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Average-case linear matrix factorization and reconstruction of low width Algebraic Branching Programs

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Abstract

Let us call a matrix X as a *linear matrix* if its entries are affine forms, i.e. degree one polynomials. What is a minimal-sized representation of a given matrix F as a product of linear matrices? Finding such a minimal representation is closely related to finding an optimal way to compute a given polynomial via an algebraic branching program. Here we devise an efficient algorithm for an average-case version of this problem. Specifically, given $w, d, n \in \mathbb{N}$ and blackbox access to the w^2 entries of a matrix product $F = X_1 \cdots X_d$, where each X_i is a $w \times w$ linear matrix over a given finite field $\mathbb{F}_q$, we wish to recover a factorization $F = Y_1 \cdots Y_{d'}$, where every Y_i is also a linear matrix over $\mathbb{F}_q$ (or a small extension of $\mathbb{F}_q$). We show that when the input F is sampled from a distribution defined by choosing random linear matrices $X_1, \ldots, X_d$ over $\mathbb{F}_q$ independently and taking their product and $n \geq 4w^2$ and the characteristic of $\mathbb{F}_q$ is at least $(ndw)^{\Omega(1)}$ then an equivalent factorization $F = Y_1 \cdots Y_d$ can be recovered in (randomized) time $(wdn \log q)^{O(1)}$. We also show that in this situation, if we are instead given a single entry of F rather than its w^2 correlated entries then the recovery can be done in (randomized) time $(d^{w^3} n \log q)^{O(1)}$.

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1 Introduction

Factorization of a given numeric matrix is an important problem in numerical linear algebra. The problem comes in various flavors, from the basic LU decomposition to singular value decomposition and non-negative matrix factorization, with wide ranging applications in signal processing, machine learning, computer vision and several other fields. In this paper, we are interested in factorization of a matrix with *polynomial entries* into linear matrices, if such a factorization exists. It is a natural generalization of factoring polynomials into linear factors. Our primary motivation for studying this matrix factorization problem comes from its close connection to reconstruction or learning of algebraic branching programs (ABPs) – a subclass of arithmetic circuits capturing determinant and iterated matrix multiplication computations. Circuit reconstruction is a notable problem in algebraic complexity theory, but unfortunately, it is hard even for simple circuit models, like set-multilinear depth-3 circuits or *tensors* [Hås90, Shi16]. Research on reconstruction has focused on restricted models (see the survey [SY10] and the references in [KNST17]), and on the average-case complexity of this problem. In [GKQ13], an average-case reconstruction algorithm was given for formulas under an intuitive input distribution. Algebraic branching programs being more powerful than formulas, the problem of efficient average-case reconstruction of ABPs is posed in [KNST17] [1] under a natural distribution. Our work fits in this agenda of average-case ABP reconstruction. Furthermore, there is a deeper reason for studying this problem which comes from the recent study of limitations of certain lower bound methods and pseudorandom polynomial families. We discuss this motivation in some details in order to put our result into perspective.

In the ensuing discussion, n-variate polynomials of degree at most d over a field $\mathbb{F}$ are referred to as (n, d)-polynomials, and arithmetic circuits as circuits. We will assume that $\mathbb{F}$ is a sufficiently large finite field, although this requirement is not necessary for the most part of the arguments.

1.1 Lower bound proofs and pseudorandom polynomials

It has been observed in some recent works [FSV17, GKSS17, Gro14, Aar08] that many known circuit lower bound proofs have the following features: Such a proof, for an circuit class $\mathcal{C}$, gives a non-zero *distinguisher* polynomial $D_\mathcal{C}$ in $N = \binom{n+d}{d}$ variables that are supposed to take values as the coefficients of a (n, d)-polynomial. Polynomial $D_\mathcal{C}$ is computable by a circuit of size $N^{O(1)}$, and has the property that it is zero when evaluated at the coefficients of a (n, d)-polynomial computed by a circuit in $\mathcal{C}$. Since, $D_\mathcal{C}$ is nonzero (with *high* probability) when evaluated at a random (n, d)-polynomial[2] [Sch80, Zip79], such a $D_\mathcal{C}$ *efficiently* distinguishes the coefficients of a random polynomial from the coefficients of a polynomial computed by a circuit in $\mathcal{C}$. These features resemble the *usefulness*, *largeness* and *constructivity* attributes of natural proofs [RR97] for Boolean circuits – and in the same vein, the above kind of lower bound proofs for arithmetic circuits are called *natural* lower bound proofs/methods. As observed in [FSV17, GKSS17], if the coefficient vectors of (n, d)-polynomials computed by circuits in $\mathcal{C}$ form a hitting set for $N^{O(1)}$ size circuits on N inputs then natural lower bound methods will not work for $\mathcal{C}$ [3]. To make our discussion

[1] It is worth noting that an average-case reconstruction algorithm for ABPs does not necessarily subsume a result on average-case reconstruction of formulas as the distributions of the inputs may be incomparable.

[2] The coefficients of a random $(n.d)$-polynomial are chosen independently and uniformly at random from $\mathbb{F}$.

[3] In a recent interesting development [EGdOW18], limitations of rank based lower bound methods have been shown *unconditionally* towards achieving *strong* lower bounds for set-multilinear depth-3 circuits and diagonal depth-3 circuits.

precise, let us take the example of arithmetic formulas.

Suppose $\mathcal{C}_n$ is the set of formulas on n $\mathbf{x}$-variables of size at most $s(n) = n^{10}$, and $\mathcal{C} = \cup_{n\geq 1}\mathcal{C}_n$. Consider the task of showing the existence of an explicit family $\{p_n\}_{n\geq 1}$, where p_n is an (n,d)-polynomial and $d = n^5$, such that p_n cannot be computed by formulas in $\mathcal{C}_n$, for every sufficiently large n. From [BC92], if f is computed by a formula in $\mathcal{C}_n$ then it is also computed by a width-3 ABP $X_1 \cdots X_{\ell(n)}$ of *length* $\ell(n)$, where $\ell(n) = s(n)^{O(1)}$ (say, $\ell(n) = s(n)^{10} = n^{100}$). For brevity, we will refer to $s(n), \ell(n)$ as s, ℓ respectively. Consider a *formal* width-3 ABP $Z_1 \cdots Z_\ell$ of length-ℓ on $\mathbf{x}$-variables, where the coefficients of the affine forms in the $\mathbf{x}$-variables are distinct $\mathbf{y}$-variables. The number of $\mathbf{y}$-variables is $m = \Theta(n\ell)$. The ABP $X_1 \cdots X_\ell$ is an instantiation of the formal ABP $Z_1 \cdots Z_\ell$ obtained by assigning field values to the $\mathbf{y}$-variables. Let $g_\ell(\mathbf{x}, \mathbf{y})$ be the polynomial computed by this formal length-ℓ ABP, and $g_\ell^{\leq d}(\mathbf{x}, \mathbf{y})$ the sum of those monomial in g_ℓ that have $\mathbf{x}$-degree at most d. As g_ℓ is computable by a formula on $\mathbf{x}$ and $\mathbf{y}$ variables of size $(n\ell)^{O(1)}$[4], a simple interpolation argument[5] shows that $g_\ell^{\leq d}$ is also computable by a formula $\mathtt{U}_n$ on $\mathbf{x}$ and $\mathbf{y}$ variables of size $(nd\ell)^{O(1)} = n^{O(1)}$. The formula $\mathtt{U}_n$ is *universal* for $\mathcal{C}_n$ in the following sense: If f is computed by a formula in $\mathcal{C}_n$ then there is an assignment $\mathbf{a} \in \mathbb{F}^m$ to the $\mathbf{y}$-variables such that $f = g_\ell^{\leq d}(\mathbf{x}, \mathbf{a})$; moreover, size of $\mathtt{U}_n$ is $n^{O(1)}$. The quantity ℓ captures the price in size we pay in going from $\mathcal{C}$ to the universal formulas $\{\mathtt{U}_n\}_{n\geq 1}$ computing the polynomial family $\{g_\ell^{\leq d}\}_{n\geq 1}$. Observe that the coefficient vector of $g_\ell^{\leq d}$, when treated as a polynomial in $\mathbf{x}$, lies in $\mathbb{F}[\mathbf{y}]^N$. If the coefficient vectors of (n,d)-polynomials computable by formulas in $\mathcal{C}$ form a hitting set for size $N^{O(1)}$ circuits on N variables then the coefficient vectors of polynomials in the set $G^{\leq d} \stackrel{\text{def}}{=} \{g_\ell^{\leq d}(\mathbf{x}, \mathbf{b}) : \mathbf{b} \in \mathbb{F}^m\}$ must also form a hitting set for the same kind of circuits.

Thus, for natural lower bound methods to fail for $\mathcal{C}$, the coefficient vectors coming from $G^{\leq d}$ must essentially 'appear random' to size $N^{O(1)}$ circuits. We might like to know at first if the coefficient vectors coming from $G^{\leq d}$ appear random to $N^{O(1)} = \exp(n)$ time *algorithms* – which leads us to the notion of pseudorandom polynomial families. A family of (n,d)-polynomials $H = \{h(\mathbf{x}, \mathbf{b}) : \mathbf{b} \in \mathbb{F}^m\}$, where $m = n^{O(1)} >> n$ and $d = n^{O(1)}$, is *pseudorandom* if $h(\mathbf{x}, \mathbf{y})$ is computable by a $n^{O(1)}$-size circuit and any algorithm that distinguishes a coefficient vector of a polynomial in H (picked according to the distribution defined by $\mathbf{b} \in_r \mathbb{F}^m$) from that of a random (n,d)-polynomial, with noticeable probability, must take time $\exp(m^{\Omega(1)}) >> \exp(n)$. In other words, if $G^{\leq d}$ is a pseudorandom family then natural lower bound methods will not work for $\mathcal{C}$. A 'weaker question' is, whether or not the family of (n,ℓ)-polynomials $G \stackrel{\text{def}}{=} \{g_\ell(\mathbf{x}, \mathbf{b}) : \mathbf{b} \in \mathbb{F}^m\}$ is pseudorandom[6]. The answer to this, as pointed out in one of the remarks after Theorem 2, is negative whenever $n = \omega(1)$. However, one can raise a similar question starting from $\mathcal{C}$ consisting of ABPs instead of formulas, and this brings us to the candidate family offered in [Aar08, Aar17].

Unlike the Boolean world where we have strong evidence for the existence of pseudorandom

[4]This is because a polynomial computed by a constant width ABP is easily computed by a formula, using a simple divide-and-conquer argument, with only a polynomial blowup in size. Together with [BC92], it implies the polynomial family $\{\mathsf{IMM}_{w,\ell}\}_{\ell\geq 1}$ (see Definition 1.1) is complete under p-projections for polynomial size formulas, for every constant $w \geq 3$.

[5]assuming $|\mathbb{F}| > \ell$

[6]It is weaker as G is pseudorandom implies $G^{\leq d}$ is pseudorandom.

functions [HILL99], in the arithmetic world pseudorandom polynomials are not well studied. A candidate family is offered in [Aar08, Aar17] using affine projections of the symbolic determinant. Just like $\{\mathsf{IMM}_{3,\ell}\}_{\ell \geq 1}$ is complete for poly-size formulas, $\{\mathsf{Det}_\ell\}_{\ell \geq 1}$ is complete for poly-size ABPs under p-projections [MV99], where Det_ℓ is the $\ell \times \ell$ symbolic determinant[7]. The question posed in [Aar08] is, whether or not the family $\{\mathsf{Det}_\ell(A \cdot \mathbf{x} + \mathbf{c}) : A \in \mathbb{F}^{\ell^2 \times n} \text{ and } \mathbf{c} \in \mathbb{F}^{\ell^2}\}$, where $\ell = n^{O(1)} >> n$, is pseudorandom under the distribution defined by $A \in_r \mathbb{F}^{\ell^2 \times n}$ and $\mathbf{c} \in_r \mathbb{F}^{\ell^2}$. Note the similarity between this question for ABPs and the 'weaker question' for formulas mentioned in the last paragraph. But now, the answer is far from obvious.

In the same spirit as [Aar08], and as mentioned in [KNST17], we may ask if the family $J \stackrel{\text{def}}{=} \{\mathsf{IMM}_{\ell,d}(A \cdot \mathbf{x} + \mathbf{c}) : A \in \mathbb{F}^{t \times n} \text{ and } \mathbf{c} \in \mathbb{F}^t\}$, where $t = \ell^2(d-2) + 2\ell$, $d = n^{O(1)}$ and $\ell = n^{O(1)} >> n$, is pseudorandom under the distribution defined by $A \in_r \mathbb{F}^{t \times n}$ and $\mathbf{c} \in_r \mathbb{F}^t$. The reason for considering this problem being, if f is a (n,d)-polynomial computed by size s ABP then f is also computed by a width-ℓ length-d ABP, where $\ell = s^{O(1)}$. This follows from the ABP homogenization argument in [Nis91]. In other words, the width-ℓ length-d ABP computing $\mathsf{IMM}_{\ell,d}$ is universal for size s ABPs computing (n,d)-polynomials, where $\ell = s^{O(1)}$.

Average-case ABP reconstruction (see Problem 2) is related to the above problem: If there is an efficient average-case ABP reconstruction algorithm for $\mathsf{IMM}_{\ell,d}(A \cdot \mathbf{x} + \mathbf{c})$ when A and $\mathbf{b}$ are chosen randomly from their respective domains, then J is *not* pseudorandom. In this sense, average-case reconstruction is a *harder*[8]problem than showing 'not pseudorandom'. Although, the question whether or not J is pseudorandom is primarily interesting when $\ell >> n$, both the problems – average-case reconstruction and pseudorandomness – make sense even for $\ell < n$. As noted after Theorem 2, it is easy to show that the family $\{\mathsf{IMM}_{\ell,d}(A \cdot \mathbf{x} + \mathbf{c}) : A \in \mathbb{F}^{n \times t} \text{ and } \mathbf{c} \in \mathbb{F}^t\}$ is not pseudorandom for $n = \ell^{\omega(1)}$. However, the situation for average-case reconstruction is not clear for low width ℓ. Our result fills up this gap in understanding by giving an efficient average-case reconstruction algorithm when $\ell << n^{1/3}$ (in particular, when ℓ is a constant).

We are now ready to state our problems and results more formally. In the rest of this article, the width parameter ℓ will be denoted by w, and the underlying set of n variables by $\mathbf{x} = \{x_1, \ldots, x_n\}$.

1.2 The problems

We study two related problems in this work, *average-case matrix factorization* and *average-case ABP reconstruction*. The average-case matrix factorization problem aids us in making progress on average-case ABP reconstruction (see also the remark after Problem 2). The definition of an ABP given below is quite standard and similar to the one stated in [KNST17].

Definition 1.1 (*Algebraic branching program*)**.** An *algebraic branching program* (ABP) of width w and length d is a product expression $X_1 \cdot X_2 \ldots X_d$, where X_1, X_d are row, column linear matrices over $\mathbb{F}$ of length w respectively, and X_i is a $w \times w$ linear matrix over $\mathbb{F}$ for $i \in [2, d-1]$. The polynomial

[7] In other words, there is a $\ell^{O(1)}$-size ABP computing Det_ℓ that is universal for size s ABPs, where $\ell = s^{O(1)}$.

[8] If $H = \{h(\mathbf{x}, \mathbf{b}) : \mathbf{b} \in \mathbb{F}^m\}$ is an arbitrary family of (n,d)-polynomials such that $h(\mathbf{x}, \mathbf{y})$ is computable by a size $n^{O(1)}$ circuit then it is trivial to show that $H' = \{h(\mathbf{x}, \mathbf{b})x_1 : \mathbf{b} \in \mathbb{F}^m\}$ is *not* a pseudorandom family, but it gives us no information about the existence of an efficient average-case reconstruction algorithm for H'.

computed by the ABP is the entry of the 1×1 matrix obtained from the product $\prod_{i=1}^{d} X_i$. An ABP of width w, length d, and in n variables will be called a (w, d, n)-ABP over $\mathbb{F}$.

Remarks:

(a) The *iterated matrix multiplication* polynomial ($\mathsf{IMM}_{w,d}$) is computed by a (w, d, n)-ABP where each entry in X_i is a distinct variable, for all $i \in [d]$, and hence $n = w^2(d-2) + 2w$.

(b) A polynomial computed by a (w, d, n)-ABP can be viewed as an entry of a product of d, $w \times w$ linear matrices $X_1, X_2, \ldots, X_d$. The $w \times w$ matrix $F = X_1 \cdot X_2 \ldots X_d$ is then called a (w, d, n)-*matrix product*. We note that in the matrix product formulation X_1, X_d are $w \times w$ linear matrices, while in the ABP formulation X_1, X_d are row and column linear matrices of length w respectively; hopefully, the context will make the dimensions of these matrices clear.

To study average-case reconstruction for ABP, [KNST17] defined a natural distribution on the polynomials computed by it. The distribution is expressed by a *random* (w, d, n)-ABP.

Definition 1.2 (*Random ABP and matrix product*)**.** A *random* (w, d, n)-ABP over $\mathbb{F}$ is a (w, d, n)-ABP $X_1 \cdot X_2 \ldots X_d$ over $\mathbb{F}$, where X_i is a random linear matrix chosen independently for every $i \in [d]$. Similarly, a *random* (w, d, n)-matrix product over $\mathbb{F}$ is a (w, d, n)-matrix product $F = X_1 \cdot X_2 \ldots X_d$ over $\mathbb{F}$, where X_i is a random linear matrix chosen independently for every $i \in [d]$.

Having defined the distributions, the two average-case problems can be posed as follows.

Problem 1 (*Average-case matrix factorization*)**.** Design an algorithm which when given $w, d, n \in \mathbb{N}$, and blackbox access to w^2, (n, d)-polynomials $\{f_{st}\}_{s,t \in [w]}$ that constitute the entries of a random (w, d, n)-matrix product F over $\mathbb{F}_q$, outputs d, $w \times w$ linear matrices $Y_1, \ldots, Y_d$ over $\mathbb{F}_q$ (or a small extension of $\mathbb{F}_q$) such that $F = Y_1 \cdot Y_2 \ldots Y_d$, with high probability[9]. The desired running time of the algorithm is $(wdn \log q)^{O(1)}$.

Problem 2 (*Average-case ABP reconstruction*)**.** Design an algorithm which when given $w, d, n \in \mathbb{N}$, and blackbox access to a (n, d)-polynomial f computed by a random (w, d, n)-ABP over $\mathbb{F}_q$, outputs a (w, d, n)-ABP over $\mathbb{F}_q$ (or a small extension of $\mathbb{F}_q$) computing f, with high probability. The desired running time of the algorithm is $(wdn \log q)^{O(1)}$.

Remark: In Problem 1 we have blackbox access to w^2 polynomials constituting the entries of a matrix, whereas in Problem 2 we have blackbox access to a *single* polynomial. In this sense, Problem 1 is supposedly easier than Problem 2. Still, Problem 1 is of independent interest because if the coefficients of the affine forms are chosen adversarially (instead of randomly) in $X_1, X_2, \ldots, X_d$ then even for $w = 3$ the problem becomes as hard as formula reconstruction [BC92].

1.3 Our Results

Throughout this article, $\mathbb{F}$ will denote $\mathbb{F}_q$ with $\mathrm{char}(\mathbb{F}) \geq (wdn)^7$, and $\mathbb{L}$ the field $\mathbb{F}_{q^w}$ [10]. Also, we will assume $d \geq 5$. Theorem 1 solves Problem 1 for $n \geq 2w^2$.

[9] The probability is taken over the input distribution and the random bits used by the algorithm, if it is randomized.

[10] $\mathbb{L}$ can constructed from a basis of $\mathbb{F}_q$ using a randomized algorithm running in $(w \log q)^{O(1)}$ time [vzGG03].

Theorem 1 (Average-case matrix factorization)**.** *For $n \geq 2w^2$, there is a randomized algorithm that takes as input blackbox access to w^2, (n, d)-polynomials $\{f_{st}\}_{s,t\in[w]}$ that constitute the entries of a random (w, d, n)-matrix product $F = X_1 \cdot X_2 \dots X_d$ over $\mathbb{F}$, and with probability $1 - (wdn)^{-\Omega(1)}$ returns $w \times w$ linear matrices $Y_1, Y_2, \dots, Y_d$ over $\mathbb{L}$ satisfying $F = \prod_{i=1}^{d} Y_i$. The algorithm runs in $(wdn \log q)^{O(1)}$ time and queries the blackbox at points in $\mathbb{L}^n$.*

Remarks:

- The constraint on $\mathrm{char}(\mathbb{F})$ is a bit arbitrary, the results in this paper hold as long as $|\mathbb{F}|$ and $\mathrm{char}(\mathbb{F})$ are sufficiently large polynomial functions in w, d and n.
- *Uniqueness of factorization:* The proof of the theorem shows that there are $C_i, D_i \in \mathsf{GL}(w, \mathbb{L})$ such that $Y_i = C_i \cdot X_i \cdot D_i$, for every $i \in [d]$. Moreover, there are $c_1, \dots, c_{d-1} \in \mathbb{L}^\times$ satisfying $C_1 = D_d = I_w$, $D_i \cdot C_{i+1} = c_i I_w$ for $i \in [d-1]$, and $\prod_{i=1}^{d-1} c_i = 1$. At a high level, it is this uniqueness feature of a random matrix product that guides the algorithm to find a factorization for F. In the worst-case, such a factorization need not be unique even if the determinants of the X_i's are coprime irreducible polynomials. For instance[11],

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \cdot \begin{bmatrix} 2x_3 - x_2 & x_4 \\ x_1 & x_3 \end{bmatrix} = \begin{bmatrix} x_3 & x_1 \\ x_4 & 2x_3 - x_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 2x_1x_3 & x_1x_4 + x_2x_3 \\ 2x_3^2 - x_2x_3 + x_1x_4 & 2x_3x_4 \end{bmatrix}.$$

Using Theorem 1, Theorem 2 addresses Problem 2 for $n \geq 4w^2$.

Theorem 2 (Average-case ABP reconstruction)**.** *For $n \geq 4w^2$, there is a randomized algorithm that takes as input blackbox access to a (n, d)-polynomial f computed by a random (w, d, n)-ABP over $\mathbb{F}$, and with probability $1 - (wdn)^{-\Omega(1)}$ returns a (w, d, n)-ABP over $\mathbb{L}$ computing f. The algorithm runs in time $(d^{w^3} n \log q)^{O(1)}$ and queries the blackbox at points in $\mathbb{L}^n$.*

Remarks:

1. *Comparison to* [KNST17]: [KNST17] gave an efficient randomized algorithm to solve Problem 2 when $n \geq w^2 d$ [12]. Theorem 2 improves over [KNST17] by relaxing the constraint on n to $n \geq 4w^2$, but pays in the running time which is exponential in w^3 [13]. Nevertheless, Theorem 2 gives an efficient randomized algorithm for average-case reconstruction of *low* width ABPs.
2. *Time-complexity*: There is one step in the algorithm that finds the affine forms in X_1 and X_d by solving systems of polynomial equations over $\mathbb{F}$, and this takes $d^{O(w^3)}$ field operations. Except this step, every other step runs in $(wdn \log q)^{O(1)}$ time. If the complexity of this step is improved then the overall time complexity of the algorithm will also come down.
3. *Not pseudorandom*: Consider a formal (w, d, n)-ABP where the coefficients of the affine forms are distinct $\mathbf{y}$-variables, and let $h(\mathbf{x}, \mathbf{y})$ be the polynomial computed by this ABP. Here, $|\mathbf{y}| = (n+1) \cdot (w^2(d-2) + 2w) = m$ (say). If $n = w^{\omega(1)}$, the family $H = \{h(\mathbf{x}, \mathbf{b}) : \mathbf{b} \in \mathbb{F}^m\}$ is not pseudorandom under the distribution defined by $\mathbf{b} \in_r \mathbb{F}^m$ [14]. This is because, the

[11] We thank Rohit Gurjar for showing us a similar example.

[12] The algorithm in [KNST17] works over both $\mathbb{Q}$ and $\mathbb{F}_q$, whereas ours is over $\mathbb{F}_q$.

[13] [KNST17] has running time polynomial in all the relevant parameters, and it also works if the width w is varying along the ABP.

[14] This is exactly the distribution that defines a random (w, d, n)-ABP.

w affine forms in X_1 are linearly independent with high probability. So, the variety of $f = h(\mathbf{x}, \mathbf{b})$ (denoted by $\mathbb{V}(f)$) has a subspace of dimension $n - w$ over $\mathbb{F}$; a random polynomial does not have this property with high probability. Moreover, using a randomized algorithm (Theorem 2.6 and 3.9 in [HW99]) we can check if $\mathbb{V}(f)$ has a large subspace in $(d^{w^2} n \log q)^{O(1)}$ time. Observe that $(d^{w^2} n \log q)^{O(1)}$ is much smaller than $\exp(m^{\Omega(1)})$ assuming $n = w^{\omega(1)}$. Hence, the algorithm distinguishes f from a random polynomial efficiently, when w is small.

4. *Comparison to [GKQ13]*: [GKQ13] gave an efficient average-case reconstruction algorithm for formulas. Their input is picked from a distribution defined by complete binary trees with alternating layers of $+$ and $\times$ gates and with random affine forms at the leaves. As width-3 ABPs form a complete model for formulas under p-projections [BC92], Theorem 2 can also be seen as giving another average-case reconstruction algorithm for formulas (when $w = 3$), albeit with a different input distribution. Our result does not subsume [GKQ13] as the input distributions appear incomparable to us.

The proof of Theorem 1 requires an efficient *affine equivalence test* for the determinant *over finite fields*. An n-variate polynomial f is *affine equivalent* to an m-variate polynomial g, for $n \geq m$, if there is an $A \in \mathbb{F}^{m \times n}$ of rank m and an $\mathbf{a} \in \mathbb{F}^m$ such that $f = g(A \cdot \mathbf{x} + \mathbf{a})$. Further, for $m = n$, f is *equivalent* to g if there is an $A \in \mathsf{GL}(n, \mathbb{F})$ such that $f = g(A \cdot \mathbf{x})$. Given blackbox access to a (n, w)-polynomial f, where $n \geq w^2$, the affine equivalence test problem for the determinant is to check whether f is affine equivalent to Det_w, and if yes then output a $B \in \mathbb{F}^{w^2 \times n}$ of rank w^2 and a $\mathbf{b} \in \mathbb{F}^{w^2}$ such that $f = \mathsf{Det}_w(B \cdot \mathbf{x} + \mathbf{b})$. The algorithm in the theorem below *almost* solves this problem over finite fields – it returns a $B \in \mathbb{L}^{w^2 \times n}$ of rank w^2 and a $\mathbf{b} \in \mathbb{L}^{w^2}$.

Theorem 3 (Determinant equivalence test)**.** *There is a randomized algorithm that takes as input blackbox access to a (n, w)-polynomial $f \in \mathbb{F}[\mathbf{x}]$, where $n \geq w^2$, and does the following with probability $1 - \frac{n^{O(1)}}{q}$: If f is affine equivalent to Det_w then it outputs a $B \in \mathbb{L}^{w^2 \times n}$ of rank w^2 and a $\mathbf{b} \in \mathbb{L}^{w^2}$ such that $f = \mathsf{Det}_w(B \cdot \mathbf{x} + \mathbf{b})$, else it outputs '$f$ not affine equivalent to Det_w'. The algorithm runs in $(n \log q)^{O(1)}$ time and queries the blackbox at points in $\mathbb{L}^n$.*

Remarks:

1. *Comparison to [Kay12]*: An efficient equivalence test for the determinant over $\mathbb{C}$ was given in [Kay12]. The computation model in [Kay12] assumes that arithmetic over $\mathbb{C}$ and root finding of univariate polynomials over $\mathbb{C}$ can be done efficiently. While we follow the general strategy of analyzing the Lie algebra and reduction to PS-equivalence from [Kay12], our algorithm is somewhat *simpler*: Unlike [Kay12], our algorithm does not involve the Cartan subalgebras and is almost the same as the simpler equivalence test for the permanent polynomial in [Kay12]. The simplification is achieved by showing that the characteristic polynomial of a random element of the Lie algebra of Det_w splits completely over $\mathbb{L}$ with high probability (Lemma 5.2) – this is crucial for Theorem 1 as it allows the algorithm to output a matrix factorization over a *fixed* low extension of $\mathbb{F}$, namely $\mathbb{L}$.

2. *Average-case ABP reconstruction over* $\mathbb{Q}$: In our arguments, Theorem 3 is the *only* place where we need the underlying field is finite. In other words, the algorithms in Theorems 1 and 2 work over $\mathbb{Q}$ if only there is an efficient equivalence test for Det_w over $\mathbb{Q}$. Also, if there is an affine equivalence test for Det_w that outputs $B, \mathbf{b}$ over the base field ($\mathbb{Q}$ or $\mathbb{F}$) then the algorithm in Theorem 2 would output an ABP over the base field.

1.4 Algorithms and their analysis

The algorithms mentioned in Theorem 1 and 2 are given in Algorithm 1 and 2, respectively. In this section, we briefly discuss their correctness and complexity – for the missing details, we allude to the relevant parts of the subsequent sections.

1.4.1 Analysis of Algorithm 1

Since $F = X_1 \cdot X_2 \ldots X_d$ is a random (w, d, n)-matrix product, with probability $1 - (wdn)^{-\Omega(1)}$, the following property is satisfied: Every X_i is a *full rank* linear matrix (that is the affine forms in X_i are $\mathbb{F}$-linearly independent), and $\det(X_1), \det(X_2), \ldots, \det(X_d)$[15] are coprime irreducible polynomials (see Claim 2.3). We analyze Algorithm 1 assuming that this property of the input is satisfied. Algorithm 1 has three main stages:

Algorithm 1 Average-case matrix factorization

INPUT: Blackbox access to w^2, (n, d)-polynomials $\{f_{st}\}_{s,t \in [w]}$ that constitute the entries of a random (w, d, n)- matrix product $F = X_1 \cdot X_2 \ldots X_d$.
OUTPUT: Linear matrices $Y_1, Y_2, \ldots, Y_d$ over $\mathbb{L}$ such that $F = Y_1 \cdot Y_2 \ldots Y_d$.

/* Factorization of the determinant */
Compute blackbox access to $\det(F)$.
Compute blackbox access to the irreducible factors of $\det(F)$; call them $g_1, g_2, \ldots, g_d$.
if the number of irreducible factors is not equal to d **then**
  Output 'Failed'.
end if

/* Affine equivalence test for determinant */
Set $j = 1$.
while $j \leq d$ **do**
  Call the algorithm in Theorem 3 with input as blackbox access to g_j; let B_j and $\mathbf{b}_j$ be its output. Construct the $w \times w$ full-rank linear matrix Z_j over $\mathbb{L}$ determined by B_j and $\mathbf{b}_j$.
  if the algorithm outputs 'g_j not affine equivalent to Det_w' **then**
    Output 'Failed'.
  end if
  Set $j = j + 1$.
end while

/* Rearrangement of the matrices */
Call Algorithm 3 on input blackbox access to F and $Z_1, \ldots, Z_d$, and let $Y_1, \ldots, Y_d$ be its output.
if Algorithm 3 outputs 'Rearrangement not possible' **then**
  Output 'Failed'.
end if

Output $Y_1, Y_2, \ldots, Y_d$.

[15] $\det(X_i)$ is the determinant of the $w \times w$ matrix X_i.

1. *Computing the irreducible factors of* $\det(F)$ *(Steps 2–6):* From blackbox access to the entries of F, a blackbox access to $\det(F)$ is computed in $(wdn \log q)^{O(1)}$ time using Gaussian elimination. Subsequently, using Kaltofen-Trager's factorization algorithm [KT90], blackbox access to the irreducible factors $g_1, g_2, \ldots, g_d$ of $\det(F)$ are constructed in $(wdn \log q)^{O(1)}$ time (see Lemma 2.1). Since $\det(X_1), \ldots, \det(X_d)$ are coprime irreducible polynomials, there is a permutation σ of $[d]$, and $c_i \in \mathbb{F}^\times$ for all $i \in [d]$, such that $c_i \cdot \det(X_i) = g_{\sigma(i)}$ and $\prod_{i=1}^d c_i = 1$. For the next two stages, assume $w > 1$ as the $w = 1$ case gets solved readily at this stage.

2. *Affine equivalence test (Steps 9–16):* Let $j = \sigma(i)$ and X_i' be the matrix X_i with the affine forms in the first row multiplied by c_i. Then, $g_j = \det(X_i') = c_i \cdot \det(X_i)$, which is affine equivalent to Det_w. At step 11, the algorithm[16] in Theorem 3 finds a $B_j \in \mathbb{L}^{w^2 \times n}$ of rank w^2 and $\mathbf{b}_j \in \mathbb{L}^{w^2}$ such that $g_j = \mathsf{Det}_w(B_j \cdot \mathbf{x} + \mathbf{b}_j)$, with probability $1 - (wdn)^{-\Omega(1)}$. Let Z_j be the matrix obtained by appropriately replacing the entries of the $w \times w$ symbolic matrix with the affine forms in $B_j \cdot \mathbf{x} + \mathbf{b}_j$ such that $\det(Z_j) = g_j = \det(X_i')$. This certifies that there are matrices $C_i, D_i \in \mathsf{SL}(\mathbb{L}, w)$ satisfying, $Z_j = C_i \cdot X_i' \cdot D_i$ or $Z_j^T = C_i \cdot X_i' \cdot D_i$ (see Fact 1 in Section 5.1). Multiplying the first column of C_i with c_i, and calling the resulting matrix C_i again, we see that there are matrices $C_i, D_i \in \mathsf{GL}(w, \mathbb{L})$ satisfying, $Z_j = C_i \cdot X_i \cdot D_i$ or $Z_j^T = C_i \cdot X_i \cdot D_i$. Observe that such C_i, D_i are *unique* up to multiplications by elements in $\mathbb{L}^\times$ [17].

3. *Rearrangement of the retrieved matrices (Steps 19–22):* At step 19, Algorithm 3 constructs the matrices $Y_1, Y_2, \ldots, Y_d$ by determining the permutation σ and whether $Z_{\sigma(i)} = C_i \cdot X_i \cdot D_i$ or $Z_{\sigma(i)}^T = C_i \cdot X_i \cdot D_i$. Internally, Algorithm 3 uses Algorithm 4 which when given blackbox access to $F_d = F$ and a Z (that is either Z_k or Z_k^T for some $k \in [d]$), does the following with probability $1 - (wdn)^{-\Omega(1)}$: If $Z = C_d \cdot X_d \cdot D_d$ then it outputs a $\tilde{D}_d = a_d D_d$ for some $a_d \in \mathbb{L}^\times$. For all other cases – if $Z = C_i \cdot X_i \cdot D_i$ or $Z^T = C_i \cdot X_i \cdot D_i$ for $i \in [d-1]$, or $Z^T = C_d \cdot X_d \cdot D_d$ – it outputs 'Failed'. Algorithm 4 uses the critical fact that F is a *random* matrix product to accomplish the above and locate the *unique* last matrix. The running time of the algorithm, which is $(wdn \log q)^{O(1)}$, and its proof of correctness[18] are discussed in Section 3.2. Algorithm 3 calls Algorithm 4 on inputs F, Z_k and F, Z_k^T for all $k \in [d]$. If Algorithm 4 returns a matrix $\tilde{D}_d$ for some $k \in [d]$ on either inputs F, Z_k or F, Z_k^T then it sets $M_d = Z_k$ or $M_d = Z_k^T$ respectively, and $\sigma(d) = k$. Subsequently, Algorithm 3 computes blackbox access to a length $d-1$ matrix product $F_{d-1} = F \cdot \tilde{D}_d \cdot M_d^{-1} = X_1 \cdots X_{d-2} \cdot (X_{d-1} \cdot a_d C_d^{-1})$, and repeats the above process to compute M_{d-1} and $\sigma(d-1)$ with the inputs F_{d-1} and $\{Z_1, \ldots, Z_d\} \backslash Z_{\sigma(d)}$. Thus, using Algorithm 4 repeatedly, Algorithm 3 iteratively determines σ and $M_d, M_{d-1}, \ldots, M_2$: At the $(d-t+1)$-th iteration, for $t \in [d-1, 2]$, it computes a matrix $\tilde{D}_t = a_t(C_{t+1} \cdot D_t)$ for some $a_t \in \mathbb{L}^\times$, sets M_t and $\sigma(t)$ accordingly, creates blackbox access to $F_{t-1} = F_t \cdot \tilde{D}_t \cdot M_t^{-1}$ and prepares the list $\{Z_1, \ldots, Z_d\} \backslash \{Z_{\sigma(d)}, Z_{\sigma(d-1)}, \ldots, Z_{\sigma(t)}\}$ for the next iteration. Finally, setting $Y_1 = F_1$ and $Y_i = M_i \cdot \tilde{D}_i^{-1}$, for all $i \in [2, d]$, we have $F = \prod_{i=1}^d Y_i$.

[16] Given in Section 5.

[17] i.e., if $C_i \cdot X_i \cdot D_i = C_i' \cdot X_i \cdot D_i'$, where X_i is a full rank matrix, then $C_i' = \alpha C_i$ and $D_i' = \alpha^{-1} D_i$ for some $\alpha \in \mathbb{L}^\times$

[18] which also gives the uniqueness of factorization mentioned in the remark after Theorem 1

1.4.2 Analysis of Algorithm 2

Let f be the polynomial computed by a (w,d,n)-ABP $X_1 \cdot X_2 \ldots X_d$. We can assume that f is a homogeneous degree-d polynomial and the entries in each X_i are linear forms (i.e., affine forms with constant term zero), owing to the following simple homogenization trick.

Homogenization of ABP: Consider the $(n+1)$-variate homogeneous degree-d polynomial

$$f_{\text{hom}} = x_0^d \cdot f\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

The polynomial f_{hom} is computable by the (w,d,n)-ABP $X_1' \cdot X_2' \ldots X_d'$, where X_i' is equal to X_i but with the constant term in the affine forms multiplied by x_0. If we construct an ABP for f_{hom} then an ABP for f is obtained by setting $x_0 = 1$.

We give an overview of the three main stages in Algorithm 2. As in Algorithm 1, the matrices $X_1, X_2, \ldots, X_d$ are assumed to be full rank linear matrices and further, for a similar reason, the $2w$ linear forms in X_1 and X_d are assumed to be $\mathbb{F}$-linearly independent.

1. *Computing the corner spaces (Steps 2–6):* Polynomial f is zero modulo each of the two w-dimensional $\mathbb{F}$-linear spaces $\mathcal{X}_1$ and $\mathcal{X}_d$ spanned by the linear forms in X_1 and X_d respectively [19]. We show in Lemma 4.1, if $n \geq 4w^2$ then with probability $1 - (wdn)^{-\Omega(1)}$ the following holds: Let $\mathbb{K} \supseteq \mathbb{F}$ be any field. If $f = 0 \mod \langle l_1, \ldots, l_w \rangle$, where l_i's are linear forms in $\mathbb{K}[\mathbf{x}]$, then the l_i's either belong to the $\mathbb{K}$-span of the linear forms in X_1 or belong to the $\mathbb{K}$-span of the linear forms in X_d. In this sense, the spaces $\mathcal{X}_1$ and $\mathcal{X}_d$ are *unique*. The algorithm invokes Algorithm 5 which computes bases of $\mathcal{X}_1$ and $\mathcal{X}_d$ by solving $O(n)$ systems of polynomial equations over $\mathbb{F}$. Such a system has $d^{O(w^2)}$ equations in $m = O(w^3)$ variables and the degree of the polynomials in the system is at most d; we intend to find all the solutions in $\mathbb{F}^m$. It turns out that owing to the uniqueness of $\mathcal{X}_1$ and $\mathcal{X}_d$, the variety over $\overline{\mathbb{F}}$ [20]defined by such a system has exactly two points and these points lie in $\mathbb{F}^m$. From the two solutions, bases of $\mathcal{X}_1$ and of $\mathcal{X}_d$ can be derived. The two solutions of the system are computed by a randomized algorithm running in $(d^{w^3} \log q)^{O(1)}$ time ([Ier89, HW99], see Lemma 2.2) – the algorithm exploits the fact that the variety over $\overline{\mathbb{F}}$ is zero-dimensional. Thus, at step 2, the two spaces are either equal to $\mathcal{X}_1$ and $\mathcal{X}_d$ or $\mathcal{X}_d$ and $\mathcal{X}_1$ respectively. Without loss of generality, we assume the former. Once bases of the corner spaces $\mathcal{X}_1$ and $\mathcal{X}_d$ are computed, an invertible transformation A maps the linear forms in the bases to distinct variables (as the linear forms in X_1 and X_d are $\mathbb{F}$-linearly independent).

2. *Computing the coefficients of the* $\mathbf{r}$ *variables (Steps 9–13)*: There is an ABP $X_1' \cdot X_2' \ldots X_d'$ computing $f' = f(A \cdot \mathbf{x})$, where X_1' and X_d' are equal to $(y_1\ y_2\ \ldots\ y_w)$ and $(z_1\ z_2\ \ldots\ z_w)^T$ respectively. For $k \in [2, d-1]$, let $R_k = (X_k')_{\mathbf{y}=0,\mathbf{z}=0}$ [21] and $F = R_2 \cdot R_3 \ldots R_{d-1}$. As $X_1 \cdot X_2 \ldots X_d$ is a random (w,d,n)-ABP, $R_2 \cdot R_3 \ldots R_{d-1}$ is a random $(w, d-2, n-2w)$-matrix product over $\mathbb{F}$. The (s,t)-th entry of F is equal to $\left(\frac{\partial f'}{\partial y_s z_t}\right)_{\mathbf{y}=0,\mathbf{z}=0}$, for $s,t \in [w]$. Blackbox access to each

[19]For a field $\mathbb{K} \supseteq \mathbb{F}$, we say f is zero modulo a $\mathbb{K}$-linear space $\mathcal{X} = \text{span}_{\mathbb{K}}\{l_1, \ldots, l_w\}$, where l_i's are linear forms in $\mathbb{K}[\mathbf{x}]$, if f is in the ideal of $\mathbb{K}[\mathbf{x}]$ generated by $\{l_1, \ldots, l_w\}$. This is also denoted by $f = 0 \mod \langle l_1, \ldots, l_w \rangle$.

[20]the algebraic closure of $\mathbb{F}$

[21]The matrix X_i' with the $\mathbf{y}$ and $\mathbf{z}$ variables in its linear forms substituted to zero.

Algorithm 2 Average-case ABP reconstruction

INPUT: Blackbox access to a (n, d)-polynomial f computed by a random (w, d, n)-ABP.
OUTPUT: A (w, d, n)-ABP over $\mathbb{L}$ computing f.

/* Computing the corner spaces */
Call Algorithm 5 on f to compute bases of the two *unique* w-dimensional $\mathbb{F}$-linear spaces $\mathcal{X}_1$ and $\mathcal{X}_d$, spanned by linear forms in $\mathbb{F}[\mathbf{x}]$, such that f is zero modulo each of $\mathcal{X}_1$ and $\mathcal{X}_d$.
if Algorithm 5 outputs 'Failed' **then**
  Output 'Failed to construct an ABP'.
end if
Compute a transformation $A \in \mathsf{GL}(n, \mathbb{F})$ that maps the bases of $\mathcal{X}_1$ and $\mathcal{X}_d$ to distinct variables $\mathbf{y} = \{y_1, y_2, \ldots, y_w\}$ and $\mathbf{z} = \{z_1, z_2, \ldots z_w\}$ respectively, where $\mathbf{y}, \mathbf{z} \subseteq \mathbf{x}$. Let $\mathbf{r} = \mathbf{x} \setminus (\mathbf{y} \uplus \mathbf{z})$, $X'_1 = (y_1\ y_2\ \ldots\ y_w)$, $X'_d = (z_1\ \ z_2\ \ \ldots\ z_w)^T$ and $f' = f(A \cdot \mathbf{x})$.

/* Computing the coefficients of the $\mathbf{r}$ variables */
Construct blackbox access to the w^2 polynomials that constitute the entries of the $w \times w$ matrix $F = (\frac{\partial f'}{\partial y_s z_t} |_{\mathbf{y}=0, \mathbf{z}=0})_{s,t \in [w]}$.
Call Algorithm 1 on input F to compute a factorization of F as $S_2 \cdot S_3 \ldots S_{d-1}$.
if Algorithm 1 outputs 'Failed' **then**
  Output 'Failed to construct an ABP'.
end if

/* Computing the coefficients of the $\mathbf{y}$ and $\mathbf{z}$ variables */
Call Algorithm 6 on inputs f' and $\{S_2, S_3, \ldots, S_{d-1}\}$ to compute matrices $T_2, T_3, \ldots, T_{d-1}$ such that f' is computed by the ABP $X'_1 \cdot T_2 \cdots T_{d-1} \cdot X'_d$.
if Algorithm 6 outputs 'Failed' **then**
  Output 'Failed to construct an ABP'.
end if
Apply the transformation A^{-1} on the $\mathbf{x}$ variables in the matrices X'_1, X'_d, and T_k for $k \in [2, d-1]$. Call the resulting matrices Y_1, Y_d, and Y_k for $k \in [2, d-1]$ respectively.
Output $Y_1 \cdot Y_2 \ldots Y_d$ as the ABP computing f.

of the w^2 entries of F are constructed in $(wdn \log q)^{O(1)}$ time using Claim 2.1. From F, Algorithm 1 computes linear matrices $S_2, \ldots, S_{d-1}$ over $\mathbb{L}$ in $\mathbf{r} = \mathbf{x} \setminus (\mathbf{y} \uplus \mathbf{z})$ variables such that $F = S_2 \cdot S_3 \ldots S_{d-1}$. Moreover, the uniqueness of factorization implies there are linear matrices $T_2, \ldots, T_{d-1}$ over $\mathbb{L}$ in the $\mathbf{x}$-variables, satisfying $(T_k)_{\mathbf{y}=0,\mathbf{z}=0} = S_k$, such that f' is computed by the ABP $X'_1 \cdot T_2 \cdots T_{d-1} \cdot X'_d$.

3. *Computing the coefficients of* $\mathbf{y}$ *and* $\mathbf{z}$ *variables in* T_k *(Steps 16–20)*: Algorithm 6 finds the coefficients of the $\mathbf{y}$ and $\mathbf{z}$ variables in the linear forms present in $T_2, \ldots, T_{d-1}$ in $(wdn \log q)^{O(1)}$ time. We present the idea here; the detail proof of correctness is given in Section 4.2. In the following discussion, $M(i,j)$ denotes the (i,j)-th entry, $M(i,*)$ the i-th row, and $M(*,j)$ the j-th column of a linear matrix M. Let us focus on finding the coefficients of y_1 in the linear forms present in $T_2(1,*), T_3, \ldots, T_{d-2}, T_{d-1}(*,1)$. There are $w^2(d-4) + 2w$ linear forms in these matrices and these would be indexed by $[w^2(d-4) + 2w]$. Let c_e be the coefficient of y_1 in the e-th linear form l_e for $e \in [w^2(d-4) + 2w]$. We associate a polynomial $h_e(\mathbf{r})$ in $\mathbf{r}$ variables with l_e as follows: If l_e is the (i,j)-th entry of T_k then $h_e \stackrel{\text{def}}{=} [S_2(1,*) \cdot S_3 \cdots S_{k-2} \cdot S_{k-1}(*,i)] \cdot [S_{k+1}(j,*) \cdot S_{k+2} \cdots S_{d-2} \cdot S_{d-1}(*,1)]$ [22]. Observe that if f' is treated as a polynomial in $\mathbf{y}$ and $\mathbf{z}$ variables with coefficients in $\mathbb{L}(\mathbf{r})$ then the coefficient of $y_1^2 z_1$ is exactly $\sum_{e \in [w^2(d-4)+2w]} c_e \cdot h_e(\mathbf{r})$. On the other hand, this coefficient is $\left(\frac{\partial f'}{\partial y_1^2 z_1}\right)_{\mathbf{y}=0,\mathbf{z}=0}$, for which we can obtain blackbox access using Claim 2.1. This allows us to write the equation,
$$\sum_{e=1}^{w^2(d-4)+2w} c_e \cdot h_e(\mathbf{r}) = \left(\frac{\partial f'}{\partial y_1^2 z_1}\right)_{\mathbf{y}=0,\mathbf{z}=0}. \quad (1)$$
We show in Lemma 4.2 and Corollary 4.1 that the polynomials h_e, for $e \in [w^2(d-4) + 2w]$, are $\mathbb{L}$-linearly independent with probability[23] $1 - (wdn)^{-\Omega(1)}$. By substituting random values to the $\mathbf{r}$ variables in the above equation, we can set up a system of $w^2(d-4) + 2w$ linear equations in the c_e's. The linear independence of the h_e's ensures that we can solve for c_e (by Claim 2.2).

1.4.3 Proof strategy for Theorem 3

The algorithm in Theorem 3 has three stages:

1. *Reduction to equivalence testing*: Applying known techniques – 'variable reduction' (Claim 5.1) and 'translation equivalence' (Claim 5.2) – the affine equivalence testing problem is efficiently reduced to *equivalence testing* for Det_w with high probability. An equivalence test takes blackbox access to a w^2-variate polynomial $g(\mathbf{y})$ as input and does the following with high probability: If g is equivalent to Det_w then it outputs a $Q \in \mathsf{GL}(w^2, \mathbb{L})$ such that $g = \mathsf{Det}_w(Q \cdot \mathbf{y})$ else it outputs 'g not equivalent to Det_w'.

2. *Reduction to PS-equivalence*: The reduction is given in Algorithm 7. The algorithm proceeds by computing an $\mathbb{F}$-basis of the Lie algebra of the group of symmetries of g (denoted as $\mathfrak{g}_g$, see Claim 5.3). It then picks an element F uniformly at random from $\mathfrak{g}_g$ and computes its

[22] by identifying the 1×1 matrix of the R.H.S with the entry of the matrix
[23] over the randomness of the input f

characteristic polynomial $h(x)$. Since $F \in \mathfrak{g}_g$, it is similar to a $L \in \mathfrak{g}_{\mathsf{Det}_w}$ (see Fact 3 in Section 5.1), implying that their characteristic polynomials are equal. As F is a random element of $\mathfrak{g}_g$, L is also a random element of $\mathfrak{g}_{\mathsf{Det}_w}$. In Lemma 5.2, we show that the characteristic polynomial h of a $L \in_r \mathfrak{g}_{\mathsf{Det}_w}$ is square-free and splits completely over $\mathbb{L}$, with high probability[24]. The roots of h are computed in randomized $(w \log q)^{O(1)}$ time ([CZ81], see also [vzGG03]). From the roots, a $D \in \mathsf{GL}(w^2, \mathbb{L})$ can be computed such that $D^{-1}FD$ is diagonal[25]. Thereafter, the structure of the group of symmetries of Det_w and its Lie algebra helps argue, in Section 5.2, that $f(D \cdot \mathbf{x})$ is PS-equivalent to Det_w.

3. *Doing the PS-equivalence*: This step follows directly from [Kay12] (see Lemma 5.1).

2 Preliminaries

2.1 Notations

$\mathsf{GL}(w, \mathbb{F})$ is the set of $w \times w$ invertible matrices over $\mathbb{F}$, and $\mathsf{SL}(w, \mathbb{F})$ the set of $w \times w$ matrices over $\mathbb{F}$ with determinant one. Bold letters $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \mathbf{w}$ are used to represent either column vectors (or sets) of variables or column vectors of field elements, calligraphic letters like $\mathcal{X}$ to represent vector spaces, capital letters like A, B, C, S, T for matrices or sets – the context of a usage of any of these symbols would hopefully make its purpose clear. The derivative of a polynomial f with respect to a monomial μ is denoted as $\frac{\partial f}{\partial \mu}$ or $\partial_\mu f$.

2.2 Algorithmic preliminaries

The following result on blackbox polynomial factorization is proved in [KT90].

Lemma 2.1 ([KT90]). *There is a randomized algorithm that takes as input blackbox access to a (n, d)-polynomial f over $\mathbb{F}$, and constructs blackbox access to the irreducible factors of f over $\mathbb{F}$ in $(nd \log q)^{O(1)}$ time with success probability $1 - \frac{(nd)^{O(1)}}{q}$.*

Let I be an ideal of $\mathbb{F}[\mathbf{x}]$ generated by (n, d)-polynomials $g_1, \ldots, g_m$, and $\mathbb{V}_{\overline{\mathbb{F}}}(I)$ the variety or the algebraic set defined by I over $\overline{\mathbb{F}}$. $\mathbb{V}_{\overline{\mathbb{F}}}(I)$ is zero-dimensional if it has finitely many points. We say a point $\mathbf{a} \in \mathbb{V}_{\overline{\mathbb{F}}}(I)$ is $\mathbb{F}$-rational if $\mathbf{a} \in \mathbb{F}^n$. The proof of the next result follows from [Ier89] (see also [HW99]).

Lemma 2.2 ([Ier89]). *There is a randomized algorithm that takes input m, (n, d)-polynomials $g_1, g_2, \ldots, g_m$ generating an ideal I of $\mathbb{F}[\mathbf{x}]$. If $\mathbb{V}_{\overline{\mathbb{F}}}(I)$ is zero-dimensional and all points in it are $\mathbb{F}$-rational then the algorithm computes all the points in $\mathbb{V}_{\overline{\mathbb{F}}}(I)$ with probability $1 - \exp(-mnd \log q)$. The running time of the algorithm is $(md^n \log q)^{O(1)}$.* [26]

[24] This lemma makes our reduction to PS-equivalence simpler than [Kay12], enabling the equivalence test to work over finite fields.

[25] In [Kay12], a basis of the centralizer of F in $\mathfrak{g}_g$ is computed first and then a $D \in \mathsf{GL}(w^2, \mathbb{C})$ is obtained that simulaneously diagonalizes this basis.

[26] A similar result, but for homogeneous $g_1, \ldots, g_m$, follows from [Laz01].

2.3 A few useful facts

We list down three claims (without proofs) that will be used in the later sections. A proof of the first can be given using interpolation. Proofs of the last two follow from applications of the Schwartz-Zippel lemma [Sch80, Zip79].

Claim 2.1. *There is a deterministic algorithm that given blackbox access to a (n, d)-polynomial $f \in \mathbb{F}[\mathbf{x}]$, and a monomial μ of constant degree in $\mathbf{x}$, computes blackbox access to $\partial_\mu f$ in $(nd \log q)^{O(1)}$ time.*

Claim 2.2. *Let $f_1, f_2, \ldots, f_m$ be $\mathbb{F}$-linearly independent (n, d)-polynomials in $\mathbb{F}[\mathbf{x}]$. If $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ are points in $\mathbb{F}^n$ chosen independently and uniformly at random, then the matrix $(f_t(\mathbf{a}_s))_{s,t\in[m]}$ has rank m over $\mathbb{F}$ with probability at least $1 - \frac{dm}{q}$.*

Claim 2.3. *Let $X_1 \cdot X_2 \ldots X_d$ be a random (w, d, n)-matrix product over $\mathbb{F}$. If $n \geq w^2$ then $X_1, X_2, \ldots, X_d$ are full rank linear matrices and $\det(X_1), \det(X_2), \ldots, \det(X_d)$ are coprime irreducible polynomials with probability $1 - (wdn)^{-\Omega(1)}$.*

3 Average-case matrix factorization: Proof of Theorem 1

The algorithm in Theorem 1 is presented in Algorithm 1. To complete the analysis, given in Section 1.4.1, we need to argue the correctness of the key step of rearrangement of the matrices (Algorithm 3) by finding the last matrix (Algorithm 4). As the functioning of Algorithm 3 is already sketched out in Section 1.4.1, the reader may skip to Section 3.2. For completeness, we include an analysis of Algorithm 3 in the following subsection.

3.1 Rearranging the matrices

Recall, we have assumed F is a (w, d, n)-matrix product $X_1 \cdot X_2 \ldots X_d$, where $X_1, X_2, \ldots, X_d$ are full rank linear matrices, and $\det(X_1), \det(X_2), \ldots, \det(X_d)$ are coprime irreducible polynomials. The inputs to Algorithm 3 are d full rank linear matrices $Z_1, Z_2, \ldots, Z_d$ over $\mathbb{L}$ such that there are matrices $C_i, D_i \in \mathsf{GL}(w, \mathbb{L})$ and a permutation σ of $[d]$ satisfying $Z_{\sigma(i)} = C_i \cdot X_i \cdot D_i$ or $Z^T_{\sigma(i)} = C_i \cdot X_i \cdot D_i$ for every $i \in [d]$. Algorithm 3 iteratively determines σ (implicitly) by repeatedly using Algorithm 4. The behavior of Algorithm 4 is summarized in the lemma below. For the lemma statement, assume $n \geq 2w^2$, Z is a full rank linear matrix over $\mathbb{L}$, and F_t is a (w, t, n)-matrix product $R_1 \cdot R_2 \ldots R_t$ over $\mathbb{L}$, where $t \leq d$. Also, $R_1, R_2, \ldots, R_t$ are full rank linear matrices, and $\det(R_1), \det(R_2), \ldots, \det(R_t)$ are coprime irreducible polynomials. Further, there are matrices $C, D \in \mathsf{GL}(w, \mathbb{L})$ and $i \in [t]$ such that $Z = C \cdot R_i \cdot D$ or $Z^T = C \cdot R_i \cdot D$.

Lemma 3.1. *Algorithm 4 takes input Z and blackbox access to the w^2 entries of F_t, and with probability $1 - (wdn)^{-\Omega(1)}$ does this: If $Z = C \cdot R_t \cdot D$ then it outputs a $\tilde{D} = aD$ for an $a \in \mathbb{L}^\times$, and for all other cases – $Z = C \cdot R_i \cdot D$ or $Z^T = C \cdot R_i \cdot D$ for $i \in [t-1]$, or $Z^T = C \cdot R_t \cdot D$ – it outputs 'Failed'.*

Algorithm 4 and the proof of Lemma 3.1 are presented in Section 3.2. We analyze Algorithm 3 below by tracing its steps:

Algorithm 3 Rearrangement of the matrices

INPUT: Blackbox access to F, and $w \times w$ full rank linear matrices $Z_1, Z_2, \ldots, Z_d$ over $\mathbb{L}$.

OUTPUT: Linear matrices $Y_1, Y_2, \ldots, Y_d$ over $\mathbb{L}$ such that $F = Y_1 \cdot Y_2 \cdots Y_d$.

Set $t = d, k = 1$, and $F_d = F$.
while $t > 1$ **do**
  while $k \leq t$ **do**
    Call Algorithm 4 on inputs F_t and Z_k.
    if Algorithm 4 outputs $\tilde{D}$ **then**
      Rename Z_k as Z_t and Z_t as Z_k, and set $\tilde{D}_t = \tilde{D}$. /* σ is determined implicitly. */
      Set $M_t = Z_t$ and $F_{t-1} = F_t \cdot \tilde{D}_t \cdot M_t^{-1}$.
      Set $k = 1$ and $t = t - 1$.
      Exit the inner loop.
    end if
    Call Algorithm 4 on inputs F_t and Z_k^T.
    if Algorithm 4 outputs a $\tilde{D}$ **then**
      Rename Z_k as Z_t and Z_t as Z_k, and set $\tilde{D}_t = \tilde{D}$. /* σ is determined implicitly. */
      Set $M_t = Z_t^T$ and $F_{t-1} = F_t \cdot \tilde{D}_t \cdot M_t^{-1}$.
      Set $k = 1$ and $t = t - 1$.
      Exit the inner loop.
    end if
    Set $k = k + 1$.
  end while
  if $k = t + 1$ **then**
    Exit the outer loop.
  end if
end while
if $t \geq 2$ **then**
  Output 'Rearrangement not possible'.
else
  Set $Y_1 = F_1$, and $Y_t = M_t \cdot \tilde{D}_t^{-1}$ for all $t \in [2, d]$. Output $Y_1, \ldots, Y_d$.
end if

Step 2: The algorithm enters an outer loop and iterates from $t = d$ to $t = 2$. For a fixed $t \in [d, 2]$, at the start of the loop the algorithm ensures F_t is a (w, t, n)-matrix product $R_1 \cdot R_2 \ldots R_t$[27] over $\mathbb{L}$, where $R_1, R_2, \ldots, R_t$ are full rank linear matrices and $\det(R_1), \det(R_2), \ldots, \det(R_t)$ are coprime irreducible polynomials. Further, there is a permutation σ_t of $[t]$, and for every $i \in [t]$ there are matrices $C_i, D_i \in \mathsf{GL}(w, \mathbb{L})$ such that either $Z_{\sigma_t(i)} = C_i \cdot R_i \cdot D_i$ or $Z^T_{\sigma_t(i)} = C_i \cdot R_i \cdot D_i$. In the loop, the algorithm determines $\sigma_t(t)$ and whether $Z_{\sigma_t(t)} = C_t \cdot R_t \cdot D_t$ or $Z^T_{\sigma_t(t)} = C_t \cdot R_t \cdot D_t$.

Steps 4–21: Inside the inner loop, the algorithm calls Algorithm 4 on inputs F_t, Z_k (step 5) and F_t, Z_k^T (step 13) for all $k \in [t]$. By Lemma 3.1, only when $k = \sigma_t(t)$, Algorithm 4 returns a $\tilde{D} = a_t D_t$ for some $a_t \in \mathbb{L}^\times$. The renaming of Z_k and Z_t (in steps 7 and 15) ensures that we have a suitable permutation σ_{t-1} of $[t-1]$ in the next iteration of the outer loop. The setting of M_t (in steps 8 and 16) implies that $M_t = C_t \cdot R_t \cdot D_t$. Hence,

$$F_{t-1} = F_t \cdot \tilde{D}_t \cdot M_t^{-1} = (R_1 \cdot R_2 \ldots R_{t-1}) \cdot (a_t C_t^{-1}).$$

By reusing symbols and calling $R_{t-1} \cdot (a_t C_t^{-1})$ as R_{t-1}, and $a_t^{-1} C_t \cdot D_{t-1}$ as D_{t-1}, we observe that the setup at step 2 is maintained in the next iteration of the outer loop.

Step 32: As $F_{t-1} = F_t \cdot \tilde{D}_t \cdot M_t^{-1}$ at every iteration of the outer loop, setting $Y_t = M_t \cdot \tilde{D}_t^{-1}$ implies $F_{t-1} = F_t \cdot Y_t^{-1}$ for every $t \in [d, 2]$. Therefore, $F = F_d = Y_1 \cdots Y_d$.

3.2 Determining the last matrix: Proof of Lemma 3.1

We give an overview of the proof by first assuming that Z is the 'last' matrix in the product F_t. The correctness of the idea is then made precise by tracing the steps of Algorithm 4.

Overview: Suppose $Z = C \cdot R_t \cdot D$, where $C, D \in \mathsf{GL}(w, \mathbb{L})$. As Z is a full rank linear matrix, we can assume the entries of Z are distinct variables, by applying an invertible linear transformation. For any polynomial $h \in \mathbb{L}[\mathbf{x}]$, $h \mod \det(Z)$ can be identified with an element of $\mathbb{L}(\mathbf{x})$ [28]. Let $Z', F_t' \in \mathbb{L}(\mathbf{x})^{w \times w}$ be obtained by reducing the entries of Z and F_t, respectively, modulo $\det(Z)$. The coprimality of the determinants of $R_1, \ldots, R_t$ and their full rank nature imply,

$$D \cdot \text{Kernel}_{\mathbb{L}(\mathbf{x})}(Z') = \text{Kernel}_{\mathbb{L}(\mathbf{x})}(F_t'),$$

and these two kernels have dimensions one. A basis of $\text{Kernel}_{\mathbb{L}(\mathbf{x})}(Z')$ can be easily derived as Z is known explicitly. However, we only have blackbox access to F_t'. To leverage the above relation, we compute bases of $\text{Kernel}_{\mathbb{L}}(F_t'(\mathbf{a}))$ and $\text{Kernel}_{\mathbb{L}}(Z'(\mathbf{a}))$ for several random $\mathbf{a} \in_r \mathbb{F}^n$, and form two matrices $U, V \in \mathsf{GL}(w, \mathbb{L})$ from these bases so that D equals $U \cdot V^{-1}$ (up to scaling by elements in $\mathbb{L}^\times$). Hereafter, $\text{Kernel}_{\mathbb{L}}$ will be denoted as Ker in the analysis of Algorithm 4.

Applying an invertible linear map (Step 2): The invertible linear transformation lets us assume that $Z = (z_{lk})_{l,k \in [w]}$, where z_{lk}'s are distinct variables in $\mathbf{x}$.

[27] For $t = d$, $R_i = X_i$ for all $i \in [d]$.

[28] $\det(Z)$ being multilinear, there is an injective ring homomorphism from $\mathbb{L}[\mathbf{x}]/(\det(Z))$ to $\mathbb{L}(\mathbf{x})$ via a simple substitution map taking a variable to a rational function.

Algorithm 4 Determining the last matrix

INPUT: Blackbox access to a (w, t, n)-matrix product F_t and a full rank linear matrix Z over $\mathbb{L}$.
OUTPUT: A matrix $\tilde{D} \in \mathsf{GL}(w, \mathbb{L})$, if Z is the 'last' matrix of the product F_t.

/* Applying an invertible linear map */
Let the first w^2 variables in $\mathbf{x}$ be $\mathbf{z} = \{z_{lk}\}_{l,k\in[w]}$. Compute an invertible linear map A that maps the affine forms in Z to distinct $\mathbf{z}$ variables, and apply A to the w^2 blackbox entries of F_t. Reusing symbols, $Z = (z_{lk})_{l,k\in[w]}$ and F_t is the matrix product after the transformation.

/* Reducing Z and F_t modulo $\det(Z)$ */
Let N_{lk} be the (l,k)-th cofactor of Z, for $l, k \in [w]$. Substitute $z_{11} = \frac{-\sum_{k=2}^{w} z_{1k} N_{1k}}{N_{11}}$ in Z and in the blackbox for F_t. Call the matrices Z' and F_t' respectively after the substitution.

/* Computing the kernels */
for $k = 1$ **to** $w + 1$ **do**
    Choose $\mathbf{a}_k, \mathbf{b}_k \in_r \mathbb{F}^n$. Compute bases of $\mathrm{Ker}(F_t'(\mathbf{a}_k))$, $\mathrm{Ker}(Z'(\mathbf{a}_k))$, $\mathrm{Ker}(F_t'(\mathbf{b}_k))$, $\mathrm{Ker}(Z'(\mathbf{b}_k))$. Pick non-zero $\mathbf{u}_k \in \mathrm{Ker}(F_t'(\mathbf{a}_k))$, $\mathbf{v}_k \in \mathrm{Ker}(Z'(\mathbf{a}_k))$, $\mathbf{w}_k \in \mathrm{Ker}(F_t'(\mathbf{b}_k))$, $\mathbf{s}_k \in \mathrm{Ker}(Z'(\mathbf{b}_k))$. If the computation fails (i.e., $N_{11}(\mathbf{a}_k) = 0$ or $N_{11}(\mathbf{b}_k) = 0$), or any of the kernels is empty, output 'Failed'.
end for

/* Extracting D from the kernels */
Compute $\alpha_k, \beta_k, \gamma_k, \delta_k \in \mathbb{L}$ for $k \in [w]$ such that $\mathbf{u}_{w+1} = \sum_{k=1}^{w} \alpha_k \mathbf{u}_k$, $\mathbf{v}_{w+1} = \sum_{k=1}^{w} \beta_k \mathbf{v}_k$, $\mathbf{w}_{w+1} = \sum_{k=1}^{w} \gamma_k \mathbf{w}_k$ and $\mathbf{s}_{w+1} = \sum_{k=1}^{w} \delta_k \mathbf{s}_k$. If the computation fails, or any of α_k, β_k, γ_k, δ_k is zero for some $k \in [w]$, output 'Failed'.

Set $U, V, W, S \in \mathbb{L}^{w\times w}$ such that the k-th column of U, V, W, S are $\frac{\alpha_k \cdot \mathbf{u}_k}{\beta_k}$, $\mathbf{v}_k$, $\frac{\gamma_k \cdot \mathbf{w}_k}{\delta_k}$, $\mathbf{s}_k$ respectively. If any of $U, V, W, S \notin \mathsf{GL}(w, \mathbb{L})$, output 'Failed'.

if $UV^{-1}SW^{-1}$ is a scalar matrix **then**
    Set $\tilde{D} = U \cdot V^{-1}$ and output $\tilde{D}$.
else
    Output 'Failed'.   /* The check fails w.h.p if Z is not the 'last' matrix */
end if

Reducing Z and F_t modulo $\det(Z)$ *(Step 5)*: The reduction of the entries of Z and the blackbox entries of F_t modulo $\det(Z)$ is achieved by the substitution,

$$z_{11} = -\frac{\sum_{k=2}^{w} z_{1k} \cdot N_{1k}}{N_{11}}.$$

After the substitution, the matrices become Z' and $F_t' = R_1' \cdot R_2' \dots R_t'$ respectively. As there is an $i \in [t]$ and $C, D \in \mathsf{GL}(w, \mathbb{L})$ such that either $Z = C \cdot R_i \cdot D$ or $Z^T = C \cdot R_i \cdot D$, we have either $Z' = C \cdot R_i' \cdot D$ or $(Z')^T = C \cdot R_i' \cdot D$ and hence $\det(Z') = \det(R_i') = \det(F_t') = 0$.

Observation 3.1. *1.* $\text{Kernel}_{\mathbb{L}(\mathbf{x})}(Z') = \text{span}_{\mathbb{L}(\mathbf{x})}\{(N_{11}\ N_{12}\ \dots N_{1w})^T\}$*,*

2. $\text{Kernel}_{\mathbb{L}(\mathbf{x})}((Z')^T) = \text{span}_{\mathbb{L}(\mathbf{x})}\{(N_{11}\ N_{21}\ \dots N_{w1})^T\}$*.*

Hence, $\text{Kernel}_{\mathbb{L}(\mathbf{x})}(Z')$ has dimension one, and the observation below implies $\text{Kernel}_{\mathbb{L}(\mathbf{x})}(F_t')$ is also one dimensional. The proof follows from the coprimality of $\det(R_1), \det(R_2), \dots, \det(R_t)$.

Observation 3.2. *For all $j \in [t]$ and $j \neq i$, $\det(R_j') \neq 0$, and so the dimension of* $\text{Kernel}_{\mathbb{L}(\mathbf{x})}(F_t')$ *is one.*

Computing the kernels (Steps 8–10): The following observation shows that the algorithm does not fail at step 9 with high probability. The proof is immediate from the above two observations and an application of the Schwartz-Zippel lemma.

Observation 3.3. *Let $\mathbf{a}_k, \mathbf{b}_k \in_r \mathbb{F}^n$ for $k \in [w+1]$. Then, for every $k \in [w+1]$, and $\mathbf{a} = \mathbf{a}_k$ or $\mathbf{b}_k$,*

1. $\text{Ker}(Z'(\mathbf{a})) = \text{span}_{\mathbb{L}}\{(N_{11}(\mathbf{a})\ N_{12}(\mathbf{a})\ \dots N_{1w}(\mathbf{a}))^T\}$*,*

2. $\text{Ker}((Z'(\mathbf{a}))^T) = \text{span}_{\mathbb{L}}\{(N_{11}(\mathbf{a})\ N_{21}(\mathbf{a})\ \dots N_{w1}(\mathbf{a}))^T\}$*,*

and $\text{Ker}(F_t'(\mathbf{a}_k))$*,* $\text{Ker}(F_t'(\mathbf{b}_k))$ *are one dimensional subspaces of $\mathbb{L}^w$, with probability $1 - (wdn)^{-\Omega(1)}$.*

Extracting D from the kernels (Steps 13 – 21): We analyse these steps for three separate cases. The analysis shows that if Z is the 'last' matrix then the algorithm succeeds with high probability, otherwise the test at step 17 fails with high probability.

Case a $[Z = C \cdot R_t \cdot D]$: From Observation 3.2, $\det(R_j'(\mathbf{a}_k))$ and $\det(R_j'(\mathbf{b}_k))$ are nonzero with high probability, for all $j \in [t-1]$ and $k \in [w+1]$. Assuming this, the following holds for all $k \in [w+1]$:

$$\begin{aligned} D \cdot \text{Ker}(Z'(\mathbf{a}_k)) &= \text{Ker}(F_t'(\mathbf{a}_k)) \ , \\ D \cdot \text{Ker}(Z'(\mathbf{b}_k)) &= \text{Ker}(F_t'(\mathbf{b}_k)) \ . \end{aligned} \tag{2}$$

Hence, at step 9, there are $\lambda_k, \rho_k \in \mathbb{L}^{\times}$ such that

$$D \cdot \mathbf{v}_k = \lambda_k \mathbf{u}_k, \quad D \cdot \mathbf{s}_k = \rho_k \mathbf{w}_k \qquad \text{for } k \in [w+1].$$

Step 13 also succeeds with high probability due to the following claim (proof in Appendix A).

Claim 3.1. *With probability $1 - (wdn)^{-\Omega(1)}$, any subset of w vectors in any of the sets $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{w+1}\}$, $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{w+1}\}$, $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_{w+1}\}$, or $\{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_{w+1}\}$ are $\mathbb{L}$-linearly independent.*

At this step, $\mathbf{v}_{w+1} = \sum_{k=1}^{w} \beta_k \mathbf{v}_k$ and $\mathbf{s}_{w+1} = \sum_{k=1}^{w} \delta_k \mathbf{s}_k$, and so by applying D on both sides,

$$\lambda_{w+1}\mathbf{u}_{w+1} = \sum_{k=1}^{w} \beta_k \lambda_k \mathbf{u}_k, \quad \rho_{w+1}\mathbf{w}_{w+1} = \sum_{k=1}^{w} \delta_k \rho_k \mathbf{w}_k \ .$$

Also, $\mathbf{u}_{w+1} = \sum_{k=1}^{w} \alpha_k \mathbf{u}_k$ and $\mathbf{w}_{w+1} = \sum_{k=1}^{w} \gamma_k \mathbf{w}_k$. By Claim 3.1, none of the $\alpha_k, \beta_k, \gamma_k, \delta_k$ is zero and

$$\frac{\lambda_k}{\lambda_{w+1}} = \frac{\alpha_k}{\beta_k} \ , \quad \frac{\rho_k}{\rho_{w+1}} = \frac{\gamma_k}{\delta_k}, \qquad \text{for all } k \in [w].$$

From the construction of U, V, W and S at step 15,

$$D \cdot V = \lambda_{w+1} U \ , \quad D \cdot S = \rho_{w+1} W,$$

and $U, V, W, S \in \mathsf{GL}(w, \mathbb{L})$ (by Claim 3.1). Therefore, $UV^{-1}SW^{-1}$ is a scalar matrix.

Case b $[Z^T = C \cdot R_t \cdot D]$: In this case, the check at step 17 fails with high probability. Suppose the algorithm passes steps 13 and 15, and reaches step 17. We show that $UV^{-1}SW^{-1}$ being a scalar matrix implies an event $\mathcal{E}$ that happens with a low probability. The event $\mathcal{E}$ can be derived as follows:

Let $M \stackrel{\text{def}}{=} U \cdot V^{-1}$, and $c \in \mathbb{L}^{\times}$ such that $M = cW \cdot S^{-1}$. Assuming the invertibility of $R'_j(\mathbf{a}_k)$ and $R'_j(\mathbf{b}_k)$ for $j \in [t-1]$ (Observation 3.2), and as in Equation 2, the following holds for all $k \in [w+1]$.

$$\begin{aligned} D \cdot \mathrm{Ker}((Z'(\mathbf{a}_k))^T) &= \mathrm{Ker}(F'_t(\mathbf{a}_k)) \ , \\ D \cdot \mathrm{Ker}((Z'(\mathbf{b}_k))^T) &= \mathrm{Ker}(F'_t(\mathbf{b}_k)) \ . \end{aligned}$$

By Observation 3.3, we can assume the above four kernels are one-dimensional. Hence, at step 9 there are $\mathbf{p}_k \in \mathrm{Ker}((Z'(\mathbf{a}_k))^T)$ and $\mathbf{q}_k \in \mathrm{Ker}((Z'(\mathbf{b}_k))^T)$ satisfying $D \cdot \mathbf{p}_k = \mathbf{u}_k$ and $D \cdot \mathbf{q}_k = \mathbf{w}_k$, for every $k \in [w+1]$. Consider the $w \times w$ matrices P and Q such that the k-th column of these matrices are $\frac{\alpha_k}{\beta_k}\mathbf{p}_k$ and $\frac{\gamma_k}{\delta_k}\mathbf{q}_k$ respectively, where $\alpha_k, \beta_k, \gamma_k, \delta_k$ are the constants computed at step 13. Clearly, $D \cdot P = U$ and $D \cdot Q = W$, where U, W are the matrices computed at step 15.

As $M = cW \cdot S^{-1}$ (by assumption), we have $D^{-1}MS = cD^{-1}W = cQ$. Hence, for $k \in [w]$,

$$D^{-1}M \cdot \mathbf{s}_k = \frac{c\gamma_k}{\delta_k}\mathbf{q}_k.$$

At step 13, $\mathbf{w}_{w+1} = \sum_{k=1}^{w} \gamma_k \mathbf{w}_k$ and $\mathbf{s}_{w+1} = \sum_{k=1}^{w} \delta_k \mathbf{s}_k$. Multiplying D^{-1} on both sides and $D^{-1}M$ on both sides of these two equations respectively,

$$\mathbf{q}_{w+1} = \sum_{k=1}^{w} \gamma_k \mathbf{q}_k, \quad \text{and} \quad D^{-1}M \cdot \mathbf{s}_{w+1} = \sum_{k=1}^{w} c\gamma_k \mathbf{q}_k \ .$$

$$\Rightarrow \quad D^{-1}M \cdot \mathbf{s}_{w+1} = c\mathbf{q}_{w+1}. \tag{3}$$

From Observation 3.3, there are $\lambda_1, \lambda_2 \in \mathbb{L}^{\times}$ such that

$$\begin{aligned} \mathbf{s}_{w+1} &= \lambda_1 \cdot (N_{11}(\mathbf{b}_{w+1}) \ \ N_{12}(\mathbf{b}_{w+1}) \ \ldots \ N_{1w}(\mathbf{b}_{w+1}))^T \ , \\ \mathbf{q}_{w+1} &= \lambda_2 \cdot (N_{11}(\mathbf{b}_{w+1}) \ \ N_{21}(\mathbf{b}_{w+1}) \ \ldots \ N_{w1}(\mathbf{b}_{w+1}))^T \ . \end{aligned}$$

Let $D^{-1}M = (m_{lk})_{l,k\in[w]}$. Using the above values of $\mathbf{s}_{w+1}$ and $\mathbf{q}_{w+1}$ in Equation 3 and restricting to the first two entries of the resulting column vectors, we have

$$\lambda_1 \left(\sum_{k=1}^{w} m_{1k} N_{1k}(\mathbf{b}_{w+1}) \right) = c\lambda_2 N_{11}(\mathbf{b}_{w+1}) \ , \quad \lambda_1 \left(\sum_{k=1}^{w} m_{2k} N_{1k}(\mathbf{b}_{w+1}) \right) = c\lambda_2 N_{21}(\mathbf{b}_{w+1}) \ .$$

Thus we get the following relation,

$$N_{21}(\mathbf{b}_{w+1}) \left(\sum_{k=1}^{w} m_{1k} N_{1k}(\mathbf{b}_{w+1}) \right) = N_{11}(\mathbf{b}_{w+1}) \left(\sum_{k=1}^{w} m_{2k} N_{1k}(\mathbf{b}_{w+1}) \right).$$

Event $\mathcal{E}$ is defined by the above equality, i.e. we say $\mathcal{E}$ has happened whenever the above equality holds. Now observe that $D^{-1}M$ is *independent*[29] of the random bits used to choose $\mathbf{b}_{w+1}$. Hence, it is sufficient to show that the above equality happens with low probability over the randomness of $\mathbf{b}_{w+1}$, for any arbitrarily fixed $m_{11}, \ldots, m_{1w}$ and $m_{21}, \ldots, m_{2w}$ from $\mathbb{L}$. Moreover, as $D^{-1}M$ is invertible, we can assume – not all in $\{m_{11}, \ldots, m_{1w}\}$ or $\{m_{21}, \ldots, m_{2w}\}$ are zero. The following observation and Schwartz-Zippel lemma complete the proof in this case.

Observation 3.4. *$N_{21}(\mathbf{z}) \left(\sum_{k=1}^{w} m_{1k} \cdot N_{1k}(\mathbf{z})\right) \neq N_{11}(\mathbf{z}) \left(\sum_{k=1}^{w} m_{2k} \cdot N_{1k}(\mathbf{z})\right)$ as polynomials in $\mathbb{F}[\mathbf{z}]$.*

Proof. Suppose the two sides are equal. As $N_{21}(\mathbf{z})$ and $N_{11}(\mathbf{z})$ are irreducible and coprime polynomials, $N_{21}(\mathbf{z})$ must divide $\sum_{k=1}^{w} m_{2k} \cdot N_{1k}(\mathbf{z})$. But the two polynomials have the same degree and they are monomial disjoint, thereby giving us a contradiction. □

Case c [$Z = C \cdot R_i \cdot D$ or $Z^T = C \cdot R_i \cdot D$ for some $i \in [t-1]$]: Assume $Z = C \cdot R_i \cdot D$ for some $i \in [t-1]$. The case $Z^T = C \cdot R_i \cdot D$ can be argued similarly. Similar to Case b, we show that if the algorithm passes steps 13 and 15, and reaches step 17 then $UV^{-1}SW^{-1}$ being a scalar matrix implies an event $\mathcal{E}$ that happens with very low probability. Hence, the check at step 17 fails with high probability. The event $\mathcal{E}$ can be derived as follows:

Let $M \stackrel{\text{def}}{=} U \cdot V^{-1}$, and $c \in \mathbb{L}^\times$ be such that $M = c \cdot WS^{-1}$. From the construction of W and S,

$$\frac{c\gamma_k}{\delta_k}\mathbf{w}_k = M \cdot \mathbf{s}_k \ , \quad \text{for all} \quad k \in [w],$$

where γ_k, δ_k are as computed at step 13. Since $\mathbf{w}_{w+1} = \sum_{k=1}^{w} \gamma_k \mathbf{w}_k$ and $\mathbf{s}_{w+1} = \sum_{k=1}^{w} \delta_k \cdot \mathbf{s}_k$,

$$c \cdot \mathbf{w}_{w+1} = M \cdot \mathbf{s}_{w+1}.$$

Let $H \stackrel{\text{def}}{=} D^{-1} \cdot R'_{i+1} \ldots R'_t$. From Observation 3.2, the following holds,

$$H^{-1} \cdot \text{Kernel}_{\mathbb{L}(\mathbf{x})}(Z') = \text{Kernel}_{\mathbb{L}(\mathbf{x})}(F'_t).$$

Let $\mathbf{n} = (N_{11}(\mathbf{b}_{w+1}) \quad N_{12}(\mathbf{b}_{w+1}) \quad \ldots \quad N_{1w}(\mathbf{b}_{w+1}))^T$. From Observation 3.3, and as $H(\mathbf{b}_{w+1})$ is invertible with high probability over the random choice of $\mathbf{b}_{w+1}$, there are $\lambda_1, \lambda_2 \in \mathbb{L}^\times$ such that

$$\mathbf{w}_{w+1} = \lambda_1 H^{-1}(\mathbf{b}_{w+1}) \cdot \mathbf{n}$$
$$\mathbf{s}_{w+1} = \lambda_2 \mathbf{n}.$$

[29] One way of seeing this is that $D^{-1}M$ is already fixed before $\mathbf{b}_{w+1}$ is chosen.

Substituting the above values of $\mathbf{w}_{w+1}$ and $\mathbf{s}_{w+1}$ in $c \cdot \mathbf{w}_{w+1} = M \cdot \mathbf{s}_{w+1}$, we have

$$c\lambda_1 H^{-1}(\mathbf{b}_{w+1}) \cdot \mathbf{n} = \lambda_2 M \cdot \mathbf{n}\,, \quad \Rightarrow \quad c\lambda_1 \mathbf{n} = \lambda_2 H(\mathbf{b}_{w+1}) \cdot M \cdot \mathbf{n}.$$

Let $H \cdot M = (h_{lk})_{l,k\in[w]}$. Restricting to the first two entries of the vectors in the above equality, we have

$$c\lambda_1 N_{11}(\mathbf{b}_{w+1}) = \lambda_2 \left(\sum_{k=1}^{w} h_{1k}(\mathbf{b}_{w+1}) \cdot N_{1k}(\mathbf{b}_{w+1}) \right),$$
$$c\lambda_1 N_{12}(\mathbf{b}_{w+1}) = \lambda_2 \left(\sum_{k=1}^{w} h_{2k}(\mathbf{b}_{w+1}) \cdot N_{1k}(\mathbf{b}_{w+1}) \right).$$

Hence, we get the following relation

$$N_{11}(\mathbf{b}_{w+1}) \cdot \left(\sum_{k=1}^{w} h_{2k}(\mathbf{b}_{w+1}) \cdot N_{1k}(\mathbf{b}_{w+1}) \right) = N_{12}(\mathbf{b}_{w+1}) \cdot \left(\sum_{k=1}^{w} h_{1k}(\mathbf{b}_{w+1}) \cdot N_{1k}(\mathbf{b}_{w+1}) \right). \quad (4)$$

Event $\mathcal{E}$ is defined by the above equality, that is $\mathcal{E}$ happens if the above equality is satisfied. Observe that the entries of the matrix product $H \cdot M = (h_{lk})_{l,k\in[w]}$ are rational functions in $\mathbf{x}$ variables and are *independent* of the random bits used to choose $\mathbf{b}_{w+1}$. We show next the probability that the above equality holds is low over the randomness of $\mathbf{b}_{w+1}$.

The only implications of the average-case nature of F_t that we have used in the proofs so far are: every R_i is full rank and $\det(R_1), \ldots, \det(R_t)$ are mutually coprime with high probability. However, these two properties are not sufficient to ensure the uniqueness of the last matrix in the product (as mentioned in a remark after Theorem 1). In the following claim, we use one more effect of F_t being a random matrix product which ensures the desired uniqueness of the last matrix.

Claim 3.2. *If $E = Q_1 \cdots Q_\ell$ is a random (w, ℓ, m)-matrix product over $\mathbb{F}$, where $w^2 + 1 \leq m \leq n$ and $\ell \leq d$, then the entries of E are $\mathbb{F}$-linearly independent with probability $1 - (wdn)^{-\Omega(1)}$.*

If the entries of E are $\mathbb{F}$-linearly independent then they are also $\mathbb{L}$-linearly independent. We conclude the proof of Case c using the above claim (proof given in Appendix A).

Observation 3.5. *Let $n \geq 2w^2$. Then all the entries of $H \cdot M$ are nonzero polynomials after setting the variables in $\mathbf{z}_1 \stackrel{\text{def}}{=} \{z_{11}, z_{21}, z_{31}, \ldots, z_{w1}\}$ to zero, with probability $1 - (wdn)^{-\Omega(1)}$.*

Proof. $H \cdot M = D^{-1} \cdot R'_{i+1} \ldots R'_t \cdot M = (h_{lk})_{l,k\in[w]}$. Recalling the substitution $z_{11} = \frac{-\sum_{k=2}^{w} z_{1k} N_{1k}}{N_{11}}$ at step 5, we observe that the rational function h_{lk} becomes a polynomial under the setting $z_{11} = z_{21} = \ldots = z_{w1} = 0$ [30]. Let $Q_j = (R_j)_{\mathbf{z}_1=0}$. By observing $(R_j)_{\mathbf{z}_1=0} = (R'_j)_{\mathbf{z}_1=0}$, it follows that $(H \cdot M)_{\mathbf{z}_1=0} = D^{-1} \cdot Q_{i+1} \ldots Q_t \cdot M$. Moreover, $Q_{i+1} \cdot Q_{i+2} \ldots Q_t$ is a random $(w, t-i, n-w)$-matrix product. By Claim 3.2, the entries of $Q_{i+1} \ldots Q_t$ are $\mathbb{L}$-linearly independent with high probability. Hence, none of the entries of $D^{-1} \cdot Q_{i+1} \ldots Q_t \cdot M$ is zero as $D, M \in \mathsf{GL}(\mathbb{L}, w)$. □

Observation 3.6. $N_{11}(\mathbf{x}) \cdot (\sum_{k=1}^{w} h_{2k}(\mathbf{x}) N_{1k}(\mathbf{x})) \neq N_{12}(\mathbf{x}) \cdot (\sum_{k=1}^{w} h_{1k}(\mathbf{x}) N_{1k}(\mathbf{x}))$ *as rational functions in $\mathbb{L}(\mathbf{x})$, with probability $1 - (wdn)^{-\Omega(1)}$.*

[30] z_{11} does not even appear in h_{lk}.

Proof. Suppose $N_{11}(\mathbf{x}) \cdot (\sum_{k=1}^{w} h_{2k}(\mathbf{x}) N_{1k}(\mathbf{x})) = N_{12}(\mathbf{x}) \cdot (\sum_{k=1}^{w} h_{1k}(\mathbf{x}) N_{1k}(\mathbf{x}))$. By substituting $\mathbf{z}_1 = 0$ in the equation, the R.H.S becomes zero whereas the L.H.S reduces to $N_{11}^2 \cdot (h_{21})_{\mathbf{z}_1=0} \neq 0$ with high probability (from Observation 3.5). □

Noting that the degrees of the numerator and the denominator of h_{lk} are upper bounded by wd, we conclude that the equality in Equation 4 happens with a low probability over the randomness of $\mathbf{b}_{w+1}$.

4 Average-case ABP reconstruction: Proof of Theorem 2

The algorithm for average-case ABP reconstruction is presented in Algorithm 2, Section 1.4.2. The algorithm uses Algorithm 5 and Algorithm 6 during its execution – we present and analyze these two algorithms in the following subsections.

4.1 Computing the corner spaces

Let f be the polynomial computed by a random (w,d,n)-ABP $X_1 \cdot X_2 \ldots X_d$ over $\mathbb{F}$, where $n \geq 4w^2$.

Lemma 4.1. *With probability $1-(wdn)^{-\Omega(1)}$ over the randomness of f, the following holds: Let $\mathbb{K} \supseteq \mathbb{F}$ be any field and $f = 0 \mod \langle l_1, \ldots, l_k \rangle$, where l_i's are linear forms in $\mathbb{K}[\mathbf{x}]$. Then $k \geq w$ and for $k = w$, the space $\text{span}_{\mathbb{K}}\{l_1, \ldots, l_w\}$ equals the $\mathbb{K}$-span of either the linear forms in X_1 or the linear forms in X_d.*

The above uniqueness of the corner spaces, $\mathcal{X}_1$ and $\mathcal{X}_d$ (defined in Section 1.4.2), helps compute them in Algorithm 5. The proof of the lemma is given at the end of this subsection.

Canonical bases of $\mathcal{X}_1$ and $\mathcal{X}_d$: For a set of variables $\mathbf{y} \subseteq \mathbf{x}$ and a linear form g in $\mathbb{F}[\mathbf{x}]$, define $g(\mathbf{y}) \stackrel{\text{def}}{=} g_{\mathbf{x} \setminus \mathbf{y} = 0}$. We say $g(\mathbf{y})$ is the linear form g projected to the $\mathbf{y}$ variables. Let $x_1, \ldots, x_w$ and v be a designated set of $w+1$ variables in $\mathbf{x}$, and $\mathbf{u} = \mathbf{x} \setminus \{x_1, \ldots, x_w, v\}$. With $n \geq 4w^2$, a random (w,d,n)-ABP $X_1 \cdot X_2 \ldots X_d$ satisfies the following condition with probability $1-(wdn)^{-\Omega(1)}$:
(*a) The linear forms in X_1 (similarly, X_d) projected to $x_1, \ldots, x_w$ are $\mathbb{F}$-linearly independent.
If the above condition is satisfied then there is a $C \in \mathsf{GL}(w, \mathbb{F})$ such that the linear forms in $X_1 \cdot C$ are of the kind:

$$x_i - \alpha_i v - g_i(\mathbf{u}), \quad \text{for } i \in [w], \tag{5}$$

where each $\alpha_i \in \mathbb{F}$ and g_i is a linear form in $\mathbb{F}[\mathbf{u}]$. Thus, we can assume without loss of generality, the linear forms in X_1 are of the above kind. Similarly, the linear forms in X_d are also of the kind:

$$x_i - \beta_i v - h_i(\mathbf{u}), \quad \text{for } i \in [w], \tag{6}$$

where each $\beta_i \in \mathbb{F}$ and h_i is a linear form in $\mathbb{F}[\mathbf{u}]$. Moreover, with probability $1-(wdn)^{-\Omega(1)}$ over the randomness of the ABP, the following condition is satisfied:
(*b) $\alpha_1, \ldots, \alpha_w$ and $\beta_1, \ldots, \beta_w$ are distinct elements in $\mathbb{F}$.
The task at hand for Algorithm 5 is to solve for α_i, g_i and β_j, h_j, for $i, j \in [w]$, assuming that conditions (*a) and (*b) are satisfied. The bases defined by Equations 5 and 6 are canonical for $\mathcal{X}_1$ and $\mathcal{X}_d$.

We analyze the three main steps of Algorithm 5 next:

Algorithm 5 Computing the corner spaces

INPUT: Blackbox access to a f computed by a random (w, d, n)-ABP.
OUTPUT: Bases of the two corner spaces $\mathcal{X}_1$ and $\mathcal{X}_d$ modulo which f is zero.

1. /* Partitioning the variables */
2. Choose $w+1$ designated variables $x_1, x_2, \ldots, x_w, v$, and let $\mathbf{u} = \mathbf{x} \setminus \{x_1, \ldots, x_w, v\}$. Partition $\mathbf{u}$ into sets $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$, each of size $4w^2 - (w+1)$. .
3.
4. /* Reduction to solving m systems of polynomial equations */
5. **for** $\ell = 1$ **to** m **do**
6.   Set $f_\ell = f_{\mathbf{u} \setminus \mathbf{u}_\ell = 0}$.
7.   Solve for *all possible* $(\alpha_1, \ldots, \alpha_w, g_1(\mathbf{u}_\ell), \ldots, g_w(\mathbf{u}_\ell))$, where each $\alpha_i \in \mathbb{F}$ and $g_i(\mathbf{u}_\ell)$ is a linear form in $\mathbb{F}[\mathbf{u}_\ell]$ such that
$$f_\ell = 0 \mod \langle x_1 - \alpha_1 v - g_1(\mathbf{u}_\ell), \ldots, x_w - \alpha_w v - g_w(\mathbf{u}_\ell) \rangle.$$
8.   **if** Step 7 does not return *exactly two* solutions for $(\alpha_1, \ldots, \alpha_w, g_1(\mathbf{u}_\ell), \ldots, g_w(\mathbf{u}_\ell))$ **then**
9.   Output 'Failed'.
10.   **else**
11.   The solutions be $(\alpha_{\ell 1}, \ldots, \alpha_{\ell w}, g_1(\mathbf{u}_\ell), \ldots, g_w(\mathbf{u}_\ell))$ and $(\beta_{\ell 1}, \ldots, \beta_{\ell w}, h_1(\mathbf{u}_\ell), \ldots, h_w(\mathbf{u}_\ell))$.
12.   **end if**
13. **end for**
14.
15. /* Combining the solutions */
16. **if** $|\cup_{\ell \in [m]} \{(\alpha_{\ell 1}, \ldots, \alpha_{\ell w}), (\beta_{\ell 1}, \ldots, \beta_{\ell w})\}| \neq 2$ **then**
17.   Output 'Failed'.
18. **else**
19.   Without loss of generality, $(\alpha_{\ell 1}, \ldots, \alpha_{\ell w}) = (\alpha_1, \ldots, \alpha_w)$ and $(\beta_{\ell 1}, \ldots, \beta_{\ell w}) = (\beta_1, \ldots, \beta_w)$ for every $\ell \in [m]$. Set $g_i(\mathbf{u}) = \sum_{\ell \in [w]} g_i(\mathbf{u}_\ell)$ and $h_i(\mathbf{u}) = \sum_{\ell \in [w]} h_i(\mathbf{u}_\ell)$ for every $i \in [w]$.
20.   Return $\{x_i - \alpha_i v - g_i(\mathbf{u})\}_{i \in [w]}$ and $\{x_i - \beta_i v - h_i(\mathbf{u})\}_{i \in [w]}$ as the bases of $\mathcal{X}_1$ and $\mathcal{X}_d$.
21. **end if**

1. *Partitioning the variables (Step 2)*: The only thing to note here is, if $n-(w+1)$ is not divisible by $4w^2-(w+1)$ then we allow the last two sets $\mathbf{u}_{m-1}$ and $\mathbf{u}_m$ to overlap – the algorithm can be suitably adjusted in this case.

2. *Reduction to solving systems of polynomial equations (Steps 5–13)*: At step 7, the task of computing $(\alpha_1,\ldots,\alpha_w,g_1(\mathbf{u}_\ell),\ldots,g_w(\mathbf{u}_\ell))$ such that

$$f_\ell = 0 \mod \langle x_1-\alpha_1 v - g_1(\mathbf{u}_\ell),\ldots,x_w-\alpha_w v-g_w(\mathbf{u}_\ell)\rangle,$$

can be reduced to solving for all $\mathbb{F}$-rational points of a system of polynomial equations over $\mathbb{F}$ as follows: Treat $\alpha_1,\ldots,\alpha_w$ and the $4w^3-w(w+1)$ coefficients of $g_1(\mathbf{u}_\ell),\ldots,g_w(\mathbf{u}_\ell)$, say $\mathbf{w}$, as formal variables. Substitute $x_i=\alpha_i v+g_i(\mathbf{u}_\ell)$ for every $i\in[w]$ in the blackbox for f_ℓ, and interpolate the resulting polynomial p in the variables $\alpha_1,\ldots,\alpha_w,\mathbf{w},v,\mathbf{u}_\ell$ with coefficients in $\mathbb{F}$. The interpolation, which can be done in $(d^{w^3}\log q)^{O(1)}$ time[31], gives p in dense representation (i.e. as a sum of monomials). Now by treating p as a polynomial in the variables $v,\mathbf{u}_\ell$ with coefficients in $\mathbb{F}(\alpha_1,\ldots,\alpha_w,\mathbf{w})$, and equating these coefficients to zero, we get a system of $d^{O(w^2)}$ polynomial equations in $O(w^3)$ variables with degree of each polynomial equation bounded by d. By Lemma 4.1, such a system has exactly two solutions over $\overline{\mathbb{F}}$ and moreover, these two solution points are $\mathbb{F}$-rational. Hence, by applying Lemma 2.2, we can compute the two solutions for $(\alpha_1,\ldots,\alpha_w,\mathbf{w})$ at step 7, in $(d^{w^3}\log q)^{O(1)}$ time.

3. *Combining the solutions (Steps 16–21):* The correctness of the steps follows from condition (*b).

Uniqueness of the corner spaces: Proof of Lemma 4.1

As $n\geq 4w^2$, a random (w,d,n)-ABP $X_1\cdots X_d$ satisfies the following condition with probability $1-(wdn)^{-\Omega(1)}$:
(**) The linear forms in X_1,X_d and any three or less of the other X_i's are $\mathbb{F}$-linearly independent.
So, it is sufficient to prove the following restatement of the lemma.

Lemma 4.1. *Suppose f is computed by a (w,d,n)-ABP $X_1\cdot X_2\cdots X_d$ satisfying the above condition (**). If $f=0 \mod \langle l_1,\ldots,l_k\rangle$, where l_i's are linear forms over $\mathbb{K}\supseteq\mathbb{F}$, then $k\geq w$ and for $k=w$, the space $\text{span}_{\mathbb{K}}\{l_1,\ldots,l_w\}$ equals the $\mathbb{K}$-span of either the linear forms in X_1 or the linear forms in X_d.*

We prove the lemma first for $d=3$, and then use this case to prove it for $d>3$.

Case $[d=3]$: There is an $A\in\mathsf{GL}(n,\mathbb{F})$ such that $f(A\cdot\mathbf{x})$ is computed by $(y_1\,y_2\,\ldots y_w)\cdot(r_{ij})_{i,j\in[w]}\cdot(z_1\,z_2\,\ldots z_w)^T$, where $\mathbf{y}=\{y_i\}_{i\in[w]}$, $\mathbf{r}=\{r_{ij}\}_{i,j\in[w]}$ and $\mathbf{z}=\{z_j\}_{j\in[w]}$ are distinct variables in $\mathbf{x}$. If $f=0 \mod \langle l_1,\ldots,l_k\rangle$, then $f(A\cdot\mathbf{x})=0 \mod \langle l_1(A\cdot\mathbf{x}),\ldots,l_k(A\cdot\mathbf{x})\rangle$. Next, we show that if $f(A\cdot\mathbf{x})=0$ modulo k' linear forms $h_1,\ldots,h_{k'}\in\mathbb{K}[\mathbf{y}\uplus\mathbf{z}\uplus\mathbf{r}]$ then $k'\geq w$, and for $k'=w$, the space $\text{span}_{\mathbb{K}}\{h_1,\ldots,h_w\}$ equals either $\text{span}_{\mathbb{K}}\{y_1,\ldots,y_w\}$ or $\text{span}_{\mathbb{K}}\{z_1,\ldots,z_w\}$. It follows that $k\geq k'\geq w$, and for $k=w$, the linear forms $l_1(A\cdot\mathbf{x}),\ldots,l_w(A\cdot\mathbf{x})$ must belong to $\mathbb{K}[\mathbf{y}\uplus\mathbf{z}\uplus\mathbf{r}]$[32] and hence $\text{span}_{\mathbb{K}}\{l_1,\ldots,l_w\}$ equals the $\mathbb{K}$-span of either the linear forms in X_1 or the linear forms in X_d.

[31] As the individual degrees of the variables in p are bounded by d, we only need $|\mathbb{F}|>d$ to carry out this interpolation.
[32] Otherwise, we will have $f(A\cdot\mathbf{x})=0$ modulo less than w linear forms in $\mathbb{K}[\mathbf{y}\uplus\mathbf{z}\uplus\mathbf{r}]$.

Reusing symbols, assume that f is computed by $X_1 \cdot X_2 \cdot X_3$, where $X_1 = (y_1\ y_2\ \ldots y_w)$, $X_2 = (r_{ij})_{i,j\in[w]}$ and $X_3 = (z_1\ z_2\ \ldots z_w)^T$, and $f = 0 \mod \langle l_1, \ldots, l_k\rangle$, where l_i's are linear forms in $\mathbb{K}[\mathbf{y} \uplus \mathbf{z} \uplus \mathbf{r}]$. Suppose $k \leq w$; otherwise, we have nothing to prove. Consider the reduced Gröbner basis[33] G of the ideal $\langle l_1, \ldots, l_k\rangle$ with respect to the lexicographic monomial ordering defined by $\mathbf{y} \succ \mathbf{z} \succ \mathbf{r}$. There are sets $S_{\mathbf{y}}, S_{\mathbf{z}} \subseteq [w]$ and $S_{\mathbf{r}} \subseteq [w] \times [w]$, satisfying $|S_{\mathbf{y}}| + |S_{\mathbf{z}}| + |S_{\mathbf{r}}| \leq k$, such that G consists of linear forms of the kind:

$$\begin{aligned} y_i - g_i(\mathbf{y}, \mathbf{z}, \mathbf{r}) &\qquad \text{for } i \in S_{\mathbf{y}}, \\ z_j - h_j(\mathbf{z}, \mathbf{r}) &\qquad \text{for } j \in S_{\mathbf{z}}, \\ r_{\ell e} - p_{\ell e}(\mathbf{r}) &\qquad \text{for } (\ell, e) \in S_{\mathbf{r}}, \end{aligned}$$

where g_i, h_j and $p_{\ell e}$ are linear forms over $\mathbb{K}$ in their respective sets of variables. Let X_1', X_2', X_3' be the linear matrices obtained from X_1, X_2, X_3 respectively, by replacing y_i by $g_i(\mathbf{y}, \mathbf{z}, \mathbf{r})$, $r_{\ell e}$ by $p_{\ell e}(\mathbf{r})$ and z_j by $h_j(\mathbf{z}, \mathbf{r})$, for $i \in S_{\mathbf{y}}$, $(\ell, e) \in S_{\mathbf{r}}$ and $j \in S_{\mathbf{z}}$. Then,

$$X_1' \cdot X_2' \cdot X_3' = 0. \tag{7}$$

The dimension of the $\mathbb{K}$-span of the linear forms of X_1' is at least $(w - |S_{\mathbf{y}}|)$, that of X_2' is at least $(w^2 - |S_{\mathbf{r}}|)$, and of X_3' is at least $(w - |S_{\mathbf{z}}|)$. Also, there are $C, D \in \mathsf{GL}(w, \mathbb{K})$ such that $X_1' \cdot C, D \cdot X_3'$ are obtained[34] from X_1, X_3 respectively, by replacing y_i by $g_i(0, \mathbf{z}, \mathbf{r})$ and z_j by $h_j(0, \mathbf{r})$, for $i \in S_{\mathbf{y}}$ and $j \in S_{\mathbf{z}}$. Consider the following equation,

$$(X_1'C) \cdot (C^{-1}X_2'D^{-1}) \cdot (DX_3') = 0. \tag{8}$$

By examining the L.H.S, we can conclude that for $s \in [w] \setminus S_{\mathbf{y}}$ and $t \in [w] \setminus S_{\mathbf{z}}$, the coefficient of the monomial $y_s z_t$ over $\mathbb{K}(\mathbf{r})$ is the (s,t)-th entry of $C^{-1}X_2'D^{-1}$ which must be zero. Hence, the dimension of the $\mathbb{K}$-span of the linear forms in $C^{-1}X_2'D^{-1}$ is at most $w^2 - (w - |S_{\mathbf{y}}|)(w - |S_{\mathbf{z}}|)$. As the dimension of the $\mathbb{K}$-span of the linear forms in X_2' remains unaltered under left and right multiplications by elements in $\mathsf{GL}(w, \mathbb{K})$, we get the relation

$$\begin{aligned} w^2 - |S_{\mathbf{r}}| &\leq w^2 - (w - |S_{\mathbf{y}}|)(w - |S_{\mathbf{z}}|) \\ \Rightarrow (w - |S_{\mathbf{y}}|)(w - |S_{\mathbf{z}}|) &\leq |S_{\mathbf{r}}| \\ \Rightarrow w^2 - (|S_{\mathbf{y}}| + |S_{\mathbf{z}}|)w + |S_{\mathbf{y}}| \cdot |S_{\mathbf{z}}| &\leq |S_{\mathbf{r}}| \\ \Rightarrow w^2 - (w - |S_{\mathbf{r}}|)w + |S_{\mathbf{y}}| \cdot |S_{\mathbf{z}}| &\leq |S_{\mathbf{r}}|, \quad \text{as } |S_{\mathbf{y}}| + |S_{\mathbf{z}}| + |S_{\mathbf{r}}| \leq k \leq w \\ \Rightarrow |S_{\mathbf{r}}|w + |S_{\mathbf{y}}| \cdot |S_{\mathbf{z}}| &\leq |S_{\mathbf{r}}|. \end{aligned}$$

As $|S_{\mathbf{y}}|, |S_{\mathbf{z}}|, |S_{\mathbf{r}}| \geq 0$, we must have $|S_{\mathbf{r}}| = 0$, and either $|S_{\mathbf{y}}| = 0$ or $|S_{\mathbf{z}}| = 0$.

Suppose $|S_{\mathbf{r}}| = |S_{\mathbf{z}}| = 0$ (the case for $|S_{\mathbf{r}}| = |S_{\mathbf{y}}| = 0$ is similar). Then, Equation 8 simplifies to

$$(X_1'C) \cdot (C^{-1}X_2) \cdot X_3 = 0.$$

[33] See [CLO07]. Equivalently, think of the set of linear forms obtained from a reduced row echelon form of the coefficient matrix of $l_1, \ldots, l_k$.

[34] via row and column operations on X_1' and X_3', respectively

If $k < w$ then there is a y_s in X_1 that is not replaced while forming $X'_1 C$ from X_1. By examining the coefficient of y_s over $\mathbb{K}(\mathbf{r}, \mathbf{z})$ in the L.H.S of the above equation, we arrive at a contradiction. Hence, $k = w$, in which case Equation 7 simplifies to

$$X'_1 \cdot X_2 \cdot X_3 = 0.$$

The entries of X'_1 are linear forms in $\mathbf{z}$ and $\mathbf{r}$, and so $X'_1 = X'_1(\mathbf{z}) + X'_1(\mathbf{r})$ where the entries of $X'_1(\mathbf{z})$ (similarly, $X'_1(\mathbf{r})$) are linear forms in $\mathbf{z}$ (respectively, $\mathbf{r}$). The above equation implies

$$X'_1(\mathbf{z}) \cdot X_2 \cdot X_3 = 0 \quad \text{and} \quad X'_1(\mathbf{r}) \cdot X_2 \cdot X_3 = 0,$$

as the two L.H.S above are monomial disjoint. It is now easy to argue that $X'_1(\mathbf{z}) = X'_1(\mathbf{r}) = 0$, implying $X'_1 = 0$ and hence the reduced Gröbner basis G is in fact $\{y_1, \ldots, y_w\}$.

Case [$d > 3$]: As before, by applying an invertible transformation, we can assume that $X_1 = (y_1\ y_2\ \ldots\ y_w)$, $X_2 = (r_{ij})_{i,j \in [w]}$ and $X_d = (z_1\ z_2\ \ldots\ z_w)^T$. Let $\mathbf{u} = \mathbf{x} \setminus (\mathbf{y} \uplus \mathbf{z} \uplus \mathbf{r})$ and $k \leq w$. Consider the reduced Gröbner basis G of the ideal $\langle l_1, l_2, \ldots, l_k \rangle$ with respect to the lexicographic monomial ordering defined by $\mathbf{u} \succ \mathbf{y} \succ \mathbf{z} \succ \mathbf{r}$. There are sets $S_{\mathbf{u}} \subseteq [n - w^2 - 2w]$, $S_{\mathbf{y}}, S_{\mathbf{z}} \subseteq [w]$ and $S_{\mathbf{r}} \subseteq [w^2]$, satisfying $|S_{\mathbf{u}}| + |S_{\mathbf{y}}| + |S_{\mathbf{z}}| + |S_{\mathbf{r}}| \leq k$, such that G consists of linear forms of the kind:

$$\begin{aligned} u_m - t_m(\mathbf{u}, \mathbf{y}, \mathbf{z}, \mathbf{r}) & \qquad \text{for } m \in S_{\mathbf{u}}, \\ y_i - g_i(\mathbf{y}, \mathbf{z}, \mathbf{r}) & \qquad \text{for } i \in S_{\mathbf{y}}, \\ z_j - h_j(\mathbf{z}, \mathbf{r}) & \qquad \text{for } j \in S_{\mathbf{z}}, \\ r_{\ell e} - p_{\ell e}(\mathbf{r}) & \qquad \text{for } (\ell, e) \in S_{\mathbf{r}}, \end{aligned}$$

where t_m, g_i, h_j and $p_{\ell e}$ are linear forms over $\mathbb{K}$ in their respective sets of variables. Let X' be the matrix obtained from X by replacing u_m by $t_m(\mathbf{u}, \mathbf{y}, \mathbf{z}, \mathbf{r})$, y_i by $g_i(\mathbf{y}, \mathbf{z}, \mathbf{r})$, z_j by $h_j(\mathbf{z}, \mathbf{r})$, and $r_{\ell e}$ by $p_{\ell e}(\mathbf{r})$, for $m \in S_{\mathbf{u}}$, $i \in S_{\mathbf{y}}$, $j \in S_{\mathbf{z}}$, and $(\ell, e) \in S_{\mathbf{r}}$. Then,

$$X'_1 \cdot X'_2 \cdot X'_3 \ldots X'_d = 0.$$

Let $X(\mathbf{u}) \stackrel{\text{def}}{=} (X)_{\mathbf{y}=\mathbf{z}=\mathbf{r}=0}$. By treating the L.H.S of the above equation as a polynomial in $\mathbf{u}$-variables with coefficients from $\mathbb{K}(\mathbf{y}, \mathbf{z}, \mathbf{r})$ and focusing on the degree-$(d-3)$ homogeneous component of this polynomial, we have

$$X'_1 \cdot X'_2 \cdot X'_3(\mathbf{u}) \ldots X'_{d-1}(\mathbf{u}) \cdot X'_d = 0. \tag{9}$$

If $X'_3(\mathbf{u}) \cdots X'_{d-1}(\mathbf{u}) \in \mathsf{GL}(w, \mathbb{K}(\mathbf{u}))$ then there is a $\mathbf{c} \in \mathbb{F}^{|\mathbf{u}|}$ such that $C = X'_3(\mathbf{c}) \cdots X'_{d-1}(\mathbf{c}) \in \mathsf{GL}(w, \mathbb{K})$. Define

$$f_1 = X_1 \cdot X_2 \cdot C \cdot X_d,$$

and observe that Equation 9 implies f_1 is zero modulo the linear forms,

$$\begin{aligned} y_i - g_i(\mathbf{y}, \mathbf{z}, \mathbf{r}) & \qquad \text{for } i \in S_{\mathbf{y}}, \\ z_j - h_j(\mathbf{z}, \mathbf{r}) & \qquad \text{for } j \in S_{\mathbf{z}}, \\ r_{\ell e} - p_{\ell e}(\mathbf{r}) & \qquad \text{for } (\ell, e) \in S_{\mathbf{r}}. \end{aligned}$$

By applying Case [d=3] on f_1, we get the desired conclusion, i.e. $k = w$ and the $\mathbb{K}$-span of the above linear forms (hence also that of $\{l_1, \ldots, l_k\}$) is either $\text{span}_{\mathbb{K}}\{y_1, \ldots, y_w\}$ or $\text{span}_{\mathbb{K}}\{z_1, \ldots, z_w\}$.

So, suppose $X'_3(\mathbf{u}) \cdots X'_{d-1}(\mathbf{u}) \notin \mathsf{GL}(w, \mathbb{K}(\mathbf{u}))$ in Equation 9. Then, there is a $j \in [3, d-1]$ such that $\det(X'_j(\mathbf{u})) = 0$. Observe that $X'_i(\mathbf{u})$ can be obtained from $X_i(\mathbf{u})$ by replacing u_m by $t_m(\mathbf{u}, 0, 0, 0)$ for $m \in S_{\mathbf{u}}$. That is,

$$X'_i(\mathbf{u}) = X_i(\mathbf{u}) \mod \langle \{u_m - t_m(\mathbf{u}, 0, 0, 0)\}_{m \in S_{\mathbf{u}}} \rangle, \quad \text{for every } i \in [3, d-1].$$

As $X_j(\mathbf{u})$ is full rank[35] and $\det(X'_j(\mathbf{u})) = 0$, the fact below implies $|S_{\mathbf{u}}| = w$, $|S_{\mathbf{y}}| = |S_{\mathbf{z}}| = |S_{\mathbf{r}}| = 0$.

Observation 4.1. *If the symbolic determinant* Det_w *is zero modulo s linear forms then $s \geq w$.*

Hence, Equation 9 simplifies to

$$\begin{aligned} X_1 \cdot X_2 \cdot X'_3(\mathbf{u}) \ldots X'_{d-1}(\mathbf{u}) \cdot X_d &= 0, \\ \Rightarrow X'_3(\mathbf{u}) \cdots X'_{d-1}(\mathbf{u}) &= 0. \end{aligned} \tag{10}$$

The above equality can not happen and this can be argued by applying induction on the number of matrices in the L.H.S of Equation 10:

Base case: ($d = 4$) The L.H.S of Equation 10 has one matrix $X'_3(\mathbf{u})$. As $X_3(\mathbf{u})$ is full rank[35], it cannot vanish modulo w linear forms.

Induction hypothesis: Equation 10 does not hold if the L.H.S has at most $d-4$ matrices.

Inductive step: ($d > 4$) Suppose Equation 10 is true. As the $2w^2$ linear forms in $X_3(\mathbf{u})$ and $X_{d-1}(\mathbf{u})$ are linearly independent[35], by Observation 4.1, at least one of $X'_3(\mathbf{u})$ and $X'_{d-1}(\mathbf{u})$ is invertible. This gives a shorter product where we can apply the induction hypothesis to get a contradiction.

4.2 Finding the coefficients in the intermediate matrices

Following the notations in Section 1.4.2, $\mathbf{y} = \{y_1, \ldots, y_w\}$ and $\mathbf{z} = \{z_1, \ldots, z_w\}$ are subsets of $\mathbf{x}$, $\mathbf{r} = \mathbf{x} \setminus (\mathbf{y} \uplus \mathbf{z})$, $X'_1 = (y_1\ y_2\ \ldots\ y_w)$ and $X'_d = (z_1\ z_2\ \ldots\ z_w)^T$. When Algorithm 2 reaches the third and final stage, it has blackbox access to a $f' \in \mathbb{F}[\mathbf{x}]$ and linear matrices $S_2, \ldots, S_{d-1} \in \mathbb{L}[\mathbf{r}]^{w \times w}$ returned by Algorithm 1, such that $S_2 \cdot S_3 \ldots S_{d-1}$ is the linear matrix factorization of a random $(w, d-2, n-2w)$-matrix product $R_2 \cdot R_3 \ldots R_{d-1}$ over $\mathbb{F}$. Further, there exist linear matrices $T_2, \ldots, T_{d-1} \in \mathbb{L}[\mathbf{x}]^{w \times w}$ satisfying $(T_k)_{\mathbf{y}=0, \mathbf{z}=0} = S_k$ for every $k \in [2, d-1]$, such that f' is computed by the ABP $X'_1 \cdot T_2 \ldots T_{d-1} \cdot X'_{d-1}$. The task for Algorithm 6 is to efficiently compute the coefficients of the $\mathbf{y}$ and $\mathbf{z}$ variables in T_k. At a high level, this is made possible because of the uniqueness of such T_k matrices: Indeed the analysis of Algorithm 6 shows that with high probability the coefficients of $\mathbf{y}$ and $\mathbf{z}$ in $T_3, \ldots, T_{d-2}$ are uniquely determined, and (if a certain canonical form is assumed then) the same is true for matrices T_2 and T_{d-1}.

Canonical form for T_2 and T_{d-1}: Matrix T_2 is said to be in canonical form if for every $l \in [w]$ the coefficient of y_l is zero in the linear form at the (i, j)-th entry of T_2, whenever $i > l$. Similarly, T_{d-1} is in canonical form if for every $l \in [w]$ the coefficient of z_l is zero in the linear form at the (i, j)-th entry of T_{d-1} whenever $j > l$. It can be verified (see [KNST17]), if f' is computed by an ABP $X'_1 \cdot T_2 \ldots T_{d-1} \cdot X'_{d-1}$ then it is computed by another ABP where the corresponding T_2 and T_{d-1} are in canonical form, and the other matrices remain unchanged.

[35] which follows from condition (**)

Linear independence of minors of a random ABP: The lemma given below is the reason Algorithm 6 is able to reduce the task of finding the coefficients of the $\mathbf{y}$ and $\mathbf{z}$ variables to solving linear equations. In the following discussion, the i-th row and j-th column of a matrix M will be denoted by $M(i,*)$ and $M(*,j)$ respectively.

Let $R_2 \cdot R_3 \ldots R_{d-1}$ be a random $(w, d-2, n-2w)$-matrix product in $\mathbf{r}$-variables over $\mathbb{F}$. For every $s,t \in [w]$, $R_2(s,*) \cdot R_3 \ldots R_{d-2} \cdot R_{d-1}(*,t)$ is a random $(w, d-2, n-2w)$-ABP having a total of $w^2(d-4)+2w$ linear forms in all the R_k matrices. Let us index the linear forms[36] by $[w^2(d-4)+2w]$. We associate a polynomial $g_e^{(s,t)}$ with the e-th linear form, for every $e \in [w^2(d-4)+2w]$, as follows: If the e-th linear form is the (ℓ,m)-th entry of R_k then

$$g_e^{(s,t)}(\mathbf{r}) \stackrel{\text{def}}{=} [R_2(s,*) \cdot R_3 \ldots R_{k-2} \cdot R_{k-1}(*,\ell)] \cdot [R_{k+1}(m,*) \cdot R_{k+2} \ldots R_{d-2} \cdot R_{d-1}(*,t)].^{37}$$

The polynomials $\{g_e^{(s,t)} : e \in [w^2(d-4)+2w]\}$, will be called the *minors* of the ABP $R_2(s,*) \cdot R_3 \ldots R_{d-2} \cdot R_{d-1}(*,t)$.

Lemma 4.2. *With probability $1-(wdn)^{-\Omega(1)}$ over the randomness of $R_2 \cdots R_{d-1}$ the following holds: For every $s,t \in [w]$, the minors $\{g_e^{(s,t)} : e \in [w^2(d-4)+2w]\}$, are $\mathbb{F}$-linearly independent.*

The proof of the lemma is given at the end of this section. Due to the uniqueness of factorization, the matrices $S_2, \ldots, S_{d-1}$ in Algorithm 2 are related to $R_2, \ldots, R_{d-1}$ as follows: There are $C_i, D_i \in \mathsf{GL}(w, \mathbb{L})$ such that $S_i = C_i \cdot R_i \cdot D_i$, for every $i \in [2, d-1]$; moreover, there are $c_2, \ldots, c_{d-2} \in \mathbb{L}^\times$ satisfying $C_2 = D_{d-1} = I_w$, $D_i \cdot C_{i+1} = c_i I_w$ for $i \in [2, d-2]$, and $\prod_{i=2}^{d-2} c_i = 1$. Define minors of the ABP $S_2(s,*) \cdot S_3 \ldots S_{d-2} \cdot S_{d-1}(*,t)$, for every $s,t \in [w]$, like above. The edges of the ABP are indexed by $[w^2(d-4)+2w]$ and a polynomial $h_e^{(s,t)}$ is associated with the e-th linear form as follows: If the e-th linear form is the (ℓ,m)-th entry of S_k then

$$h_e^{(s,t)}(\mathbf{r}) \stackrel{\text{def}}{=} [S_2(s,*) \cdot S_3 \ldots S_{k-2} \cdot S_{k-1}(*,\ell)] \cdot [S_{k+1}(m,*) \cdot S_{k+2} \ldots S_{d-2} \cdot S_{d-1}(*,t)]. \tag{11}$$

It is a simple exercise to derive the following corollary from the lemma above.

Corollary 4.1. *With probability $1-(wdn)^{-\Omega(1)}$ the following holds: For every $s,t \in [w]$, the minors $\{h_e^{(s,t)} : e \in [w^2(d-4)+2w]\}$ are $\mathbb{L}$-linearly independent.*

We are now ready to argue the correctness of Algorithm 6 by tracing its steps.

1. *Computing the partial derivatives (Step 2)*: In this step, we compute all the third order partial derivatives of f' using Claim 2.1.

2. *Computing almost all the coefficients of the* $\mathbf{y}$ *and* $\mathbf{z}$ *variables (Steps 6–13)*: Equations 12 and 13 are justified by treating f' as a polynomial in the $\mathbf{y}$ and $\mathbf{z}$ variables with coefficients from $\mathbb{L}(\mathbf{r})$, and examining the coefficients of $y_s^2 z_t$ and $y_s z_t^2$ respectively. A linear system obtained at step 9 or step 11 has $w^2(d-4)+2w$ variables and the same number of linear equations. Corollary 4.1, together with Claim 2.2, ensure that the square coefficient matrix of the linear

[36] by picking an arbitrarily fixed ordering among the linear forms

[37] by identifying the 1×1 matrix of the R.H.S with the entry of the matrix

Algorithm 6 Computing the coefficients of $\mathbf{y}$ and $\mathbf{z}$ variables in T_k

INPUT: Blackbox access to f' and linear matrices $S_2, \ldots, S_{d-1} \in \mathbb{L}[\mathbf{r}]^{w \times w}$.
OUTPUT: Linear matrices $T_2, T_3, \ldots, T_{d-1} \in \mathbb{L}[\mathbf{x}]^{w \times w}$ such that f' is computed by $\mathbf{y} \cdot T_2 \cdot T_3 \ldots T_{d-1} \cdot \mathbf{z}^T$, satisfying $(T_k)_{\mathbf{y}=0,\mathbf{z}=0} = S_k$ for every $k \in [2, d-1]$.

/* Computing the partial derivatives */
Compute blackbox access to $(\frac{\partial f'}{\partial y_s y_l z_t})_{\mathbf{y}=0,\mathbf{z}=0}$ and $(\frac{\partial f'}{\partial y_s z_l z_t})_{\mathbf{y}=0,\mathbf{z}=0}$ for all $s, l, t \in [w]$.
For every $s, t \in [w]$, let $\{h_e^{(s,t)} : e \in [w^2(d-4) + 2w]\}$ be the minors of the ABP $S_2(s, *) \cdot S_3 \ldots S_{d-2} \cdot S_{d-1}(*, t)$, as defined in Equation 11.

/* Computing *almost all* the coefficients of the $\mathbf{y}$ and $\mathbf{z}$ variables in T_k*/
Set $E = w^2(d-4) + 2w$.
for every $s, t \in [w]$ **do**
  Pick $\mathbf{a}_1, \ldots, \mathbf{a}_E \in_r \mathbb{F}^{|\mathbf{r}|}$ independently.
  Solve the linear system over $\mathbb{L}$ defined by

$$\sum_{e \in [E]} c_e \cdot h_e^{(s,t)}(\mathbf{a}_i) = \left(\frac{\partial f'}{\partial y_s^2 z_t}\right)_{\mathbf{y}=0,\mathbf{z}=0}(\mathbf{a}_i), \qquad \text{for } i \in [E], \tag{12}$$

  for a *unique* solution of $\{c_e\}_{e \in [E]}$. If the coefficient matrix is not invertible, output 'Failed'.
  For every $e \in [E]$, set the solution value of c_e as the coefficient of y_s in the e-th linear form of the ABP $T_2(s, *) \cdot T_3 \ldots T_{d-2} \cdot T_{d-1}(*, t)$.
  Solve the linear system over $\mathbb{L}$ defined by

$$\sum_{e \in [E]} d_e \cdot h_e^{(s,t)}(\mathbf{a}_i) = \left(\frac{\partial f'}{\partial y_s z_t^2}\right)_{\mathbf{y}=0,\mathbf{z}=0}(\mathbf{a}_i), \qquad \text{for } i \in [E], \tag{13}$$

  for a *unique* solution of $\{d_e\}_{e \in [E]}$.
  For every $e \in [E]$, set the solution value of d_e as the coefficient of z_t in the e-th linear form of the ABP $T_2(s, *) \cdot T_3 \ldots T_{d-2} \cdot T_{d-1}(*, t)$.
end for

/* Computing the remaining $\mathbf{y}$ and $\mathbf{z}$ coefficients in T_2 and T_{d-1} */
for every $s, t \in [w]$ **do**
  For every $l > s$, compute the coefficients of y_l in the linear forms in $T_2(s, *)$ by setting up a linear system similar to Equation 12, but with the R.H.S replaced by $\frac{\partial f'}{\partial y_s y_l z_1}$.
  For every $l > t$, compute the coefficients of z_l in the linear forms in $T_{d-1}(*, t)$ by setting up a linear system similar to Equation 13, but with the R.H.S replaced by $\frac{\partial f'}{\partial y_1 z_l z_t}$.
end for

The coefficients of the $\mathbf{r}$ variables in the linear forms in T_k remain the same as that in S_k, for all $k \in [2, d-1]$. Output $T_2, T_3, \ldots T_{d-1}$.

system is invertible (with high probability), and hence the solution computed is unique. The uniqueness implies that the solutions obtained across multiple iterations of the loop do not conflict with each other[38]. This also shows that the matrices $T_3, \dots, T_4$ are unique. By the end of this stage, the coefficients of $\mathbf{y}$ and $\mathbf{z}$ variables are computed for all the linear forms, except for the coefficients of y_l in $T_2(s, *)$ for $l > s$, and the coefficients of z_l in $T_{d-1}(*, t)$ for $l > t$. These coefficients are retrieved in the next stage.

3. *Computing the remaining* $\mathbf{y}$ *and* $\mathbf{z}$ *coefficients in* T_2 *and* T_{d-1} *(Steps 16–19)*: For an $s \in [w]$, consider the following minors of $S_2(s, *) \cdot S_3 \dots S_{d-2} \cdot S_{d-1}(*, 1)$:

$$S_3(m, *) \cdot S_4 \dots S_{d-2} \cdot S_{d-1}(*, 1) \qquad \text{for all } m \in [w].$$

Without loss of generality, let these minors be $h_1^{(s,1)}, \dots, h_w^{(s,1)}$. Let $l > s$. By treating f' as a polynomial in the $\mathbf{y}, \mathbf{z}$ variables, with coefficients from $\mathbb{L}(\mathbf{r})$, and examining the coefficient of $y_s y_l z_1$ in f', we arrive at the equation,

$$\sum_{e=1}^{w} c_e \cdot h_e^{(s,1)} + K(\mathbf{r}) = \left(\frac{\partial f'}{\partial y_s y_l z_1} \right)_{\mathbf{y}=0, \mathbf{z}=0},$$

where $c_1, \dots, c_w$ are the unknown coefficients of y_l in the linear forms of $T_2(s, *)$, and $K(\mathbf{r})$ is a *known* linear combination of some other minors. The fact that $K(\mathbf{r})$ is known at step 17 follows from this observation – while forming a monomial $y_s y_l z_1$, we either choose y_s from X_1' and y_l from $T_2(s, *)$ or $T_3, \dots, T_{d-1}(*, 1)$, or y_l from X_1' and y_s from $T_3, \dots, T_{d-1}(*, 1)$. In the latter case, we are using the fact that T_2 is in canonical form, and so y_s does not appear in $T_2(l, *)$. As the coefficients of y_s, y_l in $T_3, \dots, T_{d-1}(*, 1)$ are known from the computation in steps 6–13, we conclude that $K(\mathbf{r})$ in known. Thus, we can solve for $c_1, \dots, c_w$ by plugging in w random points in place of the $\mathbf{r}$ variables and setting up a linear system in w variables. Corollary 4.1 and Claim 2.2 imply the $w \times w$ coefficient matrix of the system is invertible, and hence the solution for $c_1, \dots, c_w$ is unique. The correctness of step 18 can be argued similarly, and this finally implies that T_2 and T_{d-1} (in canonical form) are unique.

Linear independence of minors: Proof of Lemma 4.2

We have to show that the minors of $R_2(s, *) \cdot R_3 \dots R_{d-2} \cdot R_{d-1}(*, t)$ are $\mathbb{F}$-linearly independent with high probability, for every $s, t \in [w]$, where $R_2 \cdot R_3 \dots R_{d-1}$ is a random $(w, d-2, n-2w)$-matrix product. We will prove it for a fixed $s, t \in [w]$, and then by union bound the result will follow for every $s, t \in [w]$. As $n \geq 4w^2$, we have $n - 2w \geq 3w^2$. So, it is sufficient to show the linear independence of the minors of a random (w, d, n)-ABP $X_1 \cdot X_2 \dots X_d$ in $\mathbf{x}$-variables, for $n \geq 3w^2$.

Treat the coefficients of the linear forms in $X_1, \dots, X_d$ as formal variables. In particular,

$$X_1 = \sum_{i=1}^{n} U_i^{(1)} x_i, \quad X_k = \sum_{i=1}^{n} U_i^{(k)} x_i \ \text{ for } k \in [2, d-1], \quad X_d = \sum_{i=1}^{n} U_i^{(d)} x_i, \tag{14}$$

[38] For instance, the coefficients of y_s in the linear forms in $T_2(s, *), T_3, \dots, T_{d-2}$ get computed repeatedly at step 9 for every value of $t \in [w]$ – uniqueness ensures that we always get the same values for these coefficients.

where $U_i^{(1)}$ and $U_i^{(d)}$ are row and column vectors of length w respectively, $U_i^{(k)}$ is a $w \times w$ matrix, and the entries of these matrices are distinct $\mathbf{u}$-variables. We will denote the (ℓ, m)-th entry of $U_i^{(k)}$ by $U_i^{(k)}(\ell, m)$, and the m-th entry of $U_i^{(d)}$ by $U_i^{(d)}(m)$. From the above equations, $X_1 \cdot X_2 \ldots X_d$ is a (w, d, n)-ABP over $\mathbb{F}(\mathbf{u})$. We will show in the following claim that the minors of this ABP are $\mathbb{F}(\mathbf{u})$-linearly independent. As the coefficients of the $\mathbf{x}$-monomials of these minors are polynomials (in fact, multilinear polynomials) of degree $d-1$ in the $\mathbf{u}$-variables, an application of the Schwartz-Zippel lemma implies $\mathbb{F}$-linear independence of the minors (with high probability) when the $\mathbf{u}$-variables are set randomly to elements in $\mathbb{F}$ (as is done in a random ABP over $\mathbb{F}$).

Claim 4.1. *The minors of $X_1 \cdot X_2 \ldots X_d$ are $\mathbb{F}(\mathbf{u})$-linearly independent.*

Proof. We will prove by induction on d.

Base case (d=3): Clearly, if the minors are $\mathbb{F}$-linearly independent after setting the $\mathbf{u}$-variables to some $\mathbb{F}$-elements then the minors are also $\mathbb{F}(\mathbf{u})$-linearly independent before the setting. As $n \geq w^2 + 2w$, it is possible to set the $\mathbf{u}$-variables in X_1, X_2, X_3 such that the entries of these matrices (after the setting) become distinct $\mathbf{x}$-variables. The minors of this $\mathbf{u}$-evaluated ABP $X_1 \cdot X_2 \cdot X_3$ are monomial disjoint and so $\mathbb{F}$-linearly independent.

Inductive step: Split the $w^2(d-2) + 2w$ minors of $X_1 \cdot X_2 \ldots X_d$ into two sets: The first set G_1 consists of minors g_e, for $e \in [w^2(d-3) + 2w]$, such that the e-th linear form is the (ℓ, m)-th entry of some matrix X_k satisfying $k \neq d$ and if $k = d-1$ then $m = w$. The second set G_2 consists of minors g_e, for $e \in [w^2(d-3) + 2w + 1, w^2(d-2) + 2w]$, such that the e-th linear form is either the (ℓ, m)-th entry of X_{d-1} for $m \neq w$, or the ℓ-th entry of X_d. Set G_1 has $p = w^2(d-3) + 2w$ minors and G_2 has w^2 minors.

Suppose $\mu_1, \ldots, \mu_p$ are monomials in $\mathbf{x}$-variables of degree $d-2$. Imagine a $(w^2(d-2) + 2w) \times (w^2(d-2) + 2w)$ matrix M whose rows are indexed by the minors in G_1 and G_2, and columns by monomials $\mu_1 x_1, \mu_2 x_1, \ldots, \mu_p x_1$ and $x_2^{d-1}, x_3^{d-1}, \ldots, x_{w^2+1}^{d-1}$. The (g, σ)-th entry of M contains the coefficient of the monomial σ in g, this coefficient is a multilinear polynomial in the $\mathbf{u}$-variables. In a sequence of observations, we show that there exist $\mu_1, \ldots, \mu_p$ such that $\det(M) \neq 0$.

Consider the variable $u \stackrel{\text{def}}{=} U_1^{(d)}(w)$. The following observations are easy to verify.

Observation 4.2.
1. *Variable u does not appear in any of the monomials of the (g, σ)-th entry of M if $g \in G_2$ or $\sigma \in \{x_2^{d-1}, \ldots, x_{w^2+1}^{d-1}\}$.*
2. *Variable u appears in some monomials of the (g, σ)-th entry of M if $g \in G_1$ and $\sigma \in \{\mu_1 x_1, \ldots, \mu_p x_1\}$, irrespective of $\mu_1, \ldots, \mu_p$.*

Observation 4.3. *Let $g \in G_1$ and $\sigma \in \{\mu_1 x_1, \ldots, \mu_p x_1\}$. If we treat the (g, σ)-th entry of M as a polynomial in u with coefficients from $\mathbb{F}[\mathbf{u} \setminus u]$ then the coefficient of u does not depend on the variables:*

(a) $U_i^{(d)}(j)$ *for $j \neq w$ and $i \in [n]$,*

(b) $U_i^{(d)}(w)$ *for $i \in [2, n]$,*

(c) $U_i^{(d-1)}(\ell, m)$ *for $\ell, m \in [w]$ with $m \neq w$, and $i \in [n]$.*

Denote the union of the **u**-variables specified in *(a)*, *(b)* and *(c)* of the above observation by **v**.

Observation 4.4. *The set* $\{g_{\mathbf{v}=0} : g \in G_1\}$ *equals the set* $\{h \cdot ux_1 : h \text{ is a minor of } X_1 \cdot X_2 \ldots X_{d-1}(*, w)\}$.

By the induction hypothesis, the minors of $X_1 \cdot X_2 \ldots X_{d-1}(*, w)$, say $h_1, \ldots, h_p$, are $\mathbb{F}(\mathbf{u})$-linearly independent. Hence there are p monomials in **x**-variables of degree $d-2$ such that $h_1, \ldots, h_p$, when restricted to these monomials, are $\mathbb{F}(\mathbf{u})$-linearly independent. These p monomials are our choices for $\mu_1, \ldots, \mu_p$. Let N be the $p \times p$ matrix with rows indexed by $h_1, \ldots, h_p$ and columns by $\mu_1, \ldots, \mu_p$, and $N(h, \mu)$ contains the coefficient of the monomial μ in h. Then, $\det(N) \neq 0$. Under these settings, we have the following observation (which can be derived easily from the above).

Observation 4.5. *The coefficient of* u^p *in* $\det(M)$*, when treated as a polynomial in* u *with coefficients from* $\mathbb{F}[\mathbf{u} \setminus u]$*, is* $\det(N) \cdot \det(M_0)$*, where* M_0 *is the submatrix of* M *defined by rows indexed by* $\{g : g \in G_2\}$ *and columns by* $x_2^{d-1}, \ldots, x_{w^2+1}^{d-1}$.

The next observation completes the proof of the claim by showing $\det(M) \neq 0$.

Observation 4.6. $\det(M_0) \neq 0$.

The proof of the above follows by noticing that M_0 looks like $(f_i(\mathbf{u}_j))_{i,j \in [w^2]}$, where $\mathbf{u}_1, \ldots, \mathbf{u}_{w^2}$ are some disjoint subsets of the **u**-variables and $f_1, \ldots, f_{w^2}$ are $\mathbb{F}$-linearly independent polynomials. The observation then follows from Claim 2.2.

□

5 Equivalence test for determinant over finite fields

We prove Theorem 3 in this section. It is known that the affine equivalence test can be reduced to equivalence test [Kay12], as briefly explained below.

Reduction to equivalence test: Suppose f is a (n, w)-polynomial that is affine equivalent to Det_w, where $n \geq w^2$. The following claim reduces the number of variables from n to w^2. A proof can be found in [Kay12] (see also Algorithm 8 and Claim 2.3 in [KNST17]).

Claim 5.1. *There is a randomized algorithm that takes input blackbox access to* $f(\mathbf{x})$ *and with probability* $1 - \frac{n^{O(1)}}{q}$ *outputs a matrix* $C \in \mathsf{GL}(n, \mathbb{F})$ *such that* $f(C \cdot \mathbf{x})$ *is a* (w^2, w)*-polynomial. The algorithm runs in* $(n \log q)^{O(1)}$ *time.*

Suppose $\mathbf{y} \subseteq \mathbf{x}$ is the set of w^2 variables appearing in $f(C \cdot \mathbf{x})$, and let $g(\mathbf{y})$ be the degree-w homogeneous component of $f(C \cdot \mathbf{x})$ which must be equivalent to Det_w. By using an equivalence test for Det_w, we can compute a $Q \in \mathsf{GL}(w^2, \mathbb{L})$ such that $g(\mathbf{y}) = \mathsf{Det}_w(Q \cdot \mathbf{y})$, implying $g(\mathbf{x}) = \mathsf{Det}_w(Q' \cdot \mathbf{x})$ where $Q' \in \mathbb{L}^{w^2 \times n}$ is obtained by padding Q with $(n - w^2)$ all-zero columns. Now observe that there is an $\mathbf{a} \in \mathbb{F}^n$ such that $f(C \cdot \mathbf{x}) = g(\mathbf{x} + \mathbf{a})$; the translation equivalence test in the claim below returns a $\mathbf{c} \in \mathbb{F}^n$ such that $f(C \cdot \mathbf{x}) = g(\mathbf{x} + \mathbf{c})$. Hence, $f(C \cdot \mathbf{x}) = \mathsf{Det}_w(Q'\mathbf{x} + Q' \cdot \mathbf{c})$ implying $f(\mathbf{x}) = \mathsf{Det}_w(Q'C^{-1}\mathbf{x} + Q' \cdot \mathbf{c})$. The algorithm in Theorem 3 returns $B = Q'C^{-1}$ and $\mathbf{b} = Q' \cdot \mathbf{c}$.

Claim 5.2. *Let* $f(\mathbf{x}) = g(\mathbf{x} + \mathbf{a})$*, where* f, g *are* (n, d)*-polynomials and* $\mathbf{a} \in \mathbb{F}^n$*. There is randomized algorithm that takes blackbox access to* f *and* g *and with probability* $1 - \frac{(nd)^{O(1)}}{q}$ *computes a* $\mathbf{c} \in \mathbb{F}^n$ *such that* $f(\mathbf{x}) = g(\mathbf{x} + \mathbf{c})$.

See [Kay12, DdOS14] (also Algorithm 9 and Lemma 2.1 in [KNST17]) for proofs of the claim.

For the rest of this section, set $n = w^2$. The equivalence test for Det_w is done in two steps: In the first step, the problem is reduced to the simpler problem of PS-equivalence testing. The second step then solves the PS-equivalence test. A (w^2, w)-polynomial $f \in \mathbb{L}[\mathbf{x}]$ is PS-equivalent to Det_w if there is a permutation matrix P and a diagonal matrix $S \in \mathsf{GL}(w^2, \mathbb{L})$ such that $f = \mathsf{Det}_w(PS \cdot \mathbf{x})$.

Lemma 5.1 ([Kay12]). *There is a randomized algorithm that takes input blackbox access to f, which is PS-equivalent to Det_w, and with probability $1 - \frac{w^{O(1)}}{q}$ outputs a permutation matrix P and a diagonal matrix $S \in \mathsf{GL}(w^2, \mathbb{L})$ such that $f = \mathsf{Det}_w(PS \cdot \mathbf{x})$. The algorithm runs in $(w \log q)^{O(1)}$ time.*

It is in the first step where our algorithm differs from (and slightly simplifies) [Kay12]. This reduction to PS-equivalence testing is given in Section 5.2. As in [Kay12], the algorithm uses the structure of the group of symmetries and the Lie algebra of Det_w. An estimate of the probability that a random element of the Lie algebra of g_{Det_w} has all its eigenvalues in $\mathbb{L}$ (Lemma 5.4) is key to the simplification in the first step.

5.1 Group of symmetries and Lie algebra of Determinant

We state a few well known facts and claims about the Lie algebra and the group of symmetries of Det_w. Proofs of these can be found in [Kay12, KNST17] and the references therein.

Definition 5.1. The *group of symmetries* of an n-variate polynomial f, denoted as $\mathscr{G}_f$, consists of matrices $A \in \mathsf{GL}(n, \mathbb{F})$ such that $f(\mathbf{x}) = f(A \cdot \mathbf{x})$.

$\mathsf{Det}_w(\mathbf{x})$ is the determinant of the symbolic matrix $X = (x_{ij})_{i,j \in [w]}$, where $\mathbf{x} = \{x_{ij}\}_{i,j \in [w]}$. Let $A(X)$ denote the $w \times w$ linear matrix obtained by applying a transformation $A \in \mathbb{F}^{w^2 \times w^2}$ on $\mathbf{x}$.

Fact 1. *An $A \in \mathsf{GL}(w^2, \mathbb{F})$ is in $\mathscr{G}_{\mathsf{Det}_w}$ if and only if there are two matrices $S, T \in \mathsf{SL}(w, \mathbb{F})$ such that either $A(X) = S \cdot X \cdot T$ or $A(X) = S \cdot X^T \cdot T$.*

Definition 5.2. The *Lie algebra* of a polynomial $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$, denoted as $\mathfrak{g}_f$, is the set of all $n \times n$ matrices $E = (e_{ij})_{i,j \in [n]}$ in $\mathbb{F}^{n \times n}$ satisfying

$$\sum_{i,j \in [n]} e_{ij} x_j \cdot \frac{\partial f}{\partial x_i} = 0.$$

To express the Lie algebra of Det_w, order the variables of $\mathbf{x}$ in row major fashion and call them $x_1, \ldots, x_n$. Let $\mathcal{Z}_w$ be the $\mathbb{F}$-linear space of all $w \times w$ traceless matrices over $\mathbb{F}$, $\mathcal{L}_{\text{row}}$ be the space $\mathcal{Z}_w \otimes I_w = \{Z \otimes I_w : Z \in \mathcal{Z}_w\}$, and $\mathcal{L}_{\text{col}}$ the space $I_w \otimes \mathcal{Z}_w = \{I_w \otimes Z : Z \in \mathcal{Z}_w\}$.

Fact 2. $\mathfrak{g}_{\mathsf{Det}_w} = \mathcal{L}_{\text{row}} \oplus \mathcal{L}_{\text{col}}$.

It follows that the dimension of $\mathfrak{g}_{\mathsf{Det}_w}$ over $\mathbb{F}$ is $2w^2 - 2$.

Fact 3. *Let f, g be n-variate polynomials such that there is an $A \in \mathsf{GL}(n, \mathbb{F})$ satisfying $f = g(A \cdot \mathbf{x})$. Then $\mathfrak{g}_f = A^{-1} \cdot \mathfrak{g}_g \cdot A = \{A^{-1} \cdot L \cdot A \mid L \in \mathfrak{g}_g\}$.*

Claim 5.3. *There is a randomized algorithm that given blackbox access to a (n, d)-polynomial f over $\mathbb{F}$, computes an $\mathbb{F}$-basis of $\mathfrak{g}_f$ with probability $1 - \frac{(nd)^{O(1)}}{q}$. The algorithm runs in $(nd \log q)^{O(1)}$ time.*

From Fact 2, it is easy to observe that $\mathfrak{g}_{\mathsf{Det}_w}$ contains a diagonal matrix with distinct elements on the diagonal. The next claim can be proved using this observation.

Claim 5.4. *Let $L_1, \ldots, L_{2w^2-2}$ be an $\mathbb{F}$-basis of $\mathfrak{g}_{\mathsf{Det}_w}$, and $L = \sum_{i=1}^{2w^2-2} \alpha_i \cdot L_i$, where $\alpha_1, \ldots, \alpha_{2w^2-2} \in_r \mathbb{F}$ are picked independently. Then, the characteristic polynomial of L is square-free with probability $1 - \frac{w^{O(1)}}{q}$.*

The following lemma is the main technical contribution of this section.

Lemma 5.2. *Let $L_1, \ldots, L_{2w^2-2}$ be an $\mathbb{F}$-basis of $\mathfrak{g}_{\mathsf{Det}_w}$, and $L = \sum_{i=1}^{2w^2-2} \alpha_i \cdot L_i$, where $\alpha_1, \ldots, \alpha_{2w^2-2} \in_r \mathbb{F}$ are picked independently. Then, the characteristic polynomial of L is square-free and splits completely over $\mathbb{L}$ with probability at least $\frac{1}{2w^2}$.*

Proof. Let $h(y)$ be the characteristic polynomial of L. From Claim 5.4, h is square-free with probability $1 - \frac{w^{O(1)}}{q}$. From Fact 2, $L = L_1 + L_2$ where $L_1 \in \mathcal{L}_{\text{row}}$ and $L_2 \in \mathcal{L}_{\text{col}}$. As L is uniformly distributed over $\mathfrak{g}_{\mathsf{Det}}$, so is L_1 over $\mathcal{L}_{\text{row}}$ and L_2 over $\mathcal{L}_{\text{col}}$. In other words, if $L_1 = Z_1 \otimes I_w$ and $L_2 = I_w \otimes Z_2$ then Z_1, Z_2 are both uniformly (and independently) distributed over $\mathcal{Z}_w$. If the characteristic polynomial of Z_1 (similarly Z_2) is irreducible over $\mathbb{F}$ then the eigenvalues of Z_1 (respectively, Z_2) lie in $\mathbb{L}$ and are distinct. If this happens for both Z_1 and Z_2 then there are $D_1, D_2 \in \mathsf{GL}(w, \mathbb{L})$ such that $D_1^{-1} Z_1 D_1$ and $D_2^{-1} Z_2 D_2$ are diagonal matrices. This further implies,

$$(D_1^{-1} \otimes I_w) \cdot (I_w \otimes D_2^{-1}) \cdot L \cdot (I_w \otimes D_2) \cdot (D_1 \otimes I_w)$$

is a diagonal matrix, due to the observation below.

Observation 5.1. *For any $M, N \in \overline{\mathbb{F}}^{w \times w}$, $(M \otimes I_w)$ and $(I_w \otimes N)$ commutes. Also, if $M, N \in \mathsf{GL}(w, \overline{\mathbb{F}})$ then $(M \otimes I_w)^{-1} = (M^{-1} \otimes I_w)$ and $(I_w \otimes N)^{-1} = (I_w \otimes N^{-1})$.*

Thus, if we show that the characteristic polynomial of $Z \in_r \mathcal{Z}_w$ is irreducible with probability δ then with probability at least δ^2 the characteristic polynomial of L splits completely over $\mathbb{L}$. Much like the proof of Claim 5.4, it can be shown that the characteristic polynomial of $Z \in_r \mathcal{Z}_w$ is square-free with probability $1 - \frac{w^{O(1)}}{q}$. Hence, if the characteristic polynomial of $Z \in_r \mathcal{Z}'_w$, where $\mathcal{Z}'_w \subset \mathcal{Z}_w$ consists of matrices with distinct eigenvalues in $\overline{\mathbb{F}}$, is irreducible with probability ρ then $\delta \geq \rho \cdot (1 - \frac{w^{O(1)}}{q})$. Next, we lower bound ρ.

Let $\mathcal{P}$ be the set of monic, degree-w, square-free polynomials in $\mathbb{F}[y]$ with the coefficient of y^{w-1} equal to zero. Define a map ϕ from $\mathcal{Z}'_w$ to $\mathcal{P}$,

$$\phi : \quad Z \mapsto \text{characteristic polynomial of } Z.$$

The map ϕ is onto as the companion matrix of $p(y) \in \mathcal{P}$ belongs to its pre-image under ϕ. Let $\phi^{-1}(p(y))$ be the set of matrices in $\mathcal{Z}'_w$ that map to p.

Claim 5.5. *Let $p(y) \in \mathcal{P}$. Then*

$$\frac{(q^w - 1) \cdot (q^w - q) \ldots (q^w - q^{w-1})}{q^w} \leq |\phi^{-1}(p(y))| \leq \frac{(q^w - 1) \cdot (q^w - q) \ldots (q^w - q^{w-1})}{q^w (1 - \frac{w}{q})}.$$

Proof. Let C_p be the companion matrix of $p(y)$. If the characteristic polynomial of a $Z \in \mathcal{Z}'_w$ equals $p(y)$ then there is an $E \in \mathsf{GL}(w, \mathbb{F})$ such that $Z = E \cdot C_p \cdot E^{-1}$, as the eigenvalues of C_p are distinct in $\overline{\mathbb{F}}$. Moreover, for any $E \in \mathsf{GL}(w, \mathbb{F})$, $E \cdot C_p \cdot E^{-1} \in \mathcal{Z}'_w$ has characteristic polynomial $p(y)$. Hence, $\phi^{-1}(p(y)) = \{E \cdot C_p \cdot E^{-1} \mid E \in \mathsf{GL}(w, \mathbb{F})\}$. Suppose $E, F \in \mathsf{GL}(w, \mathbb{F})$ such that $F \cdot C_p \cdot F^{-1} = E \cdot C_p \cdot E^{-1}$. Then $E^{-1}F$ commutes with C_p. Since C_p has distinct eigenvalues in $\overline{\mathbb{F}}$, $E^{-1}F$ can be expressed as a polynomial in C_p, say $h(C_p)$, of degree at most $(w-1)$ with coefficients from $\mathbb{F}$. Conversely, if $h \in \mathbb{F}[y]^{\leq (w-1)}$ [39] and $h(C_p)$ is invertible then $F = E \cdot h(C_p)$ is such that $F \cdot C_p \cdot F^{-1} = E \cdot C_p \cdot E^{-1}$. As $h_1(C_p) \neq h_2(C_p)$ for distinct $h_1, h_2 \in \mathbb{F}[y]^{\leq (w-1)}$, we have

$$|\phi^{-1}(p(y))| = \frac{|\mathsf{GL}(w, \mathbb{F})|}{|\{h \in \mathbb{F}[y] : \deg(h) \leq (w-1) \text{ and } h(C_p) \in \mathsf{GL}(w, \mathbb{F})\}|}.$$

The numerator is exactly $(q^w - 1) \cdot (q^w - q) \ldots (q^w - q^{w-1})$, and the denominator is trivially upper bounded by q^w. A lower bound on the denominator can be worked out as follows: Let $\lambda_1, \ldots, \lambda_w \in \overline{\mathbb{F}}$ be the distinct eigenvalues of C_p. If $h(y) = a_{w-1}y^{w-1} + a_{w-2}y^{w-2} + \ldots + a_0 \in \mathbb{F}[y]$, then $h(\lambda_1), \ldots, h(\lambda_w)$ are the eigenvalues of $h(C_p)$. Observe that

$$\begin{aligned} & \Pr_{h \in_r \mathbb{F}[y]^{\leq (w-1)}} \{h(\lambda_i) = 0, \text{ for some fixed } i \in [w]\} \leq \frac{1}{q}, \\ \Rightarrow\ & \Pr_{h \in_r \mathbb{F}[y]^{\leq (w-1)}} \{h(\lambda_i) = 0, \text{ for any } i \in [w]\} \leq \frac{w}{q}, \\ \Rightarrow\ & \Pr_{h \in_r \mathbb{F}[y]^{\leq (w-1)}} \{h(C_p) \in \mathsf{GL}(w, \mathbb{F})\} \geq 1 - \frac{w}{q}. \end{aligned}$$

Hence, the denominator is lower bounded by $q^w(1 - \frac{w}{q})$.

□

Let $\rho_p = \frac{|\phi^{-1}(p(y))|}{|\mathcal{Z}'_w|}$, the probability that $p(y)$ is the characteristic polynomial of $Z \in_r \mathcal{Z}'_w$. From Claim 5.5, it follows that

$$|\mathcal{Z}'_w| \leq \frac{(q^w - 1) \cdot (q^w - q) \ldots (q^w - q^{w-1})}{q^w(1 - \frac{w}{q})} \cdot |\mathcal{P}| \quad \Rightarrow \quad 1 - \frac{w}{q} \leq \rho_p \cdot |\mathcal{P}|.$$

We show in the next claim that a $p \in_r \mathcal{P}$ is irreducible over $\mathbb{F}$ with probability at least $\frac{1}{w}(1 - \frac{2}{q^{w/2}})$, implying the characteristic polynomial of $Z \in_r \mathcal{Z}'_w$ is irreducible over $\mathbb{F}$ with probability $\rho \geq \frac{1}{w}(1 - \frac{2}{q^{w/2}})(1 - \frac{w}{q})$. Therefore, the probability that the characteristic polynomial of $Z \in_r \mathcal{Z}_w$ is irreducible over $\mathbb{F}$ is $\delta \geq \frac{1}{w}(1 - \frac{2}{q^{w/2}})(1 - \frac{w}{q})(1 - \frac{w^{O(1)}}{q})$. As $q \geq w^7$, the probability that the characteristic polynomial of $L \in_r \mathfrak{g}_{\mathsf{Det}_w}$ splits completely over $\mathbb{L}$ is at least $\delta^2 \geq \frac{1}{2w^2}$.

Claim 5.6. *A polynomial $p \in_r \mathcal{P}$ is irreducible over $\mathbb{F}$ with probability at least $\frac{1}{w}(1 - \frac{2}{q^{w/2}})$.*

Proof. Let $\mathcal{F}$ be the set of monic, degree-w, square-free polynomials in $\mathbb{F}[y]$. The difference between $\mathcal{F}$ and $\mathcal{P}$ is that a polynomial in $\mathcal{P}$ additionally has coefficient of y^{w-1} equal to zero. We argue in the next paragraph that the fraction of $\mathbb{F}$-irreducible polynomials in $\mathcal{F}$ and in $\mathcal{P}$ are the same. As irreducible polynomials are square-free, the number of irreducible polynomials in $\mathcal{F}$ is at

[39] the set of polynomials in $\mathbb{F}[y]$ of degree at most $w - 1$.

least $\frac{q^w - 2q^{w/2}}{w}$ [vzGG03]. Hence, the fraction of irreducible polynomials in $\mathcal{F}$ is at least $\frac{1}{w}(1 - \frac{2}{q^{w/2}})$.

Define a map Ψ from $\mathcal{F}$ to $\mathcal{P}$ as follows: For a $u(y) = y^w + a_{w-1}y^{w-1} + \ldots + a_0 \in \mathcal{F}$, define $\Psi(u) = u(y - \frac{a_{w-1}}{w})$. Observe that the coefficient of y^{w-1} in $\Psi(u)$ is zero. It is also an easy exercise to show that $\Psi(u_1) = \Psi(u_2)$ if and only if there exists an $a \in \mathbb{F}$ such that $u_1(y) = u_2(y + a)$. As $u(y)$ is irreducible over $\mathbb{F}$ if and only if $u(y + a)$ is irreducible over $\mathbb{F}$, for $a \in \mathbb{F}$, the fraction of $\mathbb{F}$-irreducible polynomials in $\mathcal{F}$ is the same as that in $\mathcal{P}$. $\square$

This completes the proof of Lemma 5.2. $\square$

5.2 Reduction to PS-equivalence testing

Algorithm 7 gives a reduction to PS-equivalence testing for Det_w. Suppose the input to the algorithm is a blackbox access to $f = \mathsf{Det}_w(A \cdot \mathbf{x})$, where $A \in \mathsf{GL}(w^2, \mathbb{F})$. We argue the correctness of the algorithm by tracing its steps:

Algorithm 7 Reduction to PS-equivalence

INPUT: Blackbox access to a (w^2, w)-polynomial $f \in \mathbb{F}[\mathbf{x}]$ that is equivalent to Det_w over $\mathbb{F}$.
OUTPUT: A $D \in \mathsf{GL}(w^2, \mathbb{L})$ such that $f(D \cdot \mathbf{x})$ is PS-equivalent to Det_w over $\mathbb{L}$.

1. Compute an $\mathbb{F}$-basis of $\mathfrak{g}_f$. Let $\{F_1, F_2, \ldots F_{2w^2-2}\}$ be the basis. Set $j = 1$.
2.
3. **for** $j = 1$ **to** $w^3 \log q$ **do**
4.   Pick $\alpha_1, \ldots, \alpha_{2w^2-2} \in_r \mathbb{F}$ independently. Set $F = \sum_{i \in [2w^2-2]} \alpha_i \cdot F_i$.
5.   Compute the characteristic polynomial h of F. Factorize h into irreducible factors over $\mathbb{L}$.
6.   **if** h is square-free and splits completely over $\mathbb{L}$ **then**
7.   Use the roots of h to compute a $D \in \mathsf{GL}(w^2, \mathbb{L})$ such that $D^{-1} \cdot F \cdot D$ is diagonal.
8.   Exit loop.
9.   **else**
10.   Set $j = j + 1$.
11.   **end if**
12. **end for**
13.
14. **if** No D found at step 7 in the loop **then**
15.   Output 'Failed'.
16. **else**
17.   Output D.
18. **end if**

Step 1: An $\mathbb{F}$-basis of $\mathfrak{g}_f$ can be computed efficiently using Claim 5.3.

Step 3–12: At step 4 an element F of $\mathfrak{g}_f$ is chosen uniformly at random. By Fact 3, $F = A^{-1} \cdot L \cdot A$, where L is a random element of $\mathfrak{g}_{\mathsf{Det}_w}$. Lemma 5.2 implies, in every iteration of the loop, h (at step 5) is square-free and splits completely over $\mathbb{L}$ with probability at least $\frac{1}{2w^2}$. Since the loop has $w^3 \log q$ iterations, the algorithm finds an h that is square-free and splits completely over $\mathbb{L}$, with probability at least $1 - \frac{1}{q}$. Assume that the algorithm succeeds in finding such an h, and suppose

$\lambda_1, \ldots, \lambda_{w^2} \in \mathbb{L}$ are the distinct roots of h. The algorithm finds a D in step 7 by picking a random solution of the linear system obtained from the relation $F \cdot D = D \cdot \text{diag}(\lambda_1, \ldots, \lambda_{w^2})$ treating the entries of D as formal variables. We argue next that $f(D \cdot \mathbf{x})$ is PS-equivalent to Det_w over $\mathbb{L}$.

By Fact 2, $L = L_1 + L_2$ where $L_1 \in \mathcal{L}_{\text{row}}$ and $L_2 \in \mathcal{L}_{\text{col}}$. In other words, there are $Z_1, Z_2 \in \mathcal{Z}_w$ such that $L_1 = Z_1 \otimes I_w$ and $L_2 = I_w \otimes Z_2$. It is easy to verify, if L has distinct eigenvalues then so do Z_1 and Z_2. Hence, there are $D_1, D_2 \in \mathsf{GL}(w, \overline{\mathbb{F}})$ such that $D_1 Z_1 D_1^{-1}$ and $D_2 Z_2 D_2^{-1}$ are both diagonal, implying

$$M \stackrel{\text{def}}{=} (D_1 \otimes I_w) \cdot (I_w \otimes D_2) \cdot L \cdot (D_1^{-1} \otimes I_w) \cdot (I_w \otimes D_2^{-1})$$

is diagonal (by Observation 5.1) with distinct diagonal entries. Also,

$$\begin{aligned} D^{-1} \cdot F \cdot D &= (AD)^{-1} \cdot L \cdot (AD) \\ &= ((D_1 \otimes I_w) \cdot (I_w \otimes D_2) \cdot AD)^{-1} \cdot M \cdot ((D_1 \otimes I_w) \cdot (I_w \otimes D_2) \cdot AD) \end{aligned}$$

As both $D^{-1} \cdot F \cdot D$ and M are diagonal matrices with distinct diagonal entries, it must be that

$$(D_1 \otimes I_w) \cdot (I_w \otimes D_2) \cdot AD = P \cdot S,$$

where P is a permutation matrix and $S \in \mathsf{GL}(w^2, \overline{\mathbb{F}})$ is a diagonal matrix. Now observe that $\text{Det}_w((D_1 \otimes I_w) \cdot \mathbf{x}) = \beta \cdot \text{Det}_w(\mathbf{x})$ and $\text{Det}_w((I_w \otimes D_2) \cdot \mathbf{x}) = \gamma \cdot \text{Det}_w(\mathbf{x})$, for $\beta, \gamma \in \overline{\mathbb{F}} \setminus \{0\}$. Hence,

$$\begin{aligned} \text{Det}_w(P \cdot S \cdot \mathbf{x}) &= \text{Det}_w((D_1 \otimes I_w) \cdot (I_w \otimes D_2) \cdot AD \cdot \mathbf{x}) \\ &= \beta\gamma \cdot \text{Det}_w(AD \cdot \mathbf{x}) \\ &= \beta\gamma \cdot f(D \cdot \mathbf{x}) \\ \Rightarrow f(D \cdot \mathbf{x}) &= \text{Det}_w(P \cdot S' \cdot \mathbf{x}), \end{aligned}$$

where $S' \in \mathsf{GL}(w^2, \overline{\mathbb{F}})$ is also diagonal. Therefore, $f(D \cdot \mathbf{x})$ is PS-equivalent to Det_w over $\overline{\mathbb{F}}$. As $f(D \cdot \mathbf{x}) \in \mathbb{L}[\mathbf{x}]$, it is a simple exercise to show that $f(D \cdot \mathbf{x})$ must be PS-equivalent to Det_w over $\mathbb{L}$.

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A Proof of two claims in Section 3

Claim 3.1 (restated): *With probability $1-(wdn)^{-\Omega(1)}$, any subset of w vectors in any of the sets $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{w+1}\}$, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{w+1}\}$, $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{w+1}\}$, or $\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_{w+1}\}$ are $\mathbb{L}$-linearly independent.*

Proof. From Observation 3.3, for the sets $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{w+1}\}$ and $\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_{w+1}\}$ it is sufficient to show that any w columns of the $w \times (w+1)$ matrices $(N_{1i}(\mathbf{a}_j))_{i\in[w],j\in[w+1]}$ and $(N_{1i}(\mathbf{b}_j))_{i\in[w],j\in[w+1]}$ are $\mathbb{L}$-linearly independent with high probability. As the cofactors $N_{11}, \ldots, N_{1w}$ are $\mathbb{L}$-linearly independent, the above follows from Claim 2.2. For the sets $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{w+1}\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{w+1}\}$, it follows from Equation 2 that there are $\lambda_k, \rho_k \in \mathbb{L}^\times$ such that $D \cdot \mathbf{v}_k = \lambda_k \mathbf{u}_k$ and $D \cdot \mathbf{s}_k = \rho_k \mathbf{w}_k$ for all $k \in [w+1]$. Since D is invertible, the claim follows for these two sets as well. □

Claim 3.2 (restated): *If $E = Q_1 \cdots Q_\ell$ is a random (w, ℓ, m)-matrix product over $\mathbb{F}$, where $w^2 + 1 \leq m \leq n$ and $\ell \leq d$, then the entries of E are $\mathbb{F}$-linearly independent with probability $1-(wdn)^{-\Omega(1)}$.*

Proof. Treat the coefficients of the linear forms in $Q_1, Q_2, \ldots, Q_\ell$ as distinct formal variables. In particular

$$Q_k = \sum_{i=1}^{m} U_i^{(k)} x_i \quad \text{for} \quad k \in [\ell] \; ,$$

where the $U_i^{(k)}$'s are $w \times w$ matrices and the entries of these matrices are distinct **u**-variables. The entries of the matrix product E are polynomials in the **x**-variables over $\mathbb{F}(\mathbf{u})$. If we show the w^2 entries of E are $\mathbb{F}(\mathbf{u})$-linearly independent then an application of Schwartz-Zippel lemma implies the statement of the claim. On the other hand, to show that the entries of E are $\mathbb{F}(\mathbf{u})$-linearly independent, it is sufficient to show that the entries are $\mathbb{F}$-linearly independent under a setting of the **u**-variables to $\mathbb{F}$ elements. Consider such a setting: For every $k \in [\ell] \setminus \{1\}$, let $U_{w^2+1}^{(k)} = I_w$ and $U_i^{(k)} = 0$ for all $i \in [m] \setminus \{w^2+1\}$. Let $U_i^{(1)} = 0$ for all $i \geq w^2 + 1$ and set $U_1^{(1)}, \ldots, U_{w^2}^{(1)}$ in a way so that the linear forms in $\sum_{i=1}^{w^2} U_i^{(1)} x_i$ are $\mathbb{F}$-linearly independent. It is straightforward to check that the entries of E under this setting are $\mathbb{F}$-linearly independent.

□

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