

Small bias requires large formulas

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Abstract

A small-biased function is a randomized function whose distribution of truth-tables is small-biased. We demonstrate that known explicit lower bounds on the size of (1) general Boolean formulas, (2) Boolean formulas of fan-in two, (3) de Morgan formulas, as well as (4) correlation lower bounds against small de Morgan formulas apply to small-biased functions. As a consequence, any strongly explicit small-biased generator is subject to the best known explicit formula lower bounds in all these models.

On the other hand, we give a construction of a small-biased function that is tight with respect to lower bounds (1) and (2) for the relevant range of parameters. We interpret this construction as a natural-type barrier against substantially stronger lower bounds for general formulas.

1 Introduction

Formula size is one of the most thoroughly studied complexity measures of Boolean functions. A *formula* is a circuit in which every internal gate has fan-out one. The power of formulas depends on the types of gates allowed. In this work we consider two models: *General formulas* in which any gate of some pre-specified fan-in c is allowed, and *de Morgan formulas* that only use NOT gates and AND/OR gates of fan-in two.

Explicit size lower bounds for general formulas were first proved by Nečiporuk [Nec66], who showed that the selector (addressing) function requires general constant fan-in formula size $\Omega(n^2 / \log n \log \log n)$ over inputs of size n . In the same work, Nečiporuk gave an improved lower bound of $\Omega(n^2 / \log n)$ for the element distinctness function by a related but different method.

The case of de Morgan formulas had been studied even earlier. Subbotovskaya [Sub61] proved that computing parity on n bits requires formula size $\Omega(n^{3/2})$. Andreev [And87] combined the ideas of Nečiporuk and Subbotovskaya to obtain a $n^{5/2-o(1)}$ de Morgan formula size lower bound for an explicit family of functions from $\{0, 1\}^n$ to $\{0, 1\}$. Following partial improvements (Impagliazzo and Nisan [IN93], Paterson and Zwick [PZ93]), Håstad [Hås98]

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showed that Andreev’s function requires formula size $n^{3-o(1)}$, which is optimal in the exponent.¹ The same lower bound was reproved by Dinur and Meir [DM16] using different methods.

More recently, Tal gave two lower-order improvements to Håstad’s result. First, in [Tal14] he showed that Andreev’s function requires de Morgan formulas of size $\Omega(n^3/(\log n)^2 \log \log n)$, which is optimal for this function up to the doubly logarithmic term. Later, in [Tal16] he showed that another function introduced by Komargodski and Raz [KR13] requires de Morgan formula size $\Omega(n^3/(\log n)(\log \log n)^2)$.

In a related line of works, Komargodski, Raz, and Tal [KR13, KRT13, Tal14] study correlation lower bounds against small formulas. For every $k \leq n^{1/3}$, they construct two variants of an explicit function that has correlation at most 2^{-k} with any de Morgan formula of size $n^3/(\log n)^{O(1)}k^2$. Their hard functions make use of error-correcting codes with good list-decodable properties and extractors for bit-fixing sources. Weaker correlation bounds for the parity function were proved by Santhanam [San10] and, as observed in [KRT13], also follow implicitly from bounds on the approximate degree of de Morgan formulas [Rei11, BBC⁺01].

Razborov and Rudich [RR94] observed that all formula size lower bounds (known at the time) are natural, meaning that the formulas to which the bounds apply cannot compute cryptographically pseudorandom functions. On the other hand, the class NC^1 of polynomial-size logarithmic-depth bounded fan-in circuit families, which are equivalent in power to polynomial-size formula families, is believed to contain pseudorandom functions. Naor and Reingold [NR99] and Banerjee, Peikert, and Rosen [BPR12] proposed two such candidate families based on the Decisional Diffie Hellman and Learning With Errors hardness assumptions, respectively. These constructions suggest that explicit size n^C lower bounds for formulas is out of reach for current techniques for sufficiently large values of the exponent C . The values of C in these constructions (for the requisite levels of hardness) are apparently rather large, so they are unlikely to explain the perceived barriers of n^2 and n^3 for general and de Morgan formula size, respectively.

Our results Our main conceptual contribution is the realization that all known formula size lower bound techniques also apply to small-biased functions. A randomized function is (K, ε) -biased if the induced distribution over truth-tables is a (K, ε) -biased distribution.

From the perspective of natural proofs, the known properties that distinguish small formulas from random functions are local in the sense that they only make a bounded number of non-adaptive queries to the function. It is therefore reasonable to expect that the largeness condition of the relevant natural properties should continue to hold for random functions that only exhibit bounded independence. We show that these properties, in fact, merely require small bias [NN93], which is closely related to *approximate* bounded independence. In certain instances (Propositions 3 and 6), this extension (for the appropriate range of parameters) is more or less self-evident, while in others (Propositions 4, 7, and 11) it relies on less obvious properties of small-biased distributions (Lemmas 5 and 8).

As a direct consequence, in Theorem 12 we show that the best known explicit formula

¹Our discussion of formula lower bounds is based on Chapter 6 of Jukna’s book [Juk10].

lower bounds apply to any *strongly explicit* small-biased generator. In this context, a strongly explicit small-biased generator is a polynomial-time family $SB_n: \{0, 1\}^{O(n)} \times \{0, 1\}^n \rightarrow \{0, 1\}$ of functions such that for a random s , the distribution over the 2^n values $SB_n(s, x)$ as x ranges over $\{0, 1\}^n$ is a $(2^n, 2^{-2^n})$ -biased distribution. This allows us to reprove the best known formula lower bounds in a unified manner and even obtain a very minor improvement in one case.

On the other hand, in Theorem 13 we construct a (K, ε) -biased function F with fan-in two formula size $O(n(\log K)^2(\log 1/\varepsilon))$. For $\varepsilon = 2^{-2K}$, this is a $(K, 2^{-K})$ -wise independent function of fan-in two formula size $O(nK(\log K)^2)$, which matches our lower bounds for general formulas in Propositions 3 and 4 up to terms polylogarithmic in K .

In the parameter regimes that yield the explicit formula size lower bounds 1, 2, and 3 in Theorem 12, the function F has fan-in two formula size $O((n \log n)^2)$ and de Morgan formula size $O(n^4(\log n)^3)$. We view this as a barrier to proving super-quadratic lower bounds for general formulas, and super-quartic ones for de Morgan formulas.

However, these barriers are not of the natural type since our function F is not cryptographically pseudorandom: In addition to having small formula size, the function family F is computable by polynomial-size, depth 3 circuit families with AND, OR, and PARITY gates (the class $AC^0[\oplus]$), which is known not to contain cryptographic pseudorandom functions [Raz87, Smo87, RR94]. It remains open whether our bounds can be matched (or even improved in the case of de Morgan formulas) by a different construction that is plausibly secure with respect to all polynomial-size circuits, of which linear tests are a very special case.

We believe that Theorems 12 and Theorem 13 justify the study of small-biased functions as suitable candidates for formula size lower bounds. In the extreme setting of parameters $K = 2^n$, $\varepsilon = 2^{-\Theta(n)}$, known constructions of small-biased functions have seed lengths linear in n and may be plausible candidates for improved formula size lower bounds. In this regime, the general and de Morgan formula sizes of F in Theorem 13 are as large as $\Theta(n^4)$ and $\Theta(n^7)$, respectively. Do there exist, say, $(2^n, 2^{-100n})$ -biased functions of smaller formula size?

Bounded independence and small bias We will call a randomized function $F: \{0, 1\}^n \rightarrow \{0, 1\}$ (k, ε) -wise independent (in qualitative terms, *almost locally independent*) if for any k inputs x_1, \dots, x_k , the distribution $(F(x_1), \dots, F(x_k))$ is within statistical distance ε of the uniform distribution over $\{0, 1\}^k$. A random function $F: \{0, 1\}^n \rightarrow \{0, 1\}$ is (K, ε) -biased (*locally small-biased*) if for any nonempty set X of at most K distinct inputs,

$$\left| \mathbb{E} \left[\prod_{x \in X} (-1)^{F(x)} \right] \right| \leq \varepsilon.$$

When $K = 2^n$ the family is called ε -biased (*small-biased*). Small bias implies bounded independence by the following claim [NN93, Corollary 2.1].

Claim 1. *Every (K, ε) -biased function is $(K, 2^{K/2}\varepsilon)$ -wise independent.*

A family of functions $SB_{n,\varepsilon}: \{0, 1\}^{s(n,\varepsilon)} \times \{0, 1\}^n \rightarrow \{0, 1\}$ is a *strongly explicit small-biased family* if (1) $s(n, \varepsilon) = O(n + \log 1/\varepsilon)$, (2) $SB_{n,\varepsilon}$ is uniformly polynomial-time com-

putable and (3) the random function $F(x) = SB_{n,\varepsilon}(s, x)$ over uniform s is ε -biased for all n and ε .

Standard explicit constructions of small-biased sets [NN93, AGHP92, ABN⁺92, BT13, TS17] are in fact strongly explicit small-biased families.

2 Small bias requires large formulas

We are aware of two methods for proving general formula size lower bounds. Both can be traced to the work of Nečiporuk [Nec66] and are based on arguing about restrictions. We show that both methods imply lower bounds on the formula size of almost locally independent functions.

In the case of de Morgan formulas, we study three proof techniques. The first one, based on average-case shrinkage, underlies the lower bound of Andreev including improvements by Impagliazzo and Nisan, Paterson and Zwick, Håstad, and Tal. We show that this method also bounds the formula size of almost independent functions.

The second method for de Morgan formula lower bounds is due to Tal, who applies a correlation-to-computation reduction in addition to bounds on average-case shrinkage. The third method, due to Komargodski and Raz and improvements by these authors and Tal, applies a high-probability shrinkage lemma to derive strong correlation lower bounds. We show that these two methods give lower bounds on the size of small-biased functions.

Formulas with gates of arbitrary fan-in

A *restriction* $f|_\rho$ of a function f under a partial assignment ρ of its inputs is the function on the unassigned inputs obtained by fixing all the assigned variables to their values. A random \bar{k} -restriction of f is the distribution of restrictions of f under a uniform random assignment that leaves exactly k inputs unassigned.

The *size* of a formula is the number of leaves in the formula tree, namely the number of variables occurring in the formula. The following shrinkage property of formulas follows immediately from linearity of expectation:

Claim 2. *Assume $f: \{0, 1\}^n \rightarrow \{0, 1\}$ has formula size s . Then the expected formula size of a random \bar{k} -restriction of f is at most $(k/n) \cdot s$.*

We say F has formula size at most s if every function in the support of F can be computed by a formula of size at most s .

Proposition 3. *Assuming $c \leq \log \log k$, any $(2^k, 1/4)$ -wise independent function $F: \{0, 1\}^n \rightarrow \{0, 1\}$ requires fan-in c formulas of size $\Omega(n \cdot 2^k / k \log k)$.*

Proof. Suppose F has formula size s . By Claim 2 and averaging, there exists a partial assignment ρ with k unassigned variables under which the expected formula size of $F|_\rho$ is at most ks/n . By Markov's inequality,

$$\Pr_F[\text{size}(F|_\rho) \leq 2ks/n] \geq \frac{1}{2} \tag{1}$$

for any distribution of functions F , where size denotes formula size.

A formula of size \tilde{s} can be specified by listing its at most $2\tilde{s}$ gates in depth-first order. For a formula of fan-in c on k inputs, there are 2^{2^c} possible internal gates and k possible input gates, so the number of such formulas is at most $(2^{2^c} + k)^{2\tilde{s}} \leq (2k)^{2\tilde{s}}$. Therefore, setting $\tilde{s} = 2ks/n$, for a uniformly random function R it holds that

$$\Pr_R[\text{size}(R|_\rho) \leq 2ks/n] \leq \frac{(2k)^{4ks/n}}{2^{2k}}. \quad (2)$$

The event “ $\text{size}(F|_\rho) \leq 2ks/n$ ” depends on at most 2^k values of F , so if F is $(2^k, 1/8)$ -wise independent, then

$$\Pr_F[\text{size}(F|_\rho) \leq 2ks/n] \leq \Pr_R[\text{size}(R|_\rho) \leq 2ks/n] + \frac{1}{4}. \quad (3)$$

Combining (1), (2), and (3), we obtain that $(2k)^{4ks/n}/2^{2k} \geq 1/4$, from where the desired lower bound on s follows. \square

Formulas with gates of fan-in two

We now discuss the other proof of Nečiporuk, which gives a slightly stronger lower bound in the regime of $k < \log n$ and for exponentially small error. Unlike for Proposition 3, the proof of this theorem is only valid for fan-in two formulas.

Proposition 4. *For $k \leq \log n - 1$, any $(2 \cdot 2^k, 2^{-2^k})$ -wise independent function $F: \{0, 1\}^n \rightarrow \{0, 1\}$ requires fan-in two formulas of size $\Omega(n \cdot 2^k/k)$.*

The proposition is proved by showing that the number of possible restrictions of a small formula that leave the least frequently occurring inputs unrestricted is small. On the other hand, the following lemma shows that the number of distinct restrictions of an almost locally independent function is large, even when the set of unrestricted variables is fixed. A \bar{U} -restriction is a restriction under any assignment in which U is the set of free variables.

Lemma 5. *Assume F is $(2 \cdot 2^k, 2^{-2^k})$ -wise independent. For any set U of k variables, the number of distinct \bar{U} -restrictions of F is at least $\min\{2^{n-k-2}, 2^{2^k-3}\}$ with probability more than half.*

In particular, when $k \leq \log n - 1$, a $(2 \cdot 2^k, 2^{-2^k})$ -wise independent function family has at least $\frac{1}{8} \cdot 2^{2^k}$ distinct \bar{U} -restrictions with probability more than half.

Proof. Let ρ, ρ' be independent random partial assignments to the variables in \bar{U} . Then

$$\Pr_{F, \rho, \rho'}[F|_\rho = F|_{\rho'}] \leq \Pr[\rho = \rho'] + \Pr[F|_\rho = F|_{\rho'} \mid \rho \neq \rho']. \quad (4)$$

The first term equals 2^{-n+k} . To bound the second term, fix an arbitrary pair of distinct ρ, ρ' . The event that the restricted functions $F|_\rho$ and $F|_{\rho'}$ are identical depends on at most $2 \cdot 2^k$ values of F . By the almost local independence of F ,

$$\Pr_F[F|_\rho = F|_{\rho'} \mid \rho \neq \rho'] \leq \Pr[R = R'] + 2^{-2^k},$$

where $R, R': \{0, 1\}^k \rightarrow \{0, 1\}$ are independent uniformly random functions. Such functions are equal with probability at most 2^{-2^k} , and so the second term in (4) at most 2^{-2^k+1} . Therefore

$$\Pr_{F, \rho, \rho'}[F|_\rho = F|_{\rho'}] \leq 2^{-n+k} + 2^{-2^k+1}.$$

Now assume the support size of $F|_\rho$ over random ρ is less than S for at least half the functions F . Then the collision probability $\Pr_{\rho, \rho'}[F|_\rho = F|_{\rho'}]$ is at most $1/S$ for at least half the functions F and so

$$2^{-n+k} + 2^{-2^k+1} \geq \frac{1}{2S},$$

from where it follows that the larger of 2^{-n+k} and 2^{-2^k+1} is at least $1/4S$. It follows that $S \geq \min\{2^{n-k-2}, 2^{2^k-3}\}$. \square

Proof of Proposition 4. Let s be the size of F . By Claim 2 and averaging, there is a set U of size s so that the average number of such occurrences in F is at most $(k/n) \cdot s$. By Markov's inequality, at least half of the formulas in F have no more than $\tilde{s} = 2ks/n$ occurrences of variables from U .

A counting argument of Nečiporuk shows that if a formula ϕ that has at most \tilde{s} occurrences of variables from U then the number of \bar{U} -restrictions of ϕ is at most $2^{4\tilde{s}}$.

By Lemma 5, there must then exist a formula in the support of F whose number of \bar{U} -restrictions is at most $2^{4\tilde{s}} = 2^{8ks/n}$ and at least $\frac{1}{8} \cdot 2^{2^k}$. \square

Computation by de Morgan formulas

In this section we show that known proofs for de Morgan formula size also apply to small-biased functions. The following proof relies on expected shrinkage of de Morgan formulas under random restrictions [And87, IN93, PZ93, Hås98].

Proposition 6. *Assuming $k \leq n/2$, any $(2^k, 1/4)$ -wise independent function $F: \{0, 1\}^n \rightarrow \{0, 1\}$ requires de Morgan formula size $\Omega(n^2 \cdot 2^k / k^2 \log k)$.*

Proof. In a \bar{p} -random restriction, the unrestricted variables are sampled from the binomial distribution with parameter p . Tal [Tal14] showed that if f has a de Morgan formula of size s then the expected formula size of a \bar{p} -random restriction of f is $\tilde{s} = O(p^2s + \sqrt{p^2s})$. Set $p = 2k/n$. By deviation bounds, for every f in the support of F , the event that ρ has fewer than $k = \frac{1}{2}pn$ unassigned inputs or $f|_\rho$ has formula size more than $4\tilde{s}$ has probability at most $\frac{1}{2}$.

By averaging, there exists a partial assignment ρ with k unassigned inputs under which $F|_\rho$ has formula size at most $4\tilde{s}$ for at least half the functions F .

By the same argument as in the proof of Proposition 3, the $(2^k, 1/4)$ -wise independence of F then implies that $8 \cdot (9k)^{\tilde{s}}$ is at least 2^{2^k} . It follows that $\tilde{s} = \Omega(2^k / \log k)$. As $\tilde{s} = O(p^2s + \sqrt{p^2s})$ it follows that $p^2s = \Omega(2^k / \log k)$. Using the constraint $k \geq \frac{1}{2}pn$ we obtain the desired bound. \square

Tal [Tal14] recently obtained a slight improvement to the aforementioned bounds. His method also applies to small-biased functions as demonstrated in the following proposition:

Proposition 7. *Assume that $k \leq n/2$ and $\exp(-2^{k/3}) \cdot 2^{3k} \leq \varepsilon \leq 2^{-6k}$. Then every $(2^k, \varepsilon)$ -biased F requires de Morgan formula size $\Omega(n^2 \log(1/\varepsilon)/k(\log k)^2)$.*

The proof relies on the large deviation bound for small-bias distributions of Naor and Naor [NN93, Section 5]. We rework it here in more convenient notation. We say a random variable X over $\{-1, 1\}^K$ is ε -biased if $|\mathbb{E}[\prod_{i \in S} X_i]| \leq \varepsilon$ for every subset S of indices.

Lemma 8. *Let t be even and X be a $(t, (t^3/e^2K)^{t/2})$ -biased random variable over $\{-1, 1\}^K$. The probability that $|\sum X_i|$ exceeds δK is at most $3(t/\delta^2K)^{t/2}$.*

Proof. We apply a standard t -th moment calculation. By Markov's inequality,

$$\begin{aligned} \Pr\left[\left|\sum_{i=1}^K X_i\right| \geq \delta K\right] &\leq \frac{1}{(\delta K)^t} \mathbb{E}\left[\left(\sum_{i=1}^K X_i\right)^t\right] \\ &= \frac{1}{(\delta K)^t} \left(\sum_{S \in \mathcal{E}} \mathbb{E}\left[\prod_{i \in S} X_i\right] + \sum_{S \in \bar{\mathcal{E}}} \mathbb{E}\left[\prod_{i \in S} X_i\right]\right), \end{aligned}$$

where \mathcal{E} is the set of ordered terms of size t in which every index appears an even number of times. The first expectation is upper bounded by the number of such terms, which is at most $K^{t/2} \cdot t!/(t/2)! \leq 2 \cdot (tK)^{t/2}$. The second expectation is upper bounded by the number of terms times the maximum bias of each term, which is at most $\binom{K}{t} \cdot (t^3/e^2K)^{t/2} \leq (tK)^{t/2}$. The desired bound follows. \square

The following consequence of the lemma is far from tight but will be of use in the proof of Proposition 7. The *correlation* of two functions $f, \phi: \{0, 1\}^k \rightarrow \{0, 1\}$ is $\langle f, \phi \rangle = \mathbb{E}_x[(-1)^{f(x)} \cdot (-1)^{g(x)}]$, where x is uniform in $\{0, 1\}^n$.

Corollary 9. *Assume $\exp(-2^{k/3}) \cdot 2^{3k} \leq \varepsilon \leq 2^{-6k}$ and $F: \{0, 1\}^k \rightarrow \{0, 1\}$ is $(2^k, \varepsilon)$ -biased. Then for every $\phi: \{0, 1\}^k \rightarrow \{0, 1\}$, the probability that $|\langle F, \phi \rangle|$ is greater than $(e2^{-k})^{1/3}$ is at most $3\varepsilon^{1/6}$.*

Proof. We first argue that $\varepsilon \leq (t^3/e^22^k)^{t/2} \leq \varepsilon \cdot 2^{3k}$ (5) for some even $t \leq 2^{k/3}$. By our assumption on ε , the first inequality is satisfied for $t = 2$ and the second one holds for $t = 2^{k/3}$. Therefore if inequality 5 fails for all t in the range, there must exist an even $t < 2^{k/3}$ for which

$$\varepsilon 2^{3k} > \left(\frac{t^3}{e^2 2^k}\right)^{t/2} \quad \text{and} \quad \left(\frac{(t+2)^3}{e^2 2^k}\right)^{(t+2)/2} > \varepsilon.$$

Combining these two inequalities it follows that $(1 + 2/t)^{3t/2} (t+2)^{3 \cdot 2^k / e^2} > 1$, which is impossible as $t < 2^{k/3}$. Since $\varepsilon \leq 2^{-6k}$, it follows from (5) that $\varepsilon \leq (t^3/e^22^k)^{t/2} \leq \sqrt{\varepsilon}$.

We now apply Lemma 8 with parameters $K = 2^k$, $\delta = (e/K)^{1/3}$ to the truth-table of the function $(-1)^{F(x) \oplus \phi(x)}$, which is (K, ε) -biased. \square

Proof of Proposition 7. Initially we proceed as in the proof of Proposition 6 to obtain a partial assignment ρ with k unassigned inputs under which $F|_\rho$ has formula size $\tilde{s} = O((k/n)^2 s + \sqrt{(k/n)^2 s})$ for at least half the functions F . Let \mathcal{S} (for shrinkage) denote this event so that $\Pr[\mathcal{S}] \geq \frac{1}{2}$.

Tal [Tal16] showed that every formula of size \tilde{s} has correlation at least $\delta = 4(e2^{-k})^{1/3}$ (6) with some formula of size $\tilde{s}' = O(\sqrt{\tilde{s}} + \tilde{s} \log k/k)$. Let Φ be the set of all such formulas over inputs in U . Then we have

$$\mathbb{E}[|\langle F|_\rho, \Phi \rangle|] \geq \mathbb{E}[|\langle F|_\rho, \Phi \rangle| \mid \mathcal{S}] \cdot \Pr[\mathcal{S}] \geq \frac{\delta}{2},$$

where $|\langle f, \Phi \rangle|$ denotes the maximum value of $|\langle f, \phi \rangle|$ over all $\phi \in \Phi$. By Markov's inequality,

$$\Pr[|\langle F|_\rho, \Phi \rangle| \geq \delta/4] \geq \frac{\delta}{4}.$$

On the other hand, by a union bound and Corollary 9,

$$\Pr[|\langle F|_\rho, \Phi \rangle| \geq \delta/4] \leq 3|\Phi|\varepsilon^{1/6}.$$

From these two inequalities we obtain that

$$|\Phi| \geq \frac{1}{12} \cdot \varepsilon^{-1/6} \cdot \delta \geq \frac{\varepsilon^{-1/12}}{12}$$

by (6) and the assumption $\varepsilon \leq 2^{-6k}$. Since $|\Phi| \leq (9k)^{\tilde{s}'}$, it follows that $\tilde{s}' = \Omega(\log(1/\varepsilon)/\log k)$. A calculation shows that $s = \Omega(n^2 \log(1/\varepsilon)/k(\log k)^2)$ as desired. \square

Correlation with de Morgan formulas

Komargodski, Raz, and Tal [KR13, KRT13, Tal14] proved a correlation lower bound for small de Morgan formulas. Their main technical ingredient is the following high-probability shrinkage lemma for de Morgan formulas [KRT13, Tal14].

Lemma 10 (High-probability shrinkage). *For every constant $c > 0$ there exists a constant $c' > 0$ such that for every $c' \log n \leq k \leq n$ there exists a distribution over sets $U \subseteq [n]$ of size k such that for every formula f on n variables of size $s \leq n^c$, the probability that under a \bar{U} -random restriction ρ , $f|_\rho$ has formula size $\tilde{s} = (\log n)^{O(1)} \cdot (k/n)^2 \cdot s$ is at least $1 - \delta$, with $\delta = 2^{-\Omega(k)}$.*

By averaging, the existence of a distribution of sets U that achieves the desired shrinkage for every formula implies that for any given distribution on formulas of size at most s , there exists of a single set U that yields the same conclusion. We will use the lemma in this form.

Proposition 11. *Assuming $\exp(-2^{k/3}) \cdot 2^{3k} \leq \varepsilon \leq 2^{-6k}$ and $\omega(\log n) \leq k \leq n$, for every $(2^k, \varepsilon)$ -biased F , at most a $2^{-\Omega(k)}$ -fraction of F has correlation more than $2^{-\Omega(k)}$ with formulas of size at most $n^2 \log(1/\varepsilon)/(\log n)^{O(1)} k^2$.*

Proof of Proposition 11. Let \mathcal{C} (for correlating) be the event that F has correlation at least 2δ in absolute value with some formula \hat{F} of size at most s (which may depend on F) so that

$$\mathbb{E}_F[|\langle F, \hat{F} \rangle| \mid \mathcal{C}] \geq 2\delta.$$

We set $\delta = 2^{-\Omega(k)}$ and assume that $\delta \geq 2(e2^{-k})^{1/6}$ (7). For every set U of free variables and a uniformly random partial assignment ρ over the remaining variables,

$$\mathbb{E}_{F,\rho}[|\langle F|_\rho, \hat{F}|_\rho \rangle| \mid \mathcal{C}] \geq \mathbb{E}_F[|\mathbb{E}_\rho[\langle F|_\rho, \hat{F}|_\rho \rangle]| \mid \mathcal{C}] = \mathbb{E}_F[|\langle F, \hat{F} \rangle| \mid \mathcal{C}] \geq 2\delta. \quad (8)$$

By Lemma 10, there exists a set U of k variables such that

$$\Pr_{F,\rho}[\overline{\mathcal{S}} \mid \mathcal{C}] \leq \delta, \quad (9)$$

where ρ is a uniformly random partial assignment with free variables U , and \mathcal{S} is the event that $\hat{F}|_\rho$ has formula size at most $\tilde{s} = (\log n)^C \cdot (k/n)^2 \cdot s$ (10). By the formula for conditional expectations,

$$\begin{aligned} \mathbb{E}[|\langle F|_\rho, \hat{F}|_\rho \rangle| \mid \mathcal{C}] &= \mathbb{E}[|\langle F|_\rho, \hat{F}|_\rho \rangle| \mid \mathcal{CS}] \cdot \Pr[\mathcal{S} \mid \mathcal{C}] + \mathbb{E}[|\langle F|_\rho, \hat{F}|_\rho \rangle| \mid \mathcal{C}\overline{\mathcal{S}}] \cdot \Pr[\overline{\mathcal{S}} \mid \mathcal{C}] \\ &\leq \mathbb{E}[|\langle F|_\rho, \hat{F}|_\rho \rangle| \mid \mathcal{CS}] + \Pr[\overline{\mathcal{S}} \mid \mathcal{C}], \end{aligned}$$

so (8) and (9) imply that

$$\mathbb{E}_{F,\rho}[|\langle F|_\rho, \hat{F}|_\rho \rangle| \mid \mathcal{CS}] \geq \delta.$$

Let Φ be the set of all size- \tilde{s} formulas over the variables in U . Then $|\Phi| \leq (9k)^{\tilde{s}}$ (11). Since conditioned on \mathcal{S} all formulas $\hat{F}|_\rho$ are in Φ , it must be the case that

$$\mathbb{E}_{F,\rho}[|\langle F|_\rho, \Phi \rangle| \mid \mathcal{CS}] \geq \delta,$$

where $\langle f, \Phi \rangle$ denotes the maximum of $\langle f, \phi \rangle$ over all $\phi \in \Phi$. By the formula for conditional expectations, $\mathbb{E}_{F,\rho}[|\langle F|_\rho, \Phi \rangle|]$ must be at least $\delta \cdot \Pr[\mathcal{CS}]$. We can then bound $\Pr[\mathcal{CS}]$ by

$$\Pr[\mathcal{CS}] \leq \frac{1}{\delta} \cdot \mathbb{E}_{F,\rho}[|\langle F|_\rho, \Phi \rangle|] \leq \frac{1}{\delta} \left(\frac{\delta^2}{4} + \Pr_{F,\rho}[|\langle F|_\rho, \Phi \rangle| \geq \delta^2/4] \right). \quad (12)$$

By a union bound, Corollary 9, and (7),

$$\Pr_{F,\rho}[|\langle F|_\rho, \Phi \rangle| \geq \delta^2/4] \leq 3|\Phi| \cdot \varepsilon^{1/6}.$$

Using (7) and the assumption $\varepsilon \leq 2^{-6k}$, the right hand side is at most $(\delta^2/4) \cdot 12|\Phi|\varepsilon^{1/12}$. By (11) and (10), this quantity is at most $\delta^2/4$ as long as $s \leq n^2 \log(1/\varepsilon)/(\log n)^C k^2$. Plugging into (12), we conclude that $\Pr[\mathcal{CS}]$ is at most $\delta^2/2$ for formulas of the desired size.

Finally, applying (9) again, we have

$$\Pr[\mathcal{C}] = \frac{\Pr[\mathcal{CS}]}{1 - \Pr[\overline{\mathcal{S}} \mid \mathcal{C}]} \leq \frac{\delta/2}{1 - \delta} \leq \delta. \quad \square$$

3 Main theorems

Small-biased functions require large formulas

Theorem 12. *Any strongly explicit small-biased family $SB_{n,2^{-2n}}: \{0,1\}^{O(n)} \rightarrow \{0,1\}$:*

1. *requires fan-in $\log \log \log n$ formulas of size $\Omega(n^2 / \log n \log \log n)$,*
2. *requires fan-in two formulas of size $\Omega(n^2 / \log n)$,*
3. *requires de Morgan formulas of size $\Omega(n^3 / \log n (\log \log n)^2)$,*
4. *has correlation at most $2^{-\Omega(k)}$ with de Morgan formulas of size at most $n^3 / (\log n)^{O(1)} k^2$ for any k such that $\omega(\log n) \leq k \leq n$.*

Items 1 and 2 match the explicit formula size lower bounds of Nečiporuk. Item 3 matches the lower bound of Tal [Tal16]. Item 4 is a very minor improvement over the lower bound of Tal [Tal14]: His proof requires the additional assumption $k \leq n^{1/3}$.

Proof. Let F be the random function $F(x) = SB_{n,2^{-2n}}(s, x)$ for uniformly random s . To obtain item 1, we apply Proposition 3 with $k = \log n$ and Claim 1. For item 2, we apply Proposition 3 with $k = \log n - 1$ and Claim 1.

For item 3, we apply Proposition 7 with $k = 3 \log n$ and $\varepsilon = n^9 e^{-n}$. (Proposition 6 with $k = \log n$ gives the weaker bound $\Omega(n^3 / (\log n)^2 \log \log n)$.)

For item 4, we apply Proposition 11 with $\varepsilon = 2^{-2n}$, assuming without loss of generality that $k \leq n/6$. The conclusion is that at most a $2^{-\Omega(k)}$ -fraction of F can have correlation more than $2^{-\Omega(k)}$ with formulas of size s . Therefore the correlation between $SB_{n,2^{-2n}}$ and size s formulas can be at most $2^{-\Omega(k)}$. \square

Moderate formulas for small bias

Theorem 13. *For every n, k , and ε , there exists a $(2^k, \varepsilon)$ -biased $F: \{0,1\}^n \rightarrow \{0,1\}$ of fan-in two formula size $O(nk^2 \cdot \log 1/\varepsilon)$.*

Applying Claim 1 and a suitable change of parameters we obtain the following corollary to Theorem 13:

Corollary 14. *For every n, K , and ε there exist (K, ε) -wise independent functions with formula size $O(n \cdot (\log K)^2 \cdot (K + \log 1/\varepsilon))$.*

Proof of Theorem 13. Let $H_t: \{0,1\}^n \rightarrow \{0,1\}$ be the random function

$$H_t(x) = \begin{cases} \text{a random bit,} & \text{if } Ax = b, \\ 0, & \text{if not,} \end{cases}$$

where A and b are a uniformly random $t \times n$ matrix and t -dimensional boolean vector, respectively, and all algebra is over \mathbb{F}_2 . We let

$$F = F_1 \oplus F_2 \oplus \cdots \oplus F_{k+2},$$

where the F_t are independent XORs of $6 \log 1/\varepsilon$ independent copies of H_t . The formula size of F is $O(nk^2 \cdot \log 1/\varepsilon)$.

We now prove that F is $(2^k, \varepsilon)$ -biased. Let X be any nonempty set of at most 2^k distinct inputs. Set $t = \lfloor \log |X| \rfloor + 2$ and let \mathcal{U} (for unique) be the event that exactly one x in X satisfies $Ax = b$ for a random $t \times n$ matrix A and t -dimensional vector b . By the isolation lemma of Valiant and Vazirani [VV86], \mathcal{U} has probability at least $1/8$ (see for example [AB09, Lemma 17.19]). By the rule of conditional expectations,

$$\begin{aligned} \left| \mathbb{E} \left[\prod_{x \in X} (-1)^{H_t(x)} \right] \right| &\leq \left| \mathbb{E} \left[\prod_{x \in X} (-1)^{H_t(x)} \middle| \mathcal{U} \right] \right| \cdot \Pr[\mathcal{U}] + \left| \mathbb{E} \left[\prod_{x \in X} (-1)^{H_t(x)} \middle| \bar{\mathcal{U}} \right] \right| \cdot \Pr[\bar{\mathcal{U}}] \\ &\leq \left| \mathbb{E} [(-1)^{H_t(u)} \mid \mathcal{U}] \right| \cdot \Pr[\mathcal{U}] + 1 \cdot \Pr[\bar{\mathcal{U}}] \\ &= 0 \cdot \Pr[\mathcal{U}] + 1 \cdot \Pr[\bar{\mathcal{U}}] \\ &\leq 7/8. \end{aligned}$$

By independence, it follows that

$$\left| \mathbb{E} \left[\prod_{x \in X} (-1)^{F_t(x)} \right] \right| = \left| \mathbb{E} \left[\prod_{x \in X} (-1)^{H_t(x)} \right] \right|^{6 \log 1/\varepsilon} \leq \left(\frac{7}{8} \right)^{6 \log 1/\varepsilon} \leq \varepsilon,$$

so $|\mathbb{E}[\prod_{x \in X} (-1)^{F(x)}]| = \prod_{t=1}^{k-2} |\mathbb{E}[\prod_{x \in X} (-1)^{F_t(x)}]|$ is also upper bounded by ε . \square

Our small-biased function can be viewed as a simplified variant of a construction of Naor and Naor [NN93, Section 3.1.1]. The simplifications can be partly explained by a difference in objectives: Naor and Naor (and other constructions) aim to optimize the seed length, while we are interested in minimizing formula size.

By the standard simulation of fan-in two formulas by de Morgan formulas, F has de Morgan formula size at most $O((nk^2 \log 1/\varepsilon)^2)$. The de Morgan formula size analysis can be slightly improved to $O(n^2 k^3 (\log 1/\varepsilon)^2)$ by observing that the middle layer of AND gates does not suffer from the quadratic blow-up.

Specifically, in the parameter settings used in the proof of items 1 to 3 in Theorem 12, the function F has fan-in two formula size $O((n \log n)^2)$ and de Morgan formula size $O(n^4 (\log n)^3)$. We did not attempt to match the lower bound in item 4 by an explicit construction.

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