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Representations of Monotone Boolean Functions by Linear Programs

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Abstract

We introduce the notion of monotone linear programming circuits (MLP circuits), a model of computation for partial Boolean functions. Using this model, we prove the following results.

1. MLP circuits are superpolynomially stronger than monotone Boolean circuits.
2. MLP circuits are exponentially stronger than monotone span programs.
3. MLP circuits can be used to provide monotone feasibility interpolation theorems for Lovász-Schrijver proof systems, and for mixed Lovász-Schrijver proof systems.
4. The Lovász-Schrijver proof system cannot be polynomially simulated by the cutting planes proof system. This is the first result showing a separation between these two proof systems.

Finally, we discuss connections between the problem of proving lower bounds on the size of MLPs and the problem of proving lower bounds on extended formulations of polytopes.

1 Introduction

Superpolynomial lower bounds on the size of Boolean circuits computing explicit Boolean functions have only been proved for circuits from some specific families of circuits. A prominent role among these families is played by *monotone Boolean circuits.* Exponential lower bounds on monotone Boolean circuits were proved already in 1985 by Razborov [26]. In 1997 Krajíček discovered that lower bounds on monotone complexity of particular partial Boolean functions can be used to prove lower bounds on resolution proofs [18]. Incidentally, the functions used in Razborov's lower bound were just of the form needed for resolution lower bounds. Exponential lower bounds on resolution proofs had been proved before (coincidentally about at the same time as Razborov's lower bounds). However, Krajíček came

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up with a new general method, the so called *feasible interpolation,* that potentially could be used for other proof systems. Indeed, soon after his result, this method was used to prove exponential lower bounds on the cutting-planes proof system [22, 15]. That lower bound is based on a generalization of Razborov's lower bounds to a more general monotone computational model, the *monotone real circuits*. Another monotone computational model for which superpolynomial lower bounds have been obtained is the *monotone span program* model [2, 11]. An exponential lower bound on the size of monotone span programs have been recently obtained in [7]. For a long time the best known lower bound for this model of computation was of the order of $n^{\Omega(\log n)}$ [2]. Again, superpolynomial lower bounds on the size of monotone span programs can be used to derive lower bounds on the degree of Nullstellensatz proofs, as shown in [24].[1]

The results listed above suggest that proving lower bounds on stronger and stronger models of monotone computation may be a promising approach towards proving lower bounds on stronger proof systems. Indeed, in his survey article [27] Razborov presents the problem of understanding feasible interpolation for stronger systems as one of the most challenging ones in proof complexity theory.

In this work we introduce several computational models based on the notion of *monotone linear program*. In particular, we introduce the notion of *monotone linear programming gate* (MLP gate). In its most basic form, an MLP gate is a *partial* function $g : \mathbb{R}^n \to \{\mathbb{R}, *\}$ of the form $g(y) = \max\{c \cdot x \mid Ax \leq b + By, x \geq 0\}$ where y is a set of input variables, and B is a non-negative matrix. The complexity of such a gate is defined as the number of rows plus the number of columns in the matrix A. For each assignment $\alpha \in \mathbb{R}^n$ of the variables y the value $g(\alpha)$ is the optimal value of the linear program with objective function $c \cdot x$, and constraints $Ax \leq b + B\alpha$. The requirement that $B \geq 0$ guarantees monotonicity, i.e., that $g(\alpha) \leq g(\alpha')$ whenever $g(\alpha)$ is defined and $\alpha \leq \alpha'$. We note that the value $g(\alpha)$ is considered to be undefined if the associated linear program $\max\{c \cdot x \mid Ax \leq b + B\alpha\}$ has no solution. In this case, we set $g(\alpha) = *$. Other variants of MLP gates are defined in a similar way by allowing the input variables to occur in the objective function, and by allowing the corresponding linear programs to be minimizing or maximizing. We say that an MLP gate is weak if the input variables occur either in the objective function or in the constraints. We say that an MLP gate is strong if the input variables occur in both the objective function and in the constraints.

An MLP circuit is the straightforward generalization of unbounded-fan-in monotone Boolean circuits in which gates are MLP gates, instead of boolean gates. In Theorem 4.1 we show that if all gates of an MLP circuit C are weak, then this circuit can be simulated by a single weak MLP gate ℓ_C whose size is polynomial on the size of C. Since the AND and OR gates can be faithfully simulated by weak MLP gates, we have that monotone Boolean circuits can be polynomially simulated by weak MLP circuits (Theorem 5.1). In contrast, we show that weak MLP gates are super-polynomially stronger than monotone Boolean circuits. On the one hand, Razborov has shown that that any monotone Boolean circuit computing

[1] We note however that strong degree lower bounds for Nullstellensatz proofs can be proved using more direct methods [3, 6, 13, 1].

the *bipartite perfect matching function* $\mathrm{BPM}_n : \{0,1\}^{n^2} \to \{0,1\}$ must have size at least $n^{\Omega(\log n)}$. On the other hand, a classical results in linear programming theory [30] can be used to show that the same function can be computed by weak MLP gates of polynomial size.

In [2], Babai, Gál and Wigderson showed that there is a function that can be computed by span programs of linear size but which require superpolynomial-size monotone Boolean circuits. Recently, Cook et al. [7] showed that there is a function that can be computed by polynomial-size monotone Boolean circuits, but that requires exponential-size monotone span programs over the reals. Therefore, monotone span programs (which we will abbreviate by MSPs) and monotone Boolean circuits are incomparable in the sense that none of these models can polynomially simulate the other. In Theorem 5.4 we show that a particular type of weak MLP gate can polynomially simulate monotone span programs over the reals. On the other hand, by combining the results in [7] with Theorem 5.4, we have that these weak MLP gates are exponentially stronger than monotone span programs over reals. Therefore, while MBCs are incomparable with MSPs, weak MLP-gates are strictly stronger than both models of computation.

Next we turn to the problem of proving a monotone interpolation theorem for Lovász-Schrijver proof systems [20]. Currently, size lower bounds for these systems have been proved only with respect to tree-like proofs [21], and therefore, it seems reasonable that a monotone interpolation theorem for this system may be a first step towards proving size lower bounds for general LS proof systems. Towards this goal we show that MLP circuits which are constituted by strong MLP gates can be used to provide a *monotone* feasible interpolation theorem for LS proof systems. In other words, we reduce the problem of proving superpolynomial lower bounds for the size of LS proofs, to the problem of proving lower bounds on the size of MLP circuits with strong gates.

It is worth noting that we do not know how to collapse MLP circuits with strong gates into a single strong gate. Nevertheless, in Theorem 6.2 we show that a single weak MLP gate suffices in a monotone interpolation theorem for LS proofs of unsatisfiable sets of *mixed* inequalities of a certain form. Here, a mixed inequality is an inequality which involve both Boolean variables and real variables. Using this interpolation theorem together with a size lower bound for monotone real circuits due to Fu [10], we can show that MLP-circuits cannot be polynomially simulated by monotone real circuits (Theorem 6.10).

We show that the cutting-planes proof system cannot polynomially simulate the LS proof system (Corollary 6.8). Understanding the mutual relation between the power of the cutting-planes proof system and the LS proof system is a longstanding open problem in proof complexity theory. Our result solves one direction of this mutual relation by showing that for some unsatisfiable set of inequalities, LS proofs can be superpolynomially more concise than cutting-planes proofs.

Monotone linear programs programs are, in a sense, generalizations of monotone Boolean circuits and monotone span programs. The lower bounds for monotone Boolean circuits and monotone span programs were proved by two different techniques. Therefore it will be necessary to develop a new lower bound method for proving superpolynomial lower bounds

on monotone linear programs. A possible approach may be based on strengthening lower bounds on extended formulations, which is a related, but apparently easier problem. A lower bound on extended formulation is a lower bound on the number of inequalities needed to define an extension of a polytope to some higher dimension. Such lower bounds have been proven, in particular, for polytopes spanned by the 0-1 vectors representing minterms of certain monotone Boolean functions [29, 9, 4, 5]. To prove a lower bound on the size of weak MLP gates, it will be necessary to prove lower bounds on the size of extended formulations for all polytopes of a certain form that separate minterms from maxterms. This is clearly a much harder problem, but there are results on extended formulations that go in this direction [4, 5]. However, Theorem 6.10 suggests that this will surely not be easy. It gives an example of a monotone function such that the set of ones has only exponentially large extended formulation, but the minterms can be separated from a large subset of maxterms by a polynomial size dual MLP.

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2 Preliminaries

Monotone Partial Boolean Functions: A *partial Boolean function* is a mapping of the form $F : \{0,1\}^n \to \{0,1,*\}$. Intuitively, the function F should be regarded as being undefined on each point $p \in \{0,1\}^n$ for which $F(p) = *$. The support of F, which is defined as $support(F) = F^{-1}(\{0,1\})$, is the set of all points $p \in \{0,1\}^n$ for which F is defined. If p and p' are Boolean strings in $\{0,1\}^n$, then we write $p \geq p'$ to indicate that $p_i \geq p'_i$ for each $i \in \{1,...,n\}$. We say that a partial Boolean function $F : \{0,1\}^n \to \{0,1,*\}$ is *monotone* if $F(p) = 1$ whenever $p \geq p'$ and $F(p') = 1$.

Linear Programs: We use the following conventions. Variables x, y, z are used to denote real vectors, while variables p, q, r are used to denote strings of Boolean variables. $A \in \mathbb{R}^{m \times k}$ means that A is a real matrix with m rows and k columns. For vectors x and y, $x \leq y$ means $x_i \leq y_i$ for all coordinates i; the same for matrices and Boolean strings. As an abuse of notation, we write 0 (1) to denote vectors in which all coordinates are equal to 0 (1). For two vectors x and y, we will denote their scalar product by $x \cdot y$.

A linear program is an optimization problem of the form

$$\max\{c^T \cdot x \mid Ax \leq b,\ x \geq 0\}, \tag{1}$$

where $A \in \mathbb{R}^{m \times k}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^k$ for some $m, k \in \mathbb{N}$. The dual of the linear program of Equation 1 is defined as follows.

$$\min\{y^T \cdot b \mid A^T y \geq c,\ y \geq 0\}. \tag{2}$$

According to the *Linear Programming Duality*,

$$\max c^T \cdot x = \min y^T \cdot b, \tag{3}$$

provided that the maximum in Equation 1 and the minimum in Equation 2 exist.

3 Monotone Linear-Programming Gates

In this section we let A be a matrix in $\mathbb{R}^{m\times k}$, b be a vector in $\mathbb{R}^m$, c be a vector in $\mathbb{R}^k$, and B and C be matrices in $\mathbb{R}^{m\times n}$ with $B \geq 0$ and $C \geq 0$. Below, we define the notion of *monotone linear programming gate* (MLP gate).

Definition 1 (MLP Gate) *An MLP gate is a partial function $\ell : \mathbb{R}^n \to \mathbb{R} \cup \{*\}$ whose value at each point $y \in \mathbb{R}^n$ is specified via a monotone linear program. More precisely, we consider the following six types of MLP gates.*

MAX-RIGHT: $\quad \ell(y) = \max\{c^T \cdot x \mid Ax \leq b + By,\ x \geq 0\}$

MIN-RIGHT: $\quad \ell(y) = \min\{c^T \cdot x \mid Ax \geq b + By,\ x \geq 0\}$

MAX-LEFT: $\quad \ell(y) = \max\{(c + Cy)^T \cdot x \mid Ax \leq b,\ x \geq 0\}$

MIN-LEFT: $\quad \ell(y) = \min\{(c + Cy)^T \cdot x \mid Ax \geq b,\ x \geq 0\}$

MAX: $\quad \ell(y) = \max\{(c + Cy)^T \cdot x \mid Ax \leq b + By,\ x \geq 0\}$

MIN: $\quad \ell(y) = \min\{(c + Cy)^T \cdot x \mid Ax \geq b + by,\ x \geq 0\}$

Intuitively, the variables y should be regarded as input variables, while the variables x should be regarded as internal variables. If the linear program specifying a gate $\ell(y)$ has no solution when setting y to a particular point $\alpha \in \mathbb{R}^n$, then we set $\ell(\alpha) = *$. In other words, in this case we regard the value $\ell(\alpha)$ as being undefined. We note that the requirement $B \geq 0$ guarantees that the gates introduced above are monotone. More precisely, if $\alpha \leq \alpha'$, and both $\ell(\alpha)$ and $\ell(\alpha')$ are well defined, then $\ell(\alpha) \leq \ell(\alpha')$. The size $|\ell|$ of an MLP gate ℓ is defined as the number of rows plus the number of columns in the matrix A.

The gates of type MAX-RIGHT, MAX-LEFT, MIN-RIGHT and MIN-LEFT are called weak gates. Note that in these gates, the input variables y occur either only in the objective function, or only in the constraints. The gates of type MAX and MIN are called strong gates. The input variables in strong gates occur both in the constraints and in the objective function.

Definition 2 (MLP-Gate Representation) *We say that an MLP gate $\ell : \mathbb{R}^n \to \mathbb{R} \cup \{*\}$ represents a partial Boolean function $F : \{0,1\}^n \to \{0,1,*\}$ if the following can be observed for each $a \in \{0,1\}^n$.*

1. $\ell(a) > 0$ *if* $F(a) = 1$,

2. $\ell(a) \leq 0$ *if* $F(a) = 0$.

3.1 Sign Representations

We say that an MLP gate ℓ sign-represents a partial Boolean function $F : \{0,1\}^n \to \{0,1,*\}$ if the following conditions can be verified for each $a \in \{0,1\}^n$.

1. $\ell(a) > 0$ if $F(a) = 1$.

2. $\ell(a) < 0$ if $F(a) = 0$.

Proposition 3.1 *Let $F : \{0,1\}^n \to \{0,1,*\}$ be a partial Boolean function and assume that F can be represented by an MLP gate of type τ and size s. Then F can be sign-represented by an MLP gate ℓ' of type τ and size $O(s)$.*

Proof. We will prove this proposition with respect to MAX-RIGHT MLP gates. The proofs for all other types of gates is completely analogous.

Assume that F can be represented by a MAX-RIGHT MLP gate

$$\ell(y) = \max\{c \cdot x \mid Ax \leq b + By,\ x \geq 0\},$$

and let

$$\varepsilon = \min_{p \in \{0,1\}^n} \{\ell(p) \mid \ell(p) > 0\}.$$

In other words, ε is the smallest positive number which is the result of evaluating ℓ on a binary string. Let

$$\ell'(y) = \max\{c \cdot x - x' \mid x' = \varepsilon/2, Ax \leq b + By,\ x \geq 0\}.$$

Then we have $\ell'(y) = \ell(y) - \varepsilon/2$ for each $y \in \mathbb{R}^n$. This implies that, for each $y \in \mathbb{R}^n$, if $\ell(y) > 0$, then $\ell'(y) > \varepsilon/2 > 0$ and if $\ell(y) \leq 0$ then $\ell'(y) \leq -\varepsilon/2 < 0$. ∎

3.2 Weak vs Strong Gates

Recall that weak MLP gates are gates where input variables occur either only in the objective function, or only in the constraints. On the other hand, strong MLP gates are gates where input variables are allowed to occur both in the objective function and in the constraints.

The distinction between weak and strong gates is motivated by the fact that while weak gates are only able to compute piecewise-linear monotone real functions, strong gates may compute quadratic monotone real functions.

Proposition 3.2 *Let $\ell : \mathbb{R}^m \to \mathbb{R} \cup \{*\}$ be a weak MLP gate. Then the graph*

$$\{(y, \ell(y)) \mid y \in \mathbb{R}^m, \ell(y) \in \mathbb{R}\}$$

is piecewise linear.

Proof. We show that the proposition is valid for MAX-RIGHT MLP gates. The proof that it is valid for other types of weak gates is analogous. Let $\ell(y) = \max\{c \cdot x \mid Ax \leq b + By,\ x \geq 0\}$ be a MAX-RIGHT MLP gate. This gate can be alternatively represented as $\ell(y) = \max\{x_0 \mid Ax \leq b + By,\ x \geq 0,\ x_0 \leq c \cdot x\}$ where x_0 is a new variable. Let P be the polyhedron on variables x, y and x_0 defined by the inequalities $Ax \leq b + By$, $x \geq 0$ and $x_0 \leq c \cdot x$. Let P' be the polyhedron obtained by projecting P to the variables y and x_0. Then the graph of ℓ is the set $S = \{(y, x_0) \mid \forall x'_0$ such that $(y, x'_0) \in P', x'_0 \leq x_0\}$. Since S is a union of faces of P', S is piecewise linear. Note however that the set S is not necessarily convex. ■

On the other hand, the graph of strong gates may not be piecewise linear even for gates with a unique input variable.

Observation 3.3 *Strong MLP gates may compute functions whose graph is not piecewise linear.*

Proof. Consider the following MAX MLP gate ℓ and MIN MLP gate ℓ'.

$$\ell(y) = \max\{y \cdot x \mid x \leq y,\ x \geq 0\ \} \quad \ell'(y) = \min\{y \cdot x \mid x \geq y,\ x \geq 0\ \}. \tag{4}$$

Then we have that for each $y \geq \mathbb{R}^+$, $\ell(y) = y^2 = \ell'(y)$. This shows that the graphs of ℓ and ℓ' are not piecewise linear. ■

Proposition 3.2 and Observation 3.3 show that strong MLP gates are a strictly stronger model than weak gates when it comes to defining monotone *real* functions. Therefore proving lower bounds for the size of strong MLP gates computing some specific monotone Boolean function $F : \{0,1\}^n \to \{0,1,*\}$ may be harder than proving such lower bounds for the size of weak MLP gates computing F. We note however that it is still conceivable that every partial monotone Boolean function $F : \{0,1\}^n \to \{0,1,*\}$ that can be represented by strong MLP gates of size s, can be also represented by weak MLP gates of size $s^{O(1)}$.

3.3 Boolean Duality vs Linear-Programming Duality

In this section we clarify some relationships between linear-programing duality MLP representations. Towards this goal, it will be convenient to define the notions of a dual of a given type of gate. More precisely, we say that the type MAX is dual to MIN, that MAX-RIGHT is dual to MIN-LEFT, and that MAX-LEFT is dual to MIN-RIGHT. If τ is a type of gate we let τ^d denote its dual type. The following observation states that MLP gates of type τ can be simulated by MLP gates of type τ of similar complexity.

Observation 3.4 *If a partial real monotone function $f : \mathbb{R}^n \to \mathbb{R} \cup \{*\}$ can be specified via an MLP gate of type τ and size s, then f can be also specified via an MLP gate of type τ^d and size $O(s)$.*

Proof. We prove the proposition with respect to MAX-RIGHT MLP gates. The proof for other types of gates is analogous. Let $\ell(y) = \max\{c \cdot x \mid Ax \leq b,\ x \geq 0\}$ be a MAX-RIGHT MLP gate such that $f(y) = \ell(y)$ for every $y \in \mathbb{R}^n$. Consider the following MIN-LEFT MLP gate: $\ell(y) = \min\{b \cdot x \mid A^T x \geq c,\ x \geq 0\}$. Then by linear programming duality, for each $\alpha \in \mathbb{R}^n$, $\ell(\alpha)$ is defined if and only if $\ell'(y)$ is defined and $\ell'(y) = \ell(y)$. ∎

We say that the types MAX-RIGHT and MIN-RIGHT are *semi-dual* to each other. Analogously, the types MAX-LEFT and MIN-LEFT are semi-dual to each other. If τ is type of gate, we let τ^{sd} be its semi-dual type. It is not clear whether functions that can be specified via weak gates of a given type τ may be also specified by gates of type τ^{sd} without a superpolynomial increase in complexity. However, we will see next that if F is a partial Boolean function which can be represented by an MLP gate of type τ and size s, then the *Boolean-dual* of F can be represented by an MLP gate of type τ^{sd} and size $O(s)$.

We say that a partial monotone Boolean function $F : \{0,1\}^n \to \{0,1,*\}$ is *dualizable* if $F(\neg p_1, \ldots, \neg p_n)$ is well defined whenever $F(p_1, \ldots, p_n)$ is well defined. If F is dualizable, then the *Boolean dual* of F is the partial Boolean function $F^d : \{0,1\}^n \to \{0,1,*\}$ which is obtained by setting $F^d(p) = *$ for each point $p \notin \mathit{support}(F)$, and by setting $F^d(p_1, \ldots, p_n) := \neg F(\neg p_1, \ldots, \neg p_n)$ for each $p \in \mathit{support}(F)$.

Proposition 3.5 *Let $F : \{0,1\}^n \to \{0,1,*\}$ be a dualizable partial Boolean function. If F can be represented by an MLP gate ℓ of type τ and size s, then F^d can be represented by an MLP gate ℓ^{sd} of type τ^{sd} and size $O(s)$.*

Proof. We will show that if a function F can be represented by MAX-RIGHT MLP gate of size s, then F^d can be represented by a MAX-LEFT MLP gate of size $O(s)$. The proof for other types of gates follows an analogous reasoning.

Assume that F can be represented by a MAX-RIGHT MLP gate ℓ. Then by Proposition 3.1, F can be represented by a MAX-RIGHT gate ℓ' such that for each $p \in \{0,1\}^n$, $\ell'(p) > 0$ whenever $F(p) = 1$ and $\ell'(p) < 0$ whenever $F(p) = 0$. In other words, $\ell'(p)$ sign-represents F. Let

$$\ell'(p) = \max\{c^T \cdot x \mid Ax \leq b + Bp,\ x \geq 0\}$$

be such gate. Then it should be clear that the function F^d can be represented by the following MIN-RIGHT MLP gate, where $\bar{1}$ denotes the all-ones vector.

$$\begin{aligned}
\ell''(p) &= -\ell'(-p) \\
&= \min\{-c^T \cdot x \mid Ax \leq b + B(\bar{1} - p),\ x \geq 0\} \\
&= \min\{-c^T \cdot x \mid -Ax \geq -b - B\bar{1} + Bp,\ x \geq 0\}
\end{aligned}$$

∎

4 Monotone Linear Programming Circuits

Monotone linear programming circuits (MLP circuits) may be defined as the straightforward generalization of unbounded fan-in monotone Boolean circuits where monotone linear programming gates are used instead of Boolean gates. Formally, it will be convenient for us to define MLP circuits using the notation of straight-line programs, i.e., as a sequence of instructions of a suitable form.

Definition 3 (MLP Circuit) *An MLP circuit is a sequence of instructions* $C = (I_1, I_2, ..., I_r)$ *where each instruction* I_i *has one of the following forms:*

1. $I_i \equiv Input(y_i)$.
2. $I_i \equiv y_i \leftarrow c_i$, *where* y_i *is a variable and* $c_i \in \mathbb{R}$.
3. $I_i \equiv y_i \leftarrow \ell_i(y_{i_1}, ..., y_{i_{n_i}})$ *where* y_i *is a variable and* $\ell_i(y_{i_1}, ..., y_{i_{n_i}})$ *is an MLP gate with input variables* $y_{i_1}, ..., y_{i_{n_i}}$ *such that* $i_j < i$ *for each* $j \in [n_i]$.

We say that instructions of the third form are MLP instructions. We assume that the last instruction, I_r, is an MLP instruction. We say that the variable y_r, which occurs in the left-hand side of I_r is the output variable of C. For each i such that $I_i \equiv Input(y_i)$, we say that y_i is an input variable.

Let $\mathbf{y} = (y_{j_1}, y_{j_2}, ..., y_{j_n})$ be the input variables of C, and let $a \in \mathbb{R}^n$ be an assignment of the variables in $\mathbf{y}$, where $y_{j_l} = a_l$ for each $l \in \{1, ..., n\}$. For each $i \in \{1, ..., r\}$, the value induced by a on variable y_i, which is denoted by $val_a(y_i)$, is inductively defined as follows.

1. If $I_i \equiv Input(y_i)$, then $val_a(y_i) = a_i$.
2. If $I_i \equiv y_i \leftarrow c_i$, then $val_a(y_i) = c_i$.
3. If $I_i \equiv y_i \leftarrow \ell_i(y_{i_1}, ..., y_{i_{n_i}})$, and $val_a(y_{i_j}) \in \mathbb{R}$ for each $j \in \{1, ..., r\}$, then $val_a(y_i) = \ell_i(val(y_{i_1}), ..., val(y_{i_{n_i}}))$. Otherwise, $val_a(y_i) = *$.

For each assignment $a \in \mathbb{R}^n$ of the variables input variables of C, we let $C(a) = val_a(y_r)$ be the value induced by a on the output variable of C. Intuitively, the values of the variables y_i are computed instruction after instruction. If at step i, the value of the variable y_i is set to $*$ ($val_a(y) = *$), meaning that the linear program associated with the instruction I_i has no solution, then the value $*$ is propagated until the last instruction, and the circuit will output $*$.

Definition 4 (MLP-Circuit Representation) *We say that an MLP-circuit* C *represents a partial Boolean function* $F : \{0,1\}^n \to \{0, 1, *\}$ *if the following conditions are satisfied for each* $a \in \{0,1\}^n$.

1. $C(a) > 0$ *if* $F(a) = 1$.
2. $C(a) \leq 0$ *if* $F(a) = 0$.

We say that an MLP-circuit C *sharply* represents $F : \{0,1\}^m \to \{0,1,*\}$ if $C(a) = 1$ whenever $F(a) = 1$ and $C(a) = 0$ whenever $F(a) = 0$. We define the size of an MLP circuit C as the sum of the sizes of MLP gates occurring in C. The next theorem states that if all gates in an MLP circuit C are weak MLP gates with the same type τ, then this circuit can be polynomially simulated by a *single* MLP gate ℓ of type τ.

Theorem 4.1 (From Circuits to Gates) *Let $C = (I_1, ..., I_r)$ be an MLP circuit where all gates in C are weak MLP gates of type τ. Then there is an MLP gate ℓ_C of type τ and size $O(s)$ such that for each $a \in \mathbb{R}^n$ for which $C(a)$ is defined, $\ell_C(a) = C(a)$.*

Proof. First, we will prove the theorem with respect to MAX-RIGHT MLP gates. Let $C = (I_1, I_2, ..., I_r)$ be an MLP circuit in which all gates are MAX-RIGHT MLP gates. For each $i \in \{1, ..., r\}$ if I_i is an MLP instruction, then we let

$$I_i \equiv y_i \leftarrow \ell_i(y^i) = \max\{c^i \cdot x^i \mid A^i x^i \leq b^i + B^i y^i\},$$

where $y^i = (y_{i_1}, ..., y_{i_{n_i}})$ are the input variables of ℓ_i and $x^i = (x^i_1, ..., x^i_{k_i})$ are the internal variables of ℓ_i. We let $M = \{i \mid i \text{ is an MLP instruction}\}$ be the set of all i's such that I_i is an MLP instruction. We let $\mathbf{y} = (y_{j_1}, ..., y_{j_n})$ be the input variables of C, and $\mathbf{x} = x^{i_1} x^{i_2} ... x^{i_{|M|}}$ with $i_j \in M$ and $i_1 < i_2 < ... < i_{|M|}$ be a tuple containing all internal variables of MLP gates occurring in C. For each $i \in M$, let $\mathbf{A}^i\mathbf{x} \leq \mathbf{b}^i + \mathbf{B}^i\mathbf{y}$ be the system of inequalities obtained from $A^i x^i \leq b^i + B^i y^i$ by replacing each variable $y_{i_j} \in y^i$ which is not an input variable of C, with the value c_{i_j} if $I_{i_j} \equiv y_{i_j} \leftarrow c_{i_j}$, and with the expression $c^{i_j} \cdot x^{i_j}$ if I_{i_j} is an MLP instruction. Now, for $i \in M$, consider the following MAX-RIGHT MLP gate.

$$\boldsymbol{\ell}_i(\mathbf{y}) = \max\{c^i \cdot x^i \mid \mathbf{A}^j\mathbf{x} \leq \mathbf{b}^j + \mathbf{B}^j\mathbf{y},\ j \in M,\ j \leq i\} \tag{5}$$

In other words, the objective function of $\boldsymbol{\ell}_i(\mathbf{y})$ is the same as the objective function of the gate ℓ_i, but the constraints of $\boldsymbol{\ell}_i(\mathbf{y})$ are formed by all inequalities $\mathbf{A}^j\mathbf{x} \leq \mathbf{b}^j + \mathbf{B}^j\mathbf{y}$ corresponding to constraints of gates ℓ_j for $j < i$. If u is an assignment of the tuple of variables $\mathbf{x}$, then for each $j \in M$, we let $u^j \in \mathbb{R}^{k_j}$ be the assignment induced by u on the internal variables $x^j = (x_{j_1}, ..., x_{j_{k_i}})$ of gate ℓ_j. Let a be an assignment of the input variables $\mathbf{y}$, and u be an assignment of the internal variables $\mathbf{x}$. Then we say that the pair (a, u) is consistent with $\boldsymbol{\ell}_i$ if (a, u) satisfies all constraints of $\boldsymbol{\ell}_i$.

The following claim implies that for each $a \in \mathbb{R}^n$ such that $C(a)$ is defined, the value $C(a)$ is equal to the value $\boldsymbol{\ell}_r(a)$.

Claim 1 *Let $a \in \mathbb{R}^n$. If $C(a)$ is defined then the following conditions are satisfied for each $i \in M$.*

1. *There exists an assignment u of the variables $\mathbf{x}$, such that (a, u) is consistent with $\boldsymbol{\ell}_i$ and for each $j \in M$ with $j \leq i$, $c^j \cdot u^j = val_a(y_j)$.*

2. *For each assignment u of the variables $\mathbf{x}$, such that (a, u) is consistent with $\boldsymbol{\ell}_i$, and each $j \in M$ with $j \leq i$, $c^j \cdot u^j \leq val_a(y_j)$.*

3. $\boldsymbol{\ell}_i(a) = val_a(y_i)$.

We note that if $|M| = 1$ then the circuit has a unique MLP gate and the claim is trivial. Therefore, we assume that $|M| \geq 2$. Let $a \in \mathbb{R}^n$ be an assignment of the input variables $\mathbf{y}$ such that $C(a)$ is defined. The proof of Claim 1 is by induction on i. In the base case, let i be the smallest number in M. In this case, $y_i \leftarrow \ell_i(y^i)$ is the first MLP gate occurring in C, and therefore the gate $\boldsymbol{\ell}_i(\mathbf{y})$ has precisely the same objective function and constraints as $\ell_i(y^i)$. This implies that the value $\boldsymbol{\ell}_i(a)$ is equal to the value induced by a on y_i. Therefore, the claim is valid in the base case. Now, let l be an arbitrary number in M and let i be the greatest number in M which smaller than l. Let $I_l \equiv\ y_l \leftarrow \ell_l(y^l)$, where $y^l = (y_{l_1}, ..., y_{l_{n_l}})$. Then the objective function of $\boldsymbol{\ell}_l(\mathbf{y})$ is $c^l \cdot x^l$, and the constraints of $\boldsymbol{\ell}_l(\mathbf{y})$ contain all constraints of $\boldsymbol{\ell}_i(\mathbf{y})$ together with the constraints $\mathbf{A}^l\mathbf{x} \leq \mathbf{b}^l + \mathbf{B}^l\mathbf{y}$ which are obtained from $A^l x^l \leq b^l + B^l y^l$ by making the substitution $y_{l_j} \leftarrow c^{l_j} \cdot x^{l_j}$ for each $j \in \{1, ..., n_l\}$. By the induction hypothesis, Conditions 1, 2 and 3 are satisfied with respect to $\boldsymbol{\ell}_i$. Therefore by Condition 1, there is an assignment u of $\mathbf{x}$ such that $c^{l_j} \cdot u^{l_j} = val_a(y^{l_j})$ for each $j \in \{1, ..., n_l\}$. Now, since the internal variables x^l of gate ℓ_l do not occur occur with non-zero coefficient in the constraints of $\boldsymbol{\ell}_i$, we may assume that when restricted to these variables, the assignment u^l is the one that maximizes the objective function $c^l \cdot x^l$ of the linear program which defines $\ell_l(y^{l_1}, ..., y^{l_{n_l}})$ when each variable y^{l_j} is set to $c^{l_j} \cdot u^{l_j} = val_a(y^{l_j})$. When assigning this particular u to the variables $\mathbf{x}$, we have that $c^l \cdot x^l = val_a(y^l)$. This implies that Condition 1 is also satisfied with respect to $\boldsymbol{\ell}_l$. Additionally, we have that $\boldsymbol{\ell}_l(a)$ is at least $val_a(y^l)$. Now, by Condition 2, $c^{l_j} \cdot u^{l_j} \leq val_a(y_{l_j})$ for each $j \in \{1, ..., n_l\}$. Therefore, since ℓ_l is monotone, we also have that $c^l \cdot x^l \leq val_a(y^l)$. This implies that Condition 2 is also satisfied with respect to $\boldsymbol{\ell}_l$. Additionally, this shows that $\boldsymbol{\ell}_l(a)$ is at most $val_a(y^l)$. By combining the two bounds obtained for $\boldsymbol{\ell}_l(a)$, we have that $\boldsymbol{\ell}_l(a) = val_a(y^l)$. This shows that Condition 3 is also satisfied with respect to $\boldsymbol{\ell}_l$.

The proof that the theorem holds for circuits consisting of MIN-RIGHT MLP gates is analogous to the proof for the case of MAX-RIGHT MLP gates established above. If C is a circuit containing only MIN-LEFT MLP gates, then we first transform this circuit into a circuit C' consisting only of MAX-RIGHT gates using linear program duality. In other words, we replace each MIN-LEFT MLP gate in C with an equivalent MAX-RIGHT MLP gate. Then applying the proof described above, we construct a MAX-RIGHT MLP gate $\boldsymbol{\ell}_{C'}(\mathbf{y})$. Once this is done, we apply linear-programming duality one more time to convert $\boldsymbol{\ell}_{C'}(\mathbf{y})$ into an equivalent MIN-LEFT GATE. Analogously, if C is a circuit with MAX-LEFT MLP gates, then we first convert it into an equivalent circuit consisting of MIN-RIGHT gates, then transform it into a single MIN-RIGHT MLP gate in analogy with the proof described above, and finally, convert this gate back to an equivalent MAX-LEFT MLP gate. ■

5 Weak MLP Gates vs Monotone Boolean Circuits

We say that an MLP gate ℓ *sharply* represents a partial Boolean function $F : \{0,1\}^n \to \{0,1,*\}$ if $\ell(a) = 1$ whenever $F(a) = 1$, and $\ell(a) = 0$ whenever $F(a) = 0$. In this section we

show that partial Boolean functions that can be represented by monotone Boolean circuits of size s may also be sharply represented by weak MLP gates of size $O(s)$. On the other hand, we exhibit a partial function that can be represented by polynomial-size max-right MLP gates, but which require Boolean circuits of superpolynomial size.

Theorem 5.1 *Let $F : \{0,1\}^n \to \{0,1,*\}$ be a partial Boolean function, and let C be a Boolean circuit of size s representing F. Then for any weak type τ, F can be sharply represented by an MLP gate of type τ and size $O(s)$.*

Proof. The $\wedge$ gate can be sharply represented by the following MAX-RIGHT and MIN-RIGHT MLP gates respectively.

1. $\ell_{\wedge}^{max-right}(p_1, p_2) = \max\{x \mid x \leq p_1,\ x \leq p_2, x \geq 0\}$.
2. $\ell_{\wedge}^{min-right}(p_1, p_2) = \min\{x \mid x \geq p_1 + p_2 - 1, x \geq 0\}$.

Therefore, by linear-programming duality, the $\wedge$ gate can be sharply represented by constant size MIN-LEFT and MAX-LEFT MLP gates $\ell_{\wedge}^{min-left}$ and $\ell_{\wedge}^{max-left}$ respectively.

Analogously, the $\vee$ gate can be sharply represented by the following MAX-RIGHT and MIN-RIGHT MLP gates respectively.

1. $\ell_{\vee}^{max-right}(p_1, p_2) = \max\{x_1 + x_2 \mid x_1 \leq p_1,\ x_2 \leq p_2,\ x_1 + x_2 \leq 1\}$.
2. $\ell_{\vee}^{min-right}(p_1, p_2) = \min\{x \mid x \geq p_1,\ x \geq p_2, x \geq 0\}$.

Again, by linear-programming duality, the $\vee$ gate can also be sharply represented by suitable MIN-LEFT and MAX-LEFT MLP gates $\ell_{\vee}^{min-left}$ and $\ell_{\vee}^{max-left}$ of constant size.

Now let C be a Boolean circuit representing F. Then for each weak type τ we can construct an MLP circuit C^τ which sharply represents F as follows. Replace each $\wedge$ gate of C by the corresponding MLP gate $\ell_{\wedge}^{\tau}$ of type τ, and each $\vee$ gate by the corresponding MLP gate $\ell_{\vee}^{\tau}$. Then it should be clear that C^τ has size $O(s)$, and that C^τ sharply simulates F. Since all gates in C^τ have type τ, by Theorem 4.1, there is an MLP gate ℓ^τ of type τ and size $O(s)$ that sharply represents F. ∎

Let $BPM_n : \{0,1\}^{n^2} \to \{0,1\}$ be the Boolean function that evaluates to 1 on an input $p \in \{0,1\}^{n^2}$ if and only if p represents a bipartite graph with a perfect matching. The next theorem, whose proof is based on a classical result in linear programming theory (Theorem 18.1 of [30]) states that the function BPM_n has small MAX-RIGHT MLP representations.

Theorem 5.2 *The Boolean function $BPM_n : \{0,1\}^{n^2} \to \{0,1\}$ can be represented by a* MAX-RIGHT *MLP gate of size $n^{O(1)}$.*

Proof. Let I and J be subsets of $[n] = \{1, ..., n\}$, and $E \subseteq [n] \times [n]$ be a bipartite graph. We represent a subgraph of E as a 0/1 vector with n^2 coordinates, which has a 1 at position M_{ij} if and only if (i, j) is an edge of E. The bipartite perfect matching polytope associated with

E, which is denoted by $P(E)$ is the convex-hull of all vectors $M \in \{0,1\}^{n^2}$ which correspond to a perfect matching in E. Note that if E has no perfect matching then $P(E)$ is simply empty. It can be shown (Schrijver [30], Theorem 18.1) that the polytope $P(E)$ is determined by the following system of inequalities.

System 1:

1. $x \geq 0$,
2. $\sum_{j;(i,j)\in E} x_{ij} = 1$, $i \in I$,
3. $\sum_{i;(i,j)\in E} x_{ij} = 1$, $j \in J$.

In other words, if $u \in \mathbb{R}^{n^2}$ is a 0/1 vector representing a perfect matching in E, then all inequalities of System 1 are satisfied if we set $x = u$. Conversely, each vector $u \in \mathbb{R}^{n^2}$ that satisfies all inequalities in System 1 is a convex combination of 0/1 vectors corresponding to perfect matchings in E.

Now, consider the following system of inequalities.

System 2:

1. $x \geq 0$,
2. $\sum_j x_{ij} = 1$, $i \in [n]$,
3. $\sum_i x_{ij} = 1$, $j \in [n]$,
4. $x \leq p$.

If a 0/1 vector $w \in \mathbb{R}^{n^2}$ represents a graph $E \subseteq [n] \times [n]$ containing a perfect matching, then some $u \leq w$ represents a perfect matching in E. Therefore, by setting $p = w$ and $x = u$, all inequalities of System 2 are satisfied.

Now let $w \in \mathbb{R}^{n^2}$ be a 0/1 vector such that for some $u \in \mathbb{R}^{n^2}$, the assignment $p = w$ and $x = u$ satisfies all inequalities of System 2. For each subset $A \subseteq [n]$, let $\Gamma(A) = \{j \mid w_{ij} = 1\}$ be the set of neighbours of A in the graph represented by w. Then, from inequalities 1-3, we have

$$|\Gamma(A)| \geq \sum_{j\in\Gamma(A)} \sum_{i\in[n]} u_{ij} \geq \sum_{j\in\Gamma(A)} \sum_{i\in A} u_{ij} = \sum_{i\in A} \sum_{j\in[n]} u_{ij} \geq |A|. \tag{6}$$

Therefore, by Hall's marriage theorem the graph represented by w has a perfect matching.

■

In a celebrated result, Razborov proved a lower bound of $n^{\Omega(\log n)}$ for the size of monotone Boolean circuits computing the function BPM_n. By combining this result with Theorem 5.2, we have the following corollary.

Corollary 5.3 MAX-RIGHT *MLP gates cannot be polynomially simulated by monotone Boolean circuits.*

We note that the gap between the complexity of MAX-RIGHT MLP gates and the complexity of Boolean formulas computing the BPM_n function is even exponential, since Raz and Wigderson have shown a linear lower-bound on the depth of monotone Boolean circuits computing BPM_n [25].

5.1 Monotone Span Programs

Monotone span programs (MSP) were introduced by Karchmer and Wigderson [17]. Such a program, which is defined over an arbitrary field $\mathbb{F}$, is specified by a vector $c \in \mathbb{F}^k$ and a labeled matrix $A^\rho = (A, \rho)$ where A is a matrix in $\mathbb{F}^{m \times k}$, and $\rho : \{1, ..., m\} \to \{p_1, ..., p_n, *\}$ labels rows in A with variables in p_i or with the symbol $*$ (meaning that the row is unlabeled). For an assignment $p := w$, let $A^\rho_{\langle w \rangle}$ be the matrix obtained from A by deleting all rows labeled with variables which are set to 0.[2] A span program (A^ρ, c) represents a partial Boolean function $F : \{0,1\}^n \to \{0, 1, *\}$ if the following conditions are satisfied for each $w \in \{0,1\}^n$.

$$F(w) = \begin{cases} 1 \;\Rightarrow\; \exists y,\ y^T A^\rho_{\langle w \rangle} = c^T \\ 0 \;\Rightarrow\; \neg\exists y,\ y^T A^\rho_{\langle w \rangle} = c^T \end{cases} \tag{7}$$

That is, if $F(p) = 1$ then c is a linear combination of the rows of $A_{\langle w \rangle}$, while if $F(p) = 0$, then c cannot be cast as such linear combination. We define the size of a span program (A^ρ, c) as the number of rows plus the number of columns in the matrix A. The next theorem, which will be proved in Subsection 5.2, states that functions that can be represented by small MSPs over the reals can also be represented by small MIN-RIGHT MLP gates.

Theorem 5.4 *Let $F : \{0,1\}^n \to \{0,1\}$ be a Boolean function. If F can be represented by an MSP of size s over the reals, then F can be represented by a* MIN-RIGHT *MLP gate of size $O(s)$.*

It has been recently shown that there is a family of functions $\mathrm{GEN}_n : \{0,1\}^n \to \{0,1\}$ which can be computed by polynomial-size monotone Boolean circuits but which require monotone span programs over the reals of size $\exp(n^{\Omega(1)})$ [7]. On the other hand, since by Theorem 5.1, monotone Boolean circuits can be polynomially simulated by weak MLP gates of any type, we have that weak MLP gates of size polynomial in n can represent the function $\mathrm{GEN}_n : \{0,1\}^n \to \{0,1\}$. Therefore, we have the following corollary.

Corollary 5.5 *Weak MLP gates cannot be polynomially simulated by monotone span programs over the reals.*

[2] This notation is also discussed in Subsection 7.2.

5.2 Proof of Theorem 5.4

In this section we prove Theorem 5.4. As an intermediate step we define the notion of *nonnegative monotone span program* (NONNEGATIVE-MSP). Such a NONNEGATIVE-MSP is specified via a pair $(A^\rho, c)^+$ consisting of a labeled matrix $A^\rho = (A, \rho)$, and a vector c, just as in the case of monotone span programs. The only difference is in the way in which such programs are used to represent functions. We say that a NONNEGATIVE-MSP $(A^\rho, c)^+$ represents a partial Boolean function $F : \{0,1\}^n \to \{0,1,*\}$ if the following conditions are observed for each $w \in \{0,1\}^n$.

$$F(w) = \begin{cases} 1 \;\Rightarrow\; \exists y \geq 0,\ y^T A^\rho_{\langle w \rangle} = c^T \\ 0 \;\Rightarrow\; \neg\exists y \geq 0,\ y^T A^\rho_{\langle w \rangle} = c^T \end{cases} \tag{8}$$

Note that while MSP representations are defined in terms of linear combinations of rows of A^ρ, NONNEGATIVE-MSP representations are defined in terms of *nonnegative* linear combinations of rows of A^ρ.

Proposition 5.6 *Let $F : \{0,1\}^n \to \{0,1\}$ be a Boolean function. If F can be represented by an MSP of size s over the reals, then F can be represented by a* NONNEGATIVE-*MSP of size $O(s)$ over the reals.*

Proof. Let $A^\rho = (A, \rho)$ be a labeled matrix over $\mathbb{R}$, and let (A^ρ, c) be a span program over $\mathbb{R}$. Let $B = \begin{bmatrix} A \\ -A \end{bmatrix}$. In other words, for each row a_i of A, the matrix B has a row a_i, and a row $-a_i$. Now let ρ' be the function that labels the rows of B in such a way that the rows corresponding to a_i and $-a_i$ in B are labeled with the same label as row i of A. Then it should be clear that for each $w \in \{0,1\}^n$, c is equal to a linear combination of rows of $A^\rho_{\langle w \rangle}$ if and only if c is equal to a nonnegative linear combination of rows of $B^{\rho'}_{\langle w \rangle}$. Therefore, $(B^{\rho'}, c)^+$ is a NONNEGATIVE-MSP of size $O(s)$ representing F. ∎

Therefore, it is enough to show that any partial Boolean function that can be represented via NONNEGATIVE-MSPs of size s can also be represented by MIN-RIGHT MLP gates of size $O(s)$. Consider the condition

$$\exists y \geq 0,\ y^T A^\rho_{\langle w \rangle} = c^T. \tag{9}$$

In other words, the formula in Equation 11 is satisfied if and only if the row vector c^T is a positive linear combination of the rows of $A^\rho_{\langle w \rangle}$. Let $y \geq 0$ be such a non-negative vector such that $y^T A^\rho_{\langle w \rangle} = c^T$. Then we have that for each $x \in \mathbb{R}^k$ (where k is the number of columns in A), the fact that $A^\rho_{\langle w \rangle} x \geq 0$ implies that $c \cdot x \geq 0$. In particular, this is the case whenever $x \in (\mathbb{R}^+)^k$. More formally, we have the following implication.

$$\exists y \geq 0,\ y^T A^\rho_{\langle w \rangle} = c^T \;\Rightarrow\; \min\{c \cdot x \mid A^\rho_{\langle w \rangle} x \geq 0, x \geq 0\} \geq 0. \tag{10}$$

Now let $A'x \geq Bp$ be the system of inequalities obtained from $A^{\rho}_{\langle w \rangle} x \geq 0$ as follows. For each i, let a_i be the i-th row of A. For each i, if the i-th row of A is unlabeled, the system $A'x \geq Bp$ has the inequality $a_i x \geq 0$. On the other hand, if row i is labeled with variable p_j, then then $A'x \geq Bp$ has the inequality $a_i x \geq b_i$ to the system $A'x \geq Bp$. On the other hand, if the i-th row of A is labeled, then which is unlabeled For each inequality $a_i x \geq \alpha(p_j - 1)$ where $\alpha \in \mathbb{R}^+$ is a positive number that is large enough to make the inequality irrelevant when p_j is set to 0. Then it should be clear that for each $p \in \{0,1\}^n$,

$$\min\{c \cdot x \mid A'x \geq Bp, x \geq 0\} = \min\{c \cdot x \mid A^{\rho}_{\langle w \rangle} x \geq 0, x \geq 0\}. \tag{11}$$

Now let $\ell(p) = \min\{c \cdot x \mid A'x \geq Bp, x \geq 0\}$. Then for each $w \in \{0,1\}^n$, we have that

$$F(w) = \begin{cases} 1 & \Rightarrow \ell(p) \geq 0 \\ 0 & \Rightarrow \ell(p) < 0. \end{cases} \tag{12}$$

Finally, let $\varepsilon = \min_{p \in \{0,1\}^n}\{|\ell(p)| \mid \ell(p) < 0\}$ be the minimum absolute value of $\ell(p)$ where the minimum is taken over all inputs $p \in \{0,1\}^n$ which evaluate to a number strictly less than zero, and let

$$\ell'(p) = \min\{c \cdot x + x' \mid x' = \varepsilon/2,\ A'x \geq Bp, x \geq 0\}.$$

Then $\ell'(p) = \ell(p) + \varepsilon/2$ and therefore, for each $w \in \{0,1\}^n$, we have that

$$F(w) = \begin{cases} 1 & \Rightarrow \ell'(p) \geq \varepsilon/2 > 0 \\ 0 & \Rightarrow \ell'(p) < \varepsilon/2 < 0. \end{cases} \tag{13}$$

In other words, ℓ' is a MIN-RIGHT MLP representation of F.

6 Lovász-Schrijver and Cutting-Planes Proof Systems

6.1 The Lovász-Schrijver Proof System

The Lovász-Schrijver proof system is a refutation system based on the Lovász-Schrijver method for solving integer linear programs [20]. During the past two decades several variants (probably nonequivalent) of this system have been introduced. In this work we will be only concerned with the basic system LS. In Lovász-Schrijver systems the domain of variables is restricted to $\{0,1\}$, i.e., they are Boolean variables. Given an infeasible set of inequalities Φ, the goal is to use the axioms and rules of inference defined below to show that the inequality $0 \geq 1$ is implied by Φ.

- Axioms:
  1. $0 \leq p_j \leq 1$
  2. $p_i^2 - p_i = 0$ (integrality).
- Rules:

1. *Positive linear combinations of inqualities.*
2. *Multiplication:* given a linear inequality $\sum_i c_i p_i - d \geq 0$, and a variable p_j, derive

$$p_j(\sum_i c_i p_i - d) \geq 0 \quad \text{and} \quad (1 - p_j)(\sum_i c_i p_i - d) \geq 0.$$

3. *Weakening rule:*

$$\text{from } \sum_i c_i p_i - d \geq 0, \text{ derive } \sum_i c_i p_i - d' \geq 0 \text{ for any } d' < d.$$

We note that positive linear combinations may involve both linear and quadratic inequalities, but the multiplication rule can only be applied to linear inequalities. Hence, all inequalities occurring in a proof are at most quadratic. Axiom (2) corresponds to two inequalities, but it suffices to use $p_i^2 - p_i \geq 0$, since the other inequality $p_i^2 - p_i \leq 0$ follows from Axiom (1) and Rule (2). We also observe that the inequality $1 \geq 0$ can be derived from the axioms $p_i \geq 0$ and $1 - p_i \geq 0$. Therefore the weakening rule can be simulated by an application of these axioms together with linear combinations.

The LS proof system is implicationally complete. This means that if an inequality $\sum_i c_i p_i - d \geq 0$ is semantically implied by an initial set of inequalities Φ, then $\sum_i c_i p_i - d \geq 0$ can be derived from Φ by the application of a sequence of LS-rules [20].

Superpolynomial lower bounds on the size of LS proofs have been obtained only in the restricted case of tree-like proofs [21]. The problem of obtaining superpolynomial lower bounds for the size of DAG-like LS proofs remains a tantalizing open problem in proof complexity theory.

The LS proof system is stronger than Resolution. It can be shown that resolution proofs can be simulated by LS proofs with just a linear blow up in size. Additionally, the Pigeonhole principle has LS proofs of polynomial size, while this principle requires exponentially long resolution proofs [14]. On the other hand, the relationship between the power of the LS proof system and other well studied proof system is still elusive. For instance, previous to this work, nothing was known about how the LS proof system relates to the cutting-planes proof system with respect to polynomial-time simulatability. In Subsection 6.5 we will show that there is a family of sets of inequalities which have polynomial-size DAG-like LS refutations, but which require superpolynomial-size cutting-planes refutations. This shows that the cutting-planes proof system cannot polynomially simulate the LS proof system. The converse problem, of determining whether the LS proof system polynomially simulates the cutting-planes proof system, remains open.

In this paper we will consider general (i.e., DAG-like) proofs. Thus, a sequence of inequalities Π is a derivation of an inequalitity $\sum_i c_i p_i - d \geq 0$ from a set of inequalities Φ if every inequality in Π is either an element of Φ or is derived from previous ones using some LS rule. We say that Π is a refutation of the set of inequalities Φ, if the last inequality is $-d \geq 0$ for some $d > 0$.

6.2 Feasible Interpolation

Feasible interpolation is a method that can sometimes be used to translate circuit lower bounds into lower bounds for the size of refutations of Boolean formulas and linear inequalities. Let $\Psi(p, q, r)$ be an unsatisfiable Boolean formula which is a conjunction of formulas $\Phi(p, q)$ and $\Gamma(p, r)$ where q and r are disjoint sets of variables. Since $\Psi(p, q, r)$ is unsatisfiable, it must be the case that for each assignment a of the variables p, either $\Phi(a, q)$ or $\Gamma(a, r)$ is unsatisfiable, or both. Given a proof Π of unsatisfiability for $\Psi(p, q, r)$, an *interpolant* is a Boolean circuit $C(p)$ such that for every assignment a to the variables p,

1. if $C(a) = 1$, then $\Phi(a, q)$ is unsatisfiable,
2. if $C(a) = 0$, then $\Gamma(a, r)$ is unsatisfiable.

If both formulas are unsatisfiable, then $C(a)$ can be either of the two values. Krajíček has shown that given a resolution refutation Π of a CNF formula, one can construct an interpolant $C(p)$ whose size is polynomial in the size of Π [18]. Krajíček's interpolation theorem has been generalized, by himself and some other authors, to other proof systems such as the cutting-planes proof system and the Lovász-Schrijver proof system [8].

In principle, such *feasible interpolation* theorems could be used to prove lower bounds on the size of proofs if we could prove lower bounds on circuits computing some particular functions. But since we are not able to prove essentially any lower bounds on general Boolean circuits, feasible interpolation gives us only conditional lower bounds. For instance, the assumption that $\mathbf{P} \neq \mathbf{NP} \cap \mathbf{coNP}$, an apparently weaker assumption than $\mathbf{NP} \neq \mathbf{coNP}$, implies that certain tautologies require superpolynomial-size proofs on systems that admit feasible interpolation.

However, in some cases, one can show that there exist monotone interpolating circuits of polynomial size provided that all variables p appear negatively in $\Phi(p, q)$, (or positively in $\Gamma(p, r)$). In the case of resolution proofs, such circuits are simply monotone Boolean circuits [18, 19]. In the case of cutting-planes proofs, the interpolants are *monotone real circuits* [22]. Monotone real circuits are circuits with Boolean inputs and outputs, but whose gates are allowed to be arbitrary 2-input functions over the reals. Razborov's lower bound on the clique function has been generalized to monotone real circuits [22, 15]. Another proof system for which one can prove superpolynomial lower bounds using monotone feasible interpolation is the Nullstellensatz Proof System [24]. In this proof system, the monotone interpolants are given in terms of monotone span programs[3] [24].

The results mentioned above suggest that if a proof system has the feasible interpolation property, then it may also have monotone feasible interpolation property for a suitable kind of monotone computation. We will show that the Lovász-Schrijver proof system has the monotone feasible interpolation property with the interpolants computed MLP circuits with strong gates.

[3]In the context of polynomial calculus, alternative methods (e.g. [1, 16]) yield stronger lower bounds than the monotone interpolation technique.

6.3 Feasible Interpolation for the Lovász-Schrijver System

Let $F_1(q) - c_1 \geq 0, F_2(q) - c_2 \geq 0, ..., F_m(q) - c_m \geq 0$ be a sequence of linear inequalities over a set of variables q. We say that a linear inequality $F(q) - c \geq 0$ is obtained from this sequence in one *lift-and-project* step, or simply *lap*-step for short, if

$$\begin{aligned} F(q) - c \;\; = \;\; & \textstyle\sum_{ij} \alpha_{ij} q_i (F_j(q) - c_j) + \\ & \textstyle\sum_{ij} \beta_{ij} (1 - q_i)(F_j(q) - c_j) + \\ & \textstyle\sum_{j} \gamma_j (F_j(q) - c_j) + \\ & \textstyle\sum_{i} \delta_i (q_i - q_i^2) + \\ & \textstyle\sum_{j} \xi_j (F_j(q) - c_j) \end{aligned} \tag{14}$$

for some $\alpha_{ij}, \beta_{ij}, \gamma_j, \delta_i, \xi_j \geq 0$. A refutation in the LS proof system for an unsatisfiable set of inequalities $\Phi(q)$ can naturally be regarded as a sequence $L_1 \geq 0, \ldots, L_m \geq 0$ of linear inequalities where for each $i \in \{1, ..., m\}$, the inequality $L_i \geq 0$ is either in $\Phi(q)$, or is obtained from $L_1 \geq 0, ..., L_{i-1} \geq 0$ by the application of one *lap*-step. Intuitively, inequalities involving quadratic terms, obtained as instances of the integrality axiom or by the application of the multiplication rule, are regarded as intermediate steps towards the derivation of new linear inequalities.

Let p, q and r be tuples of Boolean variables. We say that an unsatisfiable set of inequalities $\Phi(p, q) \cup \Gamma(p, r)$ is *monotonically separable* if the variables in p occur in inequalities of Φ only with negative coefficients. The next theorem states that LS-proofs for monotonically separable unsatisfiable sets of inequalities can be interpolated using MLP circuits constituted of MAX MLP gates.

Theorem 6.1 *Let $\Phi(p, q) \cup \Gamma(p, r)$ be a monotonically separable unsatisfiable set of inequalities, and let $p = (p_1, ..., p_n)$. Let Π be an LS refutation of $\Phi(p, q) \cup \Gamma(p, r)$. Then one can construct an MLP circuit C containing only* MAX *MLP gates which represents a Boolean function $F : \{0, 1\}^n \to \{0, 1\}$ such that for each $a \in \{0, 1\}^n$,*

1. *if $F(a) = 1$, then $\Phi(a, q)$ is unsatisfiable,*
2. *if $F(a) = 0$, then $\Gamma(a, r)$ is unsatisfiable.*

Additionally, the size of the circuit C is polynomial in the size of Π.

Proof. We start by recalling the idea of feasible interpolation for LS in the non-monotone case as presented in [23]. For the sake of simplicity, we assume that the inequalities $0 \leq q_i \leq 1$ and $0 \leq r_i \leq 1$ are included in Φ and Γ.

Let

$$E_1(p) + F_1(q) + G_1(r) - e_1 \geq 0, \;\; \ldots \;\; , E_m(p) + F_m(q) + G_m(r) - e_m \geq 0 \tag{15}$$

be the linear inequalities of an LS refutation of $\Phi(p,q) \cup \Gamma(p,r)$. Since the last inequality is a contradiction, the linear forms E_m, F_m, G_m are zeros and $e_m > 0$. Let $a \in \{0,1\}^n$ be an assignment to variables p. Substituting a into the proof we get a refutation

$$F_1(q) + G_1(r) + E_1(a) - e_1 \geq 0, \ \ldots \ , F_m(q) + G_m(r) + E_m(a) - e_m \geq 0 \tag{16}$$

of $\Phi(a,q) \cup \Gamma(a,r)$ (note that the last inequality is $-e_m \geq 0$ as in the proof above). Our aim now is to split the restricted proof into two proofs

$$F_1(q) - c_1 \geq 0, \ \ldots \ , F_m(q) - c_m \geq 0 \quad \text{and} \quad G_1(r) - d_1 \geq 0, \ \ldots \ , G_m(r) - d_m \geq 0 \tag{17}$$

in such a way that the first sequence is the sequence of inequalities of a refutation of $\Phi(a,q)$, the second sequence is the sequence of inequalities of a refutation of $\Gamma(a,r)$, and

$$c_j + d_j \geq e_j - E_j(a) \quad \text{for } j \in \{1, \ldots, m\}. \tag{18}$$

Since $e_m > 0$, we have that either $c_m > 0$, or $d_m > 0$, or both inequalities are true. Hence, at least one of the proofs is a refutation of its initial inequalities. We now describe how such a splitting can be constructed.

First, suppose $E_j(p) + F_j(q) + G_j(r) - e_j \geq 0$ is an inequality in Φ. Then $G_j(r) \equiv 0$. This inequality will be split into

$$F_j(q) + E_j(a) - e_j \geq 0 \quad \text{and} \quad 0 \geq 0. \tag{19}$$

Since all coefficients in E_j are negative, $e_j - E_j(a)$ can be computed from a using a single MAX MLP gate (or even by a MAX-LEFT MLP gate). If $E_j(p) + F_j(q) + G_j(r) \geq e_j$ is an inequality in Γ, we split the inequality into

$$0 \geq 0 \quad \text{and} \quad G_j(r) + E_j(a) - e_j \geq 0. \tag{20}$$

Now suppose that $E_t(p) + F_t(q) + G_t(r) \geq e_t$ follows from previous inequalities and suppose we have already split the previous part of the proof. Substituting a into the j-th *lap*-step we obtain an equality of the following form.

$$\begin{gathered}
F_t(q) + G_t(r) + E_t(a) - e_t = \\
\textstyle\sum_{ij} \alpha_{ij} a_i (F_j(q) + G_j(r) + E_j(a) - e_j) + \sum_{ij} \beta_{ij} (1 - a_i)(F_j(q) + G_j(r) + E_j(a) - e_j) + \\
\textstyle\sum_{ij} \alpha'_{ij} q_i (F_j(q) + G_j(r) + E_j(a) - e_j) + \sum_{ij} \beta'_{ij} (1 - q_i)(F_j(q) + G_j(r) + E_j(a) - e_j) + \\
\textstyle\sum_{ij} \alpha''_{ij} r_i (F_j(q) + G_j(r) + E_j(a) - e_j) + \sum_{ij} \beta''_{ij} (1 - r_i)(F_j(q) + G_j(r) + E_j(a) - e_j) + \\
\textstyle\sum_i \gamma_i (a_i - a_i^2) + \\
\textstyle\sum_i \gamma'_i (q_i - q_i^2) + \sum_i \gamma''_i (r_i - r_i^2) + \\
\textstyle\sum_j \delta_j (F_j(q) + G_j(r) + E_j(a) - e_j).
\end{gathered} \tag{21}$$

In the sums, we have $j < t$ and the indices i range over the sets of indices of the corresponding variables p, q, r. All these linear combinations are nonnegative, i.e., the coefficients α_{ij}, α'_{ij}, α''_{ij}, β_{ij}, β'_{ij}, β''_{ij}, γ_i,γ'_i, γ''_i, and δ_j are nonnegative. Note that the term $\sum_i \gamma_i(a_i - a_i^2)$ is always zero, since by assumption $a_i \in \{0, 1\}$. By setting $\delta'_j = \delta_j + \sum_i(\alpha_{ij}a_i + \beta_{ij}(1 - a_j))$, for each j, and by noting that δ'_j is non-negative, Equation 21 can be simplified as follows.

$$\begin{gathered}
F_t(q) + G_t(r) + E_t(a) - e_t = \\
\textstyle\sum_{ij} \alpha'_{ij} q_i (F_j(q) + G_j(r) + E_j(a) - e_j) + \sum_{ij} \beta'_{ij}(1 - q_i)(F_j(q) + G_j(r) + E_j(a) - e_j) + \\
\textstyle\sum_{ij} \alpha''_{ij} r_i (F_j(q) + G_j(r) + E_j(a) - e_j) + \sum_{ij} \beta''_{ij}(1 - r_i)(F_j(q) + G_j(r) + E_j(a) - e_j) + \\
\textstyle\sum_i \gamma'_i(q_i - q_i^2) + \sum_i \gamma''_i(r_i - r_i^2) + \\
\textstyle\sum_j \delta'_j(F_j(q) + G_j(r) + E_j(a) - e_j).
\end{gathered} \tag{22}$$

By substituting $-c_j - d_j$ for $E_j(a) - e_j$ in Equation 22 and rearranging terms, we have the following equation.

$$\begin{gathered}
F_t(q) + G_t(r) + E_t(a) - e_t \; = \\
\begin{array}{ccc}
\textstyle\sum_{ij} \alpha'_{ij} q_i (F_j(q) - c_j) & + & \textstyle\sum_{ij} \alpha''_{ij} r_i (G_j(r) - d_j) + \\
\textstyle\sum_{ij} \beta'_{ij}(1 - q_i)(F_j(q) - c_j) & + & \textstyle\sum_{ij} \beta'_{ij}(G_j(r) - d_j) \\
\textstyle\sum_{ij} \beta''_{ij}(F_j(q) - c_j) & + & \textstyle\sum_{ij} \beta''_{ij}(1 - r_i)(G_j(r) - d_j) + \\
\textstyle\sum_i \gamma'_i(q_i - q_i^2) & + & \textstyle\sum_i \gamma''_i(r_i - r_i^2) + \\
\textstyle\sum_j \delta'_j(F_j(q) - c_j) & + & \textstyle\sum_j \delta'_j(G_j(r) - d_j) + \\
\textstyle\sum_{ij} \alpha'_{ij} q_i (G_j(r) - d_j) & + & \textstyle\sum_{ij} -\beta'_{ij} q_i (G_j(r) - d_j) + \\
\textstyle\sum_{ij} \alpha''_{ij} r_i (F_j(q) - c_j) & + & \textstyle\sum_{ij} -\beta''_{ij} r_i (F_j(q) - c_j).
\end{array}
\end{gathered} \tag{23}$$

Note that each line in the right-hand side of Equation 23, except for the last two, splits into a linear form involving only q variables and another linear form involving only r variables. Let

$$\begin{aligned}
P(q, r) \;\; = \;\; & \textstyle\sum_{ij} \alpha'_{ij} q_i (G_j(r) - d_j) + \sum_{ij} -\beta'_{ij} q_i (G_j(r) - d_j) + \\
& \textstyle\sum_{ij} \alpha''_{ij} r_i (F_j(q) - c_j) + \sum_{ij} -\beta''_{ij} r_i (F_j(q) - c_j).
\end{aligned} \tag{24}$$

be the polynomial corresponding to the two last lines of Equation 23. The key observation is that, since the inequality $F_t(q) + G_t(r) + E_t(a) - e_t \geq 0$ is linear, all quadratic terms $q_i r_k$ in the polynomial $P(q, r)$ must cancel. Therefore, $P(q, r)$ can be simplified to

$$\begin{array}{rcl} P(q,r) & = & \sum_{ij} -\alpha'_{ij} q_i d_j + \sum_{ij} \beta'_{ij} q_i d_j + \\ & & \sum_{ij} -\alpha''_{ij} r_i c_j + \sum_{ij} \beta''_{ij} r_i c_j. \end{array} \tag{25}$$

Additionally, since the inequality $F_t(q) + G_t(r) + E_t(a) - e_t \geq 0$ is implied by the inequalities $F_j(q) - c_j \geq 0$ and $G_j(r) - d_j \geq 0$ in the domain of real numbers (for $j < t$), it follows from Farkas' Lemma that the linear form $F_t(q) + G_t(r) + E_t(a) - e_t$ is a positive linear combination of linear forms $F_j(q) - c_j$ and $G_j(r) - d_j$ for $j < t$. This implies that $P(q, r)$ is also a positive linear combination of $F_j(q) - c_j$ and $G_j(r) - d_j$ for $j < t$. In other words,

$$P(q,r) = \sum_{j<t} \xi_j (F_j(q) - c_j) + \sum_{j<t} \xi'_j (G_j(r) - d_j), \tag{26}$$

for some $\xi_j, \xi'_j \geq 0$. Thus, Equation 23 can be rewritten as follows.

$$\begin{array}{c} F_t(q) + G_t(r) + E_t(a) - e_t \; = \\ \begin{array}{ccc} \sum_{ij} \alpha'_{ij} q_i (F_j(q) - c_j) & + & \sum_{ij} \alpha''_{ij} r_i (G_j(r) - d_j) + \\ \sum_{ij} \beta'_{ij} (1 - q_i)(F_j(q) - c_j) & + & \sum_{ij} \beta'_{ij} (G_j(r) - d_j) + \\ \sum_{ij} \beta''_{ij} (F_j(q) - c_j) & + & \sum_{ij} \beta''_{ij} (1 - r_i)(G_j(r) - d_j) + \\ \sum_{i} \gamma'_i (q_i - q_i^2) & + & \sum_{i} \gamma''_i (r_i - r_i^2) + \\ \sum_{j} \delta'_j (F_j(q) - c_j) & + & \sum_{j} \delta'_j (G_j(r) - d_j) + \\ \sum_{j} \xi_j (F_j(q) - c_j) & + & \sum \xi'_j (G_j(r) - d_j). \end{array} \end{array} \tag{27}$$

Now, based on the assumption that the inequalities $F_j(q) + G_j(r) + E_j(a) - e_j \geq 0$, for $j < t$, have been split into inequalities $F_j(q) - c_j \geq 0$ and $G_j(r) - d_j \geq 0$, our goal is to split the inequality $F_t(q) + G_t(r) + E_t(a) - e_t \geq 0$ into inequalities $F_t(q) - c_t \geq 0$ and $G_t(r) - d_t \geq 0$. To accomplish this goal, it is enough to find constants c'_t and d'_t such that

$$c'_t + d'_t \geq e_t - E_t(a), \tag{28}$$

and such that the following equations are satisfied.

$$
\begin{array}{lll}
F_t(q) - c'_t \;=\; & & G_t(r) - d'_t \;=\; \\
\sum_{ij} \alpha'_{ij} q_i (F_j(q) - c_j) + & & \sum_{ij} \alpha''_{ij} r_i (G_j(r) - d_j) + \\
\sum_{ij} \beta'_{ij} (1 - q_i)(F_j(q) - c_j) + & & \sum_{ij} \beta'_{ij} (G_j(r) - d_j) + \\
(a) \qquad \sum_{ij} \beta''_{ij} (F_j(q) - c_j) + & (b) & \sum_{ij} \beta''_{ij} (1 - r_i)(G_j(r) - d_j) + \qquad (29) \\
\sum_{i} \gamma'_i (q_i - q_i^2) + & & \sum_{i} \gamma''_i (r_i - r_i^2) + \\
\sum_{j} \delta'_j (F_j(q) - c_j) + & & \sum_{j} \delta'_j (G_j(r) - d_j) + \\
\sum_{j} \xi_j (F_j(q) - c_j) + & & \sum \xi'_j (G_j(r) - d_j).
\end{array}
$$

We note that to compute suitable c'_t and d'_t, it is enough to find the maximum c'_t that satisfies Equation 29.(a), and the maximum d'_t that satisfies Equation 29.(b). It turns out that computing c'_t reduces to the problem of solving a linear program whose constraints can be extracted from Equation 29.(a). Analogously, computing d'_t reduces to the problem of solving a linear program whose constraints are extracted from Equation 29.(b). We concentrate in the process of computing c'_t. The process of computing d'_t is identical (and unnecessary for the construction of the interpolant circuit).

In order to obtain an interpolant circuit constituted only of monotone gates, we will consider the process of maximizing a constant c_t satisfying the following relaxed version of Equation 29.(a), where the new variables η_{ij}, η'_{ij} satisfy $\eta_{ij} \leq c_j$ and $\eta'_{ij} \leq c_j$ for each i and each $j < t$.

$$
\begin{array}{rcl}
F_t(q) - c_t & = & \sum_{ij} \alpha'_{ij} q_i (F_j(q) - \eta_{ij}) + \\
& & \sum_{ij} \beta'_{ij} (1 - q_i)(F_j(q) - \eta'_{ij}) \\
& & \sum_{ij} \beta''_{ij} (F_j(q) - c_j) + \\
& & \sum_{i} \gamma'_i (q_i - q_i^2) + \\
& & \sum_{j} \delta'_j (F_j(q) - c_j) + \\
& & \sum_{j} \xi_j (F_j(q) - c_j).
\end{array}
\qquad (30)
$$

Note that if the inequality $F_t(q) - c'_t \geq 0$ can be derived from inequalities $F_j(q) - c_j \geq 0$ (for $j < t$) in one *lap*-step, then the inequality $F_t(q) - c_t \geq 0$ can also be derived from inequalities $F_j(q) - c_j \geq 0$ (for $j < t$) as follows. First, we apply the weakening rule to obtain the inequalities $F_j(q) - \eta_{ij} \geq 0$, and $F_j(q) - \eta'_{ij} \geq 0$, and then we apply one *lap*-step involving $F_j(q) - c_j \geq 0$, $F_j(q) - \eta_{ij} \geq 0$, and $F_j(q) - \eta'_{ij} \geq 0$. We also note that the maximum value that c_t can attain is at least as large as the maximum value that c'_t can attain, since we can always set $\eta_{ij} = \eta'_{ij} = c_j$ for each $j < t$. Therefore $F_t(q) - c_t \geq 0$ implies $F_t(q) - c'_t \geq 0$.

For each $j \leq t$ let $F_j(q) = \sum_k f_{kj} q_k$. Since the homogeneous part of the right-hand side of Equation 30 must be equal to $F_t(q)$, the following inequalities and equalities must be satisfied.

$$\eta_{ij} \leq c_j \qquad\qquad \eta'_{ij} \leq c_j$$

$$\begin{aligned} f_{kt} \;=\; & \textstyle\sum_{ij} \beta'_{ij} f_{kj} + \sum_{ij} \beta''_{ij} f_{kj} + \sum_j \delta'_j f_{kj} + \\ & \gamma'_k + \textstyle\sum_j \alpha'_{kj} \eta_{kj} + \sum_j \beta'_{kj} \eta'_{kj} + \sum_j \xi_j f_{kj}. \end{aligned} \tag{31}$$

Note that the equalities in Equation 31 are obtained by identifying the coefficient f_{kt} of the variable q_k in $F_t(q)$ with the coefficient of q_k in the right-hand side of Equation 30. Now, the process of maximizing c_t such that Equation 30 is satisfied, corresponds to solving the following maximization problem subject to the constraints given by Equation 31.

$$c_t = \max \sum_{ij} \alpha'_{ij} c_j + \sum_{ij} \beta'_{ij} c_j + \sum_{ij} \beta''_{ij} c_j + \sum_j \delta'_j c_j + \sum_j \xi_j c_j. \tag{32}$$

Or equivalently, by creating a variable x_j for each $j < t$, and by setting

$$x_j = \left(\xi_j + \delta'_j + \sum_i (\alpha'_{ij} + \beta'_{ij} + \beta''_{ij}) \right), \tag{33}$$

for each $j < t$, the maximization in Equation 32 is equivalent to the following maximization.

$$c_t = \max \sum_{j<t} c_j x_j. \tag{34}$$

Together, Equation 31, Equation 33 and Equation 34 define a MAX MLP gate whose input variables are c_j for $j < t$, and whose internal variables are $x_j, \xi_j, \eta_{ij}, \eta'_{ij}$. Note that the input variables occur both in the constraints and in the objective function.

Now, let

$$E_1(p) + F_1(q) + G_1(r) - e_1 \geq 0, \; \ldots \; , E_m(p) + F_m(q) + G_m(r) - e_m \geq 0 \tag{35}$$

be the linear inequalities occurring in a refutation for inequalities $\Phi(p,q) \cup \Gamma(p,r)$. Then we construct an interpolant circuit C as follows. For each $t \in \{1, ..., m\}$ if $E_t(p) + F_t(q) + G_t(r) - e_t \geq 0$ is an inequality in $\Phi(p,q)$, then we create a MAX MLP gate ℓ_t with inputs

p and output c_t. For each assignment $a \in \{0,1\}^n$ of the variables p, the gate ℓ_t computes the value $e_t - E_t(a)$ as already discussed in the paragraph following Equation 19. On the other hand, if $E_t(p) + F_t(q) + G_t(r) - e_t \geq 0$ is obtained from previous inequalities by the application of one *lap*-step, then we create a MAX MLP gate ℓ_t with inputs p and $c_1, ..., c_{t-1}$ and output c_t. The value of c_t is computed according to the linear program[4] defined by Equation 31, Equation 33 and Equation 34. For each assignment $a \in \{0,1\}^n$ of the variables p, if $C(a) > 0$, then we have that $c_m > 0$ and therefore $\Phi(a,q)$ is unsatisfiable. Otherwise, if $C(a) \leq 0$, then $c_m \leq 0$ and therefore $d_m > 0$. This implies that $\Gamma(a,r)$ is unsatisfiable. ■

6.4 Lovász-Schrijver Refutations of Mixed LP Problems

While proof systems for integer linear programming have been widely studied, very little is known about proof systems for mixed linear programming. In mixed linear programming part of variables range over integers and part of them range over reals. The Lovász-Schrijver system can naturally be adapted for mixed linear programming by disallowing the use of axioms and the multiplication rule for variables ranging over reals. One can easily prove that this system is complete with respect to refutations (i.e., a family of inequalities is unsatisfiable if and only if a contradiction is derivable).

We say that an unsatisfiable set of mixed inequalities $\Phi(p,q) \cup \Gamma(p,r)$ is *strongly monotonically separable* if p and q are tuples of Boolean variables, r is a tuple of real variables, and variables in p occur in $\phi(p,q)$ only with negative coefficients. Next, we will show that LS proofs for strongly monotonically separable unsatisfiable sets of mixed inequalities can be interpolated in terms of a *single* MAX-LEFT MLP *gate* (or, by linear-programming duality, a *single* MIN-RIGHT MLP *gate*).

The advantage of this interpolation theorem compared with Theorem 6.1 is that while proving lower bounds on the size of strong MLP circuits may be beyond the reach of current methods, proving a lower bound on the size of a single weak MPL gate seems to be feasible, because this problem is closely related to lower bounds on extended formulations (see Section 8).

Theorem 6.2 *Let $\Phi(p,q) \cup \Gamma(p,r)$ be a strongly monotonically separable unsatisfiable set of mixed inequalities, and let $p = (p_1, ..., p_n)$. Let Π be an LS-refutation of $\Phi(p,q) \cup \Gamma(p,r)$. Then there exists a* MAX-LEFT *MLP gate ℓ that represents a Boolean function $F : \{0,1\}^n \to \{0,1\}$ such that for every $a \in \{0,1\}^n$,*

1. *if $F(a) = 1$, then $\Phi(a,q)$ is unsatisfiable, and*
2. *if $F(a) = 0$, then $\Gamma(a,r)$ is unsatisfiable.*

Additionally, the size of the MLP gate ℓ is polynomial in the size of Π.

[4]The internal variables x_j, ξ_j, η_{ij} and η'_{ij} are distinct for each two distinct gates. Additionally, the coefficients of the variables in this linear program can be computed in polynomial time from the full LS-refutation of $\Phi(p,q) \cup \Gamma(p,r)$.

Proof. It is enough to construct a circuit C consisting of MAX-LEFT gates representing a function $F : \{0,1\}^n \to \{0,1\}$ such that for each $a \in \{0,1\}^n$, $\Phi(a,q)$ is unsatisfiable whenever $F(a) = 1$, and $\Gamma(a,r)$ is unsatisfiable whenever $F(a) = 0$. By Theorem 4.1, from the circuit C, one can construct a *single* MAX-LEFT MLP *gate* representing F whose size is linear in the size of C.

The construction of C is done in a similar way to the construction of the circuit with MAX MLP gates constructed in Theorem 6.1. The difference is that, by assuming that the LS-refutation Π is mixed, the gates used in the circuit can be restricted to MAX-LEFT MLP gates, instead of MAX MLP gates. It is enough to observe that, since the multiplication rule, and integrality axioms are only used for variables r, Equation 30 can be simplified to the following equation.

$$F_t(q) - c_t = \sum_{ij} \beta''_{ij}(F_j(q) - c_j) + \sum_j \delta'_j(F_j(q) - c_j) + \sum_j \xi_j(F_j(q) - c_j). \tag{36}$$

From Equation 36, one can extract the following constraints, where as in Equation 31, f_{kj} denotes the coefficient of q_k in the linear form $F_j(q)$.

$$f_{kt} = \sum_{ij} \beta''_{ij} f_{kj} + \sum_j \delta'_j f_{kj} + \sum_j \xi_j f_{kj}. \tag{37}$$

Finally, the objective function given in Equation 32 is simplified to

$$c_t = \max \sum_{ij} \beta''_{ij} c_j + \sum_j \delta'_j c_j + \sum_j \xi_j c_j. \tag{38}$$

Equivalently, by creating a variable x_j for each $j < t$ and by setting

$$x_j = \beta''_{ij} + \delta'_j + \xi_j, \tag{39}$$

the maximization in Equation 38 is equivalent to the following maximization.

$$c_t = \max \sum_j c_j x_j. \tag{40}$$

Together, Equation 37, Equation 39 and Equation 40 define an MLP gate with input variables c_j for $j < t$, and internal variables x_j, ξ_j. Note that the input variables c_j only appear in the objective function, and not in the constraints. Therefore, this gate is a MAX-LEFT MLP gate.

The remainder of the construction of the circuit C is completely analog to the construction in the proof of Theorem 6.1. ∎

In the next subsection we will give a natural example of a set of inequalities of the form used in the theorem. We will show that it has polynomial-size mixed LS-refutations, but it requires superpolynomial-size cutting-plane refutations.

6.5 Cutting-Planes vs. Lovász-Schrijver Refutations and Monotone Real Circuits vs MLP Gates

In this subsection we will define an unsatisfiable set of inequalities $\Phi_n(p,q) \cup \Gamma_n(p,q)$, which has polynomial-size LS-refutations, but which requires superpolynomial size refutations in the cutting-planes proof system. Additionally, we define a function $g_n : \{0,1\}^n \to \{0,1,*\}$ that has polynomial-size MLP representations, but which require superpolynomial size monotone real circuits.

We recall that the cutting-planes proof systems is defined via the following axioms and rules.

- Axioms:

  $$0 \leq p_j \leq 1.$$

- Rules:

  1. *Positive linear combinations;*
  2. *Rounding rule:* Suppose that all c_i are integers. Then

     $$\text{from } \textstyle\sum_i c_i p_i \geq d, \text{ derive } \sum_i c_i p_i \geq \lceil d \rceil.$$

A monotone real circuit is a circuit C whose gates are monotone real functions of at most two variables. The size of C is the number of gates in C. The following theorem can be used to translate superpolynomial lower bounds on the size of monotone real circuits computing certain partial Boolean functions into superpolynomial lower bounds for the size of cutting-planes proofs.

Theorem 6.3 (Monotone Interpolation for the cutting-planes Proof System [22]) *Let $\Phi(p,q) \cup \Gamma(p,r)$ be a monotonically separable unsatisfiable set of inequalities, and let $p = (p_1, ..., p_n)$. Let Π be a cutting-planes refutation for $\Phi(p,q) \cup \Gamma(p,r)$. Then one can construct a monotone real circuit C such that for every $a \in \{0,1\}^n$,*

1. *if $C(a) = 1$ then $\Phi(p,q)$ is unsatisfiable, and*
2. *if $C(a) = 0$ then $\Gamma(p,q)$ is unsatisfiable.*

Additionaly the size of the circuit C is at most a constant times the size of the refutation Π.

Let $K_n = \{\{i,j\} \mid 1 \leq i < j \leq n\}$ be the complete undirected graph with vertex set $[n] = \{1, ..., n\}$. We say that a subgraph $X \subseteq K_n$ is a perfect matching if the edges in X are vertex-disjoint and each vertex $i \in [n]$ belongs to some edge of X. We say that a subgraph $B \subseteq K_n$ is an *unbalanced complete bipartite graph* if there exist sets $V, U \subseteq [n]$ with $V \cap U = \emptyset$, $|V| > |U|$, and $B = \{\{i,j\} \mid i \in V,\ j \in U\}$. Let $W \subseteq K_n$ be a graph. We let $\mathcal{V}(W) = \{i \mid \exists j \in [n],\ \{i,j\} \in W\}$ be the vertex set of W. For each vertex $i \in \mathcal{V}(W)$,

we let $\mathcal{N}(i) = \{j \mid \{i,j\} \in W\}$ be the set of neighbours of i in W. For a subset $V \subseteq \mathcal{V}(W)$, we let $\mathcal{N}(V) = \bigcup_{v \in V} \mathcal{N}(v)$ be the set of neighbours of vertices in $\mathcal{N}(V)$. We say that W is *unbalanced* if there exists $V, U \subseteq \mathcal{V}(W)$ such that $\mathcal{N}(V) \subseteq U$ and $|V| > |U|$. Note that such an unbalanced graph W cannot contain a perfect matching X, since the existence of such a perfect matching would imply the existence of an injective mapping from V to U. We also note that unbalanced complete bipartite graphs are by definition a special case of unbalanced graphs.

Razborov proved that any monotone Boolean circuit which decides whether a graph has a perfect matching must have size at least $n^{\Omega(\log n)}$ [28]. This lower bound was generalized by Fu to the context of monotone real circuits [10]. More precisely, Fu proved that any monotone real circuit distinguishing graphs with a perfect matching from unbalanced complete bipartite graphs must have size at least $n^{\Omega(\log n)}$.

Theorem 6.4 ([10]) *Let $F : \{0,1\}^{\binom{n}{2}} \to \{0,1,*\}$ be a partial boolean function such that for each $w \in \{0,1\}^{\binom{n}{2}}$,*

- *$F(w) = 1$ if w encodes a graph with a perfect matching.*
- *$F(w) = 0$ if w encodes an unbalanced complete bipartite graph.*

Then any monotone real circuit computing F must have size at least $n^{\Omega(\log n)}$.

Since unbalanced complete bipartite graphs are a special case of unbalanced graphs, monotone real circuits distinguishing graphs with a perfect matching from unbalanced graphs must have size at least $n^{\Omega(\log n)}$ gates.

Corollary 6.5 *Let $g : \{0,1\}^{\binom{n}{2}} \to \{0,1,*\}$ be a partial boolean function such that for each $w \in \{0,1\}^{\binom{n}{2}}$,*

- *$g(w) = 1$ if w has a perfect matching.*
- *$g(w) = 0$ if w is unbalanced.*

Then any monotone real circuit computing g must have size at least $n^{\Omega(\log n)}$.

Below we will define a set Ψ_n of unsatisfiable inequalities on variables

$$p = \{w_{i,j} \mid 1 \leq i < j \leq n\} \qquad q = \{u_i, v_i \mid i \in [n]\} \qquad r = \{x_{ij} \mid 1 \leq i < j \leq n\}.$$

Intuitively each assignment of the variables in p defines a graph $W \subseteq K_n$ such that $\{i,j\} \in W$ if and only if $w_{ij} = 1$. Each assignment to the variables in q defines subsets $U, V \subseteq [n]$ where $i \in U$ if and only if $u_i = 1$, and $i \in V$ if and only if $v_i = 1$. Finally, each assignment to the variables in r defines a subset of edges X in such a way that $\{i,j\} \in X$ if and only if $x_{ij} = 1$. The set of inequalities Ψ_n would be satisfiable by an assignment α of the variables in p,q and r if and only if α defined a graph $W \subseteq K_n$ which contained, at the same time, a perfect matching X and a pair of subsets of vertices $V, U \subseteq \mathcal{V}(W)$ certifying that W is unbalanced. Since no such graph exists, the set Ψ_n is unsatisfiable.

Definition 5 (Unbalanced Graphs vs Perfect Matching Inequalities) *Let $\Phi_n(p,q) \cup \Gamma_n(p,r)$ be a set of inequalities on variables $p = \{x_{ij}\}$, $q = \{u_i, v_i\}$ and $r = \{x_i\}$ defined as follows.*

Inequalities in $\Phi(p,q)$:	*W is unbalanced.*
1) $u_j - v_i - w_{ij} + 1 \geq 0$	$\mathcal{N}(V) \subseteq U$. *If* $i \in V \wedge \{i,j\} \in W \Rightarrow j \in U$.
2) $\sum_j v_j - \sum_i u_i - 1 \geq 0$	$\lvert V\rvert > \lvert U\rvert$.
Inequalities in $\Gamma(p,r)$:	*Existence of a perfect matching.*
3) $w_{ij} - x_{ij} \geq 0$	*X is a subset of edges of W.*
4) $\sum_{i,i\neq j} x_{ij} - 1 = 0$	*X defines a perfect matching.*

Note that for each j, the equalities in 4) consist of two inequalities. Note also that the variables in $w_{ij} \in p$, which occur both in $\Phi_n(p,q)$ and in $\Gamma_n(p,r)$, only occur negatively in $\Phi_n(p,q)$. Therefore, $\Phi_n(p,q) \cup \Gamma(p,r)$ is monotonically separable.

A combination of Fu's size lower-bound for monotone real circuits (Theorem 6.4) with the monotone interpolation theorem for cutting-planes (Theorem 6.3) was used in [10] to show that a suitable unsatisfiable set of inequalities Ψ'_n requires cutting-planes refutations of size $n^{\Omega(\log n)}$. The next theorem states that a similar lower bound can be proved with respect to the inequalities introduced in Definition 5.

Theorem 6.6 *Let $\Phi_n(p,q) \cup \Gamma_n(p,r)$ be the set of inequalities of Definition 5. Then any cutting-planes refutation of $\Phi_n(p,q) \cup \Gamma_n(p,r)$ must have size at least $n^{\Omega(\log n)}$.*

Proof. If $a \in \{0,1\}^n$ represents a graph containing a perfect matching, then $\Gamma_n(a,r)$ is satisfiable, and consequently $\Phi_n(a,q)$ is unsatisfiable. Analogously, if a represents an unbalanced graph, then $\Phi_n(a,q)$ is satisfiable and consequently, $\Gamma_n(a,r)$ is unsatisfiable. Let Π be a refutation of $\Phi_n(p,q) \cup \Gamma_n(p,r)$. Then, by the interpolation theorem for monotone real circuits (Theorem 6.3), there is a monotone real circuit C of size polynomial in the size of Π such that $C(a) = 1$ if the graph represented by a has a perfect matching, and such that $C(a) = 0$ if the graph represented by a is an unbalanced graph. But by Corollary 6.5, any such circuit must have size at least $n^{\Omega(\log n)}$. Therefore, the proof Π must also have size at least $n^{\Omega(\log n)}$. ■

On the other hand, the following theorem states that the set inequalities $\Phi_n(p,q) \cup \Gamma_n(p,r)$ has LS-refutations of size polynomial in n. In fact these refutations are for the case where variables r are real, meaning that axioms and multiplication rules are never used for variables r.

Theorem 6.7 *Let $\Phi_n(p,q) \cup \Gamma_n(p,r)$ be the set of inequalities of Definition 5, Then $\Phi_n(p,q) \cup \Gamma_n(p,r)$ has an LS-refutation of size polynomial in n.*

Proof. Consider the following polynomial-size LS-refutation of $\Psi(p,q,r)$. 5. $u_j - v_i - x_{ij} + 1 \geq 0$ from 3. and 1.

6. $x_{ij}u_j - x_{ij}v_i - x_{ij}^2 + x_{ij} \geq 0$ multiplying 5. by x_{ij}
7. $x_{ij}u_j - x_{ij}v_i \geq 0$ applying $x_{ij}^2 - x_{ij} = 0$ to 6.
8. $\sum_{ij} x_{ij}u_j - \sum_{ij} x_{ij}v_i \geq 0$ sum of 7. over every i,j with $i \neq j$
9. $\sum_j u_j \sum_{i;i\neq j} x_{ij} - \sum_i v_i \sum_{j;i\neq j} x_{ij} \geq 0$ rewriting 8.
10. $\sum_j u_j - \sum_i v_j \geq 0$ from 9. and 4.
11. $-1 \geq 0$ from 2. and 10. ■

By combining Theorem 6.6 with Theorem 6.7 we have the following corollary separating cutting-planes from LS proof systems.

Corollary 6.8 *The cutting-planes proof system does not polynomially simulate the Lovász-Schrijver proof system.*

Previous to our work, the problem of determining whether the cutting-planes proof system can polynomially simulate the LS-proof system had been open for almost two decades. We note that to the best of our knowledge, the converse problem, of determining whether the LS-proof system can polynomially simulate the cutting-planes proof system remains open.

Now let $\Phi_n(p,q) \cup \Gamma'_n(p,r)$ be the set of inequalities obtained from $\Phi_n(p,q) \cup \Gamma_n(p,r)$ by assumming that the variables r range over the reals. Then $\Phi(p,q) \cup \Gamma'(p,r)$ is an natural example of strongly monotonically separable set of mixed inequalities. Note that this set express the property that the graph represented by p is at the same time balanced and contains a *fractional* perfect matching. Clearly, there is no graph that satisfies both properties simultaneously, and therefore $\Phi_n(p,q) \cup \Gamma'_n(p,r)$ is also unsatisfiable. Now, we note that the integrality axioms and the multiplication rule are never applied with respect to variables in r. Therefore, the proof of theorem 6.7 is also an LS proof of $\Phi_n(p,q) \cup \Gamma'_n(p,r)$.

Corollary 6.9 *The unsatisfiable set of mixed inequalities $\Phi_n(p,q) \cup \Gamma'_n(p,r)$ has LS-refutations of size $n^{O(1)}$.*

By combining Corollary 6.9 with Theorem 6.2, we have that MAX-LEFT MLP gates can separate graphs with a perfect matching from unbalanced graphs superpolynomially faster than monotone real circuits. In other words, monotone real circuits cannot polynomially simulate MAX-LEFT MLP gates. We leave open the question of whether MLP gates (of any type) can polynomially simulate monotone real circuits.

Theorem 6.10 *Let $g_n : \{0,1\}^{\binom{n}{2}} \to \{0,1,*\}$ be the partial Boolean function of Corollary 6.5. Then g_n can be represented by a* single MAX-LEFT *MLP gate of size polynomial in n.*

Proof. Let $\Phi_n(p,q) \cup \Gamma'_n(p,r)$ be the set of inequalities of Corollary 6.9. If $a \in \{0,1\}^n$ represents a graph containing a fractional perfect matching, then $\Gamma'_n(a,r)$ is satisfiable, and consequently $\Phi_n(a,q)$ is unsatisfiable. On the other hand, if a represents an unbalanced graph, then $\Phi_n(a,q)$ is satisfiable and consequently, $\Gamma'_n(a,r)$ is unsatisfiable.

By Theorem 6.7, $\Phi_n(p,q) \cup \Gamma'_n(p,r)$ has a mixed LS-refutation of size polynomial in n. Therefore, by Theorem 6.2, there is a MAX-LEFT MLP gate ℓ_n of size $n^{O(1)}$ such that for each $a \in \{0,1\}^{\binom{n}{2}}$, $\ell_n(a) > 0$ if a denotes a graph with a perfect matching, and such that $\ell_n(a) \leq 0$ if a denotes an unbalanced graph. Therefore, ℓ represents g_n. ∎

7 Alternative MLP Representations

In this section we describe three alternative ways of representing monotone Boolean functions via monotone linear programs. All these representations are polynomially equivalent to weak MLP gates, but they are conceptually closer to certain formalisms that have been studied in complexity theory.

7.1 Existential MLP Representations

In this section we define the notion of existential MLP representations.

Definition 6 (Existential MLP Representations) *let A be a matrix in $\mathbb{R}^{m\times k}$, b be a vector in $\mathbb{R}^m$, and B be a matrix in $\mathbb{R}^{m\times n}$ with $B \geq 0$. Let $F : \{0,1\}^n \to \{0,1,*\}$ be a partial Boolean function. We say that the triple (A,B,b) is a* MAX-EXISTENTIAL *MLP representation of F if the following conditions are satisfied for each $p \in \{0,1\}^n$.*

$$F(p) = \begin{cases} 1 \;\Rightarrow\; \exists x \geq 0,\ Ax \leq b + Bp, \\ 0 \;\Rightarrow\; \neg\exists x \geq 0,\ Ax \leq b + Bp. \end{cases} \tag{41}$$

We say that (A,B,b) is a MIN-EXISTENTIAL *representation of F if the following conditions are satisfied for each $p \in \{0,1\}^n$.*

$$F(p) = \begin{cases} 1 \;\Rightarrow\; \neg\exists x \geq 0,\ Ax \geq b + Bp, \\ 0 \;\Rightarrow\; \exists x \geq 0,\ Ax \geq b + Bp. \end{cases} \tag{42}$$

As in the case of MLP gates, the size of existential representations is measured as the number of rows plus the number of columns in the matrix A. We note that the only difference between MAX-EXISTENTIAL and MIN-EXISTENTIAL MLP representations is that while the former is defined in terms of inequalities $Ax \leq b + Bp$, the latter is defined in terms of inequalities $Ax \geq b + Bp$. We observe that these two representations are not obviously equivalent because of the requirement that $B \geq 0$. Indeed, we do not know if one representation can be transformed into the other without a superpolynomial blow up in size.

Lemma 7.1 *Let $F : \{0,1\}^n \to \{0,1,*\}$ be a partial Boolean function. Then F has a* MAX-EXISTENTIAL *(resp.* MIN-EXISTENTIAL*) MLP representation of size s if and only if F can be represented by a* MAX-RIGHT *(resp.* MIN-RIGHT*) MLP gate of size $O(s)$.*

Proof. Suppose that F can be represented by a MAX-RIGHT MLP gate ℓ of size s. Then, by Proposition 3.1, F can be sign-represented by a MAX-RIGHT MLP gate ℓ' of size $O(s)$. In other words, $\ell'(p) > 0$ whenever $F(p) = 1$ and $\ell'(p) < 0$ whenever $F(p) = 0$. Let $\ell'(p) = \max\{c \cdot x \mid Ax \leq b + Bp,\ x \geq 0\}$. Then the inequalities

$$Ax \leq b + Bp,\ x \geq 0,\ c \cdot x \geq 0$$

define a MAX-EXISTENTIAL MLP representation of F.

For the converse, assume that (A, B, b) is a MAX-EXISTENTIAL MLP representation of F, and consider the function $\ell(p) = \max\{-\bar{1} \cdot x' \mid Ax - x' \leq b + Bp, x \geq 0, x' \geq 0\}$. Let $p \in \{0,1\}^n$. If $F(p) = 1$, then the system of inequalities $Ax \leq b + Bp$ is satisfiable by some $x \geq 0$, and therefore, the maximum in the definition of $\ell(p)$ is attained when $x' = 0$. In other words, in this case $\ell(p) = 0$. On the other hand, if $F(p) = 0$, then the system $Ax \leq b + Bp$ has no solution, and the maximum in the definition of $\ell(p)$ is attained when setting x' to a vector that has at least one strictly positive coordinate. This implies that $\ell(p)$ is strictly negative. Let $\varepsilon = \min_{p \in \{0,1\}^n}\{|\ell(p)| \mid \ell(p) < 0\}$ be the minimum absolute value of $\ell(p)$ where the minimum is taken over all inputs $p \in \{0,1\}^n$ which evaluate to a number strictly less than zero. Now consider the following MAX-RIGHT MLP gate.

$$\ell'(p) = \max\{-\bar{1} \cdot x' + x'' \mid x'' = \varepsilon/2,\ Ax - x' \leq b + Bp, x \geq 0, x' \geq 0\}.$$

Then for each $p \in \{0,1\}^n$, $\ell'(p) = \ell(p) + \varepsilon/2$. This implies that $\ell'(p) \geq \varepsilon/2 > 0$ whenever $F(p) = 1$, and that $\ell'(p) \leq -\varepsilon/2 < 0$ whenever $F(p) = 0$. Therefore, ℓ' is a MAX-LEFT MLP representation of F.

Now, let $\ell(y) = \min\{c \cdot x \mid Ax \geq b + Bp,\ x \geq 0\}$ be a MIN-RIGHT MLP gate that F. Then the inequalities

$$Ax \geq b + Bp,\ x \geq 0,\ c \cdot x \leq 0$$

define a MIN-EXISTENTIAL MLP representation of F.

For the converse, let (A, B, b) be a MIN-EXISTENTIAL MLP representation of F, and let $\ell(p) = \min\{\bar{1} \cdot x' \mid Ax + x' \geq b + Bp, x \geq 0, x' \geq 0\}$. If $F(p) = 0$, then the system of inequalities $Ax \geq b + Bp$ is satisfiable by some $x \geq 0$, and therefore, the minimum is attained when setting $x' = 0$. On the other hand, if $F(p) = 1$, then the system of inequalities $Ax \geq b + Bp$ has no solution $x \geq 0$, and the minimum in the definition of $\ell(p)$ is attained when setting x' to a vector that has a strictly positive coordinate. This implies that $\ell(p)$ is strictly positive. This shows that $\ell(p)$ is a MIN-RIGHT MLP gate representing F. ■

7.2 Representation by Labeled Matrices

When considering MLP gates or EXISTENTIAL MLP representations the polyhedron defined by $Ax \leq b + Bp$ is parameterized by the input variables p via the non-negative matrix B. In this subsection we introduce a way of parameterizing polytopes in terms of the input variables. In this version, some rows of the matrix A are labeled by the Boolean variables p_j and rows may have no label. Formally a labeling for a matrix A with m rows is a function $\rho : \{1, ..., m\} \to \{p_1, ..., p_n, *\}$. For each $i \in \{1, ..., m\}$, $\rho(i) = p_j$ indicates that the i-th row is labeled with p_i, while $\rho(i) = *$ indicates that the i-th row has no label. A labeled matrix is a pair $A^\rho = (A, \rho)$ consisting of a matrix A and a labeling ρ of its rows. For each assignment $w \in \{0,1\}^n$ of the variables in p, we let $A^\rho_{[w]}$ be the sub-matrix obtained from A by deleting all rows labeled by some variable whose value was set to 1. We note that the unlabeled rows remain in $A^\rho_{[w]}$ for each $w \in \{0,1\}^n$. Analogously, we let $A^\rho_{\langle w \rangle}$ be the sub-matrix obtained from A by deleting all rows labeled by some variable whose value was set to 0. These two notations can be straightforwardly applied to vectors by considering them as $m \times 1$ matrices. That is, a labeled vector is a pair $b^\rho = (b, \rho)$ and we write $b^\rho_{[w]}$ (resp. $b^\rho_{\langle w \rangle}$) to denote the vector obtained from b by deleting coordinate i if and only if $\rho(i)$ is a variable that receives the value 1 (resp. 0).

Definition 7 (Labeled MLP Representations) *Let $F : \{0,1\}^n \to \{0,1\}$ be a Boolean function. We say that a pair (A^ρ, b^ρ) is a* MAX-LABELED *MLP representation of F if for each assignment $w \in \{0,1\}^n$ of variables $p = (p_1, ..., p_n)$,*

$$F(w) = \begin{cases} 1 \ \rightarrow \quad \exists x \geq 0, A^\rho_{[w]} x \leq b^\rho_{[w]} \\ \\ 0 \ \rightarrow \ \neg \exists x \geq 0, A^\rho_{[w]} x \leq b^\rho_{[w]} \end{cases} \tag{43}$$

We say that (A^ρ, b^ρ) is a MIN-LABELED *MLP representation of F if for each assignment $w \in \{0,1\}^n$ of variables $p = (p_1, ..., p_n)$,*

$$F(w) = \begin{cases} 1 \ \rightarrow \ \neg \exists x \geq 0, A^\rho_{\langle w \rangle} x \geq b^\rho_{\langle w \rangle} \\ \\ 0 \ \rightarrow \quad \exists x \geq 0, A^\rho_{\langle w \rangle} x \geq b^\rho_{\langle w \rangle} \end{cases} \tag{44}$$

Note that the difference between MAX-LABELED MLP representations and MIN-LABELED MLP representations is that in the former, inequalities are of the form $A^\rho_{[w]} x \leq b^\rho_{[w]}$ and the rows that are deleted from A are those corresponding to variables which the assignment w sets to 1, while in the latter, the inequalities are of the form $A^\rho_{\langle w \rangle} x \geq b^\rho_{\langle w \rangle}$ and the rows that are deleted from A are those corresponding to variables which the assignment w sets to 0.

Note that the function defined by such a representation is monotone because when setting variables in p to 1 rows of A and b are deleted, meaning that the number of constraints decreases, and the possibility of finding an x that satisfies these remaining constraints increase.

In Proposition 7.3, we will show that MAX-LABELED MLP representations and MAX-EXISTENTIAL MLP representations can be converted into each other with only a linear increase in size. Analogously, MIN-LABELED MLP representations and MIN-EXISTENTIAL MLP representations can be converted into each other with only a linear increase in size. Before showing that however, we show that the matrix B in EXISTENTIAL MLP representations can be assumed to be a 0/1 matrix with at most one 1 in each row.

Proposition 7.2 *Let (A, B, b) be a* MAX-EXISTENTIAL *(resp.* MIN-EXISTENTIAL*) MLP representation of a partial Boolean function $F : \{0,1\}^n \to \{0,1\}$ of size s. Then F has a* MAX-EXISTENTIAL *(resp.* MIN-EXISTENTIAL*) MLP presentation (A', B', b') of size $O(s)$ where B' is $0/1$ matrix with at most one 1 in each row.*

Proof. If (A, B, b) is a MAX-EXISTENTIAL MLP representation of F, then we have that for each $p \in \{0,1\}^n$, $\exists x, Ax \leq b + Bp$ if $F(p) = 1$ and $\neg\exists x, Ax \leq b + Bp$ if $F(p) = 0$. Now, consider n new variables $u = (u_1, ..., u_n)$ and the system of inequalities

$$0 \leq u \leq 1,\ u \leq p,\ Ax - Bu \leq b. \tag{45}$$

Then we have that for each $p \in \{0,1\}^n$, $\exists x Ax \leq b + Bp$ if and only if there exists x and u such that the inequalities in Equation 45 are satisfied. Therefore, the inequalities in Equation 45 also define a MAX-EXISTENTIAL MLP representation of F. Note that that each variable $p_i \in p$ occurs in at most one inequality of Equation 45, and when it does, it occurs with coefficient 1. Therefore, Equation 45 can be rewritten as $A' \begin{bmatrix} x \\ u \end{bmatrix} \leq b' + B'p$ for a suitable vector b' and suitable matrices A' and B' where B' is a 0/1 matrix with at most one 1 in each row.

Analogously, if (A, B, b) is a MIN-EXISTENTIAL MLP representation of F, then we have that for each $p \in \{0,1\}^n$, $\exists x, Ax \geq b + Bp$ if $F(p) = 1$ and $\neg\exists x, Ax \geq b + Bp$ if $F(p) = 0$. Therefore, the system of inequalities

$$0 \leq u \leq 1,\ u \geq p,\ Ax - Bu \geq b$$

defines an equivalent MIN-EXISTENTIAL MLP representation of F which can be rewritten in matrix form as $A' \begin{bmatrix} x \\ u \end{bmatrix} \geq b' + B'p$ for a suitable vector b' and suitable matrices A' and B' where B' is a 0/1 matrix with at most one 1 in each row.

■

Proposition 7.3 *Let $F : \{0,1\}^n \to \{0,1\}$. Then F has a* MAX-LABELED *(resp.* MIN-LABELED*) MLP representation of size s if and only if F has a* MAX-EXISTENTIAL *(resp.* MIN-EXISTENTIAL*) MLP representation of size $O(s)$.*

Proof. Let (A, B, b) be a MAX-EXISTENTIAL MLP representation of F and let $p = (p_1, ..., p_n)$ be the input variables of F. By Proposition 7.2, one can assume that the matrix B is a 0/1-matrix where each row has at most one 1. Now, we replace every inequality of the form $a_i \cdot x \leq b_i + p_j$ with the following two inequalities:

$$\begin{array}{lcll} a_i \cdot x & \leq & b_i & \text{with label } p_j. \\ a_i \cdot x & \leq & b_i + 1 & \text{without a label.} \end{array} \tag{46}$$

It is straightforward to check that the system of labeled inequalities obtained in this way corresponds to a MAX-LABELED MLP representation of F.

For the converse, let (A^ρ, b^ρ) be a MAX-LABELED MLP representation of F. Then just replace each inequality $a_i \cdot x \leq b_i$ labeled with p_j, by an inequality $a_i \cdot x \leq b_i + \alpha p_j$ where $\alpha \in \mathbb{R}$ is a number which is large enough to make the inequality irrelevant when $p_j = 1$. The system of inequalities obtained in this way corresponds to a MAX-EXISTENTIAL MLP representation of F.

Now let (A, B, b) be a MIN-EXISTENTIAL MLP representation of F and let $p = (p_1, ..., p_n)$ be the input variables of F. By Proposition 7.2, one can assume that the matrix B is a 0/1-matrix where each row has at most one 1. Now, we replace every inequality of the form $a_i \cdot x \geq b_i + p_j$ with the following two inequalities:

$$\begin{array}{lcll} a_i \cdot x & \geq & b_i & \text{without a label.} \\ a_i \cdot x & \geq & b_i + 1 & \text{with label } p_j. \end{array} \tag{47}$$

It is straightforward to check that the system of labeled inequalities obtained in this way corresponds to a MIN-LABELED MLP representation of F.

For the converse, let (A^ρ, b^ρ) be a MIN-LABELED MLP representation of F. Then just replace each inequality $a_i \cdot x \geq b_i$ labeled with p_j, by an inequality $a_i \cdot x \geq b_i + \alpha(p_j - 1)$ where $\alpha \in \mathbb{R}$ is a number which is large enough to make the inequality irrelevant when $p_j = 0$. The system of inequalities obtained in this way corresponds to a MIN-EXISTENTIAL MLP representation of F.

■

7.3 Representation by Zero-Sum Games

A zero-sum game is defined by a matrix $A \in \mathbb{R}^{m \times k}$. This game has two players: a *Row Player* and a *Column Player*. A strategy for the Row Player is a vector $u \in \mathbb{R}^m$, with $u \geq 0$ and $|u|_1 = 1$ (that is, a probability distribution on $\{1, \ldots, m\}$). Similarly, a strategy for the *Column Player* is a vector $v \in \mathbb{R}^k$ with $v \geq 0$ and $|v|_1 = 1$ (that is, a probability distribution on $\{1, \ldots, k\}$). Such a strategy is *pure* if all weight is placed in a unique coordinate. Given strategies u, v for the two players, the *payoff* of the game defined by A when Row Player plays strategy u and Column Player plays strategy v is defined as $u^T A v$. The payoff of a strategy u for the Row Player is defined as $\min_v u^T A v$, while the payoff of a strategy v for Column Player is defined as $\max_u u^T A v$. We say that a strategy u is a *winning strategy* for

Row Player, if for every strategy v of Column Player we have $u^T Av < 0$. On the other hand, a strategy v is a *winning strategy* for Column Player, if for every strategy u of Row Player we have $u^T Av > 0$.

Let $p = (p_1, ..., p_n)$ be Boolean variables. A double-labeled matrix is a triple $A^{\rho,\gamma} = (A, \rho, \gamma)$ where A is a $m \times k$ real matrix, $\rho : \{1, ..., m\} \to \{p_1, ..., p_n, *\}$ is a labeling of the rows of A and $\gamma : \{1, ..., k\} \to \{p_1, ..., p_n, *\}$ is a labeling of the columns of A. We say that a row i (column j) is unlabeled if $\rho(i) = *$ ($\gamma(j) = *$). For each assignment $w \in \{0,1\}^n$ of the variables in p, we denote by $A^{\rho,\gamma}_{[w]}$ the sub-matrix of A which is obtained by deleting rows labeled with variables that are set to 1, and by deleting columns labeled with variables that are set to 0 at the assignment w.

Definition 8 (Zero-Sum Representation) *Let $F : \{0,1\}^n \to \{0,1\}$ be a Boolean function on variables $p = (p_1, ..., p_n)$. We say that that a double-labeled matrix $A^{\rho,\gamma}$ is a zero-sum game representation of F if for every assignment $w \in \{0,1\}^n$,*

$$F(w) = \begin{cases} 1 \ \to \ \textit{Column Player has a winning strategy for the game } A^{\rho,\gamma}_{[w]}. \\ 0 \ \to \ \textit{Row Player has a winning strategy for the game } A^{\rho,\gamma}_{[w]}. \end{cases} \tag{48}$$

We note that the asymmetry in the way in which rows and columns are deleted guarantees that the function F is monotone. Intuitively, by setting a variable p_i to 1, Row Player is not anymore allowed to use the rows labeled with p_i and Column Player is now allowed to use the columns labeled with p_i. Therefore the space of strategies of Row Player shrinks, while the space of strategies of Column Player gets expanded. In this way, the payoff for Column player is at least as large as if the variable p_i were set to 0.

The next proposition states that zero-sum game representations are equivalent to MAX-EXISTENTIAL MLP representations. We believe that it is worth to consider zero-sum game representations as a separate concept because they seem to be more amenable for the application of lower-bound techniques based on communication complexity theory. The idea is that the variables p may be split into two disjoint groups p' and p'' in such a way that rows are only labeled with variables from p', and columns are only labeled with variables from p''. This corresponds to the setting in which we want to compute a function $F(p)$ where the variables in p' are in possession of Row player while variables in p'' are in possession of Column Player.

We note that from the point of view of expressiveness, the way in which variables are distributed among Row Player and Column Player is irrelevant. More precisely, by making appropriate modifications to a double-labeled matrix $A^{\rho,\gamma}$, one can always transform a row label into a column label, and vice versa, in such a way the resulting double-labeled matrix $A^{\rho',\gamma'}$ represents the same function. Additionally, one can always consider that each variable labels at most one row and at most one column.

Proposition 7.4 *A function $F : \{0,1\}^n \to \{0,1\}$ has a zero-sum game representation $A^{\rho,\gamma}$ of size s if and only if it has a* MAX-EXISTENTIAL *MLP representation of size $O(s)$. Additionally, the same statement holds if we assume that either all rows, or all columns are unlabeled, as well as if we assume that each variable labels at most one row and at most one column.*

Proof. Let $A^{\rho,\gamma}$ be a zero-sum game representation of size s of a function F in which no column is labeled. In other words, $\gamma(j) = *$ for each column of A. Then this matrix can be viewed as a single-labeled matrix A^{ρ}. It should be clear that the system of inequalities

$$\begin{array}{l} A^{\rho}_{[w]}x_{[w]} \leq 0, \\ \sum_{j=1}^{k} x_j = 1 \end{array} \tag{49}$$

is a MAX-LABELED MLP representation of F of size $O(s)$. Therefore, by Proposition 7.3, F has a MAX-EXISTENTIAL MLP representation of size $O(s)$.

For the converse, assume that F has a MAX-EXISTENTIAL MLP representation of size s. Then by Proposition 7.3, F has a MAX-LABELED MLP representation (A^{ρ}, b) of size $O(s)$. We may assume without loss of generality that the corresponding system of inequalities

$$A^{\rho} \leq b^{\rho} \tag{50}$$

contains the unlabeled equality $\sum_j x_j = 1$ (which is represented by two unlabeled inequalities $\sum_j x_j \leq 1$ and $\sum_j x_j \geq 1$). This assumption will be removed later. Then by adding appropriate multiples of the inequality $\sum_j x_j = 1$ to each row, System 50 can be transformed into a system of the form

$$\begin{array}{l} (A')^{\rho}x \leq 0^{\rho} \\ \\ \sum_{j=1}^{k} x_j = 1 \end{array} \tag{51}$$

Such that for each $w \in \{0,1\}^n$, the system $A^{\rho}_{[w]} \leq b^{\rho}_{[w]}$ has a solution if and only if the system

$$\begin{array}{l} (A')^{\rho}_{[w]}x \leq 0^{\rho}_{[w]}, \\ \\ \sum_{j=1}^{k} x_j = 1 \end{array} \tag{52}$$

has a solution. Additionally, we have the following immediate claim.

Claim 2 *For each $w \in \{0,1\}^n$, System 52 has a solution if and only if Column Player has a strategy to get payoff ≥ 0 in the zero-sum game defined by $(-A')^{\rho}_{[w]}$.*

Now let ε be a small enough positive number, and let A'' be the matrix obtained from by adding ε to each entry of $-A'$. Then we have that for each $w \in \{0,1\}^n$, Column Player gets a payoff ≥ 0 in the game $(-A)^{\rho}_{[w]}$ if and only if Column player gets a payoff > 0 in the game $(A'')^{\rho}_{[w]}$. Therefore, if we let γ be a labeling of the columns of A'' such that $\gamma(j) = *$ for every $j \in \{1, ..., k\}$. The double-labeled matrix $(-A')^{\rho,\gamma}$ is a zero-sum game representation of F.

Now assume that the equality $\sum_j x_j = 1$ does not belong to the system of inequalities $A^{\rho}x \leq b^{\rho}$ defined by the MAX-LABELED MLP representation (A^{ρ}, b^{ρ}). First we select a large enough positive real number α such that for every p for which there exists a solution x, we have $\sum_j x_j \leq \alpha$. Then if we take a (dummy) variable x_{k+1} and add the equality

$\sum_{j=1}^{k+1} x_j/\alpha = 1$, the new system has a solution if and only if the old one has. Finally, we make a change of variables by setting $y_j := x_j/\alpha$ for each $j \in \{1, ..., k\}$ and by setting $y_{j+1} := x_{j+1}$. Clearly, the new system of inequalities on variables y_j has a solution if and only if the old one has, and now the equation $\sum_{j=1}^{k+1} y_j = 1$ belongs to the system.

Now we show that the way in which variables are distributed among Row Player and Column Player is immaterial. Assume that p_j labels some columns of $A^{\rho,\gamma}$. Add a new row to A which has -1 on the columns labeled with p_j and 0 elsewhere. Label this new row with p_j, and remove the labels p_j from the columns. Let $(A')^{\rho',\gamma'}$ be the matrix obtained by this process. Let w be an assignment of the variables in p. If p_j is set to 1, then the new added row is not present in the matrix $(A')^{\rho',\gamma'}_{[w]}$, and therefore the column player is free to chose the columns that were labeled by p_j in the original matrix A. On the other hand, if p_j is set to zero, then the row player has a strategy with payoff < 0 for any strategy of Column Player that sets non-zero weight in some column that was previously labeled by p_j. Hence, in such case, any winning strategy for Column player must put weight 0 in these columns. A symmetric argument shows that row labels can be transformed into column labels. In this case, the difference is that in this case we create a new column which has 1 in every row labeled by p_j. Subsequently we label this new column with p_j, and remove the label p_j from the rows. Then, if $p_j = 1$, any winning strategy for Row Player must set weight 0 on all rows that were previously labeled by p_j. Note that in either case, a unique row or column labeled by p_j is created. ■

8 Monotone Linear Programs and Extended Formulations

A *polytope* is the convex hull of a nonempty finite set of vectors in $\mathbb{R}^n$; in particular, a polytope is *nonempty and bounded*. If a polytope $P \subseteq \mathbb{R}^n$ is given by a polynomial number of inequalities[5], then we can easily decide whether a vector $v \in \mathbb{R}^n$ belongs to P. An important observation is that even if P requires an exponential number of inequalities to be defined, we may still be able to test whether $v \in P$ efficiently if we can find a polytope $R \subseteq \mathbb{R}^{n+m}$ in a higher dimension with $m = n^{O(1)}$ such that P is a projection of P' and P' can be described by a polynomial number of inequalities[5].

More precisely, let $P \subseteq \mathbb{R}^n$ be a polytope, and let $P' \subseteq R^{n+m}$ be a polytope defined via a system of inequalities[6] $A(v, y) \leq b$. Then we say that the system $A(v, y) \leq b$ is an extended formulation of P if for each $v \in \mathbb{R}^n$, $v \in P \Leftrightarrow \exists y \in \mathbb{R}^m, A(v, y) \leq b$. We define the size of such extended formulation as the number of rows plus the number of columns in A. For instance, it can be shown that the permutahedron polytope $P_n \subseteq \mathbb{R}^n$, which is defined as the convex-hull of all permutations of the set $[n] = \{1, ..., n\}$, requires exponentially many inequalities to be defined. Nevertheless, P_n has extended formulations of size $O(n \log n)$ [12]. On the

[5] With coefficients specified by $n^{O(1)}$ bits.

[6] For column vectors $v \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, (v, y) denotes the column vector $(v_1, ..., v_n, y_1, ..., y_m)$.

other hand, it has been shown that for some polytopes, such as the cut polytope, the TSP polytope, etc., even extended formulations require exponentially many inequalities [9, 29].

The process of defining partial Boolean functions via linear programs is closely related, but not equivalent, to the process of defining polytopes via extended formulations. For a partial Boolean function F, let $\mathit{Ones}(F)$, and $\mathit{Zeros}(F)$ denote the set of all inputs $a \in \{0,1\}^n$ such that $F(a) = 1$, and $F(a) = 0$ respectively. Let P_F^1 denote the convex hull of $\mathit{Ones}(F)$ and P_F^0 denote the convex hull of $\mathit{Zeros}(F)$. Defining F via a linear program is equivalent to finding an extended formulation of some polyhedron Q^1 that contains P_F^1 and is disjoint from $\mathit{Zeros}(F)$, or an extended formulation of some polyhedron Q^0 that contains P_F^0 and is disjoint from $\mathit{Ones}(F)$. Finding such an extended formulation for Q^1 (resp. Q^0) with a small number of inequalities is clearly, a simpler task than finding a small extended formulation for the polyhedron P_F^1 (resp. P_F^0) itself. For instance, if F is the matching function for general graphs, then F is computable by a polynomial-size Boolean circuit (containing negation gates), and hence this function can be defined via (not necessarily monotone) linear programs of polynomial size[7]. Nevertheless, the corresponding polytope P_F^1 requires extended formulations of exponential size [29].

Let $F : \{0,1\}^n \to \{0,1,*\}$ be a partial monotone Boolean function. A minterm of F is a vector $v \in \{0,1\}^n$ such that $F(v) = 1$ and such that $F(v') \neq 1$ for each $v' \leq v$. Intuitively, a minterm is a minimal vector which causes F to evaluate to 1. Analogously, a maxterm is a vector $v \in \{0,1\}^n$ such that $F(v) = 0$ and $F(v') \neq 0$ for each $v \geq 0$. Intuitively, a maxterm is a maximal vector that causes F to evaluate to 0. We let $\hat{P}_F^1$ be the convex-hull of minterms of F, and let $\hat{P}_F^0$ be the convex-hull of maxterms of F. Let H^1 be a hyperplane containing $\hat{P}_F^1$. For each maxterm v we define the set $S_v^1 = H^1 \cap \{u \mid u \leq v\}$. Analogously, let H^0 be an hyperplane containing $\hat{P}_F^0$. We define the set $S_v^0 = H^0 \cap \{u \mid u \geq v\}$.

Definition 9 (Monotone Extension Complexity) *Let $F : \{0,1\}^n \to \{0,1,*\}$ be a partial monotone Boolean function. Below we define two notions of monotone extension complexity (mxc) for F.*

1. *We let $\mathit{mxc}_1(F)$ denote the minimum size of an extended formulation for a polytope Q^1 such that*
$$\hat{P}_F^1 \subseteq Q^1, \qquad \text{and} \qquad Q \cap \bigcup_v S_v^1 = \emptyset. \tag{53}$$

2. *We let $\mathit{mxc}_0(F)$ denote the minimum size of an extended formulation for a polytope Q^0 such that*
$$\hat{P}_F^0 \subseteq Q^0, \qquad \text{and} \qquad Q \cap \bigcup_v S_v^0 = \emptyset. \tag{54}$$

The next theorem relates the monotone extension complexity of a partial monotone Boolean function F to the size of existential MLP representations for F.

[7]Note that any function in PTIME can be defined by polynomial-size non-monotone LP programs, due to the fact that linear programming is PTIME complete.

Theorem 8.1 *Let $F : \{0,1\}^n \to \{0,1,*\}$ be a partial monotone Boolean function. Then $mxc_1(F)$ is up to a constant factor equal to the minimum size of a* MAX-EXISTENTIAL *MLP representation of F. Analogously, the $mxc_0(F)$ is up to a constant factor equal to the minimum size of a* MIN-EXISTENTIAL MLP *representation of F.*

Proof. First, we define the sets $\mathbb{R}^n_+ = \{u \in \mathbb{R}^n | u \geq 0\}$ and $\mathbb{R}^n_- = \{u \in \mathbb{R}^n \mid u \leq 0\}$. Additionally, for sets X, Y of vectors of same dimension, let $X+Y = \{v+u \mid v \in X,\ u \in Y\}$ be the Minkowski sum of X and Y.

Let (A, B, b) be a MAX-EXISTENTIAL MLP representation for F. Then for each $p \in \{0,1\}^n$ such that $F(p) = 1$, there exists an $y \geq 0$ such that all inequalities in the system $Ay \leq b + Bp$ are satisfied. Additionally, if $F(p) = 0$, then no such $y \geq 0$ exists. Therefore, the system of inequalities $Ay \leq b + Bx$ is an extended formulation for a polytope Q^1 such that $P^1_F + \mathbb{R}^n_+ \subseteq Q^1$ and $Q^1 \cap (Zeros(F) + \mathbb{R}^n_-) = \emptyset$.

For the converse, assume that the system of inequalities $A(x,y) \leq b$ defines an extended formulation for a polytope Q^1 such that $\hat{P}^1_F \subseteq Q^1$ and $Q^1 \cap \bigcup_v S^1_v = \emptyset$. Then the inequalities $A(x,y) \leq b,\ x \leq p$ define a MAX-EXISTENTIAL MLP representation for F.

Now, let (A, B, b) be a MIN-EXISTENTIAL MLP representation for F. Then for each $p \in \{0,1\}^n$ such that $F(p) = 0$, there exists an $y \geq 0$ such that all inequalities in the system $Ay \geq b + Bp$ are satisfied. Additionally, if $F(p) = 1$, then no such $y \geq 0$ exists. Therefore, the system of inequalities $Ay \geq b + Bx$ is an extended formulation for a polytope Q^0 such that $P^0_F + \mathbb{R}^n_- \subseteq Q^0$ and $Q^0 \cap (Ones(F) + \mathbb{R}^n_+) = \emptyset$.

For the converse, assume that the system of inequalities $A(x,y) \geq b$ defines an extended formulation for a polytope Q^0 such that $\hat{P}^0_F \subseteq Q^0$ and $Q^0 \cap \bigcup_v S^0_v = \emptyset$. Then the inequalities $A(x,y) \geq b,\ x \geq p$ define a MIX-EXISTENTIAL MLP representation for F. ∎

All monotone Boolean functions for which lower bounds have been proved have the property that maxterms have essentially larger weight[8] than minterms. When this happens, the sets $S^1_v = H^1 \cap \{u \mid u \leq v\}$ are simplices.

Example. Let F be the partial monotone Boolean function where minterms are k-cliques in a graph on n vertices and maxterms are complete $(k-1)$-partite graphs. Suppose $k = n^\alpha$ for some $0 < \alpha < 1$. Then we can set H^1 is as the hyperplane consisting of all vectors of weight $\binom{k}{2}$. The weight of maxterms is $\approx kn = n^{1+\alpha}$ while the weight of minterms is $\approx n^{2\alpha}$.

A possible approach to for proving superpolynomial lower bounds on the monotone extension complexity of a function is to show that any polytope Q that separates P^1_F from $\bigcup_v S^1_v$ must be close to P^1_F. If this is the case, one could then apply techniques obtained in the context of approximate extended formulations of polytopes to prove a lower bound on the size of such a Q^1.

We note however that proving lower bounds for the size of weak MLP gates using extended formulation techniques will not be an easy task. For instance, the polytope obtained as the convex-hull of points corresponding to graphs with a perfect-matching can only be described

[8]The weight of a vector $v \in \{0,1\}^n$ is the number of times that 1 occurs in v.

via extended formulations of exponential size. Nevertheless, Theorem 6.10 shows that weak MLP gates of polynomial size can be used to separate this convex-hull from the set of points corresponding to unbalanced graphs.

9 Conclusion

In this work we introduced several models of computation based on the notion of monotone linear programs. In particular, we introduced the notions of weak and strong MLP gates. We reduced the problem of proving lower bounds for the size of LS proofs to the problem of proving lower bounds for the size of MLP circuits with strong gates, and the problem of proving lower bounds on the size of mixed LS proofs to the problem of proving lower bounds on the size of single weak MLP gates.

When it comes to comparing MLP gates with other models of computation, we have shown that weak MLP gates are strictly more powerful than monotone Boolean circuits and monotone span programs. Additionally, these gates cannot be polynomially simulated by monotone real circuits. Finally, by combining some results mentioned above, we proved that the cutting-planes proof system is not powerful enough to polynomially simulate the LS proof system. This is the first result showing a separation between the power of these two systems.

The results mentioned above indicate that the study of monotone models of computation based on linear programming has the potential to shed new light on deep questions in circuit complexity and in proof complexity. We note however, that when proposing a new model of monotone computation, there is always a danger that the model is too strong. So strong that proving size lower bounds on this model for explicit Boolean functions would imply a major breakthrough in computational complexity. For instance, a *nondeterministic monotone circuit* for a Boolean function $F(p)$ is a monotone circuit $C(p, q, r)$, where q and r are strings of variables of equal length such that

$$F(p) = 1 \;\Leftrightarrow\; \exists q\; C(p, q, \neg q) = 1.$$

Note that this is a fully syntactic definition—the form of the circuit ensures that the function it computes is monotone. Yet this kind of circuits are equivalent to general nondeterministic circuits.

Proposition 9.1 *If a monotone function F is computed by a nondeterministic circuit of size s, then there exists a monotone nondeterministic circuits of size $O(s)$ that computes F*

Proof. Suppose

$$F(p) = 1 \;\Leftrightarrow\; \exists q\; C(p, \neg p, q, \neg q) = 1,$$

where C is monotone. Then we can represent F as follows

$$F(p) = 1 \;\Leftrightarrow\; \exists q, r\; C(r, \neg r, q, \neg q) \wedge \bigwedge_i (p_i \vee \neg r_i) = 1.$$

■

Nevertheless, we are confident that the models we have introduced in this work do not suffer from this excess of computational power. In particular, in Section 7 we have provided several alternative formulations of monotone models of computation that are equivalent in power to weak MLP gates, but which are conceptually close to notions that have been used before to provide lower bounds for explicit functions. Additionally, in Section 8 we have established tight connections between the problem of proving lower bounds for the size of existential MLP representations and the problem of proving lower bounds for the extension complexity of certain separating polytopes.

We conclude this work by stating some open problems whose solution could lead to the development of more powerful techniques for the obtention of explicit size lower bounds for monotone models of computation and proof systems.

1. Prove superpolynomial lower bounds for the size of weak MLP gates representing an explicit partial function F.

2. Prove superpolynomial lower bounds for the size of MLP circuits with strong gates representing an explicit partial function F.

3. In Subsection 3.3, we have shown that if a partial Boolean function F can be represented by a weak MLP gate of type τ and size s, then the function F^d (the Boolean dual of F) can be represented by a weak MLP gate of type τ^{sd} and size $O(s)$. It would be interesting to determine whether F^d can be represented by an MLP gate of type τ of size $s^{O(1)}$. We note that a similar question is still open in the context of monotone span programs. In other words, it is not known if the fact that a function F can be represented by span programs of size s implies that F^d can be represented by span programs of size $s^{O(1)}$.

4. Is it possible to bound the coefficients occurring in MLP gates without increasing too much the size of representations? More specifically, given an MLP gate ℓ of polynomial size representing a function F, can one modify it in such a way that all coefficients in the inequalities and objective function defining ℓ have polynomial magnitude? Note that a similar question is open in the context of monotone span programs.

5. Is it possible to prove exponential lower bounds in nonconstructive ways? If the answer to the previous problem is positive, such a lower bound would follow by simple counting.

6. Is there a *total* function F that can be represented by polynomial size MLP gates but such that $Ones(F)$ or $Zeros(F)$ does not have polynomial size extended formulation?

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