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More on Size and Width in QBF Resolution

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Abstract

In their influential paper 'Short proofs are narrow – resolution made simple' [3], Ben-Sasson and Wigderson introduced a crucial tool for proving lower bounds on the lengths of proofs in the resolution calculus. Over a decade later their technique for showing lower bounds on the size of proofs, by examining the width of all possible proofs, remains one of the most effective lower bound techniques in propositional proof complexity.

We continue the investigation begun in [5] into the application of this technique to proof systems for quantified Boolean formulas. We demonstrate a relationship between the size of proofs in level-ordered Q-Resolution and the width of proofs in Q-Resolution. In general, however, the picture is not positive, and for most stronger systems based on Q-Resolution, the size-width relation of [3] fails, answering an open question from [5].

1. Introduction

Proof complexity aims to understand the strength and limitations of various systems of logic. In particular, we seek upper and lower bounds on the size of proofs, and to develop general methods for finding such bounds. Resolution is a refutational system for propositional logic, with close connections to modern SAT solvers [7]. An important tool for proving lower bounds on the length of Resolution refutations was introduced in [3]. Ben-Sasson and Wigderson showed that whenever a short resolution refutation exists, a narrow refutation can be constructed from it; so conversely if every refutation of some family of formulas must contain a clause of large width, then no small refutation can exist.

The authors of [5] began the study of possible relationships between size, width and space of refutations in the context of resolution-based proof systems for quantified Boolean formulas (QBF). Understanding which lower bound techniques are effective for QBF is of great importance (cf. [4, 6]); however, the findings of [5] show that size-width relations in the spirit of [3] fail in Q-Resolution, both tree-like and DAG-like. This was shown by presenting a specific class of formulas with short proofs, but requiring large width (even when just counting existential variables).

This investigation is continued here by considering three additional QBF proof systems: level-ordered Q-Resolution, universal Q-Resolution (QU-Res), and long-distance Q-Resolution (LDQ-Res). While QU-Res is a natural counterpart to propositional Resolution, level-ordered and long-distance Q-Resolution are motivated by their connections to QBF solving [9, 10, 13].

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Implicit assumptions underlying Ben-Sasson and Wigderson's argument break down in the context of Q-Resolution due to the restrictions imposed by the quantifier prefix. We show that the original argument of [3] can be lifted to QBF in level-ordered Q-Resolution and relate the proof size in that system to the width of Q-Resolution refutations. In contrast, we lift the negative results of [5] to the stronger systems of QU-Res and LDQ-Res, thus answering a question of [5].

2. Preliminaries

2.1. *Quantified Boolean formulas*

Quantified Boolean logic is an extension of propositional logic in which variables may be universally as well as existentially quantified. We consider quantified Boolean formulas (QBFs) in closed prenex conjunctive normal form, denoted $\Phi = \mathcal{Q}\phi$. In the quantifier prefix, $\mathcal{Q} = \mathcal{Q}_1 X_1 \dots \mathcal{Q}_m X_m$, the X_i are disjoint sets of variables, and $Q_i \in \{\forall, \exists\}$. The matrix ϕ is a formula in conjunctive normal form over the variables in $\bigcup_{i=1}^{m} X_i$. A variable $x \in X_i$ is at quantification level i, written $\mathrm{lv}(x) = i$. x is existentially quantified if $\mathcal{Q}_i = \exists$, universally quantified otherwise. If C is a clause in ϕ then $\mathrm{var}(C)$ is the set of variables appearing in C.

$\Phi[x/a]$ for $a \in \{0, 1\}$ is the result of setting $x = a$ throughout ϕ and removing x from $\mathcal{Q}$, so $\Phi[x/1]$ removes all clauses from ϕ that contain x as a positive literal, and removes $\neg x$ from the clauses that contain the negative literal.

Semantically, $\forall u\ \Phi = \Phi[u/0] \wedge \Phi[u/1]$ and $\exists x\ \Phi = \Phi[x/0] \vee \Phi[x/1]$.

2.2. *Refutations of false QBFs*

A refutation π of QBF Φ derives the empty clause by application of the derivation rules. Each line of π is a clause that either appears in the input formula Φ, or is the result of applying a derivation rule to one or two clauses that already appear at an earlier line of π. The final line is the empty clause. A refutation induces a directed acyclic graph (DAG) with each internal node of the DAG associated with a clause, and edges directed from the parent(s) to the child of a single proof step. If the induced graph is a tree then the refutation is tree-like. In this case, each derived clause is only used once.

$\pi[x/a]$ for $a \in \{0, 1\}$ is the result of substituting a for x in every clause in π. This substitution may cause some clauses to be satisfied, in which case they cannot be used in future derivations. Therefore it cannot immediately be assumed that the result of this restriction is a valid proof.

2.3. *Resolution proof systems*

Search-based solving for SAT is based on the DPLL procedure [8], often augmented with clause learning. The Resolution proof system [14] is a refutational proof system acting on propositional formulas in conjunctive normal form. Refutations of false formulas generated in search-based SAT solvers can be understood as Resolution proofs. The proof system has a single inference rule deriving $C \vee D$ from $C \vee x$ and $D \vee \neg x$ where C and D are clauses and x (the pivot) is a variable, and for all variables $y \neq x$ that appear in C, the negation of y does not appear in D.

Resolution is extended to act on QBFs in prenex conjunctive normal form by the addition of the universal reduction rule, which derives C from $C \vee x$ when x

is universally quantified and is at the highest (inner-most) quantifier level of all variables appearing in C. This proof system is known as QU-Resolution (QU-Res) [15]. In Q-Resolution [12], the pivot in the resolution rule is restricted to existentially quantified variables.

Since search-based SAT solvers have proved successful, it is natural that this approach has been extended to QBF, for example in the solver depQBF [13]. The basic search procedure for QBF must assign variables according to the order of the quantifier prefix, starting from the outermost block. Therefore, as DPLL corresponds to tree-like Resolution, so QDPLL corresponds to tree-like level-ordered Q-Resolution.

Definition 1 ([11])**.** *Let π be a Q-Resolution refutation of a QBF. We say that π is level-ordered if and only if the following holds: Let $x \vee C_1$ and $\neg x \vee C_2$ be some clauses resolved in π. Then $lv(y) \leq lv(x)$ for any existential variable $y \in var(C_1 \vee C_2)$.*

We may assume without loss of generality that all proof steps with pivot at level i in the quantifier prefix must be carried out before any proof steps with pivot at level $j < i$.

Long-distance Resolution (LDQ-Res) [1] allows universal literals u and $\neg u$ to appear in the two clauses being resolved together provided that the resolution variable x is at a lower quantification level than u. The opposing literals are merged to form the special universal literal u^*. Formally,

$$\frac{C_1 \vee U_1 \vee x \qquad C_2 \vee U_2 \vee \neg x}{C_1 \vee C_2 \vee U}$$

where x is existentially quantified. C_1 and C_2 must not contain any complementary literals or special universal literals. U_1 and U_2 contain only universal literals appearing later in the quantifier prefix than the pivot x, every literal in U_1 is a special universal literal, or has its complement in U_2 (and vice versa). Then $U = \{u^* | u \in var(U_1)\}$. The literal u^* can be $\forall$-reduced in the same way as any other universal literal.

Note that while QU-Resolution and LDQ-Res are exponentially separated from Q-Resolution [15, 9], it is easy to see that they do not reduce proof size when the proof is required to be level-ordered. It has been shown that even with clause learning and other heuristics, it is impossible for a search-based QBF solver to produce a refutation that is not level-ordered on some input formulas [10].

2.4. Size and width

The size of a proof π is written $|\pi|$ and is the number of clauses in π (equivalently, the number of nodes in the associated tree or DAG). The size of deriving a clause C from Φ (in proof system P), denoted $S_P(\Phi \vdash C)$, is the minimum size of any P-proof of C from Φ. We drop the subscripts indicating the proof system under consideration if it is already clear from the context.

The width $w(C)$ of a clause C is the number of existential variables it contains. The width $w(\Phi)$ of a QBF Φ is the maximum width of a clause in Φ. Similarly, the width $w(\pi)$ of derivation π is the maximum width of any clause contained in π. The width of deriving a clause C from Φ (in proof system P), denoted $w_P(\Phi \vdash C)$, is the minimum width of any P-proof of C from Φ.

For tree-like propositional Resolution, Ben-Sasson and Wigderson [3] showed that $w(\phi \vdash \bot) \leq w(\phi) + lg(S(\phi \vdash \bot))$, and a similar relation holds for DAG-like Resolution, $w(\phi \vdash \bot) \leq w(\phi) + O(\sqrt{n \ln S(\phi)})$.

3. Negative results

We revisit the counterexample for the size-width relation in tree-like Q-Resolution from [5].

Proposition 2 ([5])**.** *There is a family of false QBF sentences Φ_n over $O(n^2)$ variables, such that $w(\Phi_n) = 3$ and in tree-like Q-Resolution $S(\Phi_n \vdash \bot) = n^{O(1)}$, and $w(\Phi_n \vdash \bot) = \Omega(n)$.*

To prove this proposition, the following QBFs, introduced in [11], are used.

$$\Phi_n = \exists x_{1,1} \dots x_{1,n} \dots x_{n,n} \forall z \exists a_1 \dots a_n, b_1 \dots b_n, y_0 \dots y_n, p_0 \dots p_n$$
$$\bigwedge_{i,j=1}^{n} (x_{i,j} \vee z \vee a_i) \wedge \bigwedge_{i,j=1}^{n} (\neg x_{i,j} \vee \neg z \vee b_j) \tag{1}$$
$$\wedge \neg y_0 \wedge \bigwedge_{i=1}^{n} (y_{i-1} \vee \neg a_i \vee \neg y_i) \wedge y_n \tag{2}$$
$$\wedge \neg p_0 \wedge \bigwedge_{j=1}^{n} (p_{j-1} \vee \neg b_j \vee \neg p_j) \wedge p_n \tag{3}$$

There are $O(n^2)$-size tree-like Q-Resolution refutations:

1. Collapse the clauses in (2) and (3) to $\bigvee_{i=1}^{n} \neg a_i$ and $\bigvee_{j=1}^{n} \neg b_j$ ($O(n)$ steps).
2. Resolve $\bigvee_{i=1}^{n} \neg a_i$ with $(x_{i,j} \vee z \vee a_i)$ for fixed j and i ranging from 1 to n. Then $\forall$-reduce z, giving $\bigwedge_{i=1}^{n} x_{i,j}$ ($O(n)$ steps for each j).
3. Resolve $\bigwedge_{i=1}^{n} x_{i,j}$ with $(\neg x_{i,j} \vee \neg z \vee b_j)$ for fixed j and $i = 1 \dots n$, now we have $\bigwedge_{j=1}^{n} (b_j \vee \neg z)$ ($O(n)$ steps for each j).
4. Resolve all $(b_j \vee \neg z)$ with $\bigvee_{j=1}^{n} \neg b_j$ and $\forall$-red $\neg z$ to reach the empty clause ($O(n)$ steps).

Then we show that any valid Q-Resolution refutation must be wide, by arguing that the clause immediately following the first $\forall$-red step in any refutation of Φ_n must have width $\Omega(n)$. We do not repeat the full argument here, the idea is that in order to perform $\forall$-red on z (w.l.o.g.), some a_i must be removed from a clause $(x_{i,j} \vee z \vee a_i)$. Doing so necessarily introduces another a literal negatively (perhaps via the intermediate introduction of a y literal), and to remove this introduces another (different) $x_{i,j}$, which must remain until after the $\forall$-red, as well as another positive a literal. This repeats and ensures that n different $x_{i,j}$ literals must collect in the clause before it is free of a_i and y_i literals and the $\forall$-red can be performed.

It was left open in [5] whether the size-width relation holds for extensions of Q-Resolution. We begin by demonstrating that simple modifications of this example, inspired by [2], show that the result also fails for the tree-like versions of QU-Res and LDQ-Res.

Proposition 3. *There is a family of false QBF sentences Φ'_n over $O(n^2)$ variables, such that $w(\Phi'_n) = 3$ and in tree-like QU-Resolution $S(\Phi'_n \vdash \bot) = n^{O(1)}$, and $w(\Phi'_n \vdash \bot) = \Omega(n)$.*

Proof. Modify Φ_n to Φ'_n by adding another universal variable z' at the same level as z. Replace (1) with $\bigwedge_{i,j=1}^{n} (x_{i,j} \vee z \vee z' \vee a_i) \wedge \bigwedge_{i,j=1}^{n} (\neg x_{i,j} \vee \neg z \vee \neg z' \vee b_j)$.

The size $O(n^2)$ refutation of Φ_n is trivially extended to a refutation of Φ'_n by performing a $\forall$-reduction step on z' immediately after any $\forall$-reduction step on z. It is simple to confirm that the proof remains valid. In particular, any resolution step that could be blocked by z' in Φ'_n would already have been blocked by z in Φ.

The duplication of the universal variable also ensures that universal resolution cannot result in narrower proofs compared to Q-Resolution. This is simply because we have ensured that every universal literal in the input formula has its complement only in clauses which also conflict on another variable. In addition, any derived clause must contain all of the universal variables from its parents unless it is derived by $\forall$-reduction or universal resolution. Therefore universal resolution is forbidden until some $\forall$-reduction step has occurred, until this point we may only use existential resolution. The argument sketched above therefore readily applies to show that the clause immediately following the first $\forall$-red step in any refutation of Φ_n must have width $\Omega(n)$. $\square$

The idea of duplicating universal variables can be applied to any formula to prevent universal resolution steps from being possible. In particular, since we know of a family of formulas with long refutations and small width in Q-Resolution, we can also construct similar formulas with long refutations and small width in QU-Resolution.

Proposition 4. *There is a family of false QBF sentences Φ''_n over $O(n^2)$ variables, such that $w(\Phi''_n) = 3$ and in tree-like LDQ-Resolution $S(\Phi''_n \vdash \bot) = n^{O(1)}$, and $w(\Phi''_n \vdash \bot) = \Omega(n)$.*

Proof. In this case Φ_n is modified by replacing lines (2) and (3) with $\neg y_0 \wedge \bigwedge_{i=1}^{n} (y_{i-1} \vee \neg a_i \vee \neg y_i \vee z) \wedge y_n$ and $\neg p_0 \wedge \bigwedge_{j=1}^{n} (p_{j-1} \vee \neg b_j \vee \neg p_j \vee \neg z) \wedge p_n$ respectively. This does not affect satisfiability since these clauses are only relevant under one or other of the assignments to z. The same $O(n^2)$ refutation of the original formula applies to give the size upper bound.

For the width lower bound note that long-distance steps are impossible in refuting this formula. The only long-distance steps that could be performed are on some $x_{i,j}$ variable prior to any $\forall$-reduction in the clauses involved. Then the parent clauses contain some a, b, y, or p literal, as well as the special universal z^*. In order for the derived clause to form part of the refutation, a_i (without loss of generality) would need to be removed via resolution at some later point. This is now impossible. The only input clause that contains $\neg a_i$ also contains z, and any derived clause containing $\neg a_i$ must also contain z because $\forall$-reduction is blocked by $\neg a_i$. None of these clauses can be resolved with a clause containing z^*, so no long-distance resolution step can apply before at least one $\forall$-red step, until then we are restricted to standard resolution steps, and so again the clause following the first $\forall$-red must have width $\Omega(n)$. $\square$

4. Relating size and width in tree-like level-ordered Q-Resolution

Suppose we have a Resolution refutation of some propositional formula ϕ. The final step in the proof resolves x and $\neg x$. So we also have a derivation from ϕ to x (and also $\neg x$), that is, ϕ implies x ($\neg x$). A crucial part of Ben-Sasson and Wigderson's argument in [3] rests on the observation that this derivation of x can be easily transformed into a refutation of $\phi[x/0]$, by simply applying the assignment $x = 0$ to every clause in the derivation.

This does not hold in general in Q-Resolution, even in the tree-like case. Indeed, it is possible to have a Q-Resolution derivation from Φ to u, a universal variable not at the outermost level in the prefix, where $\Phi[u/0]$ is not even false. The restricted derivation only remains valid if no clause has been satisfied by the assignment. Therefore, if we have a derivation of x, and so wish to restrict by $x = 0$, we must know that $\neg x$ does not appear in any clause in the derivation. We show that this property holds for tree-like level-ordered Q-Resolution proofs.

Definition 5 (Regularity). *A refutation π of Φ is regular if along any path from root to leaf no two nodes are associated with the same variable.*

Lemma 6. *Any tree-like level-ordered Q-Resolution refutation may be assumed to be regular.*

Proof. Each path must remove variables in the order of the prefix, so a contiguous (possibly empty) section of the path relates to each level. We show that any repeated pivot within such a section of the path can be removed without increasing the size of the proof.

For a section corresponding to a universal quantifier block, a series of $\forall$-reduction steps are applied successively to a single input clause, which cannot contain both u and $\neg u$, so there can be no $\forall$-reduction steps on opposing literals in a single branch. Since the only condition for carrying out $\forall$-reduction is that no literal is in the clause from a higher quantification level, the $\forall$-reduction steps in this block can be reordered and repetitions collapsed to a single step.

For resolution steps, consider two consecutive resolution steps on a branch with pivot x (i.e., between these two steps all have pivot different to x). First (closest to the leaves in the proof tree), $A \vee x$ and $B \vee \neg x$ are resolved to give $A \vee B$ which does not contain variable x. Continuing along this branch, x is reintroduced during some other resolution step. Without loss of generality assume x is reintroduced positively, and later side clause $E \vee \neg x$ is resolved with clause $D \vee x$ from our branch. Between the two resolution steps it is not possible that x appeared negatively in any clause, else some clause is a tautology or the two steps were not consecutive resolutions on x.

Remove the first resolution step and instead of deriving $A \vee B$ continue with $A \vee x$. Proceed down the branch, at each node associate a new clause C' in place of C, and a set $\mathcal{B}_C$ to be the set of literals in C but not in C'. In place of $C_1 = A \vee B$ we have $C'_1 = A \vee x$, and $\mathcal{B}_{C_1} = B \setminus A$. For C_k the result of resolving C_i and side clause C_j on pivot y, if $y \in \mathcal{B}_{C_i}$ then $C'_k = C'_i$, cut out the resolution step and the branch producing C_j, set $\mathcal{B}_{C_k} = \mathcal{B}_{C_i} \cup C_j \setminus C_i$. Otherwise, $y \in C'_i$ and $\neg y \in C_j$. By assumption C_j does not contain $\neg x$. Therefore C'_k is the valid result of resolving C'_i and C_j on y and $\mathcal{B}_{C_k} = \mathcal{B}_{C_i}$. For all nodes, $\neg x \notin C'$, $C' \setminus x$ subsumes C, and from the point at which x was reintroduced to the branch in the original refutation, the new clause properly subsumes the old.

Any consecutive nodes with identical associated clauses can be merged, and we have removed at least one resolution step to construct a new refutation no larger than the original. The new refutation remains tree like and level ordered. $\square$

Lemma 7. *Let π be a regular refutation of Φ. If the final step of π is resolution on x then $\pi[x/0]$ is a refutation of $\Phi[x/0]$ and $\pi[x/1]$ is a refutation of $\Phi[x/1]$. If the final step of π is $\forall$-reduction on x then either $\pi[x/0]$ is a refutation of $\Phi[x/0]$ or $\pi[x/1]$ is a refutation of $\Phi[x/1]$.*

Proof. In all cases consider an input to the final proof step. This will be a clause consisting of a single literal: without loss of generality, the clause $\{x\}$. In $\pi[x/0]$ this clause becomes the empty clause. Every proof step in the restricted proof remains valid unless it is a $\forall$-reduction on x (impossible by assumption of regularity), or (one of) the input(s) to the proof step has been satisfied in the restriction. If a clause in π is satisfied by the restriction then that clause originally contained $\neg x$. Since the final clause did not contain $\neg x$ this literal must have been removed from a clause in an intermediate step, but this contradicts the assumption of regularity. All input clauses to $\pi[x/0]$ therefore belong to $\Phi[x/0]$, and $\pi[x/0]$ is a valid refutation of $\Phi[x/0]$. $\square$

Lemma 8. *Let Φ be a QBF, $\mathcal{C}$ a clause, and x an existentially quantified variable in the outermost quantifier block of Φ. In Q-Resolution, if $w(\Phi[x/0] \vdash \mathcal{C}) \leq k$ then $w(\Phi \vdash \mathcal{C} \vee x) \leq k+1$.*

Proof. Let π be a width k Q-Resolution derivation of $\mathcal{C}$ from $\Phi[x/0]$. Add x into every clause of π. We claim that the result is a valid derivation of $\mathcal{C} \vee x$.

For C an initial clause in $\Phi[x/0]$, $C \vee x$ is either an initial clause of Φ or can be obtained from an initial clause by weakening. For C in π and not an initial clause, either C is the result of a resolution step or a universal reduction step.

If C is the result of resolving A and B then $C \vee x$ can be derived from $A \vee x$ and $B \vee x$. Neither A nor B can contain $\neg x$ so the resolution step is not blocked.

If C is the result of a universal reduction from A then $C \vee x$ is the result of universal reduction from $A \vee x$. Since x is in the outermost quantifier block it cannot block any universal reduction step because every universal variable is at a higher quantification level.

The width of every clause in the derivation is increased by 1, and so the width of the whole proof is increased by 1. $\square$

Similarly, if $w(\Phi[x/1] \vdash \mathcal{C}) \leq k$ then $w(\Phi \vdash \mathcal{C} \vee \neg x) \leq k+1$. Also in the following, all $[x/0]$ may be swapped for $[x/1]$ and vice versa.

Lemma 9. *Let Φ be a QBF and x an existentially quantified variable in the outermost quantifier block of Φ. In Q-Resolution, if $w(\Phi[x/0] \vdash \bot) \leq k-1$ and $w(\Phi[x/1] \vdash \bot) \leq k$ then $w(\Phi \vdash \bot) \leq \max\{k, w(\Phi)\}$. For x universally quantified in the outermost quantifier block, if $w(\Phi[x/0] \vdash \bot) \leq k$ then $w(\Phi \vdash \bot) \leq k$.*

Proof. If x is existentially quantified then since $w(\Phi[x/0] \vdash \bot) \leq k-1$ we have also that $w(\Phi \vdash x) \leq k$ by Lemma 8. Resolve x with every clause in Φ containing $\neg x$, the resulting collection of clauses are exactly those in the matrix of $\Phi[x/1]$, and from these we can derive $\bot$ in width k. The total width of the derivation is $\max\{k, w(\Phi)\}$.

If x is universally quantified then $\{x\}$ can be derived in width k and $\forall$-reduction derives $\bot$. The width of the derivation is k since universally quantified variables do not contribute to the width of a clause. $\square$

We can now state the relation between width in Q-Resolution and size in level-ordered Q-Resolution.

Theorem 10. *$w_Q(\Phi \vdash \bot) \leq w(\Phi) + \lceil \lg S_L(\Phi \vdash \bot) \rceil$, where Q is Q-Resolution and L is level-ordered tree-like Q-Resolution.*

Proof. We begin with a level-ordered refutation π of Φ. Let $b = \lceil \lg S_L(\Phi \vdash \bot) \rceil$ so that $S_L(\Phi \vdash \bot) \leq 2^b$. If $b = 0$ then the empty clause is in Φ, so both $w(\Phi)$ and $w_Q(\Phi \vdash \bot)$ are 0 and we are done.

Otherwise the last step of the proof may be a universal reduction $\frac{x}{\bot}$ or a resolution step $\frac{x \quad \neg x}{\bot}$ where x is in the outermost quantifier block.

In the case of universal reduction, consider π_x, the derivation of x. $\pi_x[x/0]$ is a level-ordered refutation of $\Phi[x/0]$ of size S_x. By induction on the number of variables in Φ we have that $w_Q(\Phi[x/0] \vdash \bot) \leq w(\Phi[x/0]) + \lceil \lg S_x \rceil$. By Lemma 9, $\Phi \vdash \bot$ has the same width as the restricted proof, and $w(\Phi[x/0]) = w(\Phi)$, so the result follows.

In the case of resolution being the last step, consider π_x and $\pi_{\neg x}$, the level-ordered derivations of x and $\neg x$, of sizes S_x and $S_{\neg x}$, respectively. $\pi_x[x/0]$ is a level-ordered refutation of $\Phi[x/0]$ of size S_x. $S_L(\Phi \vdash \bot) = S_x + S_{\neg x} + 1$. Without loss of generality, $S_x \leq 2^{b-1}$ so by induction on b, there is a (possibly not level-ordered) proof with $w_Q(\Phi[x/0] \vdash \bot) \leq w(\Phi[x/0]) + b - 1 \leq w(\Phi) + b - 1$, and by induction on the number of variables in Φ, $w_Q(\Phi[x/1] \vdash \bot) \leq w(\Phi[x/1]) + b \leq w(\Phi) + b$. x is outermost so by Lemma 9 we can use these two refutations to construct a refutation of Φ with width at most $w(\Phi) + b$ and the result follows. $\square$

In the proof of Theorem 10, we begin with a small level-ordered proof and construct another proof from it which has small width. However, during the construction, the proof loses the level-ordered property. It is not in general possible to construct a level-ordered proof with small width, as demonstrated by the following counter-example

$$\Phi = \exists x_1 \dots x_n \forall z \exists a_1 \dots a_n, y_0 \dots y_n$$
$$(\neg y_0) \wedge (y_n) \wedge \bigwedge_{i \in [n]} (\neg x_i) \wedge (z \vee a_i) \wedge (y_{i-1} \vee \neg a_i \vee x_i \vee \neg y_i)\,.$$

All clauses are needed to refute Φ. Any level-ordered proof must carry out all resolution steps on y_i variables before resolving on x_i variables, and it is simple to verify that doing so must result in a clause that contains all x_i variables. There is a short tree-like level-ordered refutation which collapses $(y_{i-1} \vee \neg a_i \vee x_i \vee \neg y_i)$ together to $(\neg a_1 \vee \ldots \vee \neg a_n \vee x_1 \vee \ldots \vee x_n)$, then resolves this with all $(z \vee a_i)$, removes z and finally refutes $\bigwedge_{i \in [n]} (\neg x_i) \wedge (x_1 \vee \ldots \vee x_n)$, all of which takes linear size.

Additionally, Theorem 10 cannot be lifted to DAG-like level-ordered Q-Resolution since for the counter-example given in [5] for DAG-like Q-Resolution, the short proof discussed in [5] is level-ordered and directly applies here. A crucial part of the argument in the propositional case is to carefully select the next variable to use in restricting the refutation, but it is not possible in general to ensure that this variable belongs to a particular level of the prefix.

5. Conclusion

We have demonstrated that the result of [3] can be lifted to relate two variants of Q-Resolution, highlighting an interesting relationship between level-ordered and non level-ordered proofs in Q-Resolution. Level ordered Q-Resolution is important since it corresponds to the QDPLL algorithm that underlies some modern QBF solving algorithms, so a mechanism to lower bound the size of proofs is useful in understanding the strength of search-based QBF solvers. Removing either the restriction that the proof must be level-ordered, or the restriction that it must be tree like, is sufficient to lose the desired behaviour. We have also answered the open question from [5] regarding extensions of Q-Resolution, by demonstrating how the counterexamples may be lifted to these stronger calculi.

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