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(Min, Plus) is Not stronger than (Or, And)

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Abstract. We observe that a known structural property of $(\min, +)$ circuits (and formulas) implies that lower bounds on the monotone circuit/formula size remain valid also for $(\min, +)$ circuits/formulas, even when only nonnegative integer weights are allowed. So, the lower bound proved in ECCC TR18-020 can be alternatively derived from known lower bounds on the monotone formula complexity of the threshold-2 function.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Tropical $(\min, +)$ circuits and formulas solve minimization problems $f : \mathbb{N}^n \to \mathbb{N}$ of the form $f(x) = \min_{a \in A} \langle a, x\rangle + c_a$, where $A \subset \mathbb{N}^n$ is a finite set of vectors, each $c_a \in \mathbb{N}$ is a constant, and $\langle a, x\rangle = a_1x_1 + \cdots + a_nx_n$; values of variables x_i are referred to as input *weights*. Such a problem is a 0-1 *minimization problem* if $A \subseteq \{0, 1\}^n$, A is an antichain (no two vectors are comparable under $\leq$), and $c_a = 0$ holds for all $a \in A$. The *boolean version* of the problem f is a monotone boolean function $\widehat{f}(x) = \bigvee_{a\in A} \bigwedge_{i:\, a_i \neq 0} x_i$.

Let $\mathrm{Min}(f)$ denote the minimum size of a $(\min, +)$ circuit solving the problem f, and $\mathrm{Bool}(\widehat{f})$ the minimum size of monotone boolean circuit computing the boolean version $\widehat{f}$ of f. Let also $\mathrm{Min}^*(f)$ denote the version of $\mathrm{Min}(f)$ restricted to *constant-free* $(\min, +)$ circuits (those without input gates holding constants $c \in \mathbb{N}$).

If the infinite weight ∞ *is* also allowed, then $\mathrm{Min}(f) \geq \mathrm{Bool}(\widehat{f})$ holds for any 0-1 minimization problem (see, e.g., [2, Lemma 11]). If the infinite weight is *not* allowed, then $\mathrm{Min}^*(f) \geq \mathrm{Bool}(\widehat{f})$ still holds for any such problem f (see [3, Appendix A]).

In ECCC TR18-020, the authors are interested in the case when: (i) ∞ is *not* allowed as a weight, and (ii) $(\min, +)$ circuits *can* use constant inputs. In this comment, we show that (after a slight modification) a structural property of constant-free $(\min, +)$ circuits given in [3, Appendix A] yields the lower bound $\mathrm{Min}(f) \geq \mathrm{Bool}(\widehat{f})$ also under conditions (i) and (ii).

Lemma 1. *If f is a 0-1 minimization problem on an antichain $\{\vec{0}\} \neq A \subset \{0, 1\}^n$, then*

$$\mathrm{Min}^*(f) = \mathrm{Min}(f) \geq \mathrm{Bool}(\widehat{f})\,. \tag{1}$$

The same also holds for $(\min, +)$ boolean formulas.

Remark 1. Together with known lower bounds on the monotone boolean formula size of threshold functions, Lemma 1 yields the Theorem 14 of [6] giving a matching lower bound $\mathrm{Min}(f_{n,r}) \geq n\lceil \log n\rceil$ for the minimization problem

$$f_{n,r} = \mathsf{MinSum}^r_n := \Big\{ \sum_{i\in S} x_i \colon S \subseteq [n], |S| = r \Big\}$$

when $r = n - 1$. Indeed, these problems $f_{n,r}$ are 0-1 minimization problems whose sets A of feasible solutions are slices of the binary n-cube and, hence, are antichains. The boolean

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versions of these problems are the boolean threshold-r functions Th_r^n. For $r = n-1$, a (matching) lower bound $\mathrm{Bool}(\mathrm{Th}_{n-1}^n) = \mathrm{Bool}(\mathrm{Th}_2^n) \geq n\lceil \log n \rceil$ is known [8, 5]; lower bounds $\Omega(n \log n)$ were earlier shown in [1, 4, 7]. So, Lemma 1 gives the same lower bound for $(\min, +)$ formulas.

Remark 2. It is also proved in [6, Theorem 10] that the function $\max\{x, y\}$ cannot be computed by a $(\min, +)$ circuit. The proof is by an application of a carefully chosen restriction to the variables, and showing that the $(\min, +)$ formula does not output the correct value of max on this restriction. Note, however, that this fact follows also from a *general* property of $(\min, +)$ circuits: functions $f : \mathbb{N}^n \to \mathbb{N}$ computable by $(\min, +)$ circuits are *superadditive*: $f(u + v) \geq f(u) + f(v)$ holds for all $u, v \in \mathbb{N}^n$, but max is not superadditive. Indeed, any $(\min, +)$ circuit computes some tropical polynomial $p(x) = \min_{b \in B}\langle b, x\rangle + c_b$, and any such polynomial computes a superadditive function, since $\min_{b\in B}\langle b, x\rangle + \min_{b\in B}\langle b, y\rangle \leq \min_{b\in B}\langle b, x + y\rangle$. On the other hand, the function $f(x_1, x_2) = \max\{x_1, x_2\}$ is not superadditive: say, for $u = (1, 0)$ and $v = (0, 1)$, we have $f(u + v) = \max\{1, 1\} = 1$ but $f(u) + f(v) = \max\{1, 0\} + \max\{0, 1\} = 1 + 1 = 2$.

Remark 3. On page 11 of [6], the authors wrote "For functions computable in a constant-free manner, it is hard to see how constants can help." The equality in Eq. (1) confirms this intuition: at least for $(\min, +)$ circuits and formulas solving 0-1 minimization problems, constant inputs *cannot* help. Note that, in the case of $(\max, +)$ circuits (and *maximization* problems f), the equality $\mathrm{Max}^*(f) = \mathrm{Max}(f)$ is trivial: the circuit must correctly compute the value $f(x) = 0$ also on the all-0 weighting $x := \vec{0}$.

1. Proof of Lemma 1

A circuit (or formula) over any commutative semiring $(R, \oplus, \otimes)$ not only computes some polynomial over this semiring, but also *produces* (purely syntactically) a unique subset $B \subset \mathbb{N}^n$ of vectors in a natural way. At an input gate holding a semiring element $c \in R$, the singleton $\{\vec{0}\}$ is produced (regardless of what this element c actually is). At an input gate holding a variable x_i, the singleton $\{\vec{e}_i\}$ is produced, where $\vec{e}_i \in \{0, 1\}^n$ is the i-th unit vector. Let now u be a gate at two inputs of which sets A and B are produced. Then the set produced at u is $A \cup B$, if u is a $\oplus$-gate, and is the Minkowski sum (or sumset) $A + B = \{a + b \colon a \in A, b \in B\}$, if u is a $\otimes$-gate. The set produced by the entire circuit is the set produced at its output gate.

Remark 4. Note that, unlike for the function *computed*, the set *produced* by a circuit depends only on the structure of this circuit, not on the underlying semiring. Note also that the set B produced by a $(\min, +)$ circuit (or formula) F is just the projection onto the first n coordinates of the set $S(F)$ defined in [6, Definition 1].

Now we turn to the actual proof of Lemma 1. Let f be a 0-1 minimization problem on an antichain $\{\vec{0}\} \neq A \subset \{0, 1\}^n$. Take a $(\min, +)$ circuit F solving this problem, and let $B \subset \mathbb{N}^n$ be the set of vectors produced by F. Then F solves some minimization problem of the form $F(x) = \min_{b\in B}\langle b, x\rangle + c_b$ for some (not necessarily zero) scalars $c_b \in \mathbb{N}$; this can be shown by an easy induction of the size of F. Say that B *lies above* A if every vector $b \in B$ *contains* some vector $a \in A$, that is, if $b_i \geq a_i$ holds for all positions i.

Claim 1 ([3, Appendix A]). *$A \subseteq B$, and B lies above A.*

Proof. The proof is almost the same as the proof of this claim in the case of constant-free circuits, given in [3, Appendix A]: just replace (on line 7 of the proof) the weights $x_i = 1$ for

$i \notin S_b$ by the weights $x_i = 1 + \max\{c_b \colon b \in B\}$ ("F" stands for the set "B" in this proof). The rest is then the same. □

Our next goal is to construct (without increasing the size) a *constant-free version* F_* of our circuit F such that F_* produces the same set B. We can clearly assume that F has no gates whose *both* inputs are constants. Now, if $v = u \circ c$ is a gate in F, where $\circ \in \{\min, +\}$ and c is a constant input gate, then contract the edge (u, v), that is, replace every edge (v, w) leaving v by the edge (u, w), and remove the gate v. Finally, remove all constant input gates together with edges leaving them.

Claim 2. *The constant-free version F_* of F produces the same set B.*

Proof. Recall that at *every* input gate holding a constant c, the same set $\{\vec{0}\}$ is produced, regardless of what this constant c actually is. So, the constant-free version F_* of F must produce either the same set B (produced by F) or the set $B \setminus \{\vec{0}\}$. But vector $\vec{0}$ cannot belong to B because otherwise we would have that $f(x) \le F(x) \le 0 + c_{\vec{0}}$ must hold for all input weightings $x \in \mathbb{N}^n$, a contradiction with our assumption that $A \neq \{\vec{0}\}$ (this assumption implies that f can take arbitrarily large values $f(x)$). So, the circuit F_* must produce the same set B as F. □

Starting from the constant-free $(\min, +)$ circuit F_*, we now construct its *boolean version* by simply replacing min-gates by OR gates, and $+$-gates by AND gates. The resulting monotone boolean circuit $\widehat{F}$ produces the same set B of vectors as F_*: sets produced by circuits do not depend on the underlying semiring (Remark 4). So, the boolean function computed by the circuit $\widehat{F}$ is of the form $g(x) = \bigvee_{b \in B} \bigwedge_{i \colon b_i \neq 0} x_i$, and it remains to verify that $g(x) = \widehat{f}(x)$ holds for all $x \in \{0,1\}^n$. This directly follows from Claim 1: if $\widehat{f}(x) = 1$, then $g(x) = 1$ because $A \subseteq B$, and if $g(x) = 1$, then $\widehat{f}(x) = 1$ because B lies above A.

This proves the inequality $\mathrm{Min}(f) \ge \mathrm{Bool}(\widehat{f})$ in Eq. (1). The equality $\mathrm{Min}^*(f) = \mathrm{Min}(f)$ follows from the following claim.

Claim 3. *The constant-free version F_* solves the same problem f.*

Proof. We know that the circuit F solves a problem $F(x) = \min_{b \in B} \langle b, x\rangle + c_b$, where B is the set of vectors produced by F, and $c_b \in \mathbb{N}$ are some (not necessarily zero) scalars. We know that the circuit F solves the problem $f(x) = \min_{a \in A} \langle a, x\rangle$. So, $F(x) = f(x)$ must hold for all input weightings $x \in \mathbb{N}^n$. Let $B' = \{b \in B \colon c_b = 0\}$. Then, for every input $x \in \mathbb{N}^n$, we have that the minimum in $F(x)$ must be achieved on some vector $b \in B'$. To show this, assume contrariwise there is an input $x \in \mathbb{N}^n$ on which $F(x) = \langle b, x\rangle + c_b$ holds for some $b \in B$ with $c_b > 0$. By Claim 1, we know that $a \le b$ must hold for some vector $a \in A$. But then $f(x) \le \langle a, x\rangle \le \langle b, x\rangle < \langle b, x\rangle + c_b = F(x)$, a contradiction with $f(x) = F(x)$. So, for all inputs $x \in \mathbb{N}^n$, we have $F(x) = \min_{b \in B'} \langle b, x\rangle = f(x)$. By Claim 2, we know that the constant-free version F_* of F produces all the vectors of the set B and, in particular, all vectors of $B' \subseteq B$. So, the circuit F_* also solves our problem f. □

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