Comment on Meir's paper "The Direct Sum of Universal Relations"

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Abstract

In [1] Meir proved that deterministic communication complexity of the m-fold direct sum of the universal relation is at least $m(n - O(\log m))$. In this comment we present a log-rank argument which improves Meir's lower bound to $m(n-1) - 1$.

Abbreviate $[k] = \{1, 2, \ldots, k\}$. By $D(T)$ we denote the deterministic communication complexity of the communication problem T.

Definition 1. *Let $m, n \in \mathbb{N}$. The m-fold direct sum of the universal relation on n bits, denoted by $U_n^{\otimes m}$, is the communication problem in which Alice and Bob get matrices X and Y from $\{0,1\}^{m\times n}$ that differ on every row. They are required to output a tuple $(j_1, \ldots, j_m) \in [n]^m$, such that for every row $i \in [m]$ it holds that $X_{i,j_i} \neq Y_{i,j_i}$.*

Definition 2. *Let $m, n \in \mathbb{N}$. Let V_n^m denote the communication problem in which Alice and Bob get two matrices X and Y from $\{0,1\}^{m\times n}$. The goal is to output 1, if X and Y differ on every row, and 0 otherwise.*

Proposition 1. $D(U_n^{\otimes m}) + m + 1 \geq D(V_n^m)$.

Proof. If π is the protocol solving $U_n^{\otimes m}$, then π can be transformed into the protocol solving V_n^m at the expense of increasing communication complexity by at most $m+1$. The transformation is as follows. Let $X, Y \in \{0,1\}^{m\times n}$ be any two matrices. Alice and Bob run π on (X, Y). Let $(j_1, \ldots, j_m)$ be the output of π. Alice sends $X_{1,j_1}, \ldots, X_{m,j_m}$ to Bob. If for every $i \in [m]$ it holds that $X_{i,j_i} \neq Y_{i,j_i}$, Bob sends 1 to Alice. Otherwise he sends 0. The last bit of Bob is the output of the protocol. □

Proposition 2. $D(U_n^{\otimes m}) \geq m(n-1) - 1$.

Proof. By proposition 1 it is enough to show that $D(V_n^m)$ is at least nm. Let A_n^m be communication matrix of V_n^m. The rows and the columns of A_n^m are labeled by $m \times n$ binary matrices. We consider each such matrix as the sequence of

m binary strings of length n. Let's assume that these sequences are ordered according to the lexicographic order. Then we have

$$A_n^m = \begin{pmatrix} 0 & A_n^{m-1} & \dots & A_n^{m-1} & A_n^{m-1} \\ A_n^{m-1} & 0 & \dots & A_n^{m-1} & A_n^{m-1} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ A_n^{m-1} & A_n^{m-1} & \dots & A_n^{m-1} & 0 \end{pmatrix}.$$

By the the induction on m this equality implies that $A_n^m = A \otimes A \otimes \ldots \otimes A$ (m times), where $\otimes$ denotes the Kronecker product and A is $2^n \times 2^n$ matrix with the zeroes on the diagonal and ones elsewhere. As the rank of A (over the reals) is 2^n, the rank of A_n^m (over the reals) is $(2^n)^m = 2^{nm}$ and hence by the log-rank lower bound $D(V_n^m) \geq nm$. □

References

[1] Meir, O. The direct sum of universal relations. In *Electronic Colloquium on Computational Complexity (ECCC)* (2017).