Comment on Meir’s paper “The Direct Sum of Universal Relations”

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September 3, 2017

Abstract

In [1] Meir proved that deterministic communication complexity of the \(m\)-fold direct sum of the universal relation is at least \(m(n - O(\log m))\).

In this comment we present a log-rank argument which improves Meir’s lower bound to \(m(n - 1) - 1\).

Abbreviate \([k] = \{1, 2, \ldots, k\}\). By \(D(T)\) we denote the deterministic communication complexity of the communication problem \(T\).

**Definition 1.** Let \(m, n \in \mathbb{N}\). The \(m\)-fold direct sum of the universal relation on \(n\) bits, denoted by \(U_n^{\otimes m}\), is the communication problem in which Alice and Bob get matrices \(X\) and \(Y\) from \([0, 1]^{m \times n}\) that differ on every row. They are required to output a tuple \((j_1, \ldots, j_m) \in [n]^m\), such that for every row \(i \in [m]\) it holds that \(X_{i,j_i} \neq Y_{i,j_i}\).

**Definition 2.** Let \(m, n \in \mathbb{N}\). Let \(V_n^m\) denote the communication problem in which Alice and Bob get two matrices \(X\) and \(Y\) from \([0, 1]^{m \times n}\). The goal is to output 1, if \(X\) and \(Y\) differ on every row, and 0 otherwise.

**Proposition 1.** \(D(U_n^{\otimes m}) + m + 1 \geq D(V_n^m)\).

**Proof.** If \(\pi\) is the protocol solving \(U_n^{\otimes m}\), then \(\pi\) can be transformed into the protocol solving \(V_n^m\) at the expense of increasing communication complexity by at most \(m + 1\). The transformation is as follows. Let \(X, Y \in \{0, 1\}^{m \times n}\) be any two matrices. Alice and Bob run \(\pi\) on \((X, Y)\). Let \((j_1, \ldots, j_m)\) be the output of \(\pi\). Alice sends \(X_{1,j_1}, \ldots, X_{m,j_m}\) to Bob. If for every \(i \in [m]\) it holds that \(X_{i,j_i} \neq Y_{i,j_i}\), Bob sends 1 to Alice. Otherwise he sends 0. The last bit of Bob is the output of the protocol. \(\square\)

**Proposition 2.** \(D(U_n^{\otimes m}) \geq m(n - 1) - 1\).

**Proof.** By proposition 1 it is enough to show that \(D(V_n^m)\) is at least \(nm\). Let \(A_n^m\) be communication matrix of \(V_n^m\). The rows and the columns of \(A_n^m\) are labeled by \(m \times n\) binary matrices. We consider each such matrix as the sequence of
$m$ binary strings of length $n$. Let’s assume that these sequences are ordered according to the lexicographic order. Then we have

$$A_n^m = \begin{pmatrix} 0 & A_{n}^{m-1} & \cdots & A_{n}^{m-1} & A_{n}^{m-1} \\ A_{n}^{m-1} & 0 & \cdots & A_{n}^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n}^{m-1} & A_{n}^{m-1} & \cdots & 0 \end{pmatrix}.$$  

By the induction on $m$ this equality implies that $A_n^m = A \otimes A \otimes \ldots \otimes A$ ($m$ times), where $\otimes$ denotes the Kronecker product and $A$ is $2^n \times 2^n$ matrix with the zeroes on the diagonal and ones elsewhere. As the rank of $A$ (over the reals) is $2^n$, the rank of $A_n^m$ (over the reals) is $(2^n)^m = 2^{nm}$ and hence by the log-rank lower bound $D(V_n^m) \geq nm$.  

References