim}(V^*, X_n, Z_n)) = 1] > \epsilon. \tag{1}$$

As mentioned, the intuition behind Lemma 1 is quite straightforward: For a successful V^*, there must exist a step i at which V^* outputs a pair of images and will complete the proof of knowledge of one pre-image at a later time in the real concurrent interaction with high probability, but without knowledge of the corresponding witness no efficient algorithm can invert one of the images, since otherwise, if for every step of V^* there is an efficient algorithm that can extract the target pre-images with overwhelming probability, we are able to show that there *exists* a simulator, incorporating all these efficient inverting algorithms as its subroutines, that will simulate the view of V^* successfully.

To formalize this intuition in the asymptotic setting, we view the behaviour of V^* as an infinite table, in which the entry in the i-th row and n-th column represents the i-th step of V^* (followed immediately by the response from the honest prover) in its concurrent interaction on input the security parameter n (cf. Fig 1).

With this table, we dissect V^* and examine its every step *across all security parameters* $n \in \mathbb{N}$, i.e., examine the set of entries $\{(n, i_n = i)\}_{n\in\mathbb{N}}$. A few terminologies follow.

Imaginary steps. Note that for the i-th row of the table (i.e., V^*'s step i), if a security parameter n satisfies $\text{poly}(n) < i$, V^* on the input security parameter n will never reach step i. To simplify the presentation, we think of the step i for each n s.t. $\text{poly}(n) < i$ as an *imaginary step* of V^* with

$$\Pr\left[h \leftarrow \mathsf{Trans}^i : \begin{array}{c} h = h' || (\beta_1^j, \beta_2^j, a^j) \wedge \\ \Pr[V^* \mid_h \rightsquigarrow (j, 2)] \geq p(n) \end{array}\right] = 0.$$

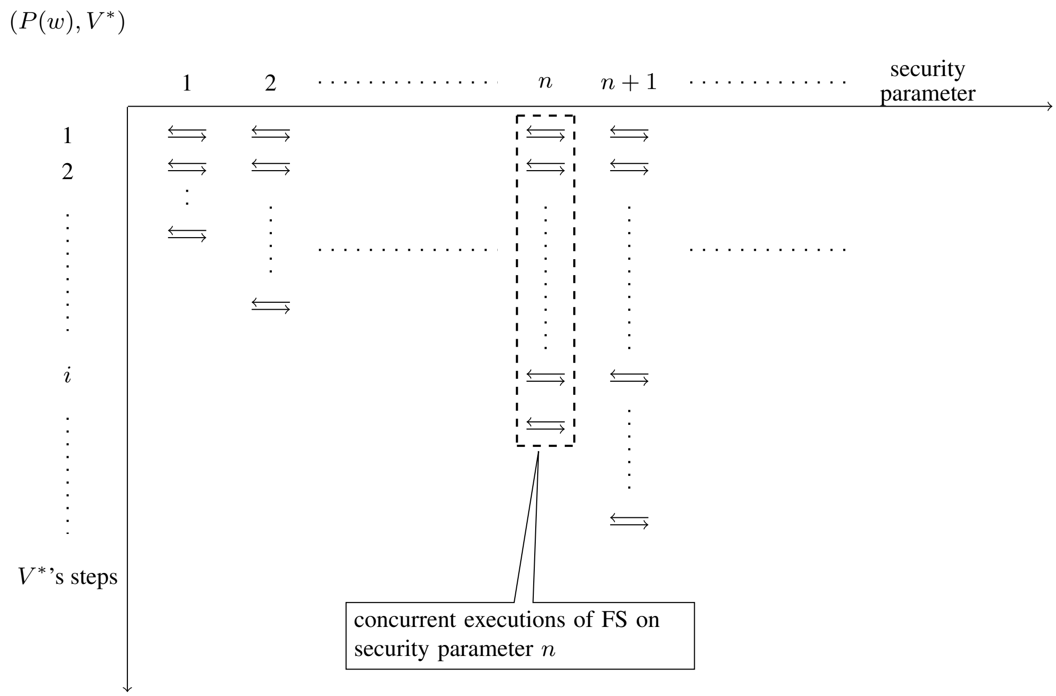


Fig. 1: V^*'s behaviour.

Significant/insignificant entries with respect to p. Given a (possibly infinite) set K of security parameters, and a set $K' = \{(n, i_n)\}_{n \in K}$, we say the entry $(n, i_n) \in K'$ is *significant* with respect to p if for which the first condition of Lemma 1 holds, i.e.,

$$\Pr\left[h \leftarrow \mathsf{Trans}^{i_n} : \begin{array}{c} h = h' || (\beta_1^j, \beta_2^j, a^j) \wedge \\ \Pr[V^* \mid_h \rightsquigarrow (j, 2)] \geq p(n) \end{array}\right] > p(n),$$

Otherwise, we call it *insignificant*.

Throughout this paper, all significant entries are significant with respect to the fixed probability p defined in Lemma 1.

Solving a set of entries with respect to $(p, \mathbb{P})$. Given a set K of security parameters, and a set (finite or infinite) $K' = \{(n, i_n)\}_{n \in K}$, we say a PPT T solves the set K' with respect to $(p, \mathbb{P})$, if for every *significant* entry $(n, i_n) \in K'$ with respect to p, T, with running time bounded by $\mathbb{P}$, breaks the second condition of Lemma 1 on (n, i_n), i.e., for all $n \in K$,

$$\Pr\left[T(h, \mathrm{PartR}_h) \in \{f^{-1}(\beta_1^j), f^{-1}(\beta_2^j)\} \middle| \begin{array}{c} h' || (\beta_1^j, \beta_2^j, a^j) = h \leftarrow \mathsf{Trans}^{i_n} \\ \wedge \Pr[V^* \mid_h \rightsquigarrow (j, 2)] \geq p(n) \end{array}\right] > 1 - p(n). \quad (2)$$

otherwise, we say T cannot solve the set $(n, i_n) \in K'$ with respect to $(p, \mathbb{P})$. Note that we don't make any requirement on T for those *insignificant* entries $(n, i_n) \in K'$ (i.e., those entries for which the first condition of Lemma 1 does not hold). To take an extreme example, if for *all* $(n, i_n) \in K'$ the first condition of Lemma 1 fails to hold, i.e.,

$$\Pr\left[h \leftarrow \mathsf{Trans}^{i_n} : \begin{array}{c} h = h' || (\beta_1^j, \beta_2^j, a^j) \wedge \\ \Pr[V^* \mid_h \rightsquigarrow (j, 2)] \geq p(n) \end{array}\right] < p(n),$$

then, by definition, any PPT algorithm with running time bounded by $\mathbb{P}$ can solve the set K' with respect to $(p, \mathbb{P})$. For simplicity, we let the algorithm that solves such a set K' to be a special dummy machine denoted by ϕ, which runs in time 0.

When the context is clear, we often simply say a PPT T with running time bounded by a polynomial $\mathbb{P}$ cannot solve *any* entry (n, i_n) in the set (finite or infinite) K' if every entry in K' is significant (with respect to p) and no PPT algorithm can solve even a single entry in K' with respect to $(p, \mathbb{P})$ (i.e., any PPT T with running time bounded by $\mathbb{P}$ does not make the inequality (2) hold for even a single entry in K').

With these definitions, we observe the following fact.

Fact 1. Fix a verifier step i and let p be as defined above. If for any polynomial $\mathbb{P}$, there does not exist an algorithm that solves the set $\{(n, i_n = i)\}_{n\in\mathbb{N}}$ with respect to $(p, \mathbb{P})$, then there is an infinite set I on which both conditions of Lemma 1 hold.

Proof. Observe first that if for any polynomial $\mathbb{P}$, there is no algorithm that solves the set $\{(n, i)\}_{n\in\mathbb{N}}$ with respect to $(p, \mathbb{P})$, then for any *finite* set K of security parameters, the same holds for the set $\{(n, i)\}_{n\in\mathbb{N}\setminus K}$. To see this, suppose for the sake of contradiction that, for some finite set K, there are a polynomial $\mathbb{P}$ and a PPT T such that T solves the set $\{(n, i_n = i)\}_{n\in\mathbb{N}\setminus K}$ with respect to $(p, \mathbb{P})$. Let c_k be the largest security parameter in K, and T' be the inverting algorithm that, upon receiving a pair of images, inverts one of them by exhausting all possible pre-images. Note that the running time of T' is bounded by a *constant* 2^{c_k}. We now have a non-uniform PPT algorithm, denoted by T_i, which applies T on the security parameters $n \in \mathbb{N} \setminus K$ and T' on $n \in K$, can solve the set $\{(n, i)\}_{n\in\mathbb{N}}$ with respect to $(p, \mathbb{P} + 2^{c_k})$, which contradicts the hypothesis of this fact since $\mathbb{P}(n) + 2^{c_k}$ is still a polynomial in n.

With this observation, we now have that, for every polynomial (monomial) n^c, $c \in \mathbb{N}$, there is an infinite set K_c of security parameters such that no algorithm can solve *any* entry (n, i) in the infinite set $\{(n, i)\}_{n\in K_c}$ with respect to (p, n^c). This is because that if an algorithm with running time bounded by n^c can solve all but a finite entries (n, i) in the i-th row, then, using the above reasoning, we will have a non-uniform PPT algorithm, with running time bounded by n^c plus some constant C, that can solve all entries in the i-th row $\{(n, i)\}_{n\in\mathbb{N}}$ with respect to $(p, n^c + \mathrm{C})$, which contradicts the hypothesis of this fact again.

Note that $K_c \subseteq K_{c-1}$ for all $c \in \mathbb{N}$. The desired infinite set I can be constructed as follows. Let $n_0 = 0$ and n_c be $\min\{K_c \setminus n_{c-1}\}$[9] for each $c \in \mathbb{N}$. We define I to be

$$I := \{(n_c, i)\}_{c\in\mathbb{N}}.$$

It is easy to verify that the first condition of Lemma 1 holds.[10] Consider an arbitrary PPT algorithm T that runs in time bounded by an arbitrary polynomial $\mathbb{P}^\dagger$, and suppose that $\mathbb{P}^\dagger(n) \leq n^{c'}$. Then T cannot solve *any* entry $(n_c, i) \in I$ (i.e., does not make the inequality (2) hold) for any $c > c'$. With the observation that $c > c'$ implies $n_c > n_{c'}$, we have that T cannot solve any entry $(n_c, i) \in I$ for any $n_c > n_{c'}$. This establishes the second condition of Lemma 1.

□

The following dissection procedure (cf. Fig 2) will yield an infinite set I as desired.

The dissection procedure.

Initially set $I_0 := \{(n_0 = 0, i_{n_0} = 0)\}$, $S_0 := \{(T_0 = \phi, \mathbb{P}_0 = 0)\}$.

[9] Note that in case K_c is identical to K_{c-1}, then $n_{c-1} \in K_c$.

[10] Note that for every $c \in \mathbb{N}$, for any entry (n, i) in $\{(n, i)\}_{n\in K_c}$, the first condition of Lemma 1 holds for (n, i), since otherwise the entry (n, i) is insignificant and therefore can be solved by any PPT algorithm with running time bounded by n^c.

For $i = 1, 2, ...$, given $I_{i-1} = \{(n_0, i_{n_0}), ..., (n_{k-1}, i_{n_{k-1}})\}$[11], $S_{i-1} = \{(T_0, \mathbb{P}_0), ..., (T_{i-1}, \mathbb{P}_{i-1})\}$ and $\mathbb{P} = \max\{\mathbb{P}_0, \mathbb{P}_1, ..., \mathbb{P}_{i-1}\}$, we check the i-th step of V^* for all $n \in \mathbb{N}$ and do the following:

1. If for any polynomial $\mathbb{P}'$ there is no PPT algorithm that solves the set $\{(n, i_n = i)\}_{n\in\mathbb{N}}$ with respect to $(p, \mathbb{P}')$, let I be as defined in the above Fact 1, and stop this process;
2. If there is PPT T_i that solves the set $\{(n, i_n = i)\}_{n\in\mathbb{N}}$ with respect to $(p, \mathbb{P})$, suppose that the running time of T_i is $\mathbb{P}_i \leq \mathbb{P}$, set $S_i \leftarrow S_{i-1} \cup (T_i, \mathbb{P}_i)$, and $I_i \leftarrow I_{i-1}$ (Note that we do not update the set I_{i-1});
3. If there is PPT T_i that solves the set $\{(n, i_n = i)\}_{n\in\mathbb{N}}$ with respect to $(p, \mathbb{P}_i)$ for some $\mathbb{P}_i > \mathbb{P}$, but no such PPT algorithm that solves the set $\{(n, i_n = i)\}_{n\in\mathbb{N}}$ for all $n \in \mathbb{N}$ with respect to $(p, \mathbb{P})$, then
   (a) set $S_i \leftarrow S_{i-1} \cup \{(T_i, \mathbb{P}_i)\}$, and,
   (b) if $i >$ poly(n_{k-1})[12], find a $n_k > n_{k-1}$ on which the first condition of Lemma 1 holds, but there is no PPT algorithm that solves the set $I_{i-1} \cup \{(n_k, i_{n_k} = i)\}$ with respect to $(p, \mathbb{P})$[13]. Set $I_i \leftarrow I_{i-1} \cup \{(n_k, i_{n_k} = i)\}$.

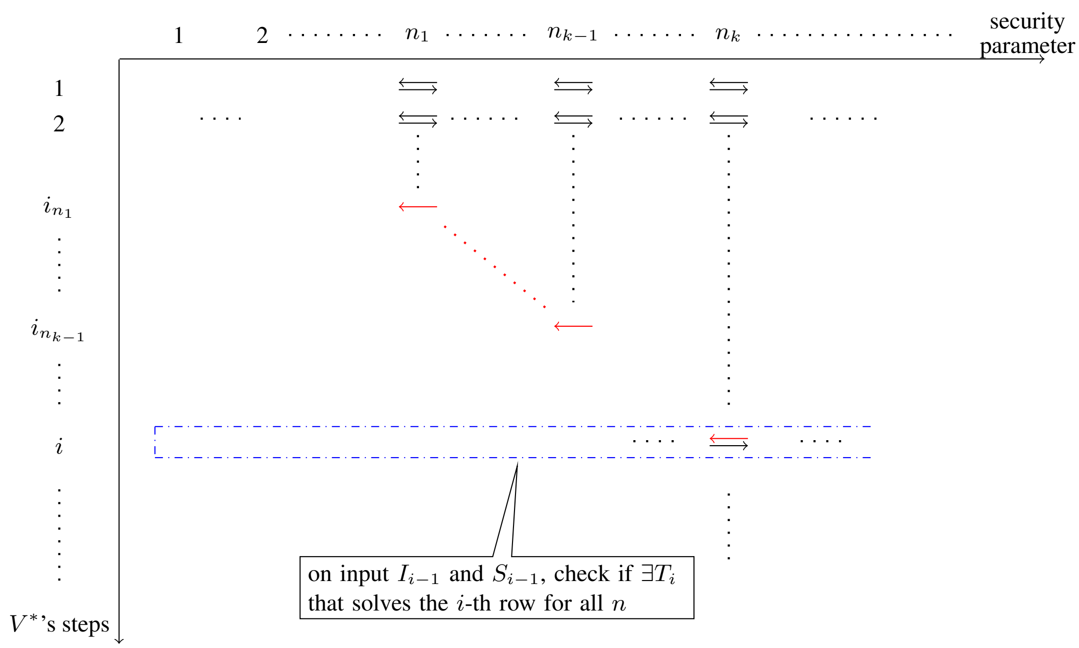


Fig. 2: The dissection procedure. For a magic adversary V^* there must exist either a single row (a step of V^*) from which we find the desired infinite set I, or infinite many rows from each of which we add a new entry to the set I.

Denote by I the set resulted from the above dissection procedure, which is either of the form $\{(n_c, i)\}_{c\in\mathbb{N}}$ (when we encounter the first case during the dissection procedure), or of the form

[11] Here $k \leq i - 1$. Note that we may not update the set I at each step i.

[12] This means that the current i-step is an imaginary step of V^* for those $n \leq n_{k-1}$.

[13] As will be showed in proof of claim 1 in the next section, we can always find such a n_k.

$\{(n_k, i_{n_k})\}$ (otherwise). Lemma 1 follows from the following two claims that we will prove in the next section.

Claim 1. If we encounter the first case during the above dissection, or there is no polynomial $\mathbb{P}$ s.t. $\mathbb{P} = \sup\{\mathbb{P}_i : i \in \mathbb{N}\}$, i.e., there is no polynomial upper-bound on the infinite set $\{\mathbb{P}_i : i \in \mathbb{N}\}$, then the set I is infinite and on which both conditions of Lemma 1 hold.

Claim 2. If we will never encounter the first case during the above dissection, and there is a polynomial $\mathbb{P}$ s.t. $\mathbb{P} = \sup\{\mathbb{P}_i : i \in \mathbb{N}\}$, then there is a PPT simulator that breaks the inequality (1).

Remark 3. (On the mere existence of T_i) Note that at each step of the dissection procedure we only ask if there *exists* a good extractor T_i, and that these algorithms may depend on specific verifier. It may be the case that these T_i exist but we cannot construct them from the code V^* efficiently, as we showed for the concrete adversary from [CKPR01].

However, as we will prove in the next section, the mere existence of good extractors T_i helps us show the *existence* of a simulator for V^* under the security definition of "$\forall V^* \exists S$" (see next section for a proof).

Remark 4. (On the dependence between T_i's) We stress that the dependence between the possible algorithms T_i's is irrelevant here. Note that at each step i, we set a clear bar $(p, \mathbb{P})$ and check if there exists an algorithm T_i with running time less than $\mathbb{P}$ that can solve the i-th row for all those significant entries in this row with respect to p. If there exists a PPT T_i that solves this row but runs in time $\mathbb{P}_i > \mathbb{P}$, we record this new $\mathbb{P}_i$ and when we enter the next step $(i+1)$, we have a higher bar on the running time for checking the existence of T_{i+1}.

Nevertheless, if one can construct a verifier V^* for which there is a deep dependence between these T_i's such that, say, the running time of T_{i-1} is twice that of T_i for many i, then we will soon find a desired set I as required by Lemma 1.

3.2 Proofs of Claim 1

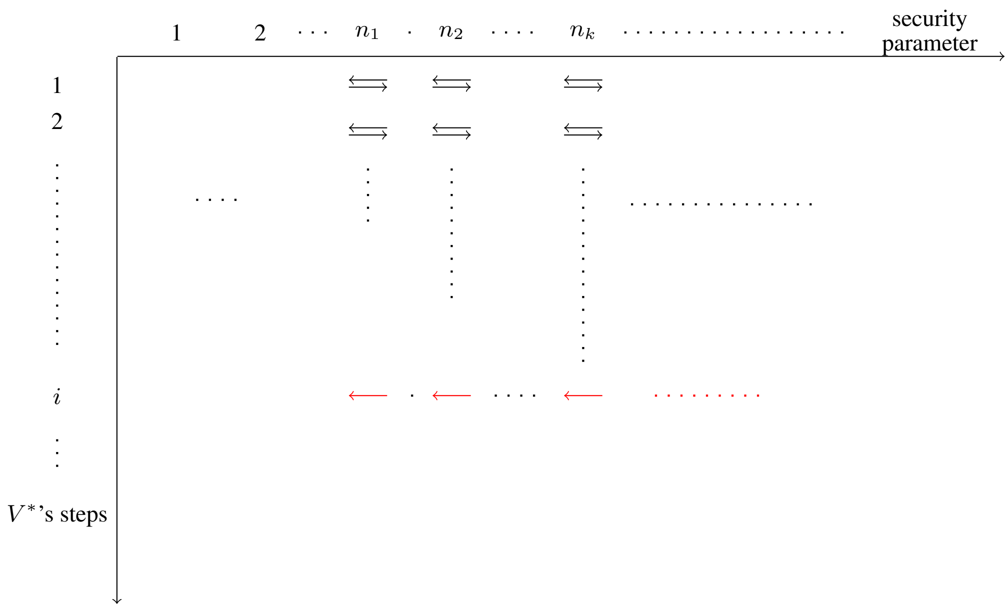
As showed in Fact 1, if we encounter the first case when checking step i of V^* (for all $n \in \mathbb{N}$), there must be an infinite set $I = \{(n, i_n = i)\}$ on which both conditions of Lemma 1 hold (cf. Fig 3(a)).

In the case that we will never encounter the first case in the dissection procedure but there is no specific polynomial that upper bounds the infinite set $\{\mathbb{P}_i\}_{i\in\mathbb{N}}$, we need to prove the following to complete the proof of Claim 1 (cf. Fig 3(b)):

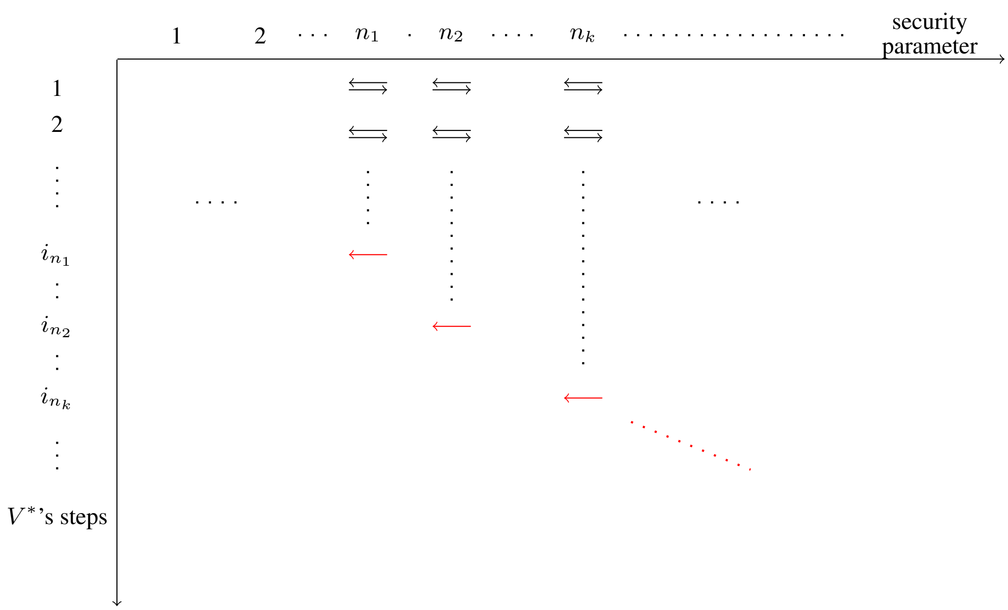
1. As i approaches infinity, the resulting set $\{(n_k, i_{n_k})\}$, denoted by $I_{i\to\infty}$, becomes infinite;
2. Both conditions of Lemma 1 hold on $I_{i\to\infty}$.

For the item 1, note that, for any $(n_{k-1}, i_{n_{k-1}}) \in I_{i\to\infty}$, there must be a step i of V^*, $i > \text{poly}(n_{k-1})$, such that the minimum running time $\mathbb{P}_i$ for a PPT algorithm to solve the set $\{(n, i_n = i)\}_{n\in\mathbb{N}}$ is strictly greater than $\mathbb{P} = max\{\mathbb{P}_1, \mathbb{P}_2 ..., \mathbb{P}_{i-1}\}$ (since otherwise we will have a specific polynomial upper bound on all $\{\mathbb{P}_i\}_{i\in\mathbb{N}}$). From such a step i, we can always find a $n_k > n_{k-1}$ on which the first condition of Lemma 1 holds, but there is no PPT T, running in time $\leq \mathbb{P}$, that solves the entry $(n_k, i_{n_k} = i)$, since otherwise, if for every $n > n_{k-1}$, there exists an algorithm T^n with running time less than $\mathbb{P}$ that solves the entry $(n, i_n = i)$, then we have a non-uniform algorithm T_i, by applying T^n to the entry $(n, i_n = i)$ for every $n > n_{k-1}$[14], solves the set $\{(n, i_n = i)\}_{n>n_{k-1}}$ with respect to $(p, \mathbb{P})$. Note also that for all $n \leq n_{k-1}$, the step i of V^* is an imaginary step and thus T_i automatically solves the set $\{(n, i_n = i)\}_{n\leq n_{k-1}}$, we conclude T_i solves the set $\{(n, i_n = i)\}_{n\in\mathbb{N}}$ with respect to $(p, \mathbb{P})$, a contradiction.

[14] One can think of T_i as a family of circuits $\{T^n\}_{n>n_{k-1}}$.



(a)



(b)

Fig. 3: There are infinite red entries in the set I on which both conditions of Lemma 1 hold: When encountering the first case during the dissection of V^*, we have a desired set I which lies in a single row, as depicted in figure (a); otherwise, we will have a desired set I of the form depicted in figure (b).

Note first that the step 3(b) of the dissection procedure guarantees that the first condition of Lemma 1 holds on the all entries in the infinite set $I_{i\to\infty}$. We now prove that the second condition of Lemma 1 also holds on $I_{i\to\infty}$. Consider an arbitrary PPT algorithm T with running time bounded by an arbitrary polynomial $\mathbb{P}^{\dagger}$. Observe that, by the hypothesis of Claim 1, there is a step i such that the running time $\mathbb{P}_i$ of T_i is strictly greater than $\mathbb{P}^{\dagger}$, and thus the PPT algorithm T with running time bounded by $\mathbb{P}^{\dagger}$ cannot solve any entry in the infinite set $\{(n_k, i_{n_k})\}_{i_{n_k}>i} \subset I_{i\to\infty}$, i.e., the set updated after the examining of the step i of V^*, since for every entry $(n_k, i_{n_k}) \in I_{i\to\infty}$, if $i_{n_k} > i$, then the minimal running time for solving the entry (n_k, i_{n_k}) is strictly greater than $\mathbb{P}_i(n_k) > \mathbb{P}^{\dagger}(n_k)$. Observe that $i_{n_k} > i_{n'_k}$ implies $n_k > n'_k$, therefore we conclude that, for a PPT T with running time bounded by an arbitrary polynomial $\mathbb{P}^{\dagger}$, there is some $N_0 = n'_k \in \mathbb{N}$ (which depends on $\mathbb{P}^{\dagger}$) such that the PPT algorithm T cannot solve any entry in the infinite set $\{(n_k, i_{n_k})\}_{n_k>n'_k}$. Thus the second condition of Lemma 1 holds on $I_{i\to\infty}$.

3.3 Proof of Claim 2

We now turn to the proof of Claim 2.

From the "if condition" of Claim 2 it follows that there exists a set of algorithms $\{T_i\}_{i\in\mathbb{N}}$ such that each T_i solves the i-th step of V^* for all $n \in \mathbb{N}$, i.e., the set $\{(n, i_n = i)\}_{n\in\mathbb{N}}$, with respect to $(p, \mathbb{P})$ (cf. Fig 4).

Fix an arbitrary security parameter $n \in \mathbb{N}$. We show a simulator Sim that breaks the inequality (1). Sim, taking the collection of algorithms $(\{T_i\}_{1\le i\le \mathrm{poly}(n)})$ as input (recall that V^* runs in at most $\mathrm{poly}(n)$ steps), runs in time at most $\mathrm{poly}(n)\mathbb{P}(n)$.

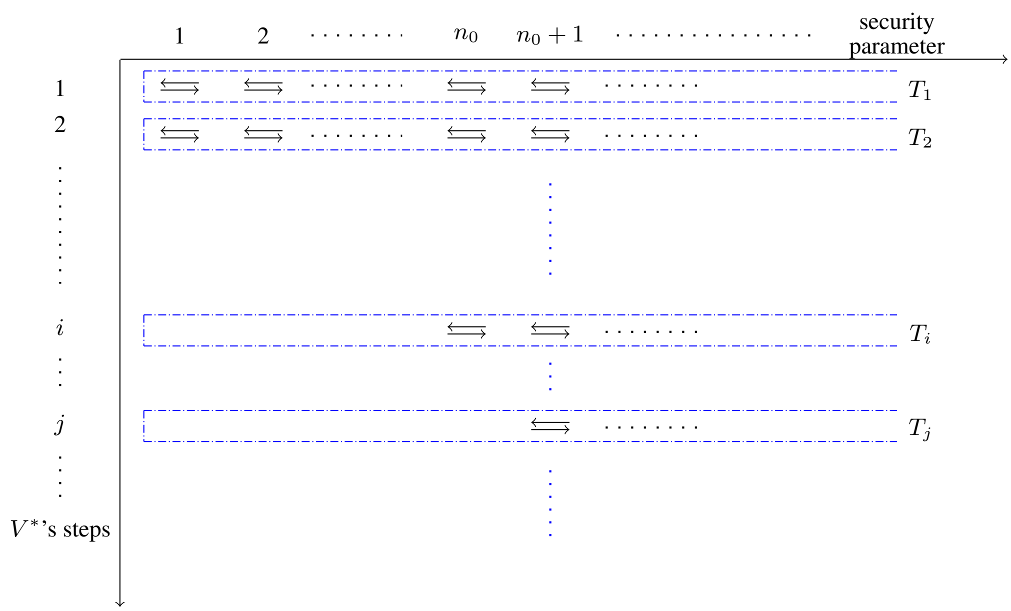


Fig. 4: If lemma 1 does not hold, then for each i, there is an algorithm T_i that solves i's step for all $n \in \mathbb{N}$ and runs in time less than a priori fixed polynomial, which leads to a good simulator.

In the real interaction, we denote by $V^*|_h^l \rightsquigarrow (j, 2)$ the event that V^*, based on the history prefix h, outputs the second verifier message of the session j at its l-th step, and by $\mathrm{Fail}_{real}^{(i,l)}$ the event that, The Simulator $\mathsf{Sim}(\{T_i\})$

input : $(x, z) \leftarrow (X_n, Z_n)$

1. If the next-scheduled-message is the first prover message in a new session, send a random string (as the honest prover) to V^* to set up a commitment scheme.
2. Upon receiving the first verifier message (β_1, β_2, a) in a session at the V^*'s i-th step, apply T_i to find one of pre-images of (β_1, β_2). If T_i succeeds, store it on a table $\mathcal{L}$ (indicating this session is solved), and send a random challenge e to V^*; if not, just send e to V^*.
3. If the next-scheduled-message is the third prover message in a session (i.e., entering the second phase in which the simulator plays the role of prover), check $\mathcal{L}$ if this session is already solved, if so, use the pre-image as a fake witness to carry out this session; if not (i.e., the simulator gets stuck), return $\perp$.

output: When V^* terminates, output (x, z) and the entire interaction.

conditioned on V^* outputting the first verifier message of a session at its i-th step, T_i, given the history prefix h up to the i-th step of V^* and PartR_h, fails to extract the corresponding pre-image but $V^* |_h^l \rightsquigarrow (j, 2)$.

By the inequality (2), and noting that the condition "$\Pr[V^* |_h \rightsquigarrow (j,2)] \geq p(n)$" implied by "$\Pr[V^* |_h^l \rightsquigarrow (j,2)] \geq p(n)$", we have[15]

$$
\begin{aligned}
&\Pr[\mathrm{Fail}_{real}^{(i,l)}] \\
=&\Pr\left[\mathrm{Fail}_{real}^{(i,l)} \middle| \begin{array}{l} h'||(\beta_1^j, \beta_2^j, a^j) = h \leftarrow \mathsf{Trans}^i \\ \wedge \ \Pr[V^* |_h^l \rightsquigarrow (j,2)] \geq p(n) \end{array}\right] \Pr\left[\begin{array}{l} h'||(\beta_1^j, \beta_2^j, a^j) = h \leftarrow \mathsf{Trans}^i \\ \wedge \ \Pr[V^* |_h^l \rightsquigarrow (j,2)] \geq p(n) \end{array}\right] \\
&+ \Pr\left[\mathrm{Fail}_{real}^{(i,l)} \middle| \begin{array}{l} h'||(\beta_1^j, \beta_2^j, a^j) = h \leftarrow \mathsf{Trans}^i \\ \wedge \ \Pr[V^* |_h^l \rightsquigarrow (j,2)] < p(n) \end{array}\right] \Pr\left[\begin{array}{l} h'||(\beta_1^j, \beta_2^j, a^j) = h \leftarrow \mathsf{Trans}^i \\ \wedge \ \Pr[V^* |_h^l \rightsquigarrow (j,2)] < p(n) \end{array}\right] \\
\leq& p \Pr\left[\begin{array}{l} h'||(\beta_1^j, \beta_2^j, a^j) = h \leftarrow \mathsf{Trans}^i \\ \wedge \ \Pr[V^* |_h \rightsquigarrow (j,2)] \geq p(n) \end{array}\right] \\
&+ p \Pr\left[\begin{array}{l} h'||(\beta_1^j, \beta_2^j, a^j) = h \leftarrow \mathsf{Trans}^i \\ \wedge \ \Pr[V^* |_h \rightsquigarrow (j,2)] < p(n) \end{array}\right] \\
\leq& p(n).
\end{aligned}
$$

In the simulation, for $1 \leq l \leq \mathrm{poly}(n)$, we denote by E_l be the event that Sim does not output $\perp$ upon receiving any message from V^* before the the step l of V^*, and define $\mathrm{Fail}_{sim}^{(i,l)}$ in a way similar to $\mathrm{Fail}_{real}^{(i,l)}$.

Note that for any $i \leq l$, conditioning on the event E_l, by standard hybrid argument (using the fact that witness indistinguishability preserves in concurrent setting), we have

$$\Pr[\mathrm{Fail}_{sim}^{(i,l)} | \mathrm{E}_l] \leq \Pr[\mathrm{Fail}_{real}^{(i,l)}] + negl(n) \leq p(n) + negl(n).$$

The probability that the simulator outputs $\perp$ upon receiving the l-th verifier message, denote by $\perp \leftarrow \mathsf{Sim}|_l$, is at most (note that $\perp \leftarrow \mathsf{Sim}|_l$ implies the event E_l)

[15] Observe that, conditioned on the probability that V^* reaches the second verifier step of session j is less than p, $\mathrm{Fail}_{real}^{(i,l)}$ happens with probability at most p.

$$\Pr[\bot\leftarrow \mathsf{Sim}|_l] = \sum_{i=1}^{l-1} \Pr[\mathrm{Fail}_{sim}^{(i,l)}|\mathrm{E}_l] \leq (l-1)p(n) + negl(n),$$

and thus the probability that the simulator outputs $\bot$ is at most

$$\Pr[\bot\leftarrow \mathsf{Sim}] = \sum_{l=1}^{\mathrm{poly}} \Pr[\bot\leftarrow \mathsf{Sim}|_l] \leq \mathrm{poly}^2(n)p(n) + negl(n).$$

Observe that, conditioning on not being $\bot$, by standard hybrid argument (using the same fact that witness indistinguishability preserves in concurrent setting) again, the output of Sim is indistinguishable from the real interaction, thus for all PPT D,

$$\begin{aligned}
&\Pr[\mathsf{D}(\mathsf{Trans}_{V^*}(P(X_n, W_n), V^*(Z_n))) = 1] - \Pr[\mathsf{D}(\mathsf{Sim}(V^*, X_n, Z_n)) = 1] \\
\leq &\Pr[\bot\leftarrow \mathsf{Sim}] + negl(n) \\
\leq &\mathrm{poly}^2(n)p(n) + negl(n) \\
\leq &\mathrm{poly}^2(n)\frac{\epsilon(n)}{2\mathrm{poly}^2(n)} + negl(n) \\
\leq &\epsilon(n),
\end{aligned}$$

which breaks the inequality (1) and thus concludes the proof of Claim 2.

4 Tuning in to the Same Channel

As showed in the previous section, the real concurrent interaction between the honest prover and a successful adversary V^* will magically generate a history prefix of the form $h'||(\beta_1, \beta_2, a)$ for which only algorithms with knowledge of the corresponding witness can extract one of the pre-images of (β_1, β_2) with overwhelming probability. However, different algorithms using different witnesses/randomness may recover different pre-images from this history. Thus, to exploit the power of V^* in our setting, we first need to make sure that all parties are in the same channel, i.e., recover the same pre-image.

In this section we construct non-interactive algorithms C and E from the magic adversary V^* such that, taking as input the witness to x, C generates a β and E can obtain the pre-image of the *same* β.

Lemma 2. *Let p, f, $\{(X_n, W_n, Z_n)\}_{n\in N}$, the infinite set I, and V^* be as in Lemma 1. Then, for every $(n, i_n) \in I$, there exist two PPT algorithms C and E such that the following conditions hold:*

1. *C generates β, α such that $\beta = f(\alpha)$ with probability*

$$\Pr[(x, w, z) \leftarrow (X_n, W_n, Z_n) : C(x, w, z) = (\beta, \alpha, aux)] \geq p^2 - negl(n)$$

.

2. *It is easy for E with knowledge of w to invert the image output by C with probability*

$$\Pr[(x,w,z)\leftarrow(X_n,W_n,Z_n):E(\beta,aux,w)=f^{-1}(\beta)|C(x,w,z)=(\beta,\alpha,aux)]\geq 1-negl(n).$$

3. *For any PPT algorithm T without knowing w, there is N_0 such that for every $n > N_0$ (s.t. $(n,\cdot)\in I$) it holds that:*

$$\Pr[(x,w,z)\leftarrow(X_n,W_n,Z_n):T(\beta,aux)=f^{-1}(\beta)|C(x,w,z)=(\beta,\alpha,aux)]\leq 1-p.$$

Proof. Fix $(n,i)\in I$ (from here on we drop the n on i_n for simplicity). Incorporating V^* and the honest prover P, (n,i) and the inverse polynomial p, the algorithm C, on input (x,w,z), plays the role of the honest prover and extracts (by rewinding) one-pre-image of the pair images of f output by V^* at its i-th step, and then outputs the pre-image extracted and the corresponding image (together with some auxiliary information). To make sure that different algorithms can extract the same pre-image, we have C repeat the extraction precedure many times and output the image corresponding to the *most-often* extracted pre-image. The detailed description of C follows.

The Algorithm C

---

input : $(x,w,z)\leftarrow(X_n,W_n,Z_n)$

1. Run P and V^* on input (x,w,z) until obtain the history prefix h up to the step i of V^*. If the V^*'s step i message v_i is the first verifier message of the form (β_1,β_2,a) in a session, say, session j, then continue; otherwise, return $\perp$.
2. Resume the interaction between P and V^* until V^* terminates. If the second accepting verifier message t in session j appears in this interaction, continue; otherwise, return $\perp$.
3. Repeat the following two steps $\frac{n}{p}$ times (there are at most $\frac{n^2}{p^2}$ iterations of step 2 within this step):
   (a) Run the above step 2 using fresh randomness (based on the same history prefix h) until either the second accepting verifier message in session j appears *twice* or the $\frac{n}{p}$-th iteration is reached. If two accepting transcripts of the first phase in session j of the Feige-Shamir protocol are obtained within these $\frac{n}{p}$ iterations (for the purpose of simplifying the analysis of the algorithm E, here we don't use the transcript obtained in step 2), compute α such that $\beta_b=f(\alpha)$ from them; otherwise, return $\perp$.
   (b) Store (β_b,α) in a list.
4. Set β to be β_b for which the corresponding pair (β_b,α) appears most often in the above list, and aux to be (h,PartR_h,x,z), where PartR_h includes only the randomness used by V^* and the randomness used by honest provers in those *incomplete* sessions in producing h.

output: (β,α,aux).

Consider the following set of history prefix (up to the step i of V^*):

$$\mathcal{H}:=\{h:h=h'||(\beta_1,\beta_2,a^j)\wedge\ \Pr[V^*|_h\rightsquigarrow(j,2)]\geq p(n)\}.$$

By the first condition of Lemma 1, the probability that the history prefix h generated in the step 1 is in $\mathcal{H}$(which implies C does not output " $\perp$ " in its first step) is greater than p. Conditioned on $h\in\mathcal{H}$, C does not output " $\perp$ " with probability at least p, and a single execution of the step 3(a) fails to extract α only with probability $(1-p)^{\frac{n}{p}}\approx e^{-n}$, which leads to the probability that all $\frac{n}{p}$ repetitions of the step 3(a) succeed is at least $(1-(1-p)^{\frac{n}{p}})^{\frac{n}{p}}>1-negl(n)$. Thus the probability that C outputs (β,α,aux) is at least $p^2(1-negl(n))>p^2-negl(n)$, as desired.

The algorithm E, taking (β, aux, w) as input, simply repeats $\frac{n}{p}$ times the step 3(a) of the algorithm C to extract the pre-image of β.

The Algorithm E

---

input : (β, aux, w)

1. Parse aux into $(h, \text{PartR}_h, x, z)$, and parse the last message v_i in h into (β_1, β_2, a).
2. Suppose that $\beta = \beta_b$. Repeat the step 3(a) of C until the pre-image α of β_b is extracted or the $\frac{n}{p}$-th iteration is reached, and if all iterations fail, return $\perp$.

output: α.

Observe that the algorithm C has to succeed in extraction in *all* $\frac{n}{p}$executions of the step 3(a) in order to output (β, α, aux). It follows from standard Chernoff bound that, except for exponentially small probability, the probability that, conditioned on outputting (β, α, aux), a single execution of the step 3(a) of C will extract one pre-image is at least $\frac{7}{8}$. Note also that the image β output by C is the one of which C extracts the corresponding pre-image more than $\frac{n}{2p}$ times, therefore (by Chernoff bound again), except for exponentially small probability, the probability that a single execution of the step 3(a) of C based on h will extract the pre-image of β is at least $\frac{1}{4}$. Thus, the probability that E fails to extract the pre-image of β, i.e.,

$$\Pr[\perp \leftarrow E(\beta, aux, w) | C(x,w,z) = (\beta,\alpha,aux)] < (\frac{1}{4})^{\frac{n}{p}},$$

which is negligible. This proves the second condition of Lemma 2.

□

5 Hardness Amplification and a Tailored Hard-Core Lemma

For our applications, we need to increase the probability that the algorithm C in Lemma 2 outputs an image β significantly while decreasing T's success probability to a negligible level. In addition, if the statement x has multiple witnesses, we also want algorithm E to work when given an arbitrary one as input.

Our basic strategy for achieving these goals is to use classic hardness amplification method with some careful modifications. We show how to transform the algorithms C and E, which work on the infinite set I, into algorithms M and Find with desired properties.

Let p be as in Lemma 1, and let $q_1 = \frac{n}{(p)^2}$, $q_2 = \frac{n}{p}$ and $q = q_1 q_2$.

The Algorithm M

---

input : (β, aux, w)

1. Arrange $\{(x_k, w_k, z_k)\}_{k=1}^{q}$ into $q_1 \times q_2$ tuples, denoted by $\{(x_i^j, w_i^j, z_i^j)\}_{i,j=1}^{q_2,q_1}$.
2. For $i = 1, 2, ..., q_2$, run C on each (x_i^j, w_i^j, z_i^j), $j \in [1, q_1]$, until C outputs (β, α, aux). If for some i all these q_1 runs of C fail, return $\perp$; otherwise, set $(\beta_i, \alpha_i, aux_i)$ to be (β, α, aux).

output: $\{(\beta_i, \alpha_i, aux_i)\}_{i=1}^{q_2}$.

The Algorithm Find

input : $\{(x_k, w_k, z_k)\}_{k=1}^{q}$, $\{(\beta_i, aux_i)\}_{i=1}^{q_2}$

1. Arrange $\{(x_k, w_k, z_k)\}_{k=1}^{q}$ in the same way as M and obtain $\{(x_i^j, w_i^j, z_i^j)\}_{i,j=1}^{q_2,q_1}$.
2. For $i = 1, 2, ..., q_2$, obtain the statement x_i from aux_i, find the j-th entry (x_i^j, w_i^j, z_i^j) from $\{(x_i^j, w_i^j, z_i^j)\}_{j=1}^{q_1}$ such that $x_i^j = x_i$ and fetch the corresponding w_i^j, set $w_i = w_i^j$ and run E on input (β_i, aux_i, w_i). If E fails, output $\perp$, otherwise, set α_i to be the output of E.

output: $\{\alpha_i\}_{i=1}^{q_2}$.

Lemma 3. *The algorithms M and Find satisfy the following properties:*

1. *The probability that M outputs $\{(\beta_i, \alpha_i, aux_i)\}_{i=1}^{q_2}$ such that $\beta_i = f(\alpha_i)$ holds for each i is negligibly close to 1.*
2. *Conditioned on M outputting $\{(\beta_i, \alpha_i, aux_i)\}_{i=1}^{q_2}$, the probability that Find inverts all these β_i's successfully is negligibly close to 1.*
3. *Conditioned on M outputting $\{(\beta_i, \alpha_i, aux_i)\}_{i=1}^{q_2}$, for any PPT T, given as input only $(\{(x_k, z_k)\}_{k=1}^{q}$, $\{(\beta_i, aux_i)\}_{i=1}^{q_2})$ (without any witnesses to the x_k's), the probability that T inverts all these β_i's successfully is negligible.*
4. *For any two inputs to Find with different witnesses, $(\{(x_k, w_k, z_k)\}_{k=1}^{q}, \{(\beta_i, aux_i)\}_{i=1}^{q_2})$ and $(\{(x_k, w_k', z_k)\}_{k=1}^{q}, \{(\beta_i, aux_i)\}_{i=1}^{q_2})$ with $\{w_k\}_{k=1}^{q} \neq \{w_k'\}_{k=1}^{q}$, Find succeeds on each input with almost (negligibly close to each other) the same probability.*

The first property follows from the fact that, for each i, the probability that C fails on all q_1 tuples (x_i^j, w_i^j, z_i^j) is less than $(1-p^2)^{q_1} = (1-p^2)^{\frac{n}{p^2}}$. Thus M succeeds on $\{(x_i^j, w_i^j, z_i^j)\}_{j=1}^{q_1}$ (i.e., C succeeds on $\{(x_i^j, w_i^j, z_i^j)\}$ for some $j \in [1, q_1]$) for all $i \in [1, q_2]$ with probability less than

$$(1-(1-p^2)^{q_1})^{q_2} = (1-(1-p^2)^{\frac{n}{p^2}})^{\frac{n}{p}} \approx e^{\frac{-n}{e^n p}} > 1 - \frac{n}{e^n p}$$

which is negligibly close to 1.

The second property directly follows from the second condition of Lemma 2. Observe that the third condition of Lemma 2 guarantees the failure probability of T on each $i \in [1, q_2]$ is greater than p, then it will succeed on *all* $i \in [1, q_2]$ with probability at most $(1-p)^{q_2} = (1-p)^{\frac{n}{p}}$, which gives us the above third property.

The last property is due to the following observation. For any two inputs $(\{(x_k, w_k, z_k)\}_{k=1}^{q}$, $\{(\beta_i, aux_i)\}_{i=1}^{q_2})$ and $(\{(x_k, w_k', z_k)\}_{k=1}^{q}, \{(\beta_i, aux_i)\}_{i=1}^{q_2})$ with $\{w_k\}_{k=1}^{q} \neq \{w_k'\}_{k=1}^{q}$, if the gap between the probabilities that Find succeeds on them is non-negligible, then there are two inputs (β_k, aux_k, w_k) and (β_k, aux_k, w_k') to E with $w_k \neq w_k'$, $(x_k, w_k), (x_k, w_k') \in R_L$ (recall that x_k is stored in aux_k), such that the gap between the probabilities that E succeeds on them is also non-negligible. This means that V^* can tell apart the real interactions in which the honest prover uses different witnesses with non-negligible probability, which breaks the concurrent witness indistinguishability of the Feige-Shamir protocol.

The algorithm M generates q_2 number of images $(\beta_1, \beta_2, ..., \beta_{q_2})$ of one-way function $f : \{0,1\}^n \to \{0,1\}^{\ell(n)}$ in a way such that they are hard for any PPT algorithm (without knowing the corresponding witnesses) to invert simultaneously. This enables us to apply Goldreich-Levin hard-core predicate for the function of $f^{\bigotimes q_2}$ *with respect to the distribution on* $(\beta_1, \beta_2, ..., \beta_{q_2})$ generated by M. Formally, we need the following form of the Goldreich-Levin theorem.

Lemma 4 (Goldreich-Levin). *Let $f : \{0,1\}^n \to \{0,1\}^{\ell(n)}$ be a function computable in polynomial time, G be a PPT algorithm. If for every PPT T,*

$$\Pr[(f(x), aux) \leftarrow G(1^n) : T(1^n, f(x), aux) \in f^{-1}(f(x))] \leq negl(n),$$

then, the inner product of x and a random r modulo 2, denoted by $\langle x, r\rangle$, is a hardcore predicate for f, i.e., for every PPT T',

$$\Pr\left[\begin{array}{c}(f(x), aux) \leftarrow G(1^n), \\ r \leftarrow \{0,1\}^{n\times q_2}\end{array} : T'(1^n, f(x), r, aux) = \langle x, r\rangle\right] \leq \frac{1}{2} + negl(n).$$

The Goldreich-Levin theorem typically states for the distribution $f(U)$, i.e., for x being drawn from uniform distribution, but its proof strategy ignores the distribution on the images of f and the auxiliary input (as long as both T and T' are given the same auxiliary string as input) completely, so the same proof applies to the above lemma (cf. [Gol01]).

In our setting, this means that the inner product (modulo 2) $\langle(\alpha_1, \alpha_2, ..., \alpha_{q_2}), r\rangle$ is a hard core predicate for $f^{\bigotimes q_2} : \{0,1\}^{n\times q_2} \to \{0,1\}^{\ell(n)\times q_2}$ against any PPT T that takes as auxiliary input $(\{(x_k, z_k)\}_{k=1}^q, \{(\beta_i, aux_i)\}_{i=1}^{q_2})$, where $\{(\beta_i, aux_i)\}_{i=1}^{q_2}$ is output by M.

6 Constructions for Public-Key Encryption and Key Agreement

In this section, we assume that, for an arbitrary inverse polynomial ϵ, V^* breaks ϵ-distributional concurrent zero knowledge of Feige-Shamir protocol for distributions over arbitrary OR NP-relations. We construct public-key encryption and key agreement from V^* and injective one-way functions. This completes the proof of Theorem 1.

Let q, q_2, M, Find and the infinite set I be as defined in previous sections. The final construction of public-key encryption scheme proceeds as follows. The receiver generates q number of YES instances together with their corresponding witnesses, $\{(x_{1,k}, w_{1,k})\}_{k=1}^q$ and publishes $\{x_{1,k}\}_{k=1}^q$ as his public key. To encrypt a bit m, the sender generates $\{(x_{2,k}, w_{2,k})\}_{k=1}^q$, and prepares a sequence of OR statements $\{(x_{1,k} \vee x_{2,k})\}_{k=1}^q$ (Note that each $\{w_{b,i}\}_{k=1}^q$, $b \in [1,2]$, are valid witnesses). Then the sender applies M using $\{w_{2,k}\}_{k=1}^q$ to generate an image of $f^{\bigotimes q_2}$ and encrypt m using Goldreich-Levin; to decrypt the cipher-text, the receiver applies Find using $\{w_{1,k}\}_{k=1}^q$ as witnesses to obtain the corresponding pre-image and then obtains the plain-text.

Formally, we need to assume the following for our constructions of public-key encryption (and key agreement):

- An arbitrary *injective* one-way function $f : \{0,1\}^n \to \{0,1\}^{\ell(n)}$ (used in the Feige-Shamir protocol). The injectiveness will be used for one party to recover the same hardcore bit that generated by the other party.
- An arbitrary efficiently samplable distribution ensemble $D = \{D_n\}_{n\in N}$ over R_L for an arbitrary NP language L.
- An arbitrary efficiently samplable distribution ensemble $\{Z_n\}_{n\in N}$ over $\{0,1\}^*$.
- A joint distribution ensemble $\{(X_n, W_n, Z_n)\}_{n\in N}$ on which the adversary V^* breaks the p_0-distributional concurrent zero knowledge of Feige-Shamir protocol, where each distribution (X_n, W_n, Z_n) defined in the following way: Sample $(x_1, w_1) \leftarrow D_n$, $(x_2, w_2) \leftarrow D_n$, $z \leftarrow Z_n$, $b \leftarrow \{1,2\}$, and output $((x_1, x_2), w_b)$.

We now construct PKE for a single bit message on every security parameter n s.t. $(n, \cdot) \in I$.

Key generation $\mathsf{Gen}(1^n)$: $\{(x_{1,k}, w_{1,k})\}_{k=1}^q \leftarrow D_n^{\bigotimes q}$, and set $pk = \{x_{1,k}\}_{k=1}^q$, $sk = \{w_{1,k}\}_{k=1}^q$.

Encryption $\mathsf{Enc}(pk = \{x_{1,k}\}_{k=1}^{q}, m)$ ($m \in \{0,1\}$):

1. $\{(x_{2,k}, w_{2,k})\}_{k=1}^{q} \leftarrow D_n^{\bigotimes q}$, $\{z_k\}_{k=1}^{q} \leftarrow Z_n^{\bigotimes q}$.
2. for $k \in [1,q]$, set x_k to be a random order of the pair $(x_{1,k}, x_{2,k})$.
3. $\{(\beta_i, \alpha_i, aux_i)\}_{i=1}^{q_2} \leftarrow \mathsf{M}(\{(x_k, w_{2,k}, z_k)\}_{k=1}^{q})$.
4. $r \leftarrow \{0,1\}^{n \times q_2}$, $h \leftarrow \langle(\alpha_1, \alpha_2, ..., \alpha_{q_2}), r\rangle \in \{0,1\}$.
5. Output $c = (\{(x_k, z_k)\}_{k=1}^{q}, \{(\beta_i, aux_i)\}_{i=1}^{q_2}, r, h \bigoplus m)$.

Decryption $\mathsf{Dec}(sk = \{w_{1,k}\}_{k=1}^{q}, c)$:

1. Parse c into $\{(x_k, z_k)\}_{k=1}^{q} || \{(\beta_i, aux_i)\}_{i=1}^{q_2} ||r||c'$.
2. $\{\alpha_i\}_{i=1}^{q_2} \leftarrow \mathsf{Find}(\{(x_k, w_{1,k}, z_k)\}_{k=1}^{q}, \{(\beta_i, aux_i)\}_{i=1}^{q_2})$.
3. $h \leftarrow \langle(\alpha_1, \alpha_2, ..., \alpha_{q_2}), r\rangle$.
4. Output $m = h \bigoplus c'$.

Notice that the input to M in the encryption algorithm can be viewed as being drawn from (X_n, W_n, Z_n) defined above. The correctness of this scheme follows from properties 1, 2, 4 of algorithms M and Find presented in the previous section. It should be noted that our scheme is not perfectly correct s-ince it is possible for M/Find to fail during the encryption/decryption process. However, this happens only with negligible probability.

It is also easy to verify the security against chosen-plaintext-attack, which is essentially due to the property 3 of M, together with the security of the hardcore bit for $f^{\bigotimes q_2}$.

Following the well-known paradigm, one can transform a public-key encryption scheme against chosen-plaintext-attack into a key agreement protocol (A, B) with security against eavesdropping adversary in a simple way: the party A generates a public/secrete key pair and send the public-key to B, and then B sends back a ciphertext of the secret session key under A's public key to A. This establishes a common session secret key between A and B.

Extensions to Multiparty Key Agreement. Our key agreement protocol can be easily extended to the multiparty setting. Roughly, if V^* is able to break ϵ-distributional concurrent zero knowledge of the Feige-Shamir protocol on a distribution on instances of the form $(x_1 \vee x_2 \vee ... \vee x_n)$, then the n parties can establish a session secret key as follows. Each party A_i generates a sequence of pairs $\{(x_{i,k}, w_{i,k})\}_{k=1}^{q})$. In their first round the parties $A_1, A_2, ..., A_{n-1}$ send their sequences of $\{(x_{i,k})\}_{i,k=1}^{n-1,q})$ to the n-th party, then the n-th party uses these sequences as a public key of the above PKE scheme to encrypt the session secret key and send the ciphertext to all $n-1$ parties. Upon receiving the ciphertext, each A_i, $i = [1, n-1]$, decrypts it and obtains the session secret key using their own $\{(w_{i,k})\}_{k=1}^{q}$.

7 Concluding Remarks

We prove a win-win result regarding the complexity of public-key encryption and the round-complexity of concurrent zero knowledge. One of the most interesting problem is to determine which one is (or both are) true. We believe that when we can prove one of these two statements, we might obtain a much stronger result (e.g., result with respect to the (nicer) standard definitions) than the ones stated here.

If we can show a reduction from one-way functions to public-key encryption (i.e., delete the "minicrypt" from the Impagliazzo's list of five worlds [Imp95]), that will be a major achievement in cryptography; if we can prove that the Feige-Shamir protocol is indeed concurrent zero knowledge, that will bring a new exciting individual reduction technique for cryptography, which permits us to prove some non-trivial structure of computation– e.g., the existence of those good extractors $\{T_i\}_{i\in\mathbb{N}}$ used by the simulator presented in section 3.3.– shared by all possible efficient adversaries.

8 Acknowledgement

We thank Yu Chen, Yanyan Liu, Shunli Ma, Hailong Wang, Bo Wu, Zhenbin Yan, Jingyue Yu and Jiang Zhang for helpful discussions.

References

[Adl78] Leonard M. Adleman. Two theorems on random polynomial time. In *Proceedings of the 19th Annual Symposium on Foundations of Computer Science - FOCS'78*, pages 75–83. IEEE Computer Society, 1978.

[Bar01] Boaz Barak. How to go beyond the black-box simulation barrier. In *Proceedings of the 42th Annual IEEE Symposium on Foundations of Computer Science - FOCS'01*, pages 106–115. IEEE Computer Society, 2001.

[BBF13] Paul Baecher, Christina Brzuska, and Marc Fischlin. Notions of black-box reductions, revisited. In *Advances in Cryptology - ASIACRYPT'13*, LNCS 8269, pages 296–315. Springer, 2013.

[BCC88] Gilles Brassard, David Chaum, and Claude Crépeau. Minimum disclosure proofs of knowledge. *Journal of Computer and System Science*, 37(2):156–189, 1988.

[BCPR14] Nir Bitansky, Ran Canetti, Omer Paneth, and Alon Rosen. On the existence of extractable one-way functions. In *Proceedings of the 45th Annual ACM Symposium on the Theory of Computing - STOC'14*, pages 505–514. ACM Press, 2014.

[BHSV98] Mihir Bellare, Shai Halevi, Amit Sahai, and Salil Vadhan. Many-to-one trapdoor functions and their relation to public-key cryptosystems. In *Advances in Cryptology - CRYPTO'98*, LNCS 1462, pages 283–298. Springer, 1998.

[Blu86] Manuel Blum. How to prove a theorem so no one else can claim it. In *Proceedings of international congress of mathematicians - ICM'86*, 1986.

[BP15] Nir Bitansky and Omer Paneth. On non-black-box simulation and the impossibility of approximate obfuscation. In *SIAM Journal on Computing*, Valume 44(5), pages 1325–1383, 20015.

[BT08] Andrej Bogdanov and Luca Trevisan. Average-case complexity. https://arxiv.org/pdf/cs/0606037.pdf, 2008.

[BU08] Michael Backes and Dominique Unruh. Limits of constructive security proofs. In *Advances in Cryptology - ASIACRYPT'08*, LNCS 5350, pages 290–307. Springer, 2008.

[CHS05] Ran Canetti, Shai Halevi, and Michael Steiner. Hardness amplification of weakly verifiable puzzles. In *Theory of Cryptography - TCC'05*, LNCS 3378, pages 17–33. Springer, 2005.

[CKPR01] Ran Canetti, Joe Kilian, Erez Petrank, and Alon Rosen. Black-box concurrent zero-knowledge requires omega(log n) rounds. In *Proceedings of the 33rd Annual ACM Symposium Theory of Computing- STOC'01*, pages 570–579. ACM press, 2001.

[CLMP13] Kai-Min Chung, Huijia Lin, Mohammad Mahmoody, and Rafael Pass. On the power of nonuniformity in proofs of security. In *ITCS 2013*, pages 389–400, 2013.

[CLOS02] Ran Canetti, Yehuda Lindell, Rafail Ostrovsky, and Amit Sahai. Universally composable two-party and multi-party computation. In *Proceedings of the 34th Annual ACM Symposium on the Theory of Computing - STOC'02*, pages 494–503. ACM Press, 2002.

[CLP13a] Ran Canetti, Huijia Lin, and Omer Paneth. Public-coin concurrent zero-knowledge in the global hash model. In *Theory of Cryptography - TCC'13*, LNCS 7785, pages 80–99. Springer, 2013.

[CLP13b] Kai-Min Chung, Huijia Lin, and Rafael Pass. Constant-round concurrent zero knowledge from p-certificates. In *Proceedings of the 54th Annual Symposium on Foundations of Computer Science - FOCS'13*, pages 50–59. IEEE Computer Society, 2013.

[CLP15a] Kai-Min Chung, Huijia Lin, and Rafael Pass. Constant-round concurrent zero-knowledge from indistinguishability obfuscation. In *Advances in Cryptology - CRYPTO'15*, LNCS 9216, pages 287–307. Springer, 2015.

[CLP15b] Kai-Min Chung, Edward Lui, and Rafael Pass. From weak to strong zero-knowledge and applications. In *Theory of Cryptography - TCC'15*, LNCS 9014, pages 66–92. Springer, 2015.

[CS99] Ronald Cramer and Victor Shoup. Signature schemes based on the strong RSA assumption. In *ACM conference on Computer and Communications Security - CCS'99*, pages 46–52. ACM Press, 1999.

[Dam91] Ivan Damgård. Towards practical public key systems secure against chosen ciphertext attacks. In *Advances in Cryptology - CRYPTO'91*, LNCS 576, pages 445–456. Springer, 1991.

[DGL+16] Yi Deng, Juan A. Garay, San Ling, Huaxiong Wang, and Moti Yung. On the implausibility of constant-round public-coin zero-knowledge proofs. In *Security in Communication Networks - SCN'16*, LNCS 9841, pages 237–253. Springer, 2016.

[DGS09] Yi Deng, Vipul Goyal, and Amit Sahai. Resolving the simultaneous resettability conjecture and a new non-black-box simulation strategy. In *Proceedings of the 50th Annual Symposium on Foundations of Computer Science - FOCS'09*, pages 251–260. IEEE Computer Society, 2009.

[DH76] Whitfield Diffie and Martin E. Hellman. New directions in cryptography. *IEEE Transactions on Information Theory*, 22(6):644–654, 1976.

[DNRS03] Cynthia Dwork, Moni Naor, Omer Reingold, and Larry J. Stockmeyer. Magic functions. *Journal of the ACM*, 50(6):852–921, 2003.

[DNS98] Cynthia Dwork, Moni Naor, and Amit Sahai. Concurrent zero-knowledge. In *Proceedings of the 30rd Annual ACM Symposium Theory of Computing- STOC'98*, pages 409–418. ACM press, 1998.

[DS16] Dana Dachman-Soled. Towards non-black-box separations of public key encryption and one way function. In *Theory of Cryptography - TCC'16B*, LNCS 9986, pages 169–191. Springer, 2016.

[FS89] Uriel Feige and Adi Shamir. Zero knowledge proofs of knowledge in two rounds. In *Advances in Cryptology - CRYPTO'89*, LNCS 435, pages 526–544. Springer, 1989.

[GGJ13] Vipul Goyal, Divya Gupta, and Abhishek Jain. What information is leaked under concurrent composition? In *Advances in Cryptology - CRYPTO'13*, LNCS 8043, pages 220–238. Springer, 2013.

[GGJS12] Sanjam Garg, Vipul Goyal, Abhishek Jain, and Amit Sahai:. Concurrently secure computation in constant rounds. In *Advances in Cryptology - Eurocrypt'12*, LNCS 7237, pages 99–116. Springer, 2012.

[GGS15] Vipul Goyal, Divya Gupta, and Amit Sahai. Concurrent secure computation via non-black box simulation. In *Advances in Cryptology - CRYPTO'15*, LNCS 9216, pages 23–42. Springer, 2015.

[GK96] Oded Goldreich and Ariel Kahan. How to construct constant-round zero-knowledge proof systems for NP. *Journal of Cryptology*, 9(3):167–190, 1996.

[GLP+15] Vipul Goyal, Huijia Lin, Omkant Pandey, Rafael Pass, and Amit Sahai. Round-efficient concurrently composable secure computation via a robust extraction lemma. In *Theory of Cryptography - TCC'15*, LNCS 9014, pages 260–289. Springer, 2015.

[GM82] Shafi Goldwasser and Silvio Micali. Probabilistic encryption and how to play mental poker keeping secret all partial information. In *Proceedings of the 14rd Annual ACM Symposium Theory of Computing- STOC'82*, pages 365–377. ACM press, 1982.

[GMR89] Shafi Goldwasser, Silvio Micali, and Charles Rackoff. The knowledge complexity of interactive proof systems. *SIAM. Journal on Computing*, 18(1):186–208, 1989.

[GMR01] Yael Gertner, Tal Malkin, and Omer Reingold. On the impossibility of basing trapdoor functions on trapdoor predicates. In *Proceedings of the 42th Annual Symposium on Foundations of Computer Science - FOCS'01*, pages 126–135. IEEE Computer Society, 2001.

[GNW95] Oded Goldreich, Noam Nisan, and Avi Wigderson. On yaos xor-lemma. In *Electronic colloquium on computational complexity, TR95-050*, 1995.

[Gol93] Oded Goldreich. A uniform-complexity treatment of encryption and zero-knowledge. *Journal of Cryptology*, 6(1):21–53, 1993.

[Gol01] Oded Goldreich. *Foundations of Cryptography*, volume Basic Tools. Cambridge University Press, 2001.

[Goy13] Vipul Goyal. Non-black-box simulation in the fully concurrent setting. In *Proceedings of the 45th Annual ACM Symposium on the Theory of Computing - STOC'13*, pages 221–230. ACM Press, 2013.

[HILL99] Johan Hastad, Russell Impagliazzo, Leonid A. Levin, and Michael Luby. A pseudorandom generator from any one-way function. *SIAM. Journal on Computing*, 28(4):1364–1396, 1999.

[HKS03] Dennis Hofheinz, Eike Kiltz, and Victor Shoup. Practical chosen ciphertext secure encryption from factoring. *Journal of Cryptology*, 26(1):102–118, 2003.

[HR07] Iftach Haitner and Omer Reingold. Statistically-hiding commitment from any one-way function. In *Proceedings of the 39rd Annual ACM Symposium Theory of Computing- STOC'07*, pages 1–10. ACM press, 2007.

[HS11] Thomas Holenstein and Grant Schoenebeck. General hardness amplification of predicates and puzzles. In *Theory of Cryptography - TCC'11*, LNCS 6597, pages 19–36. Springer, 2011.

[IL89] Russell Impagliazzo and Michael Luby. One-way functions are essential for complexity based cryptography. In *Proceedings of the 30th Annual Symposium on Foundations of Computer Science - FOCS'89*, pages 230–235. IEEE Computer Society, 1989.

[Imp95] Russell Impagliazzo. A personal view of average-case complexity. In *Proceedings of the 10th Annual IEEE Structure in Complexity Theory Conference*, pages 134–147. IEEE Computer Society, 1995.

[IR89] Russell Impagliazzo and Steven Rudich. Limits on the provable consequences of one-way permutations. In *Proceedings of the 21th Annual ACM Symposium on the Theory of Computing - STOC'89*, pages 44–61. ACM Press, 1989.

[KL07] Jonathan Katz and Yehuda Lindell. *Introduction to Modern Cryptography*. Chapman and Hall/CRC Press, 2007.

[KP01] Joe Kilian and Erez Petrank. Concurrent and resettable zero-knowledge in poly-loalgorithm rounds. In *Proceedings of the 33rd Annual ACM Symposium Theory of Computing- STOC'01*, pages 560–569. ACM press, 2001.

[Lin03a] Yehuda Lindell. Bounded-concurrent secure two-party computation without setup assumptions. In *Proceedings of the 35rd Annual ACM Symposium Theory of Computing- STOC'03*, pages 683–692. ACM press, 2003.

[Lin03b] Yehuda Lindell. General composition and universal composability in secure multi-party computation. In *Proceedings of the 44th Annual Symposium on Foundations of Computer Science - FOCS'03*, pages 394–403. IEEE Computer Society, 2003.

[Lin08] Yehuda Lindell. Lower bounds and impossibility results for concurrent self composition. *Journal of Cryptology*, 21(2):200–249, 2008.

[NY90] Moni Naor and Moti Yung. Public-key cryptosystems provably secure against chosen ciphertext attacks. In *Annual ACM Symposium on the Theory of Computing - STOC'90*, pages 427–437. ACM Press, 1990.

[Pas04] Rafael Pass. Bounded-concurrent secure multi-party computation with a dishonest majority. In *Proceedings of the 36th Annual ACM Symposium on the Theory of Computing - STOC'04*, pages 232–241. ACM Press, 2004.

[PR03] Rafael Pass and Alon Rosen. Bounded-concurrent secure two-party computation in a constant number of rounds. In *Proceedings of the 44th Annual Symposium on Foundations of Computer Science - FOCS'03*, pages 404–413. IEEE Computer Society, 2003.

[PR05] Rafael Pass and Alon Rosen. Concurrent non-malleable commitments. In *Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science - FOCS'05*, pages 563–572. IEEE Computer Society, 2005.

[PRS02] Manoj Prabhakaran, Alon Rosen, and Amit Sahai. Concurrent zero knowledge with logarithmic round-complexity. In *Proceedings of the 43th Annual IEEE Symposium on Foundations of Computer Science - FOCS'02*, pages 366–375. IEEE Computer Society, 2002.

[Rab79] Michael Rabin. Digitalized signatures and public-key encryptions as intractable as factorization. Technical Report MIT/LCS/TR-212, MIT Laboratory for Computer Science, 1979.

[Reg09] Oded Regev. On lattices, learning with errors, random linear codes, and cryptography. *Journal of the ACM*, 56(6), 2009.

[Rom90] John Rompel. One-way functions are necessary and sufficient for secure signatures. In *Proceedings of the 22rd Annual ACM Symposium Theory of Computing- STOC'90*, pages 387–394. ACM press, 1990.

[RSA78] Ronald L. Rivest, Adi Shamir, and Leonard M. Adleman. A method for obtaining digital signatures and public-key cryptosystems. *Communications of the ACM*, 21(2):120–126, 1978.

[RTV04] Omer Reingold, Luca Trevisan, and Salil Vadhan. Notions of reducibility between cryptographic primitives. In *Theory of Cryptography - TCC'04*, LNCS 2951, pages 1–20. Springer, 2004.

[Sah99] Amit Sahai. Non-malleable non-interactive zero knowledge and adaptive chosen-ciphertext security. In *Proceedings of the 40th Annual Symposium on Foundations of Computer Science - FOCS'99*, pages 543–553. IEEE Computer Society, 1999.

ECCC ISSN 1433-8092

https://eccc.weizmann.ac.il