Electronic Colloquium on Computational Complexity, Report No. 7 (2018)

A Generalized Turán Problem and its Applications

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Abstract

Our first theorem in this papers is a hierarchy theorem for the query complexity of testing graph properties with 1-sided error; more precisely, we show that for every super-polynomial f, there is a graph property whose 1-sided-error query complexity is $f(\Theta(1/\varepsilon))$. No result of this type was previously known for *any* f which is super polynomial. Goldreich [ECCC 2005] asked to exhibit a graph property whose query complexity is $2^{\Theta(1/\varepsilon)}$. Our hierarchy theorem partially resolves this problem by exhibiting a property whose 1-sided-error query complexity is $2^{\Theta(1/\varepsilon)}$. We also use our hierarchy theorem in order to resolve a problem raised by the second author and Alon [STOC 2005] regarding testing relaxed versions of bipartiteness.

Our second theorem states that for any function f there is a graph property whose 1-sided-error query complexity is $f(\Theta(1/\varepsilon))$ while its 2-sided-error query complexity is only $\text{poly}(1/\varepsilon)$. This is the first indication of the surprising power that 2-sided-error testing algorithms have over 1-sided-error ones, even when restricted to properties that are testable with 1-sided error. Again, no result of this type was previously known for any f that is super polynomial.

The above two theorems are derived from a graph theoretic result which we think is of independent interest, and might have further applications. Alon and Shikhelman [JCTB 2016] recently introduced the following generalized Turán problem: for fixed graphs H and T, and an integer n, what is the maximum number of copies of T, denoted by $\text{ex}(n, T, H)$, that can appear in an n-vertex H-free graph? This problem received a lot of attention recently, with an emphasis on $\text{ex}(n, C_3, C_{2\ell+1})$. Our third theorem in this paper gives tight bounds for $\text{ex}(n, C_k, C_\ell)$ for *all* the remaining values of k and ℓ.

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ISSN 1433-8092

1 Introduction

1.1 Background and motivation

Property testers are fast randomized algorithms which can quickly determine if an object satisfies some predetermined property $\mathcal{P}$ or is "far" from satisfying $\mathcal{P}$. The systematic study of such problems began with the seminal papers of Rubinfeld and Sudan [36] and Goldreich, Goldwasser and Ron [25]. In the past two decades, problems of this type have been studied in so many areas, that it will be impossible to survey them even briefly in this extended abstract. We refer the reader to the upcoming book of Goldreich [24] for more background and references on the subject.

Our focus in this paper will be testing graph properties in the *dense graph model*, introduced in the aforementioned [25], which was the first model in which property testing problems have been systematically studied. In this model, the input graph G is given via its $n \times n$ adjacency matrix, and we assume that there is an oracle that can answer queries of the form: is (i, j) an edge of G? We say that an n-vertex graph G is ε-far from satisfying property $\mathcal{P}$ if one should add/remove at least εn^2 edges in order to turn G into a graph satisfying $\mathcal{P}$. An ε-tester for $\mathcal{P}$ is an algorithm that can distinguish with high probability (say, 2/3) between the case that G satisfies $\mathcal{P}$ and the case that G is ε-far from satisfying it. By a result of Goldreich and Trevisan [26], we can always assume that an ε-tester for $\mathcal{P}$ works by sampling a random subset of vertices S of a prescribed size and making its decision based on (the isomorphism class of) $G[S]$ (the subgraph of G induced by S). We denote by $q = q'_{\mathcal{P}}(\varepsilon, n)$ the smallest integer for which there is an ε-tester for n-vertex graphs that works by randomly selecting a set of vertices S of size q. We say that $\mathcal{P}$ is *testable* if $q'_{\mathcal{P}}(\varepsilon, n) \le q_{\mathcal{P}}(\varepsilon)$, that is, if there is an ε-tester which inspects a subgraph of size that depends only on ε and not on $|V(G)|$.

In this paper we will only consider *monotone* properties, that is, properties closed under removal of edges and vertices. Let $w_{\mathcal{P}}(\varepsilon)$ be the smallest integer so that if G is ε-far from satisfying $\mathcal{P}$, then a random subset $S \subseteq V(G)$ of $w_{\mathcal{P}}(\varepsilon)$ vertices is such that $G[S]$ does not satisfy $\mathcal{P}$ with probability at least 2/3. In other words, $w_{\mathcal{P}}(\varepsilon)$ tells us how many vertices we should sample from a graph that is ε-far from satisfying $\mathcal{P}$ in order to find a *witness* (hence the notation $w_{\mathcal{P}}(\varepsilon)$) to this fact.

A result of Alon and the second author [5] states that a function $w_{\mathcal{P}}(\varepsilon)$ indeed exists for every monotone property $\mathcal{P}$. Note that this immediately implies that every monotone graph property is testable, since we trivially have $q_{\mathcal{P}}(\varepsilon) \le w_{\mathcal{P}}(\varepsilon)$. In fact, it means that every such property is testable with 1-sided error, where a tester has 1-sided error if it accepts graphs satisfying the property with probability 1 (and rejects those that are ε-far from the property with probability at least 2/3). Actually, it is easy to see that a 1-sided-error tester of a monotone property $\mathcal{P}$ cannot reject an input if $G[S]$ satisfies $\mathcal{P}$. Hence $w_{\mathcal{P}}(\varepsilon)$ actually equals the query complexity of the optimal 1-sided-error tester for $\mathcal{P}$. Hence, from now on we use $w_{\mathcal{P}}(\varepsilon)$ as the measure of the optimal query complexity of 1-sided-error testers of a monotone property $\mathcal{P}$.

We now turn to describe the main problems we will investigate in this paper. The main shortcoming of the result of [5] is that the upper bounds it supplied were of the form $w_{\mathcal{P}}(\varepsilon) \le \text{tower}(\text{poly}(1/\varepsilon))$ (or worse) where tower(x) is a tower of exponents of height x. Since for many natural graph properties such as k-colorability one can in fact show that $w_{\mathcal{P}}(\varepsilon) = \text{poly}(1/\varepsilon)$ [25], this raised the natural problem of determining for which properties one has $w_{\mathcal{P}}(\varepsilon) = \text{poly}(1/\varepsilon)$. As it turns out, this is not an easy task since determining $w_{\mathcal{P}}(\varepsilon)$ even for "simple" properties $\mathcal{P}$ such as triangle-freeness, is equivalent to determining the best bounds for the well-known triangle-removal lemma [35], a famous open problem in extremal graph theory [13]. To date, the best known results [19, 35] for the triangle removal lemma, as well as for many similar properties, are $(1/\varepsilon)^{c \log 1/\varepsilon} \le w_{\mathcal{P}}(\varepsilon) \le \text{tower}(O(\log 1/\varepsilon))$, that is, there is still an enormous gap between the best known lower/upper bounds. Nonetheless, prior to this work there was no example of a property that was shown to have query complexity $f(\varepsilon)$ for some super-polynomial[1] f. See also Subsection 8.5 of [24] for more on this issue. Motivated by the above, Goldreich [23, 24] raised the following challenge.

Problem 1 ([23, 24]). *Exhibit a (natural) graph property $\mathcal{P}$ satisfying $q_{\mathcal{P}}(\varepsilon) = 2^{\Theta(1/\varepsilon)}$.*

Given the fundamental time/space hierarchy theorems in complexity theory, it is even natural to go one step further and ask if there are hierarchy theorems for the query complexity of testing graph properties with 1-sided and 2-sided error. More precisely:

Problem 2. *Is it the case that for any $f : (0,1) \to \mathbb{N}$ there is a graph property $\mathcal{P}$ satisfying $q_{\mathcal{P}}(\varepsilon) = f(\varepsilon)$? Is there always a $\mathcal{P}$ satisfying $w_{\mathcal{P}}(\varepsilon) = f(\varepsilon)$?*

It is easy to show (see [25, 24]) that there are properties that can be tested with 2-sided error but cannot be tested with 1-sided error with *any* query complexity independent[2] of n. It is thus more natural to restrict ourselves to graph properties that *can* be tested with 1-sided error, and ask:

Problem 3. *To what extent are 2-sided testers more powerful than 1-sided testers?*

Another motivation to look at this problem (see [24]) is the observation that 1-sided testers have (for the most part) no algorithmic ideas behind them, and are essentially equivalent to (usually very hard to prove) statements in extremal combinatorics. On the other hand, 2-sided testers are usually much more algorithmic in nature. So another motivation for Problem 3 can be colloquially stated as "are algorithms more powerful than combinatorics in the setting of testing dense graphs"?

We now turn to describe a problem raised by the first author and Alon [5]. For what follows, we use $w_k(\varepsilon)$ instead of $w_{\mathcal{P}}(\varepsilon)$, where $\mathcal{P}$ is the k-colorability property. Erdős [16] (implicitly) conjectured that k-colorability is testable with 1-sided error, that is, that $w_k(\varepsilon)$ is well defined. This was proved for $k = 2$ by Bollobás, Erdős, Simonovits and Szemerédi [10] and for general k by Rödl and Duke [34]. The proof of [34] relied on the regularity lemma [40] and thus supplied very weak bounds for $w_k(\varepsilon)$. A much better bound was obtained by Goldreich, Goldwasser and Ron [25] who proved that $w_k(\varepsilon) = \text{poly}(1/\varepsilon)$. In a recent breakthrough, Sohler [39] obtained the nearly tight bound $w_k(\varepsilon) = \tilde{\Theta}(1/\varepsilon)$, as well as similar results for some related problems.

As we mentioned earlier, it was shown by [5] that in fact, *every* monotone graph property is testable with 1-sided error, where the bounds for $w_{\mathcal{P}}(\varepsilon)$ are of tower-type. Goldreich [24] and Alon and Fox [3] asked to characterize the properties for which $w_{\mathcal{P}}(\varepsilon) = \text{poly}(1/\varepsilon)$. Since this problem currently seems to be out of reach, the following (very) special case was raised as an open problem in [5]: given a set of integers L, let $\mathcal{P}(L)$ be the property of being C_ℓ-free for every $\ell \in L$. The problem of [5] then asks the following

Problem 4 ([5]). *Characterize the sets of integers L for which $w_{\mathcal{P}(L)}(\varepsilon) = \text{poly}(1/\varepsilon)$.*

The result of [25] stating that $w_2(\varepsilon) = \text{poly}(1/\varepsilon)$ is then equivalent to the statement that if L consists of *all* odd integers then $w_{\mathcal{P}(L)}(\varepsilon) = \text{poly}(1/\varepsilon)$. Another related result is due to Alon [1] who proved that $w_{\mathcal{P}(L)}(\varepsilon)$ is super-polynomial whenever L is a *finite* set of odd integers and that $w_{\mathcal{P}(L)}(\varepsilon) = \text{poly}(1/\varepsilon)$ if L contains at least one even integer. Thus, the remaining open cases of Problem 4 are when L is an infinite set of odd integers.

[1]There are examples of properties that were shown to have query complexity $f(\varepsilon)$ for certain polynomials f. See [7] and its references.

[2]Consider the property $\mathcal{P}$ of having $n^2/4$ edges. It is easy to see that $q_{\mathcal{P}}(\varepsilon) \leq \text{poly}(1/\varepsilon)$ as one can just estimate the edge density of the input. On the other hand, it is also easy to see that $\mathcal{P}$ is not testable with 1-sided error using a number of queries that is independent of n.

1.2 New results regarding testing graph properties

In this subsection we describe our main results related to the problems discussed in the previous subsection. The first theorem gives a positive answer to Problem 2 by establishing a (nearly) tight query complexity hierarchy for 1-sided-error query complexity. No result of this type was previously known, for any super-polynomial f.

Theorem 1. *There is an absolute constant c such that for every decreasing function* $f\colon (0,1) \to \mathbb{N}$ *satisfying* $f(x) \geq 1/x$*, there is a monotone graph property* $\mathcal{P}$ *satisfying* $f(\varepsilon) \leq w_{\mathcal{P}}(\varepsilon) \leq \varepsilon^{-14} f(\varepsilon/c)$.

We now describe two applications of Theorem 1. The following immediate corollary gives a partial positive answer to Problem 1 raised by Goldreich [23, 24].

Corollary 1.1. *There is a monotone graph property* $\mathcal{P}$ *satisfying* $w_{\mathcal{P}}(\varepsilon) = 2^{\Theta(1/\varepsilon)}$.

We believe that the property $\mathcal{P}$ in the above corollary passes the "naturalness" test (pun intended!) asked for in Problem 1, since it is just the property of not containing cycles of certain (carefully chosen) lengths. The problem of establishing a hierarchy theorem for 2-sided-error query complexity, and in particular Problem 1 with respect to 2-sided testers, remains open.

Our second application of Theorem 1 (actually, this will be an application of its proof) gives a complete answer to Problem 4 raised by the second author and Alon [5]. As mentioned after Problem 4, we can assume that L is an infinite set of odd integers.

Corollary 1.2. *Let* $L = \{\ell_1, \ell_2, \ldots\}$ *be an infinite increasing sequence of odd integers. Then*

$$w_{\mathcal{P}(L)}(\varepsilon) = \mathrm{poly}(1/\varepsilon) \quad \textit{if and only if} \quad \limsup_{j \longrightarrow \infty} \frac{\log \ell_{j+1}}{\log \ell_j} < \infty.$$

By the above corollary, as long as ℓ_j does not grow faster than 2^{2^j}, we have $w_{\mathcal{P}(L)}(\varepsilon) = \mathrm{poly}(1/\varepsilon)$, while for any (significantly) faster growing ℓ_j this is not the case.

Our second theorem in this paper addresses Problem 3. It is natural to guess that at least for monotone properties $\mathcal{P}$, 2-sided testers should not have any advantage over 1-sided testers, since the only way to test $\mathcal{P}$ is to find a witness to the fact that the input graph does not satisfy $\mathcal{P}$. As Theorem 2 below shows, this intuition turns out to be false in a very strong sense. This theorem shows that 2-sided-error property testers can be *arbitrarily* stronger than 1-sided-error testers, even for monotone graph properties. Prior to this work, it was not even known that 2-sided-error testers can be super-polynomially stronger than 1-sided-error testers.

Theorem 2. *For every decreasing function* $f\colon (0,1) \to \mathbb{N}$ *satisfying* $f(x) \geq 1/x$*, there is a monotone graph property* $\mathcal{P}$ *so that*

- $\mathcal{P}$ *has* 1*-sided error query complexity* $w_{\mathcal{P}}(\varepsilon) \geq f(\varepsilon)$.
- $\mathcal{P}$ *has* 2*-sided error query complexity* $q_{\mathcal{P}}(n, \varepsilon) = \mathrm{poly}(1/\varepsilon)$ *for every* $n \geq n_0(\varepsilon)$.

We note that the first item in the above theorem holds even if one assumes that n (the size of the input graph) is large enough as a function of ε.

1.3 A tight bound for a Turán-type problem

We now turn to describe the third theorem of this paper, which gives a tight bound for a Turán-type problem in extremal graph theory. This theorem (and some of the lemmas related to it) will be the main tool we will use in order to prove the results stated in the previous subsection.

Very recently, Alon and Shikhelman [6] introduced the following problem; for fixed graphs H and T, estimate $\mathrm{ex}(n,T,H)$, which is the maximum number of copies[3] of T in an n-vertex graph that contains no copy of H. Note that $\mathrm{ex}(n,K_2,H)$ is just $\mathrm{ex}(n,H)$, the classical Turán function, which is the maximum number of edges in an n-vertex H-free graph. Estimating $\mathrm{ex}(n,H)$ for various graphs H is one of the most well-studied problems in graph theory. We refer the reader to [6] for more background, motivation, and several examples of well-studied problems which fall into this framework.

Let C_k denote the k-cycle, that is, the cycle of length k. One of the most well-studied problems in extremal combinatorics is the estimation of $\mathrm{ex}(n,C_k)$. While for odd k it is known [37] that $\mathrm{ex}(n,C_k)=\lfloor n^2/4\rfloor$ (for large enough n), the problem of estimating $\mathrm{ex}(n,C_k)$ for even k is still open with many recent results, see the survey [42] and its references. As discussed in [6], cycles have also been studied in the setting of $\mathrm{ex}(n,T,H)$. Bollobás and Győri [11] proved that $\mathrm{ex}(n,C_3,C_5)=\Theta\big(n^{\frac{3}{2}}\big)$. Győri and Li [28] extended this result by considering $\mathrm{ex}(n,C_3,C_{2\ell+1})$. Their bound was subsequently improved upon by Alon and Shikhelman [6]. At the moment, the best known bounds are

$$\Omega(\mathrm{ex}(n,\{C_4,C_6,\dots,C_{2\ell}\}))\le \mathrm{ex}(n,C_3,C_{2\ell+1})\le O(\ell\cdot \mathrm{ex}(n,C_{2\ell})), \tag{1}$$

where $\mathrm{ex}(n,\{C_4,C_6,\dots,C_{2\ell}\})$ is the maximal number of edges in an n-vertex graph with no copy of C_{2t} for any $2\le t\le \ell$. The lower bound above was proved in [28], and the upper bound in [6]. The bounds in (1) were also independently obtained by Füredi and Özkahya [20]. The lower and upper bounds in (1) are known to be of the same order of magnitude, $\Theta\big(n^{1+1/\ell}\big)$, for $\ell\in\{2,3,5\}$ (see e.g. [9, 43]). Another notable recent result is the *exact* determination of $\mathrm{ex}(n,C_5,C_3)$ by [27, 29].

Our third theorem in this paper, stated as Theorem 3, significantly extends the above results of [11, 28, 6] by giving asymptotically tight bounds for $\mathrm{ex}(n,C_k,C_\ell)$ for all fixed k,ℓ. This theorem will be the key tool in the proofs of Theorems 1 and 2. We believe this result to be of independent interest, and hope it will find other applications. In this theorem, as well as later on in the paper, we write $O_k/\Omega_k/\Theta_k$ to indicate that the notation hides constants which depend on k. When we write $O/\Omega/\Theta$, we mean that the implicit constants are absolute.

Theorem 3. *For distinct k,ℓ we have*

$$\mathrm{ex}(n,C_k,C_\ell)=\begin{cases}\Theta_k\big(n^{k/2}\big) & k\ge 5, \ell=4,\\ \Theta_k\big(\ell^{\lceil k/2\rceil}n^{\lfloor k/2\rfloor}\big) & \ell\ge 6 \text{ even, } k\ge 4,\\ \Theta_k\big(\ell^{\lceil k/2\rceil}n^{\lfloor k/2\rfloor}\big) & k,\ell \text{ odd, } 5\le k<\ell.\end{cases}$$

Observe that in the above theorem, our bounds are tight also when only k is fixed. A tight dependence on ℓ will be important due to the way we apply Theorem 3 in Section 4. Let us see which cases are not covered by Theorem 3 or by (1). Observe that if k is even and ℓ is odd, or if k and ℓ are both odd and $k>\ell$, then a blow-up of C_k does not contain copies of C_ℓ. Therefore, in these cases we have $\mathrm{ex}(n,C_k,C_\ell)=\Theta_k(n^k)$. Thus, the only remaining case is $\mathrm{ex}(n,C_3,C_{2\ell})$, for which we will prove the following.

[3]When counting copies of T in G we always mean unlabeled copies.

Proposition 1.3. *For $\ell \geq 2$ we have $\Omega\big(\mathrm{ex}(n, C_4, C_6, \ldots, C_{2\ell})\big) \leq \mathrm{ex}(n, C_3, C_{2\ell}) \leq O_\ell\big(\mathrm{ex}(n, C_{2\ell})\big)$.*

As in the case of (1), the lower and upper bounds in Proposition 1.3 are known to be of the same order of magnitude for $\ell \in \{2, 3, 5\}$.

Paper organization: In Section 2 we give a tight upper bound for $\mathrm{ex}(n, C_{2k+1}, C_{2k+3})$ where $k \geq 2$, which turns out to require a different argument than the one needed to handle all other cases of Theorem 3. This problem appears to be significantly harder than $\mathrm{ex}(n, C_3, C_5)$ which was resolved by Bollobás–Győri [11]. This is best evidenced by the fact that while $\mathrm{ex}(n, C_3, C_5) = \Theta(n^{3/2})$, for the general problem we have $\mathrm{ex}(n, C_{2k+1}, C_{2k+3}) = \Theta_k(n^k)$ for $k \geq 2$. Section 3 contains the proof of Theorem 3 and Proposition 1.3. In this section we also prove a tight bound $\mathrm{ex}(n, P_k, C_\ell)$ for all values of $k \geq 2$ and ℓ, where P_k is the path with k edges (see Theorem 4). In Section 4 we apply our bounds for $\mathrm{ex}(n, C_k, C_\ell)$ in order to prove Theorems 1 and 2 and Corollary 1.2. Lemma 3.1, which is the key lemma in the proof of Theorem 3, is proved in Section 5. The main tool used in its proof is a bound for the skew version of the even-cycle Turán problem, due to Naor and Verstraëte [33]. Finally, in Section 6 we prove the lower bounds in Theorem 3 and Proposition 1.3.

The dependence of our bounds on ℓ is important due to the way we apply them in Section 4. We made little effort, however, to optimize their dependence on k. Finally, since in most arguments the parity of the cycle lengths will be important, we will use $2k$ or $2k+1$ (and analogously 2ℓ or $2\ell+1$) to denote the lengths of the cycles.

2 The Case $\mathrm{ex}(n, C_{2k+1}, C_{2k+3})$

In this section we give a tight upper bound for $\mathrm{ex}(n, C_{2k+1}, C_{2k+3})$ when $k \geq 2$. Let us introduce some notation that we will use throughout the paper. For a graph G and disjoint sets $X, Y \subseteq V(G)$, we denote by $E(X, Y)$ the set of edges with one endpoint in X and one endpoint in Y, and set $e(X, Y) = |E(X, Y)|$. For $v \in V(G)$ and $X \subseteq V(G)$, denote $N_X(v) = \{x \in X : (v, x) \in E(G)\}$.

Let $U_1, \ldots, U_s$ be disjoint vertex sets in a graph. A $(U_1, \ldots, U_s)$-*path* is a path $u_1, \ldots, u_s$ with $u_i \in U_i$. Similarly, a $(U_1, \ldots, U_s)$-*cycle* is a cycle $u_1, \ldots, u_s, u_1$ with $u_i \in U_i$. Let $p(U_1, \ldots, U_s)$ denote the number of $(U_1, \ldots, U_s)$-paths and let $c(U_1, \ldots, U_s)$ denote the number of $(U_1, \ldots, U_s)$-cycles. We denote by P_k the path of length k, where the length of a path is the number of edges in it. We will frequently use the following simple averaging argument.

Claim 2.1. *Let G be a graph. If for every partition $V(G) = U_1 \cup \cdots \cup U_k$ it holds that $c(U_1, \ldots, U_k) \leq r$, then the number copies of C_k in G is at most $\frac{1}{2}k^{k-1}r$. Similarly, if for every partition $V(G) = U_1 \cup \cdots \cup U_k$ it holds that $p(U_1, \ldots, U_k) \leq r$, then the number of copies of P_{k-1} in G is at most $\frac{1}{2}k^k r$.*

Proof. Let $V(G) = U_1 \cup \cdots \cup U_k$ be a random partition, generated according to $\mathbb{P}[v \in U_i] = \frac{1}{k}$ for each $v \in V(G)$ and $1 \leq i \leq k$, independently. Then $\mathbb{E}\left[c(U_1, \ldots, U_k)\right] = \#C_k(G) \cdot 2k \cdot k^{-k}$ and $\mathbb{E}\left[p(U_1, \ldots, U_k)\right] = \#P_{k-1}(G) \cdot 2 \cdot k^{-k}$, where $\#C_k(G)$ (resp. $\#P_{k-1}(G)$) denotes the number of copies of C_k (resp. P_{k-1}) in G. Since these expectations are not larger than r, the claim follows. ∎

In what follows, let us denote the vertices of C_{2k+1} (the $(2k+1)$-cycle) by $1, \ldots, 2k+1$, with edges $\{1, 2\}, \ldots, \{2k, 2k+1\}, \{2k+1, 1\}$. For a graph G, denote by $\mathcal{I}(G)$ the set of all non-empty independent sets of G. We will need the following trivial (yet somewhat complicated to state) claim.

Claim 2.2. *Let J be a non-empty independent set of C_{2k+1}. Then there is $I \in \mathcal{I}(C_{2k+1})$ which contains J and satisfies the following. Let $i_1, \ldots, i_r$ be the elements of I in the order they appear* *when traversing the cycle* $1, \dots, 2k+1$. *Then for every* $1 \le j \le r$, i_j *and* i_{j+1} *are at distance either* 2 *or* 3, *namely* $i_{j+1} - i_j \equiv 2, 3 \pmod{2k+1}$, *and if* i_j *and* i_{j+1} *are at distance* 3 *then either* $i_j \in J$ *or* $i_{j+1} \in J$.

Proof. If $|J| = 1$, say without loss of generality $J = \{1\}$, then $I = \{2j-1 : 1 \le j \le k\}$ is easily seen to satisfy the requirements of the claim. Assume then that $|J| \ge 2$, and let $j_1, \dots, j_r$ be the elements of J, as they appear when traversing the $(2k+1)$-cycle $1, \dots, 2k+1$. For each $1 \le i \le r$, we greedily pick an independent set I_i in the path connecting j_i and j_{i+1}, which contains both j_i and j_{i+1}, as follows. In addition to j_i and j_{i+1}, we add to I_i the elements $j_i + 2, j_i + 4, \dots$ until we reach j_{i+1} or $j_{i+1} - 1$. If we reached j_{i+1}, then the distance between every pair consecutive elements of I_i is 2, and if we reached $j_{i+1} - 1$ then this true for all pairs except for $j_{i+1} - 3, j_{i+1}$. It is now easy to see that $I = \bigcup_{i=1}^{r} I_i$ satisfies the requirements of the claim. ■

Lemma 2.3. *For every* $k \ge 2$ *it holds that* $\mathrm{ex}(n, C_{2k+1}, C_{2k+3}) \le (2k+1)^{2k} 2^{2k+1} n^k$.

Proof. Let G be an n-vertex C_{2k+3}-free graph. By claim 2.1 it is sufficient to prove that for every partition $V(G) = U_1 \cup \dots \cup U_{2k+1}$ we have $c(U_1, \dots, U_{2k+1}) \le 2^{2k+1} n^k$. We will actually prove that

$$c(U_1, \dots, U_{2k+1}) \le \sum_{I \in \mathcal{I}(C_{2k+1})} \prod_{i \in I} |U_i|. \tag{2}$$

This will be sufficient, as C_{2k+1} has at most 2^{2k+1} independent sets, and each of these sets contributes at most n^k to the above sum. Assume by contradiction that (2) is false. Let $\mathcal{C}$ denote the set of all $(U_1, \dots, U_{2k+1})$-cycles in G. We first show that there is $C = (u_1, \dots, u_{2k+1}) \in \mathcal{C}$ such that for every $I \in \mathcal{I}(C_{2k+1})$ there is $C' \in \mathcal{C} \setminus \{C\}$ which contains $\{u_i : i \in I\}$. We find C greedily as follows. As long as there is $C = (u_1, \dots, u_{2k+1}) \in \mathcal{C}$ and $I \in \mathcal{I}(C_{2k+1})$ such that C is the only $(U_1, \dots, U_{2k+1})$-cycle containing $\{u_i : i \in I\}$, we remove C from $\mathcal{C}$, and we say that C was removed due to $\{u_i : i \in I\}$. Fixing any $I \in \mathcal{I}(C_{2k+1})$ and $u_i \in U_i$ for $i \in I$, observe that at most one cycle from $\mathcal{C}$ was removed due to $\{u_i : i \in I\}$. Thus, the overall number of cycles removed is not larger than the right-hand side of (2). Since by our assumption (2) is false, there is a cycle $C = (u_1, \dots, u_{2k+1}) \in \mathcal{C}$ which had not been removed by the end of the process. Then C satisfies our requirement. Let us fix such a $C = (u_1, \dots, u_{2k+1})$ for the rest of the proof.

Let J be the set of all $1 \le i \le 2k+1$ such that there is $u'_i \in U_i \setminus \{u_i\}$ which is adjacent to u_{i-1} and u_{i+1}. We claim that J is a non-empty independent set (of the $(2k+1)$-cycle). To show that J is an independent set, assume by contradiction that there is $1 \le i \le 2k+1$ such that $i, i+1 \in J$, and let $u'_i \in U_i \setminus \{u_i\}$ and $u'_{i+1} \in U_{i+1} \setminus \{u_{i+1}\}$ be witnesses to $i, i+1 \in J$. Then $u'_i, u_{i+1}, u_i, u'_{i+1}, u_{i+2}, \dots, u_{i-1}, u'_i$ is a $(2k+3)$-cycle, a contradiction. We now show that $J \ne \emptyset$. Set $I' = \{2j : 2 \le j \le k\} \cup \{1\}$ and $I'' = \{2j : 3 \le j \le k\} \cup \{1, 3\}$ and note that they are both independent sets. By our choice of $C = (u_1, \dots, u_{2k+1})$, there is $C' = (u'_1, \dots, u'_{2k+1}) \in \mathcal{C} \setminus \{C\}$ which contains u_i for every $i \in I'$. Since $C' \ne C$, one of the following holds: either $u'_i \ne u_i$ for some $i \in \{2j+1 : 2 \le j \le k\}$, implying that $i \in J$ and we are done, or $(u'_2, u'_3) \ne (u_2, u_3)$. If $u'_2 = u_2$ or $u'_3 = u_3$ then $3 \in J$ or $2 \in J$, respectively, and again we are done. We deduce that $u'_2 \ne u_2$ and $u'_3 \ne u_3$. By repeating the same argument with respect to I'', we get a cycle $C'' = (u''_1, \dots, u''_{2k+1}) \in \mathcal{C} \setminus \{C\}$ such that either $u''_i \ne u_i$ for some $i \in \{2j+1 : 3 \le j \le k\} \cup \{2\}$, implying that $i \in J$ and we are done, or $u''_4 \ne u_4$ and $u''_5 \ne u_5$. But now $u_1, u'_2, u'_3, u_4, u_3, u''_4, u''_5, u_6, \dots, u_{2k+1}, u_1$ is a $(2k+3)$-cycle, a contradiction. See the top drawing in Figure 1 for an illustration.

We thus proved that J is a non-empty independent set. Apply Claim 2.2 to J to get $I \in \mathcal{I}(C_{2k+1})$ with the properties stated in the claim. By our choice of $C = (u_1, \dots, u_{2k+1})$, there is

$C' = (u'_1, \dots, u'_{2k+1}) \in \mathcal{C} \setminus \{C\}$ which contains u_i for every $i \in I$. Let $i_1, \dots, i_r$ be the elements of I in the order they appear when traversing the cycle $1, \dots, 2k+1$. Since $C' \neq C$, there is $1 \leq j \leq r$ such that $(u'_{i_j+1}, \dots, u'_{i_{j+1}-1}) \neq (u_{i_j+1}, \dots, u_{i_{j+1}-1})$ [4]. Assume without loss of generality that $j = 1$ and $i_1 = 2$ (so in particular, $2 \in I$). By the guarantees of Claim 2.2, we have $i_2 - i_1 \equiv 2, 3 \pmod{2k+1}$, so either $i_2 = 4$ or $i_2 = 5$. Assume first that $i_2 = 4$. Then $u'_3 \neq u_3$, implying that $3 \in J$, which is impossible as $2 \in I$, $J \subseteq I$ and I is an independent set. Assume now that $i_2 = 5$. If $u'_3 = u_3$ then $u'_4 \neq u_4$ and so $4 \in J$, which is again impossible as $5 \in I$, $J \subseteq I$ and I is an independent set. So $u'_3 \neq u_3$ and similarly $u'_4 \neq u_4$. By the guarantees of Claim 2.2, we have that either $2 \in J$ or $5 \in J$, say without loss of generality that $2 \in J$. Then by the definition of J, there is $u''_2 \in U_2 \setminus \{u_2\}$ adjacent to u_1 and u_3. But now $u_1, u''_2, u_3, u_2, u'_3, u'_4, u_5, \dots, u_{2k+1}, u_1$ is a $(2k+3)$-cycle, a contradiction. See the bottom drawing in Figure 1 for an illustration. This completes the proof. ■

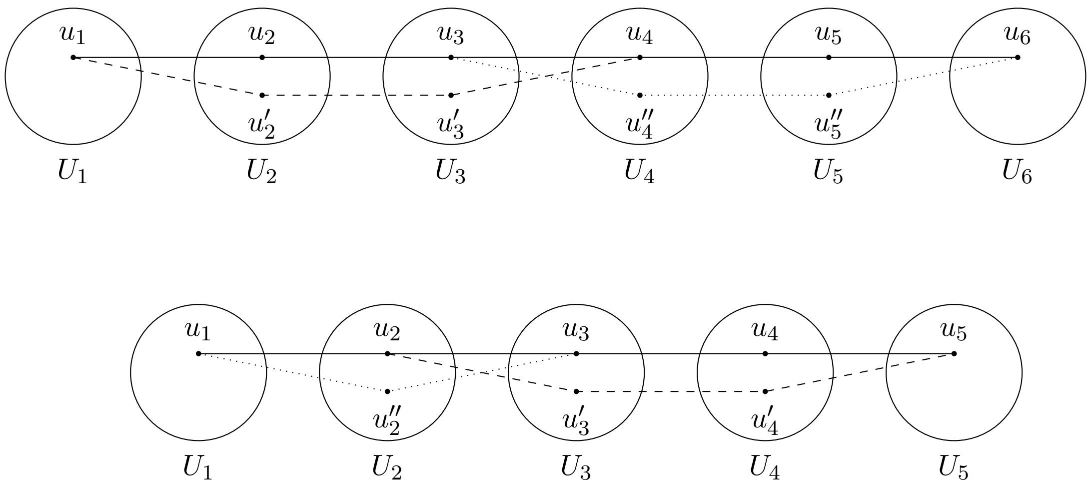

Figure 1: Illustrations for the proof of Lemma 2.3

3 Proof of Turan-type Results

In this section we prove the upper bounds for all cases in Theorem 3, except for the case of two consecutive odd integers which was handled in Section 2. The lower bounds will be proven in Section 6. At the end of this section, we give the proof of Proposition 1.3.

3.1 Preliminary Lemmas

Here we introduce several lemmas which will be used in the proof of Theorem 3. We start with the following key lemma, which is the most important ingredient in the proof of Theorem 3, as it allows us to obtain tight bounds in terms of n and ℓ. The proof of this lemma appears in Section 5.

Lemma 3.1. *Let $\ell \geq 3$, let G be an n-vertex graph, let $X, Y, Z, W \subseteq V(G)$ be pairwise-disjoint vertex-sets and assume that the bipartite graphs (X, Y), (Y, Z) and (Z, W) are $C_{2\ell}$-free. Then there are subsets $Y' \subseteq Y$ and $Z' \subseteq Z$ such that*

[4] Here subscripts are taken modulo $2k + 1$, while double subscripts are taken modulo r.

1. $e(Y', X), e(Y', Z), e(Z', Y), e(Z', W) = O(\ell n)$.

2. $p(X, Y \setminus Y', Z \setminus Z', W) = O(\ell^2 n^2)$.

At the end of Section 5, we explain why the sets Y' and Z' in the statement of Lemma 3.1 are required, and why Lemma 3.1 is false for $\ell = 2$. The falsity of Lemma 3.1 for $\ell = 2$ is the reason we need a separate proof for the case $\mathrm{ex}(n, C_{2k+1}, C_{2k+3})$ (see Section 2).

In what follows we will need a special case of the following theorem, which gives a tight bound on $\mathrm{ex}(n, P_k, C_{2\ell})$ for every $k \geq 2$. The proof of this theorem appears at the end of this section.

Theorem 4. *For every $k \geq 2$, we have*

$$\mathrm{ex}(n, P_k, C_{2\ell}) = \begin{cases} \Theta_k(n^{k/2+1}) & \ell = 2, \\ \Theta_k(\ell^{\lfloor (k+1)/2 \rfloor} n^{\lceil (k+1)/2 \rceil}) & \ell \geq 3. \end{cases}$$

To complement Theorem 4, note that $\mathrm{ex}(n, P_k, C_{2\ell+1}) = \Theta_k(n^{k+1})$, since a blowup of P_k does not contain odd cycles. The following lemma also plays a key role in the proof of Theorem 3.

Lemma 3.2. *Let $s \geq 2$ and $\lambda \geq 1$, let G be an n-vertex graph and let $U_1, \dots, U_s \subseteq V(G)$ be pairwise-disjoint sets such that $e(U_1, U_2) \leq \lambda(|U_1| + |U_2|)$ and $e(N_{U_{i+1}}(u_i), U_{i+2}) \leq \lambda(\left|N_{U_{i+1}}(u_i)\right| + |U_{i+2}|)$ for every $1 \leq i \leq s-2$ and $u_i \in U_i$. Then*

$$p(U_1, \dots, U_s) \leq \begin{cases} \lambda^{(s-1)/2} n^{(s-3)/2} \left(|U_1||U_s| + \lambda n\right) & s \text{ is odd}, \\ \lambda^{s/2} n^{s/2-1} (|U_1| + |U_2|) & s \text{ is even}. \end{cases}$$

Proof. The proof is by induction on s. The base case $s = 2$ is given by our assumption that $e(U_1, U_2) \leq \lambda(|U_1| + |U_2|)$. Let then $s \geq 3$. Note that for every $u_1 \in U_1$, the sets $N_{U_2}(u_1), U_3, \dots, U_s$ satisfy the assumptions of the lemma, so we may apply the induction hypothesis to them. Suppose first that s is odd. We have

$$\begin{aligned} p(U_1, \dots, U_s) &= \sum_{u_1 \in U_1} p(N_{U_2}(u_1), U_3, \dots, U_s) \leq \sum_{u_1 \in U_1} \lambda^{(s-1)/2} n^{(s-3)/2} (|N_{U_2}(u_1)| + |U_s|) \\ &= \lambda^{(s-1)/2} n^{(s-3)/2} \cdot (e(U_1, U_2) + |U_1||U_s|) \leq \lambda^{(s-1)/2} n^{(s-3)/2} \cdot (\lambda(|U_1| + |U_2|) + |U_1||U_s|) \\ &\leq \lambda^{(s-1)/2} n^{(s-3)/2} \cdot (|U_1||U_s| + \lambda n), \end{aligned}$$

where in the first inequality we used the induction hypothesis for $s-1$, and in the second inequality we used the assumption $e(U_1, U_2) \leq \lambda(|U_1| + |U_2|)$. The induction step for even s is similar. Indeed,

$$\begin{aligned} p(U_1, \dots, U_s) &= \sum_{u_1 \in U_1} p(N_{U_2}(u_1), U_3, \dots, U_s) \leq \sum_{u_1 \in U_1} \lambda^{(s-2)/2} n^{(s-4)/2} (|N_{U_2}(u_1)||U_s| + \lambda n) \\ &= \lambda^{(s-2)/2} n^{(s-4)/2} \cdot e(U_1, U_2) \cdot |U_s| + \lambda^{s/2} n^{s/2-1} \cdot |U_1| \\ &\leq \lambda^{(s-2)/2} n^{(s-4)/2} \cdot \lambda(|U_1| + |U_2|) \cdot |U_s| + \lambda^{s/2} n^{s/2-1} \cdot |U_1| \leq \lambda^{s/2} n^{s/2-1} \cdot (|U_1| + |U_s|), \end{aligned}$$

where in the first inequality we used the induction hypothesis for $s-1$, in the second inequality we used the assumption $e(U_1, U_2) \leq \lambda(|U_1| + |U_2|)$, and in the last inequality we used the trivial bound $|U_1| + |U_2| \leq n$. ■

We now derive two important corollaries of Lemma 3.2, stated as Lemmas 3.3 and 3.4 below. In their proof we will use the following well-known theorem of Erdős and Gallai.

Theorem 5 ([17])**.** *For every $t \geq 1$ we have* $\mathrm{ex}(n, P_t) \leq \frac{t-1}{2}n$.

Lemma 3.3. *Let $2 \leq s < t$ be integers having the same parity, let G be an n-vertex graph and let $U_1, \dots, U_s \subseteq V(G)$ be pairwise-disjoint vertex-sets such that there is no path of length $t-1$ inside $U_1 \cup \dots \cup U_s$ between a vertex in U_1 and a vertex in U_s. Then*

$$p(U_1, \dots, U_s) \leq \begin{cases} \left(\frac{t-s}{2}\right)^{(s-1)/2} n^{(s-3)/2} \left(|U_1||U_s| + \frac{t-s}{2}n\right) & s \text{ is odd,} \\ \left(\frac{t-s}{2}\right)^{s/2} n^{s/2-1} \left(|U_1| + |U_2|\right) & s \text{ is even.} \end{cases}$$

Proof. We may and will assume that every edge in G is on some $(U_1, \dots, U_s)$-path (as deleting all other edges does not change $p(U_1, \dots, U_s)$). It is sufficient to show that the conditions of Lemma 3.2 hold for $\lambda = \frac{t-s}{2} \geq 1$. We prove the stronger statement that for every $1 \leq i \leq s-1$ and for every $U'_i \subseteq U_i$ and $U'_{i+1} \subseteq U_{i+1}$, it holds that $e(U'_i, U'_{i+1}) \leq \frac{t-s}{2}\left(|U'_i| + |U'_{i+1}|\right)$. If, by contradiction, this does not hold, then by Theorem 5 there is a path $P = v_1, \dots, v_{t-s+2}$ of length $t-s+1$ in the bipartite graph (U'_i, U'_{i+1}). Since $t-s+1$ is odd, we may assume without loss of generality that $v_1 \in U'_i$ and $v_{t-s+2} \in U'_{i+1}$. By our assumption, the edge (v_1, v_2) is on some $(U_1, \dots, U_s)$-path, implying that there is a path $P' \subseteq U_1 \cup \dots \cup U_i$ between[5] U_1 and v_1. Similarly, since the edge (v_{t-s+1}, v_{t-s+2}) is on some $(U_1, \dots, U_s)$-path, there is a path $P'' \subseteq U_{i+1} \cup \dots \cup U_s$ between v_{t-s+2} to U_s. Then $P'PP''$ is a path of length $t-1$ inside $U_1 \cup \dots \cup U_s$ between U_1 and U_s, in contradiction to our assumption. ■

Lemma 3.4. *Let $s, \ell \geq 2$, let G be an n-vertex $C_{2\ell}$-free graph, let $\{u_0\}, U_1, \dots, U_s \subseteq V(G)$ be pairwise-disjoint vertex-sets, and suppose that u_0 is adjacent to every vertex in U_1. Then*

$$p(U_1, \dots, U_s) \leq \begin{cases} (\ell-1)^{(s-1)/2} n^{(s-3)/2} \left(|U_1||U_s| + (\ell-1)n\right) & s \text{ is odd,} \\ (\ell-1)^{s/2} n^{s/2-1} (|U_1| + |U_2|) & s \text{ is even.} \end{cases}$$

Proof. It is sufficient to show that the conditions of Lemma 3.2 hold with $\lambda = \ell - 1 \geq 1$. If $e(U_1, U_2) > (\ell-1)(|U_1| + |U_2|)$ then by Theorem 5, there is a path of length $2\ell - 1$ in the bipartite graph (U_1, U_2). This path contains a subpath of length $2\ell-2$ with both endpoints in U_1, which closes a 2ℓ-cycle with u_0, in contradiction to the assumption of the lemma. Similarly, if $e(N_{U_{i+1}}(u_i), U_{i+2}) > (\ell-1)(\left|N_{U_{i+1}}(u_i)\right| + |U_{i+2}|)$ for some $1 \leq i \leq s-2$ and $u_i \in U_i$, then by Theorem 5 there is a path of length $2\ell-1$ in the bipartite graph with sides $N_{U_{i+1}}(u_i)$ and U_{i+2}. This path contains a subpath of length $2\ell-2$ with both endpoints in $N_{U_{i+1}}(u_i)$, which closes a 2ℓ-cycle with u_i, in contradiction to the assumption of the lemma. ■

The construction in Claim 6.1 shows that the bounds in the above two lemmas, as well as in Lemma 3.2, are tight (up to the constants depending on the parameters λ, s, t, ℓ). We now derive the following corollary of the above two lemmas, which will be used later on.

Lemma 3.5. *Let $k, \ell \geq 2$, let G be an n-vertex graph and assume either that G is $C_{2\ell}$-free or that G is $C_{2\ell+1}$-free and $\ell > k$. Then for every partition $V(G) = V_1 \cup \dots \cup V_{2k+1}$ we have*

$$c(V_1, \dots, V_{2k+1}) \leq \ell^{k-1} n^{k-2} \cdot \left[p(V_1, V_2, V_3, V_4) + p(V_{2k+1}, V_1, V_2, V_3)\right].$$

Proof. Fix any (V_1, V_2, V_3)-path v_1, v_2, v_3. We claim that

$$p\left(N_{V_4}(v_3), V_5, \dots, V_{2k}, N_{V_{2k+1}}(v_1)\right) \leq \ell^{k-1} n^{k-2} \cdot \left(|N_{V_4}(v_3)| + |N_{V_{2k+1}}(v_1)|\right). \tag{3}$$

[5] It might be the case that $v_1 \in U_1$ (if $i = 1$), in which case P' has no edges.

Indeed, if G is $C_{2\ell}$-free then (3) follows from Lemma 3.4, applied with $s = 2k-2$, $u_0 = v_3$ and the sets $N_{V_4}(v_3), V_5, \ldots, V_{2k}, N_{V_{2k+1}}(v_1)$ as $U_1, \ldots, U_s$. If G is $C_{2\ell+1}$-free and $\ell > k$ then there is no path of length $2\ell-3$ inside $V_4 \cup \cdots \cup V_{2k+1}$ between a vertex in $N_{V_4}(v_3)$ and a vertex in $N_{V_{2k+1}}(v_1)$, as such a path would close a $(2\ell+1)$-cycle with the path $v_1v_2v_3$. So (3) follows from Lemma 3.3, applied with $s = 2k-2$, $t = 2\ell-2$, and the sets $N_{V_4}(v_3), V_5, \ldots, V_{2k}, N_{V_{2k+1}}(v_1)$ as $U_1, \ldots, U_s$. By summing (3) over all (V_1, V_2, V_3)-paths we get

$$\begin{aligned} c(V_1, \ldots, V_{2k+1}) &= \sum_{v_1,v_2,v_3} c(v_1, v_2, v_3, V_4, \ldots, V_{2k+1}) = \sum_{v_1,v_2,v_3} p(N_{V_4}(v_3), V_5, \ldots, V_{2k}, N_{V_{2k+1}}(v_1)) \\ &\le \ell^{k-1} n^{k-2} \cdot \sum_{v_1,v_2,v_3} \left(|N_{V_4}(v_3)| + |N_{V_{2k+1}}(v_1)|\right) \\ &= \ell^{k-1} n^{k-2} \cdot [p(V_1, V_2, V_3, V_4) + p(V_{2k+1}, V_1, V_2, V_3)] \ , \end{aligned}$$

thus completing the proof. ■

3.2 Proof of Theorem 3

Here we prove Theorem 3. The proof is split into several parts: Lemma 3.6 handles the case that both cycle lengths are even; Lemma 3.7 handles the case where the forbidden cycle is even and the cycle whose number of copies is maximized is odd; finally, Lemma 3.8 handles the case where the cycle lengths are non-consecutive odd integers. For convenience, we rephrase each of the cases, denoting the cycle lengths by $2k$ or $2k+1$ and 2ℓ or $2\ell+1$ (rather than k and ℓ).

Lemma 3.6. *For every $k, \ell \ge 2$ we have $\mathrm{ex}(n, C_{2k}, C_{2\ell}) = O_k(\ell^k n^k)$.*

Proof. Let G be an n-vertex $C_{2\ell}$-free graph. By Claim 2.1, it is enough to prove that $c(V_1, \ldots, V_{2k}) = O(\ell^k n^k)$ for every partition $V(G) = V_1 \cup \cdots \cup V_{2k}$. Consider one such partition. Fixing $v_1 \in V_1$, we apply Lemma 3.4 with $s = 2k-1$, $u_0 = v_1$ and the sets $N_{V_2}(v_1), V_3, \ldots, V_{2k-1}, N_{V_{2k}}(v_1)$ as $U_1, \ldots, U_s$, to get

$$c(v_1, V_2, \ldots, V_{2k}) = p(N_{V_2}(v_1), V_3, \ldots, V_{2k-1}, N_{V_{2k}}(v_1)) \le \ell^{k-1} n^{k-2} \cdot (|N_{V_2}(v_1)| \cdot |N_{V_{2k}}(v_1)| + \ell n) \, .$$

By summing over all $v_1 \in V_1$, we get

$$\begin{aligned} c(V_1, \ldots, V_{2k}) &= \sum_{v_1 \in V_1} c(v_1, V_2, \ldots, V_{2k}) \le \ell^{k-1} n^{k-2} \cdot \left(\sum_{v_1 \in V_1} |N_{V_2}(v_1)| \cdot |N_{V_{2k}}(v_1)| \right) + \ell^k n^{k-1} \cdot |V_1| \\ &= \ell^{k-1} n^{k-2} \cdot p(V_{2k}, V_1, V_2) + \ell^k n^{k-1} \cdot |V_1| = O(\ell^k n^k) \ , \end{aligned}$$

where in the last inequality we used Theorem 4, which gives $p(V_{2k}, V_1, V_2) = O(\ell n^2)$. ■

Lemma 3.7. *For every $k \ge 2$ we have*

$$\mathrm{ex}(n, C_{2k+1}, C_{2\ell}) \le \begin{cases} O_k(n^{k+1/2}) & \ell = 2, \\ O_k(\ell^{k+1} n^k) & \ell \ge 3. \end{cases}$$

Proof. We start with the case that $\ell \ge 3$. Let G be an n-vertex $C_{2\ell}$-free graph. By Claim 2.1, it is enough to prove that for every partition $V(G) = V_1 \cup \cdots \cup V_{2k+1}$ we have $c(V_1, \ldots, V_{2k+1}) = O(\ell^{k+1} n^k)$.

By Theorem 4 we have $p(V_{2k+1}, V_1, V_2, V_3), p(V_1, V_2, V_3, V_4) \le O(\ell^2 n^2)$. Plugging these estimates into Lemma 3.5 gives $c(V_1, \ldots, V_{2k+1}) = O(\ell^{k+1} n^k)$, as required.

The proof for the case $\ell = 2$ is similar. As in the previous case, we consider a partition $V(G) = V_1 \cup \cdots \cup V_{2k+1}$ of an n-vertex C_4-free graph. The only difference is that for $\ell = 2$, Theorem 4 gives $p(V_{2k+1}, V_1, V_2, V_3), p(V_1, V_2, V_3, V_4) = O(n^{5/2})$. Plugging this into Lemma 3.5 gives the required bound $c(V_1, \ldots, V_{2k+1}) = O_k(n^{k+1/2})$. ■

Lemma 3.8. *For every $2 \le k < \ell - 1$ we have $\mathrm{ex}(n, C_{2k+1}, C_{2\ell+1}) = O((2k+1)^{2k} \ell^{k+1} n^k)$.*

Proof. Let G be an n-vertex $C_{2\ell+1}$-free graph. By Claim 2.1, we only need to prove that the bound $c(V_1, \ldots, V_{2k+1}) \le O(\ell^{k+1} n^k)$ holds for every partition $V(G) = V_1 \cup \cdots \cup V_{2k+1}$. Fix one such partition. We may and will assume that for every $1 \le i \le 2k+1$, every edge in $E(V_i, V_{i+1})$ is on some $(V_1, \ldots, V_{2k+1})$-cycle. We claim that the bipartite graph (V_i, V_{i+1}) is $C_{2\ell-2k+2}$-free for every $1 \le i \le 2k+1$ (with indices taken modulo $2k+1$). Assume by contradiction that there is a $(2\ell - 2k + 2)$-cycle C in the bipartite graph (V_i, V_{i+1}), and let $e \in E(V_i, V_{i+1})$ be an arbitrary edge of C. By our assumption, there is a $(V_1, \ldots, V_{2k+1})$-cycle C' containing e. But now $C \cup C' \setminus \{e\}$ is a $(2\ell+1)$-cycle, a contradiction.

In light of the above, we may apply Lemma 3.1 to $(V_{2k+1}, V_1, V_2, V_3)$ with $\ell - k + 1 \ge 3$ in place of ℓ and thus obtain subsets $V_1' \subseteq V_1$, $V_2' \subseteq V_2$ satisfying $e(V_1', V_{2k+1}), e(V_1', V_2), e(V_2', V_1), e(V_2', V_3) = O(\ell n)$ and $p(V_{2k+1}, V_1 \setminus V_1', V_2 \setminus V_2', V_3) = O(\ell^2 n^2)$. Similarly, applying Lemma 3.1 to V_1, V_2, V_3, V_4 gives subsets $V_2'' \subseteq V_2$ and $V_3'' \subseteq V_3$ such that $e(V_2'', V_1), e(V_2'', V_3), e(V_3'', V_2), e(V_3'', V_4) = O(\ell n)$ and $p(V_1, V_2 \setminus V_2'', V_3 \setminus V_3'', V_4) = O(\ell^2 n^2)$. Setting $W_1 = V_1 \setminus V_1'$, $W_2 = V_2 \setminus (V_2' \cup V_2'')$ and $W_3 = V_3 \setminus V_3''$, we see that

$$\begin{aligned} c(V_1, \ldots, V_{2k+1}) \le\ & c(W_1, W_2, W_3, V_4, \ldots, V_{2k+1}) + c(V_1', V_2, \ldots, V_{2k+1}) + \\ & c(V_1, V_2', V_3, \ldots, V_{2k+1}) + c(V_1, V_2'', V_3, \ldots, V_{2k+1}) + c(V_1, V_2, V_3'', V_4, \ldots, V_{2k+1}). \end{aligned} \tag{4}$$

By our choice of V_1', V_2', V_2'', V_3'' via Lemma 3.1 and by the definition of the sets W_1, W_2, W_3, we have $p(V_{2k+1}, W_1, W_2, W_3) = O(\ell^2 n^2)$ and $p(W_1, W_2, W_3, V_4) = O(\ell^2 n^2)$. Plugging these bounds into Lemma 3.5 gives

$$c(W_1, W_2, W_3, V_4, \ldots, V_{2k+1}) \le \ell^{k-1} n^{k-2} \cdot O(\ell^2 n^2) \le O(\ell^{k+1} n^k).$$

It remains to bound the other four terms in (4). Consider the term $c(V_1', V_2, \ldots, V_{2k+1})$. Fixing any $v_1 \in V_1'$, note that there is no path of length $2\ell - 1$ inside $V_2 \cup \cdots \cup V_{2k+1}$ between a vertex in $N_{V_2}(v_1)$ and a vertex in $N_{V_{2k+1}}(v_1)$, as such a path would close a $(2\ell+1)$-cycle with v_1. Thus, we may apply Lemma 3.3 with $s = 2k$, $t = 2\ell$ and $N_{V_2}(v_1), V_3, \ldots, V_{2k}, N_{V_{2k+1}}(v_1)$ as $U_1, \ldots, U_s$, to get

$$c(v_1, V_2, \ldots, V_{2k+1}) = p(N_{V_2}(v_1), V_3, \ldots, V_{2k}, N_{V_{2k+1}}(v_1)) \le \ell^k n^{k-1} \left(|N_{V_2}(v_1)| + |N_{V_{2k+1}}(v_1)| \right).$$

By summing the above over all $v_1 \in V_1'$ we obtain

$$\begin{aligned} c(V_1', V_2, \ldots, V_{2k+1}) &= \sum_{v_1 \in V_1'} c(v_1, V_2, \ldots, V_{2k+1}) \le \ell^k n^{k-1} \cdot \sum_{v_1 \in V_1'} \left(|N_{V_2}(v_1)| + |N_{V_{2k+1}}(v_1)| \right) \\ &= \ell^k n^{k-1} \cdot \left(e(V_1', V_2) + e(V_1', V_{2k+1}) \right) \le O(\ell^{k+1} n^k)\ , \end{aligned}$$

where the last equality relies on the guarantees of Lemma 3.1. The remaining three terms in (4) are shown to be $O(\ell^{k+1} n^k)$ in the same manner. This completes the proof. ■

Having proven Theorem 3, we summarize our upper bounds on $\mathrm{ex}(n, C_{2k+1}, C_{2\ell+1})$ in Lemma 3.9 below. This lemma will be used in Section 4. We need the well-known Even Cycle Theorem of Bondy and Simonovits:

Theorem 6 ([12])**.** *For every $\ell \geq 2$ we have $\mathrm{ex}(n, C_{2\ell}) \leq O(\ell n^{1+1/\ell})$.*

Lemma 3.9. *There is an absolute constant c such that for every $1 \leq k < \ell$ we have the following.*

$$\mathrm{ex}(n, C_{2k+1}, C_{2\ell+1}) \leq \begin{cases} c\ell^2 n^{1+1/\ell} & k = 1, \\ c(2k+1)^{2k}(2\ell+1)^{k+1}n^k & k \geq 2. \end{cases}$$

Proof. The case $k = 1$ follows immediately by combining (1) with Theorem 6. As for the case $k \geq 2$, recall that by Lemma 3.8 we have $\mathrm{ex}(n, C_{2k+1}, C_{2\ell+1}) \leq O((2k+1)^{2k}(2\ell+1)^{k+1}n^k)$ for every $2 \leq k < \ell-1$. In light of Lemma 2.3, this bound holds for $\ell = k+1$ as well (as $2\ell+1 = 2k+3 > 4$). ■

3.3 Proof of Theorem 4 and Proposition 1.3

Here we prove Theorem 4 and Proposition 1.3. For Theorem 4 we will need the following lemma.

Lemma 3.10. *Let $\ell \geq 2$ and let G be an n-vertex $C_{2\ell}$-free graph. Then every $v \in V(G)$ is the endpoint of at most $4(\ell-1)n$ paths of length 2.*

Proof. Let $v \in V(G)$ and assume, by contradiction, that v is the endpoint of $r > 4(\ell-1)n$ paths of length 2. Let $V(G) \setminus \{v\} = V_1 \cup V_2$ be a random partition, obtained by putting each $u \in V(G) \setminus \{v\}$ in one of the sets V_1, V_2 with probability $\frac{1}{2}$, independently. Since $\mathbb{E}\left[p(v, V_1, V_2)\right] = \frac{1}{4}r$, there is a choice of V_1, V_2 for which $e(N_{V_1}(v), V_2) = p(v, V_1, V_2) \geq \frac{1}{4}r > (\ell-1)n > (\ell-1)\left(|N_{V_1}(v)| + |V_2|\right)$. This stands in contradiction to Lemma 3.4, applied with $s = 2, u_0 = v, U_1 = N_{V_1}(v), U_2 = V_2$. ■

Proof of Theorem 4. The lower bounds are proved in Section 6: the lower bound for $\ell = 2$ is given by Lemma 6.4, and the lower bound for $\ell \geq 3$ is given by Corollary 6.3. Thus, it remains to prove the upper bounds. We prove both cases simultaneously by induction on k. The base cases are $k = 2, 3$. For $k = 2$, Lemma 3.10 implies that $\mathrm{ex}(n, P_2, C_{2\ell}) = O(\ell n^2)$, as required.

Suppose now that $k = 3$. We first handle the case $\ell \geq 3$. By Claim 2.1, it is enough to show that $p(X, Y, Z, W) \leq O(\ell^2 n^2)$ for every vertex-partition $X \cup Y \cup Z \cup W$ of an n-vertex $C_{2\ell}$-free graph. Let $Y' \subseteq Y$ and $Z' \subseteq Z$ be as in Lemma 3.1. In light of Item 2 in Lemma 3.1, it is enough to prove that $p(X, Y', Z, W) = O(\ell^2 n^2)$ and $p(X, Y, Z', W) = O(\ell^2 n^2)$. Fix any $y \in Y'$. By Lemma 3.10, we have $p(y, Z, W) = O(\ell n)$, and hence $p(X, y, Z, W) \leq O(\ell n) \cdot |N_X(y)|$. By summing over all $y \in Y'$ and using the guarantees of Lemma 3.1, we get

$$p(X, Y', Z, W) = \sum_{y \in Y'} p(X, y, Z, W) \leq O(\ell n) \cdot \sum_{y \in Y'} |N_X(y)| = O(\ell n) \cdot e(Y', X) \leq O(\ell^2 n^2).$$

The bound $p(X, Y, Z', W) = O(\ell^2 n^2)$ is proven similarly.

Now we handle the case $\ell = 2$. Let G be an n-vertex C_4-free graph. Observe that the number of paths of length 3 in a graph G is at most $\sum_{v \in V(G)} \#P_1(v) \cdot \#P_2(v)$, where $\#P_i(v)$ is the number of paths of length i having v as an endpoint (so $\#P_1(v)$ is just the degree of v). By combining Lemma 3.10 with Theorem 6, we get that $\sum_{v \in V(G)} \#P_1(v) \cdot \#P_2(v) \leq O(n) \cdot 2e(G) \leq O(n^{5/2})$, as required.

Let now $k \geq 4$. Let G be an n-vertex $C_{2\ell}$-free graph, and observe that the number of paths of length k in G is at most

$$\sum_{v \in V(G)} \#P_{k-2}(v) \cdot \#P_2(v) \leq O(\ell n) \sum_{v \in V(G)} \#P_{k-2}(v) \leq O(\ell n) \cdot \mathrm{ex}(n, P_{k-2}, C_{2\ell}),$$

where in the first inequality we used Lemma 3.10. Thus, $\mathrm{ex}(n, P_k, C_{2\ell}) \leq O(\ell n) \cdot \mathrm{ex}(n, P_{k-2}, C_{2\ell})$. It is now easy to see that the theorem follows by induction on k, with the base cases $k = 2, 3$. ■

Proof of Proposition 1.3. We start with the lower bound. For $\ell \geq 3$, this is the statement of Claim 6.6. For $\ell = 2$, we get it from Lemma 6.4 and the well-known fact that $\mathrm{ex}(n, C_4) = O(n^{3/2})$ (see Theorem 6). For the upper bound, let G be an n-vertex $C_{2\ell}$-free graph, and observe that for every $v \in V(G)$, the neighbourhood of v does not contain a path of length $2\ell - 2$; indeed, such a path would close a copy of $C_{2\ell}$ with v. By Theorem 5 we have $e(N(v)) \leq \frac{2\ell-3}{2} \cdot |N(v)|$. On the other hand, the number of triangles containing v is exactly $e(N(v))$, so the number of triangles in G is

$$\frac{1}{3} \sum_{v \in V(G)} e(N(v)) \leq \frac{2\ell - 3}{6} \sum_{v \in V(G)} |N(v)| = \frac{2\ell - 3}{3} \cdot e(G) \leq \frac{2\ell - 3}{3} \cdot \mathrm{ex}(n, C_{2\ell}) \;,$$

thus completing the proof. ■

4 Proof of Property Testing Results

In this section we prove Theorems 1 and 2 and Corollary 1.2. Given a monotone graph property $\mathcal{P}$ and $\varepsilon \in (0,1)$, recall that $w_{\mathcal{P}}(\varepsilon)$ is the minimal positive integer such that for every sufficiently large graph G which is ε-far from satisfying $\mathcal{P}$, a randomly-chosen induced subgraph of G of order $w_{\mathcal{P}}(\varepsilon)$ does not satisfy $\mathcal{P}$ with probability at least $\frac{2}{3}$. Recall that for a set of integers L, $\mathcal{P}(L)$ is the property of being L-*free*, that is, being C_ℓ-free for every $\ell \in L$. In this section we will only consider the properties $\mathcal{P}(L)$, where L is an infinite set of odd integers. To simplify the notation, we will write $w_L(\varepsilon)$ instead of $w_{\mathcal{P}(L)}(\varepsilon)$. In what follows, $c, c', c'', c_1, c_2, \ldots$ are absolute constants which are implicitly assumed to be large enough.

The following theorem is a special case of the main result of Alon et al. [2]. For a graph G, denote by $\mathrm{maxcut}(G)$ the largest size of a cut in G.

Theorem 7. *For every $\varepsilon \in (0, 1/2)$, for every n-vertex graph G and for every $q \geq c\varepsilon^{-4}\log(1/\varepsilon)$, a uniformly chosen set $Q \in \binom{V(G)}{q}$ satisfies $\left|\frac{\mathrm{maxcut}(G)}{n^2} - \frac{\mathrm{maxcut}(G[Q])}{q^2}\right| < \varepsilon$ with probability at least $\frac{5}{6}$.*

We now derive the following lemma from Theorem 7.

Lemma 4.1. *For every $\varepsilon \in (0,1)$ and for every graph G which is ε-far from bipartiteness, it holds that with probability at least $\frac{2}{3}$, a random induced subgraph of G of order $c\varepsilon^{-5}$ is $\frac{\varepsilon}{2}$-far from bipartiteness.*

Proof. Let G be a graph which is ε-far from bipartiteness. Then clearly

$$\mathrm{maxcut}(G) \leq e(G) - \varepsilon n^2 = \left(\frac{e(G)}{n^2} - \varepsilon\right) n^2.$$

Set $q = c\varepsilon^{-5}$ and let $Q \in \binom{V(G)}{q}$ be chosen uniformly at random. Then with probability at least $\frac{5}{6}$ we have $\mathrm{maxcut}(G[Q]) \leq (\frac{e(G)}{n^2} - \frac{3\varepsilon}{4})q^2$, where we applied Theorem 7 with $\frac{\varepsilon}{4}$ in place of ε. By a standard second-moment-method argument one can easily show that a randomly chosen induced subgraph of order at least $c\varepsilon^{-2}$ has the same edge density as G, up to an additive error of ε. Thus, (by applying this argument with $\varepsilon/4$ in place of ε), the inequality

$$\left|\frac{e(G)}{n^2} - \frac{e(G[Q])}{q^2}\right| < \frac{\varepsilon}{4}$$

holds with probability at least $\frac{5}{6}$. Thus, with probability at least $\frac{2}{3}$ we have

$$\text{maxcut}(G[Q]) \leq \left(\frac{e(G)}{n^2} - \frac{3\varepsilon}{4}\right) q^2 \leq e(G[Q]) - \frac{\varepsilon}{2} q^2,$$

which implies that $G[Q]$ is $\frac{\varepsilon}{2}$-far from bipartiteness. This completes the proof. ■

The next lemma we will need is Lemma 4.2 below, which relies on Lemma 4.1 and on the following theorem of Komlós.

Theorem 8 ([30])**.** *For every $\varepsilon \in (0, 1/2)$, every graph which is ε-far from bipartiteness contains an odd cycle of length at most $c\varepsilon^{-1/2}$.*

Lemma 4.2. *Let $\varepsilon \in (0, 1)$, suppose that $n \geq q \geq c_1\varepsilon^{-11}$ and let G be an n-vertex graph. If G is ε-far from being bipartite then there is an odd $3 \leq s \leq c_1\varepsilon^{-1/2}$ such that with probability at least $\frac{2}{3}$, a random induced subgraph of G of order q contains at least $\left(\varepsilon^6 q/c_1\right)^s$ copies of C_s.*

Proof. By Theorem 4.1, a uniformly chosen $P \in \binom{V(G)}{c\varepsilon^{-5}}$ induces a graph which is $\frac{\varepsilon}{2}$-far from bipartiteness with probability at least $2/3$. By Theorem 8, such an induced subgraph contains an odd cycle of length at most $c(\varepsilon/2)^{-1/2}$. Thus, there is $3 \leq s \leq c(\varepsilon/2)^{-1/2}$ such that a random P as above contains an s-cycle with probability at least $\varepsilon^{1/2}/c'$. Set $d = 4c'\varepsilon^{-1/2}$ and let $P_1, \dots, P_d \in \binom{V(G)}{c\varepsilon^{-5}}$ be chosen uniformly at random and independently. Setting $R = P_1 \cup \dots \cup P_d$, we see that $G[R]$ contains an s-cycle with probability at least $1 - \left(1 - \varepsilon^{1/2}/c'\right)^{4c'\varepsilon^{-1/2}} \geq 1 - e^{-4} \geq 11/12$. Moreover, the probability that there are $1 \leq i < j \leq d$ for which $P_i \cap P_j \neq \emptyset$ is at most $\binom{d}{2} \cdot n \cdot (c\varepsilon^{-5}/n)^2 \leq c''\varepsilon^{-11}/n \leq \frac{1}{2}$, where in the last inequality we used the assumption that $n \geq c_1\varepsilon^{-11}$. Thus, setting $r = d \cdot c\varepsilon^{-5} = 4cc'\varepsilon^{-11/2}$, we see that $\mathbb{P}[|R| = r] \geq \frac{1}{2}$. Since $G[R]$ contains an s-cycle with probability at least $\frac{11}{12}$, we infer that at least a $\frac{5}{6}$-fraction of all sets $R' \in \binom{V(G)}{r}$ are such that $G[R']$ contains an s-cycle. Let $\mathcal{R}$ be the set of all $R' \in \binom{V(G)}{r}$ having this property, and note that $|\mathcal{R}| \geq \frac{5}{6}\binom{n}{r}$.

Fix any $q \geq r$. For $Q \in \binom{V(G)}{q}$, define the random variable $Z(Q) = |\binom{Q}{r} \cap \mathcal{R}|$ (namely, $Z(Q)$ is the number of sets in $\mathcal{R}$ which are contained in Q), and let $\mathcal{Q} = \left\{Q \in \binom{V(G)}{q} : Z(Q) \geq \frac{1}{2}\binom{q}{r}\right\}$. By linearity of expectation, we have $\mathbb{E}[Z] = |\mathcal{R}| \cdot \binom{q}{r}/\binom{n}{r} \geq \frac{5}{6}\binom{q}{r}$. Since $0 \leq Z \leq \binom{q}{r}$, it is now easy to deduce (by averaging) that $\mathbb{P}[Z \geq \frac{1}{2}\binom{q}{r}] \geq \frac{2}{3}$, implying that $|\mathcal{Q}| \geq \frac{2}{3}\binom{n}{q}$.

Now let $Q \in \mathcal{Q}$. By the definition of $\mathcal{Q}$, there are at least $\frac{1}{2}\binom{q}{r}$ r-sets $R \subseteq Q$ such that $G[R]$ contains a copy of C_s. On the other hand, a copy of C_s in $G[Q]$ is contained in exactly $\binom{q-s}{r-s}$ such r-sets. Thus, $G[Q]$ contains at least

$$\frac{\binom{q}{r}}{2\binom{q-s}{r-s}} = \frac{\binom{q}{s}}{2\binom{r}{s}} \geq \frac{1}{2}\left(\frac{q}{er}\right)^s \geq \left(\varepsilon^6 q/c_1\right)^s.$$

copies of C_s. This completes the proof. ■

Lemma 4.4, stated below, is the main lemma in this section. Its proof uses Lemma 3.9, Lemma 4.2 and the following lemma from [4].

Lemma 4.3 ([4])**.** *Let K be a k-vertex graph, let F be an f-vertex graph which has a homomorphism into K and let G be the $\frac{n}{k}$-blowup of K where $n \geq n_0(k, f)$. Then G is $\frac{1}{2k^2}$-far from being F-free.*

Lemma 4.4. *There is a constant $c_2 \geq c_1$ (where c_1 is from Lemma 4.2) such that the following holds. Let $(\ell_i)_{i \geq 1}$ be an infinite increasing sequence of odd integers with $\ell_1 \geq 3$, and set $L = \{\ell_i : i \geq 1\}$. Then the following holds.*

1. *Let $\varepsilon \in (0, 1)$ be small enough so that $c_1\varepsilon^{-1/2} \geq \ell_1$. Let ℓ_i be the maximal element of L not larger than $c_1\varepsilon^{-1/2}$, let $n \geq q \geq c_2\varepsilon^{-13} \cdot \ell_1^2 \cdot \ell_{i+1}$, and let G be an n-vertex graph which is ε-far from being bipartite. Then with probability at least $\frac{2}{3}$, a random induced subgraph of G of order q is not L-free. Thus, $w_L(\varepsilon) \leq c_2\varepsilon^{-13} \cdot \ell_1^2 \cdot \ell_{i+1}$.*
2. *For every $i \geq 1$ we have $w_L(\frac{1}{2(\ell_i+2)^2}) \geq \ell_{i+1}$.*

Proof. We start by proving the first assertion of Item 1. Let G be an n-vertex graph which is ε-far from bipartiteness. By Lemma 4.2, there is an odd $3 \leq s \leq c_1\varepsilon^{-1/2}$ such that for a randomly chosen $Q \in \binom{V(G)}{q}$, the graph $G[Q]$ contains at least $(\varepsilon^6 q/c_1)^s$ copies of C_s with probability at least $\frac{2}{3}$. We claim that if $G[Q]$ has this property then $G[Q]$ is not L-free. This will show that a random induced subgraph of G of order q is not L-free with probability at least $\frac{2}{3}$. This will also prove the upper bound on $w_L(\varepsilon)$ stated in Item 1, since every graph which is ε-far from being L-free is also ε-far from bipartiteness (as L contains only odd integers).

Assume first that $s = 3$. If $\ell_1 = 3$ then $G[Q]$ is clearly not L-free, as it contains at least one triangle. So we may assume that $\ell_1 = 2\ell + 1 > 3$. It is easy to see that for c_2 large enough, our choice of q guarantees that

$$(\varepsilon^6 q/c_1)^3 > c\ell_1^2 q^{3/2} > c\ell^2 q^{1+1/\ell} \geq \mathrm{ex}(q, C_3, C_{2\ell+1}) ,$$

where in the last inequality we use Lemma 3.9. This means that $G[Q]$ contains more triangles than $\mathrm{ex}(q, C_3, C_{2\ell+1})$. So $G[Q]$ contains a cycle of length $\ell_1 = 2\ell + 1$ and hence is not L-free.

Assume from now on that $s > 3$. Observe that for a large enough c_2 we have

$$(\varepsilon^6 q/c_1)^s > c \cdot (c_1\varepsilon^{-1/2})^s \cdot \ell_{i+1}^{s/2} \cdot q^{s/2} \geq cs^s \ell_{i+1}^{s/2} q^{s/2} \geq cs^s \ell_{i+1}^{(s+1)/2} q^{(s-1)/2} \geq \mathrm{ex}(q, C_s, C_{\ell_{i+1}}) ,$$

where in the first and third inequalities we use our choice of q, in the second inequality we use $s \leq c_1\varepsilon^{-1/2}$ and in the last inequality we use Lemma 3.9 with $2k + 1 = s$ and $2\ell + 1 = \ell_{i+1}$, noting that $s < \ell_{i+1}$ by our choice of ℓ_i and by $s \leq c_1\varepsilon^{-1/2}$. As $G[Q]$ contains more s-cycles than $\mathrm{ex}(q, C_s, C_{\ell_{i+1}})$, it must contain a cycle of length ℓ_{i+1}. Thus, $G[Q]$ is not L-free.

We now prove the second Item. Fixing $i \geq 1$, let n be large enough so that Lemma 4.3 is applicable to $k = \ell_i + 2$ and $f = \ell_{i+1}$, and let G be the $\frac{n}{\ell_i+2}$-blowup of C_{ℓ_i+2}. Note that $C_{\ell_{i+1}}$ has a homomorphism into C_{ℓ_i+2}, as $\ell_{i+1} \geq \ell_i + 2$. Thus, by applying Lemma 4.3 with $K = C_{\ell_i+2}$ and $F = C_{\ell_{i+1}}$, we conclude that G is $\frac{1}{2(\ell_i+2)^2}$-far from being $C_{\ell_{i+1}}$-free and hence also $\frac{1}{2(\ell_i+2)^2}$-far from being L-free. On the other hand, there is no homomorphism from C_k to C_{ℓ_i+2} for any odd $k \leq \ell_i$. Thus, every subgraph of G on less than ℓ_{i+1} vertices is L-free. Item 2 of the lemma follows. ■

The proofs of Theorem 1 and Corollay 1.2 now follow quite easily from the above lemma.

Proof of Theorem 1. Set $\ell_1 = 3$ and $\ell_{i+1} = 2f(\frac{1}{2(\ell_i+2)^2}) + 1$. Then ℓ_i is odd for every $i \geq 1$, and $(\ell_i)_{i\geq 1}$ is increasing as f satisfies $f(x) \geq 1/x$. Setting $L = \{\ell_i : i \geq 1\}$, we will show that the property of L-freeness satisfies the assertion of the theorem. More precisely, we will show that there is an absolute constant $\varepsilon_0 > 0$ such that $w_L(\varepsilon) \leq \varepsilon^{-14} f(\varepsilon/c)$ for every $\varepsilon < \varepsilon_0$, and that $w_L(\varepsilon) \geq f(\varepsilon)$ for an infinite sequence of values of ε which tends to 0. Let $\varepsilon \in (0, 1)$ be small enough so that $c_1\varepsilon^{-1/2} \geq 3 = \ell_1$, and let ℓ_i be the maximal element of L not larger than $c_1\varepsilon^{-1/2}$. Item 1 of Lemma 4.4 implies that

$$w_L(\varepsilon) \leq c_2\varepsilon^{-13} \cdot \ell_1^2 \cdot \ell_{i+1} = 9c_2\varepsilon^{-13} \cdot \ell_{i+1} \leq 27c_2\varepsilon^{-13} \cdot f(\frac{1}{2(\ell_i + 2)^2}) \leq \varepsilon^{-14} \cdot f(\varepsilon/c) \ ,$$

where in the last inequality we used that $\ell_i \leq c_1\varepsilon^{-1/2}$, that f is decreasing, and that $1/\varepsilon > 27c_2$ (which can be guaranteed by appropriately choosing ε_0). The second part of Lemma 4.4 implies that for every $i \geq 1$, $w_L(\frac{1}{2(\ell_i+2)^2}) \geq \ell_{i+1} > f(\frac{1}{2(\ell_i+2)^2})$. So there is a decreasing sequence $(\varepsilon_i)_{i\geq 1}$ with $\varepsilon_i \to 0$ (namely $\varepsilon_i = \frac{1}{2(\ell_i+2)^2}$) such that $w_L(\varepsilon_i) \geq f(\varepsilon_i)$. The theorem follows. ■

Proof of Corollary 1.2. The first part of Lemma 4.4 implies that for a sufficiently small ε we have $w_L(\varepsilon) \leq \text{poly}(1/\varepsilon) \cdot \ell_{i+1}$, where ℓ_i is the maximal element of L not larger than $c_1\varepsilon^{-1/2}$. Thus, if $\ell_{i+1} \leq \ell_i^d$ for some $d = d(L)$ and every sufficiently large i, then $w_L(\varepsilon) \leq \text{poly}(1/\varepsilon)$ for every sufficiently small ε. On the other hand, the second part of Lemma 4.4 implies that unless $\ell_{i+1} \leq \ell_i^d$ for some $d = d(L)$ and for every large enough i, the function $w_L(\varepsilon)$ is super-polynomial in $1/\varepsilon$ for infinitely many values of ε. We conclude that $w_L(\varepsilon) = \text{poly}(1/\varepsilon)$ if and only if $\ell_{i+1} \leq \ell_i^d$ for every large enough i, which is equivalent to having $\limsup_{j\longrightarrow\infty} \frac{\log \ell_{j+1}}{\log \ell_j} \leq d < \infty$. ■

Lemma 4.5. *Let $(\ell_i)_{i\geq 1}$ be an infinite increasing sequence of odd integers with $\ell_1 \geq 3$, and set $L = \{\ell_i : i \geq 1\}$. Then every L-free graph is $o(1)$-close to bipartiteness.*

Proof. Our goal is to show that for every sufficiently small ε there is $n_0(\varepsilon)$ such that every L-free graph on $n \geq n_0(\varepsilon)$ vertices is ε-close to being bipartite. So fix $\varepsilon > 0$ small enough to satisfy $c_1\varepsilon^{-1/2} \geq \ell_1$, and let ℓ_i be the maximal element of L not larger than $c_1\varepsilon^{-1/2}$. By (the contrapositive of) Item 1 in Lemma 4.4, every n-vertex L-free graph is ε-close to bipartiteness, provided that n is large enough to satisfy $n \geq c_2\varepsilon^{-13} \cdot \ell_1^2 \cdot \ell_{i+1}$. This completes the proof. ■

The quantitative version of Lemma 4.5 states that L-free n-vertex graphs are roughly $\Theta(\ell_i^{-2})$-close to bipartiteness, where i is the maximal integer satisfying $n \geq \ell_{i+1}$ (here we assume that the sequence $(\ell_i)_{i\geq 1}$ grows fast enough). Let us explain why this dependence on the sequence $(\ell_i)_{i\geq 1}$ is unavoidable. For $n = \ell_{i+1}$, let G be the $\frac{n-1}{\ell_i+2}$-blowup of C_{ℓ_i+2}, plus an isolated vertex. Then G is L-free; it contains neither an odd cycle of length at most ℓ_i (as such a cycle is not homomorphic to C_{ℓ_i+2}), nor an odd cycle of length at least ℓ_{i+1} (as $\ell_{i+1} > n - 1$ and G has an isolated vertex). Nonetheless, it is easy to see that G is $\Theta(\ell_i^{-2})$-far from bipartiteness. This shows that the $o(1)$-term in Lemma 4.5 may tend to zero arbitrarily slowly, depending on the family L. For example, if $\ell_i = \text{tower}(i)$ then $\ell_i = \log_2(\ell_{i+1})$, so every L-free n-vertex graph is roughly $\Theta(\frac{1}{\log^2 n})$-close to bipartiteness, and this is tight.

Proof of Theorem 2. By (the proof of) Theorem 1, there is an increasing sequence of odd integers $L = \{\ell_1 = 3, \ell_2, \ell_3, \dots\}$ such that $w_L(\varepsilon) \geq f(\varepsilon)$. Thus, it remains to present a 2-sided tester for L-freeness which has query complexity $\text{poly}(1/\varepsilon)$. Our ε-tester works as follows: it samples a random induced subgraph of the input of order $q = q(\varepsilon) = c\varepsilon^{-5}$ and accepts if and only if this subgraph is $\frac{\varepsilon}{2}$-close to bipartiteness. Let us prove that this algorithm is indeed a valid ε-tester for graphs of order $n \geq n_0(\varepsilon)$, where $n_0(\varepsilon)$ will be (implicitly) chosen later. Let G be an n-vertex input graph. If G is ε-far from L-freeness then it is also ε-far from bipartiteness, so Lemma 4.1 implies that with probability at least $\frac{2}{3}$, G is rejected. Assume now that G is L-free. By Lemma 4.5, if n is large enough then G is $\frac{\varepsilon}{12}$-close to bipartiteness. Hence, there is a set $E \subseteq E(G)$ of size $|E| \leq \frac{\varepsilon}{12}n^2$ such that $G \setminus E$ (the graph obtained from G by deleting the edges in E) is bipartite. Let $Q = \{x_1, \dots, x_q\}$ denote the vertex-set sampled by the tester. The expected number of pairs $1 \leq i < j \leq q$ for which $\{x_i, x_j\} \in E$ is $\binom{q}{2} \cdot \frac{2|E|}{n(n-1)} \leq \frac{\varepsilon}{6}q^2$. By Markov's inequality, we have $|E(G[Q]) \cap E| \leq \frac{\varepsilon}{2}q^2$ with probability at least $\frac{2}{3}$. Thus, with probability at least $\frac{2}{3}$, $G[Q]$ is $\frac{\varepsilon}{2}$-close to bipartiteness (as deleting the edges in $E(G[Q]) \cap E$ makes $G[Q]$ bipartite), and G is accepted by the tester. ■

5 Proof of Lemma 3.1

We will need an upper bound on Zarankiewicz numbers for even cycles, proved by Naor and Verstraëte [33]. For integers $n, m \geq 1$ and $\ell \geq 2$, let $z(n, m, C_{2\ell})$ denote the maximal number of edges in a $C_{2\ell}$-free bipartite graph with sides of size n and m.

Theorem 9 ([33]). *For $m \leq n$ it holds that*

$$z(n, m, C_{2\ell}) \leq \begin{cases} (2\ell - 3)\left((nm)^{1/2+1/(2\ell)} + 2n\right) & \ell \text{ is odd,} \\ (2\ell - 3)\left(n^{1/2}m^{1/2+1/\ell} + 2n\right) & \ell \text{ is even.} \end{cases}$$

The following lemma is an easy corollary of Theorem 9.

Lemma 5.1. *Let $\ell \geq 2$, let G be an n-vertex graph and let $X, Y \subseteq V(G)$ be disjoint sets such that the bipartite graph (X, Y) is $C_{2\ell}$-free. Let Y' be the set of all vertices in Y having at least d neighbours in X. Then*

$$|Y'| \leq \begin{cases} \max\{(6\ell/d)^{2\ell/(\ell-1)}n^{(\ell+1)/(\ell-1)}, 6\ell n/d\} & \ell \text{ is odd,} \\ \max\{(6\ell/d)^{2\ell/(\ell-2)}n^{\ell/(\ell-2)}, 6\ell n/d\} & \ell \text{ is even and } \ell \geq 4, \\ 2n/(d - n^{1/2}) & \ell = 2 \text{ and } d > n^{1/2}. \end{cases}$$

Proof. Note that
$$d|Y'| \leq e(Y', X) \leq z(n, |Y'|, C_{2\ell}). \tag{5}$$
Suppose first that ℓ is odd. We apply Theorem 9 with parameter $m = |Y'|$. If $(|Y'|n)^{1/2+1/(2\ell)} \geq n$ then Theorem 9 gives $z(n, |Y'|, C_{2\ell}) \leq 6\ell(|Y'|n)^{1/2+1/(2\ell)}$, and if $(|Y'|n)^{1/2+1/(2\ell)} \leq n$ then Theorem 9 gives $z(n, |Y'|, C_{2\ell}) \leq 6\ell n$. By combining these inequalities with (5) we get that either $|Y'| \leq (6\ell/d)^{2\ell/(\ell-1)}n^{(\ell+1)/(\ell-1)}$ or $|Y'| \leq 6\ell n/d$, as required.

Suppose now that ℓ is even and $\ell \geq 4$. By Theorem 9, we have $z(n, |Y'|, C_{2\ell}) \leq 6\ell n^{1/2}|Y'|^{1/2+1/\ell}$ if $n^{1/2}|Y'|^{1/2+1/\ell} \geq n$ and $z(n, |Y'|, C_{2\ell}) \leq 6\ell n$ otherwise. By combining these inequalities with (5) we get that either $|Y'| \leq (6\ell/d)^{2\ell/(\ell-2)}n^{\ell/(\ell-2)}$ or $|Y'| \leq 6\ell n/d$, as required.

Finally, suppose that $\ell = 2$ and that $d > n^{1/2}$. Theorem 9 gives $z(n, |Y'|, C_4) \leq n^{1/2}|Y'| + 2n$. By combining this with (5) we get that $|Y'| \leq 2n/(d - n^{1/2})$, as required. ■

We are now ready to prove Lemma 3.1.

Proof of Lemma 3.1. We start by considering the case of even $\ell \geq 4$. Define the sets $Y' = \{y \in Y : |N_X(y)| \geq \ell n^{2/(\ell+2)}\}$ and $Z' = \{z \in Z : |N_W(z)| \geq \ell n^{2/(\ell+2)}\}$. Apply Lemma 5.1 with $d = \ell n^{2/(\ell+2)}$ to get $|Y'|, |Z'| \leq O(n^{\ell/(\ell+2)})$. By plugging these bounds into Theorem 9, one can check that $e(Y', X), e(Y', Z), e(Z', Y), e(Z', W) \leq z(n, O(n^{\ell/(\ell+2)}), C_{2\ell}) = O(\ell n)$. Next, note that by the definitions of Y' and Z' we have

$$\begin{aligned} p(X, Y \setminus Y', Z \setminus Z', W) &< e(Y \setminus Y', Z \setminus Z') \cdot \ell n^{2/(\ell+2)} \cdot \ell n^{2/(\ell+2)} \\ &\leq z(n, n, C_{2\ell}) \cdot \ell^2 n^{4/(\ell+2)} \leq O(\ell^3 n^{1+1/\ell+4/(\ell+2)}), \end{aligned}$$

where in the last inequality we used Theorem 9. So if $\ell^3 n^{1+1/\ell+4/(\ell+2)} \leq \ell^2 n^2$ then we get the required bound $p(X, Y \setminus Y', Z \setminus Z', W) = O(\ell^2 n^2)$, and the proof is complete (for even ℓ). Otherwise, we have $\ell^3 n^{1+1/\ell+4/(\ell+2)} > \ell^2 n^2$ and hence $n < \ell^{\ell(\ell+2)/(\ell^2-3\ell-2)} = \ell \cdot \ell^{(5\ell+2)/(\ell^2-3\ell-2)} \leq O(\ell)$. Since $p(X, Y, Z, W) \leq n^4$, we have $p(X, Y, Z, W) \leq n^4 = O(\ell^2 n^2)$, and again we are done.

We now consider the case of odd $\ell \geq 3$. We define $Y' = \{y \in Y : |N_X(y)| \geq \ell n^{2/(\ell+1)}\}$ and $Z' = \{z \in Z : |N_W(z)| \geq \ell n^{2/(\ell+1)}\}$. Similarly to the previous case, we apply Lemma 5.1 with $d = \ell n^{2/(\ell+1)}$ to obtain $|Y'|, |Z'| \leq O(n^{(\ell-1)/(\ell+1)})$. We then plug these bounds into Theorem 9 to get $e(Y', X), e(Y', Z), e(Z', Y), e(Z', W) \leq z(n, O(n^{(\ell-1)/(\ell+1)}), C_{2\ell}) = O(\ell n)$. It remains to bound $p(X, Y \setminus Y', Z \setminus Z', W)$. Assume first that $\ell \geq 5$. By the definitions of Y' and Z' we have

$$\begin{aligned} p(X, Y \setminus Y', Z \setminus Z', W) &< e(Y \setminus Y', Z \setminus Z') \cdot \ell n^{2/(\ell+1)} \cdot \ell n^{2/(\ell+1)} \\ &\leq z(n, n, C_{2\ell}) \cdot \ell^2 n^{4/(\ell+1)} \leq O(\ell^3 n^{1+1/\ell+4/(\ell+1)}), \end{aligned}$$

where in the last inequality we used Theorem 9. If $\ell^3 n^{1+1/\ell+4/(\ell+1)} \leq \ell^2 n^2$ then by the above we have $p(X, Y \setminus Y', Z \setminus Z', W) = O(\ell^2 n^2)$, as required. Otherwise, we have $\ell^3 n^{1+1/\ell+4/(\ell+1)} > \ell^2 n^2$ and hence $n < \ell^{\ell(\ell+1)/(\ell^2-4\ell-1)} = \ell \cdot \ell^{(5\ell+1)/(\ell^2-4\ell-1)} = O(\ell)$. But then $p(X, Y, Z, W) \leq n^4 = O(\ell^2 n^2)$, and again we are done.

Thus, it remains to show that $p(X, Y \setminus Y', Z \setminus Z', W) = O(n^2)$ when $\ell = 3$. Recall that in this case we defined $Y' = \{y \in Y : |N_X(y)| \geq 3n^{1/2}\}$ and similarly $Z' = \{z \in Z : |N_W(z)| \geq 3n^{1/2}\}$. We need some additional definitions. Define $Y_{low} = \{y \in Y : |N_X(y)| < n^{1/3}\}$ and similarly $Z_{low} = \{z \in Z : |N_W(z)| < n^{1/3}\}$. Define $\mathcal{I} = \{i : \frac{1}{2}n^{1/3} \leq 2^i < 3n^{1/2}\}$, and for each $i \in \mathcal{I}$ set $Y_i = \left\{y \in Y : 2^i \leq |N_X(y)| < 2^{i+1}\right\}$ and $Z_i = \left\{z \in Z : 2^i \leq |N_W(z)| < 2^{i+1}\right\}$. It is immediate from these definitions that $Y \setminus Y' \subseteq Y_{low} \cup \bigcup_{i \in I} Y_i$ and similarly $Z \setminus Z' \subseteq Z_{low} \cup \bigcup_{i \in I} Z_i$. Note that

$$p(X, Y_{low}, Z_{low}, W) < e(Y_{low}, Z_{low}) \cdot n^{1/3} \cdot n^{1/3} \leq z(n, n, C_6) \cdot n^{2/3} \leq O(n^2),$$

where in the last inequality we used Theorem 9. Hence, in order to finish the proof we need to bound $p(X, \bigcup_{i \in I} Y_i, Z_{low}, W)$, $p(X, Y_{low}, \bigcup_{i \in I} Z_i, W)$ and $p(X, \bigcup_{i \in I} Y_i, \bigcup_{i \in I} Z_i, W)$. We start with the first two terms. Fix any $i \in \mathcal{I}$. By Lemma 5.1 with $d = 2^i$, we have $|Y_i| \leq \max\{18^3 \cdot 2^{-3i} \cdot n^2, 18 \cdot 2^{-i} \cdot n\} = O(2^{-3i} \cdot n^2)$, where we used the fact that $9 \cdot 2^{-3i} n^2 > 2^{-i} n$, which follows from $2^i < 3n^{1/2}$. So we get

$$e(Y_i, Z_{low}) \leq z(|Y_i|, n, C_6) \leq 3 \cdot \left((|Y_i| n)^{2/3} + 2n\right) \leq 3 \cdot \left(O(n^2 2^{-2i}) + 2n\right) \leq O(n^2 \cdot 2^{-2i}),$$

where in the second inequality we used Theorem 9, and in the last inequality we used $n^2 \cdot 2^{-2i} > n/9$ which follows from $2^i < 3n^{1/2}$. Now we have

$$p(X, \bigcup Y_i, Z_{low}, W) = \sum_{i \in \mathcal{I}} p(X, Y_i, Z_{low}, W) < \sum_{i \in \mathcal{I}} e(Y_i, Z_{low}) \cdot 2^{i+1} \cdot n^{1/3} \leq \sum_{i \in \mathcal{I}} O(n^2 \cdot 2^{-2i}) \cdot 2^{i+1} \cdot n^{1/3}$$

$$= O(n^{7/3}) \cdot \sum_{i \in \mathcal{I}} 2^{-i} \leq O(n^{7/3}) \cdot \sum_{i:\, 2^i \geq \frac{1}{2} n^{1/3}} 2^{-i} = O(n^{7/3}) \cdot O(n^{-1/3}) = O(n^2),$$

where in the first inequality we used the definitions of Z_{low} and Y_i, and in the last inequality we used the definition of $\mathcal{I}$. The bound $p(X, Y_{low}, \bigcup Z_i, W) = O(n^2)$ is proved similarly.

Finally, we bound $p(X, \bigcup Y_i, \bigcup Z_i, W)$. To this end, fix any $i, j \in \mathcal{I}$. We showed above that $|Y_i| \leq O(n^2 \cdot 2^{-3i})$. By the same argument we get $|Z_j| \leq O(n^2 \cdot 2^{-3j})$. Thus we have

$$\begin{aligned} e(Y_i, Z_j) &\leq z(|Y_i|, |Z_j|, C_6) \leq 3 \cdot \Big((|Y_i||Z_j|)^{2/3} + |Y_i| + |Z_j| \Big) \\ &\leq O(n^{8/3}) \cdot 2^{-2i} \cdot 2^{-2j} + O(n^2) \cdot (2^{-3i} + 2^{-3j}) \leq O(n^{8/3}) \cdot 2^{-2i} \cdot 2^{-2j}, \end{aligned}$$

where in the second inequality we used Theorem 9, and in the last inequality we used the fact that $18n^{8/3} \cdot 2^{-2i} \cdot 2^{-2j} \geq \max\{n^2 2^{-3i}, n^2 2^{-3j}\}$, which follows from $\frac{1}{2} n^{1/3} \leq 2^i, 2^j < 3n^{1/2}$. Now we get

$$\begin{aligned} p(X, \bigcup Y_i, \bigcup Z_i, W) &= \sum_{i,j \in \mathcal{I}} p(X, Y_i, Z_j, W) \leq \sum_{i,j \in \mathcal{I}} e(Y_i, Z_j) \cdot 2^{i+1} \cdot 2^{j+1} \\ &\leq \sum_{i,j \in \mathcal{I}} O(n^{8/3}) \cdot 2^{-2i} \cdot 2^{-2j} \cdot 2^{i+1} \cdot 2^{j+1} = O(n^{8/3}) \cdot \sum_{i,j \in \mathcal{I}} 2^{-i} \cdot 2^{-j} \\ &\leq O(n^{8/3}) \cdot \sum_{2^i, 2^j \geq \frac{1}{2} n^{1/3}} 2^{-i} \cdot 2^{-j} = O(n^{8/3}) \cdot O(n^{-2/3}) = O(n^2), \end{aligned}$$

where in the first inequality we used the definitions of Y_i and Z_j, and in the last inequality we used the definition of $\mathcal{I}$. This completes the proof. ■

Let us explain why the sets Y' and Z' in Lemma 3.1 are required, (namely, that the statement $p(X, Y, Z, W) = O_\ell(n^2)$ is generally false). Note that by Theorem 9, the average degree between the four sets in Lemma 3.1 is $O(n^{1/3})$. One might thus guess that $p(X, Y, Z, W) = O(n \cdot (n^{1/3})^3) = O(n^2)$. To see that this is not the case, we can take Y to be a single vertex connected to all the vertices of X and Z, distribute all other vertices equally among X, Z and W, and take the bipartite graph between Z, W to be an extremal graph with no $C_{2\ell}$. Although this example satisfies $p(X, Y, Z, W) \gg n^2$, by removing the single vertex of Y we can make sure that $p(X, Y, Z, W) = O(n^2)$. This is precisely what Lemma 3.1 states. What we see in the proof of Theorem 4 is that if one assumes that the *entire* graph is $C_{2\ell}$-free (and not just the 3 bipartite graphs between the 4 sets) then one no longer needs to remove vertices in order to guarantee that $p(X, Y, Z, W) = O_\ell(n^2)$.

Let us note that Lemma 3.1 does not hold for $\ell = 2$. Indeed, in the proof of Lemma 6.4 we construct an n-vertex C_4-free graph, in which every vertex has degree $\Theta(n^{1/2})$ and lies on $\Theta(n^{3/2})$ paths of length 3. Taking a random vertex partition of this graph into four sets X, Y, Z, W, we see that with high probability, every vertex $y \in Y$ (resp. $z \in Z$) has $\Theta(n^{1/2})$ neighbours in X (resp. W), and every vertex in the graph lies on $\Theta(n^{3/2})$ (X, Y, Z, W)-paths. Suppose now, by contradiction, that the assertion of Lemma 3.1 holds for the sets X, Y, Z, W. Since every $y \in Y$ has $\Theta(n^{1/2})$ neighbours in X, and since $e(Y', X) = O(n)$, we must have $|Y'| = O(n^{1/2})$. Similarly, $|Z'| = O(n^{1/2})$. As every vertex lies on $\Theta(n^{3/2})$ (X, Y, Z, W)-paths, we have $p(X, Y, Z, W) = \Theta(n^{5/2})$ and $p(X, Y', Z, W), p(X, Y, Z', W) = O(n^2)$. But this implies that $p(X, Y \setminus Y', Z \setminus Z', W) = \Theta(n^{5/2})$, in contradiction to the statement of Lemma 3.1.

6 Lower Bound for $\mathrm{ex}(n, C_k, C_\ell)$ and $\mathrm{ex}(n, P_k, C_\ell)$

In this section we prove all lower bounds in Theorem 3 and Proposition 1.3. We start with the following two claims, which handle the case where the forbidden cycle is *not* C_4. Claim 6.1 gives lower bounds on $\mathrm{ex}(n, C_k, C_\ell)$ and $\mathrm{ex}(n, P_k, C_\ell)$ with the correct dependence on n, whenever $\ell \neq 4$. To get the correct dependence on ℓ for $\ell \gg k$, we need Claim 6.2, which gives a general lower bound for $\mathrm{ex}(n, T, H)$, but is only applicable when H (that is, C_ℓ) is somewhat larger than T (that is, C_k or P_k). To prove the lower bound for all values of k and $\ell \neq 4$, we need to combine these two claims, which is done in Corollary 6.3. For a graph G, denote by $\alpha(G)$ the independence number of G.

Claim 6.1. *For a pair of distinct $k \geq 3$ and $4 \neq \ell \geq 3$ we have* $\mathrm{ex}(n, C_k, C_\ell) = \Omega_k\big(n^{\lfloor k/2 \rfloor}\big)$. *For $k \geq 2$ and $4 \neq \ell \geq 3$ we have* $\mathrm{ex}(n, P_k, C_\ell) = \Omega_k\big(n^{\lceil (k+1)/2 \rceil}\big)$.

Proof. We start with the first part of the claim. Let I be a maximum independent set of the k-cycle $1, \dots, k$. Replace each $i \in I$ with a vertex-set of size m, where different vertices are replaced with disjoint sets and all of these sets are disjoint from $[k] \setminus I$. Edges of C_k are replaced with complete bipartite graphs. In other words, we take a blowup of C_k in which vertices $i \in [k] \setminus I$ are not blown up, while vertices $i \in I$ are blown up to size m. As $|I| = \alpha(C_k) = \lfloor k/2 \rfloor$, the resulting graph has $n := \lfloor k/2 \rfloor \cdot m + \lceil k/2 \rceil$ vertices and $m^{|I|} = m^{\lfloor k/2 \rfloor} = \Omega_k\big(n^{\lfloor k/2 \rfloor}\big)$ copies of C_k. It is easy to check that this graph is C_ℓ-free by our assumptions that $\ell \neq k$ and $\ell \neq 4$.

We now prove the second part of the claim using a similar construction. Let I be a maximum independent set of the path P_k on the vertices $1, \dots, k+1$. Replace each $i \in I$ with a vertex-set of size m, where different vertices are replaced with disjoint sets and all of these sets are disjoint from $[k+1] \setminus I$. Edges of P_k are replaced with complete bipartite graphs. As $|I| = \alpha(P_k) = \lceil (k+1)/2 \rceil$, the resulting graph has $n := \lceil (k+1)/2 \rceil \cdot m + \lfloor (k+1)/2 \rfloor$ vertices and $m^{|I|} = m^{\lceil (k+1)/2 \rceil} = \Omega_k(n^{\lceil (k+1)/2 \rceil})$ copies of P_k. It is easy to check that this graph is C_ℓ-free by our assumptions that $\ell \neq 4$. ■

Claim 6.2. *Let T, H be graphs on t and h vertices, respectively, such that $h - \alpha(H) - 1 \geq t - \alpha(T)$. Then for every $n \geq h - \alpha(H) - 1 + \alpha(T)$, it holds that* $\mathrm{ex}(n, T, H) \geq \Omega_t\left((h - \alpha(H))^{t - \alpha(T)} n^{\alpha(T)}\right)$.

Proof. Suppose that $V(T) = \{1, \dots, t\}$ and let I be a maximum independent set of T. Let $U_1, \dots, U_t$ be disjoint vertex-sets such that $|U_1| + \dots + |U_t| = n$ and such that the following holds: $\sum_{i \in V(T) \setminus I} |U_i| = h - \alpha(H) - 1$, these $h - \alpha(H) - 1$ vertices are divided as equally as possible among the $t - \alpha(T)$ sets $(U_i)_{i \in V(T) \setminus I}$, and the $n - h + \alpha(H) + 1$ vertices of $\bigcup_{i \in I} U_i$ are divided as equally as possible among $(U_i)_{i \in I}$. Then none of $U_1, \dots, U_t$ is empty by the assumptions of the claim. Define a graph G on $U_1 \cup \dots \cup U_t$ by making (U_i, U_j) a complete bipartite graph if $\{i, j\} \in E(T)$, and an empty bipartite graph otherwise (there are no edges inside the sets $U_1, \dots, U_t$). Then G has $\Omega_t\left((h - \alpha(H))^{t - \alpha(T)} n^{\alpha(T)}\right)$ copies of T. It remains to show that G is H-free. Assume by contradiction that there is a copy of H in G. Then this copy contains two adjacent vertices which are both in $\bigcup_{i \in I} U_i$, since $\sum_{i \in V(T) \setminus I} |U_i| < h - \alpha(H)$. But $\bigcup_{i \in I} U_i$ is an independent set in G, as I is an independent set in T and G is a blowup of T, a contradiction. ■

We are now ready to prove the lower bounds in the last two items of Theorem 3 and in the second item of Theorem 4. In other words, we handle all cases in which the forbidden cycle is not C_4.

Corollary 6.3. *For a pair of distinct $k \geq 3$ and $4 \neq \ell \geq 3$ we have* $\mathrm{ex}(n, C_k, C_\ell) = \Omega_k\big(\ell^{\lceil k/2 \rceil} n^{\lfloor k/2 \rfloor}\big)$. *For $k \geq 2$ and $4 \neq \ell \geq 3$ we have* $\mathrm{ex}(n, P_k, C_\ell) = \Omega_k(\ell^{\lfloor (k+1)/2 \rfloor} n^{\lceil (k+1)/2 \rceil})$.

Proof. Note that since our bound hides constants that depend on k, if $\ell < k+3$ then the assertion of the corollary follows from Claim 6.1. So we may assume that $\ell \geq k+3$, which implies that $\lceil \ell/2 \rceil \geq \lceil k/2 \rceil + 1$. Under this assumption, Claim 6.2 is applicable to $(T,H) = (C_k, C_\ell)$, giving $\mathrm{ex}(n, C_k, C_\ell) = \Omega_k(\ell^{\lceil k/2 \rceil} n^{\lfloor k/2 \rfloor})$, and to $(T,H) = (P_k, C_\ell)$, giving $\mathrm{ex}(n, P_k, C_\ell) = \Omega_k(\ell^{\lfloor (k+1)/2 \rfloor} n^{\lceil (k+1)/2 \rceil})$. ■

When excluding C_4, a different construction is required. The construction we use is due to Erdős and Rényi [18]. The case of $\mathrm{ex}(n, C_3, C_4)$ was handled (using the same construction) in [6]. Via the following lemma, we get the lower bound in the first item of Theorem 3 and of Theorem 4.

Lemma 6.4. *Let q be a prime power and set $n = q^2 - 1$. Then there is an n-vertex C_4-free graph which contains at least $\left(\frac{1}{2k} - o(1)\right) n^{\frac{k}{2}}$ copies of C_k for every $4 \neq k \geq 3$, and at least $\left(\frac{1}{2} - o(1)\right) n^{\frac{k}{2}+1}$ copies of P_k for every $k \geq 1$. Here, the $o(1)$ term is a function which depends on k and tends to 0 as n tends to infinity. Hence, $\mathrm{ex}(n, C_k, C_4) \geq \left(\frac{1}{2k} - o(1)\right) n^{\frac{k}{2}}$ for every $4 \neq k \geq 3$, and $\mathrm{ex}(n, P_k, C_4) \geq \left(\frac{1}{2} - o(1)\right) n^{\frac{k}{2}+1}$ for every $k \geq 1$.*

Proof. The last part of the theorem is deduced from the first part as follows. It is known that for every large enough x there is a prime in the interval $[x - x^\theta, x]$ for an absolute constant $\theta \in [\frac{1}{2}, 1)$, see e.g. [8]. Fixing a large enough n, let p be a prime in $[x - x^\theta, x]$ for $x = n^{1/2}$. Now take the construction from the first part of the theorem on $p^2 - 1$ vertices and add isolated vertices to get a graph on n vertices. This graph gives the required lower bounds on $\mathrm{ex}(n, C_k, C_4)$ and $\mathrm{ex}(n, P_k, C_4)$.

From now on we assume that $n = q^2 - 1$, where q is a prime power. Let $\mathbb{F}$ be the field with q elements. The vertex set of G is $\mathbb{F}^2 \setminus \{(0,0)\}$ and a pair of vertices $(a,b), (c,d)$ are adjacent if and only if $ac + bd = 1$. Note that $(a,b) \in V(G)$ has a loop if and only if $a^2 + b^2 = 1$. The number of solutions to $x^2 + y^2 = 1$ is at most $2q$, since for every fixed $x \in \mathbb{F}$ there are at most 2 solutions for y. This implies that the number of loops is at most $2q$. Note that for every $(a,b) \in V(G)$ there are q solutions (x,y) to $ax + by = 1$. Thus, the degree of every $(a,b) \in V(G)$ is either $q-1$ or q, depending on whether or not (a,b) has a loop. This implies that for every $k \geq 1$, G contains at least $\frac{1}{2} n(q-1)(q-2)\dots(q-k) = \left(\frac{1}{2} - o(1)\right) n^{\frac{k}{2}+1}$ paths of length k.

Observe that for every pair of vertices $(a,b), (c,d) \in V(G)$, there is at most one solution to the system $ax + by = cx + dy = 1$, implying that (a,b) and (c,d) have at most one common neighbour. This shows that G is C_4-free. To finish the proof, it remains to show that the number of k-cycles in G is as stated. Since this was proved for $k = 3$ in [6], we may assume from now on that $k \geq 5$.

Note that if $(a,b), (c,d) \in V(G)$ are linearly independent and have no loops then they have a common neighbour. Indeed, by linear independence there is a (unique) solution to the system $ax + by = cx + dy = 1$. As (a,b) and (c,d) do not have loops, this solution is neither (a,b) nor (c,d), and hence it is a common neighbour of (a,b) and (c,d). As the number of loops in G is at most $2q$, the number of pairs of vertices $(a,b), (c,d) \in V(G)$ for which either (a,b) or (c,d) has a loop is at most $2qn$. Furthermore, the number of collinear pairs $(a,b), (c,d) \in V(G)$ is $\frac{(q-1)n}{2}$. Therefore, all but $2qn + \frac{(q-1)n}{2} \leq 3nq$ of the pairs of vertices are linearly independent and do not have loops, and hence have a common neighbour. We have thus proven the following.

Fact 6.5. *All but $3nq$ of the pairs of vertices in G have a common neighbour.*

Note that for every $t \geq 2$ and $v_1, v_{t+1} \in V(G)$, the number of paths of length t between v_1 and v_{t+1} is at most q^{t-2}. Indeed, consider a path $v_1, \dots, v_{t+1}$. Since the maximal degree in G is q, the number of choices of $v_2, \dots, v_{t-1}$ is at most q^{t-2}. Since v_t is a common neighbour of v_{t-1} and v_{t+1}, there is at most one choice for v_t given $v_2, \dots, v_{t-1}$.

A path is *good* if its endpoints have a common neighbour which is not on the path, and otherwise it is *bad*. To complete the proof, it is enough to show that for every $t \geq 3$, the number of bad paths of length t is $O(nq^{t-1})$. Indeed, we already proved that G contains at least $\left(\frac{1}{2} - o(1)\right) n^{\frac{k}{2}}$ paths of length $k-2$. Since the number of bad paths of length $k-2$ is $O(nq^{k-3}) = O\left(n^{\frac{k-1}{2}}\right)$, the number of good paths of length $k-2$ is at least $\left(\frac{1}{2} - o(1)\right) n^{\frac{k}{2}}$. A good path of length $k-2$ can be made into a k-cycle by adding the (unique) common neighbour of the endpoints of the path. Since every cycle contains k subpaths of length $k-2$, the lemma follows.

It thus remains to show that for every $t \geq 3$, the number of bad paths of length t is $O(nq^{t-1})$. There are two types of bad paths: those whose endpoints do not have a common neighbour, and those whose endpoints have a common neighbour which is on the path. First, by Fact 6.5, the number of pairs of vertices $u, v \in V(G)$ which do not have a common neighbour is at most $3nq$. We proved that for each such u, v there are at most q^{t-2} paths of length t between u and v. Thus, there are at most $O(nq^{t-1})$ paths of length t whose endpoints do not have a common neighbour. Second, let $u, v \in V(G)$ be vertices having a common neighbour and let w be their unique common neighbour. The number of paths of length t from u to v in which w is at distance i from u (and hence at distance $t-i$ from v) is at most q^{t-3} if $i \in \{1, t-1\}$ and at most $q^{i-2}q^{t-i-2} = q^{t-4}$ if $2 \leq i \leq t-2$. By summing over $1 \leq i \leq t-1$ we get that the number of paths of length t from u to v which contain w is at most $2q^{t-3} + (t-3)q^{t-4} = O(q^{t-3})$. Since the number of choices for u, v is at most $\binom{n}{2}$, the total number of paths of length t that contain the common neighbour of their endpoints is $O(n^2 q^{t-3}) = O(nq^{t-1})$. In conclusion, the number of bad paths is $O(nq^{t-1})$, as required. ■

We end this section by proving the lower bound in Proposition 1.3.

Claim 6.6. *For every* $\ell \geq 3$ *we have* $\mathrm{ex}(n, C_3, C_{2\ell}) = \Omega\big(\mathrm{ex}(n, \{C_4, C_6, \ldots, C_{2\ell}\})\big)$.

Proof. We use an argument similar to the one used in [28]. Let $G' = (A \cup B, E)$ be a maximum size $n \times n$ bipartite graph with no $C_4, C_6, \ldots, C_{2\ell}$. Let G be the graph obtained from G' by replacing every vertex of A by an edge (and replacing edges of G' by copies of $K_{2,1}$). Then G has $3n$ vertices, and one triangle per each edge of G'; so G contains $e(G') \geq \frac{1}{2} \cdot \mathrm{ex}(2n, \{C_4, C_6, \ldots, C_{2\ell}\})$ triangles. Now assume by contradiction that C is a copy of $C_{2\ell}$ in G. By contracting the edges of C inside A, we get a closed walk C' in G' of length at most 2ℓ. For each $a \in A$, let a_1 and a_2 denote the two "copies" of a in G. If for every $a \in C' \cap A$, only one of the copies of a is in C, then $C' = C$, in contradiction to the $C_{2\ell}$-freeness of G'. So there is some $a \in A$ such that $a_1, a_2 \in C$. In the cycle C there are two paths between a_1 and a_2, and since $|C| = 2\ell \geq 6$, one of these paths must have length at least 3. Hence, there are distinct $b_1, b_2 \in B$ such that $(a_1, b_1), (a_2, b_2) \in E(G)$, and there is a path P in G between b_1 and b_2 which does not go through a_1 or a_2. Contracting P gives a path P' in G' between b_1 and b_2, which does not go through a. Then a, b_1, P, b_2, a is a cycle in G' of length at most 2ℓ, in contradiction to the choice of G'. ■

Note added: After posting the paper to the Arxiv, weve learned that Proposition 1.3 was obtained earlier by Füredi and Özkahya [20], and that Lemma 3.6 was obtained independently by Gerbner, Győri, Methuku and Vizer [21].

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ECCC ISSN 1433-8092

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