

# Complexity of Unordered CNF Games

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## Abstract

The classic TQBF problem is to determine who has a winning strategy in a game played on a given CNF formula, where the two players alternate turns picking truth values for the variables in a given order, and the winner is determined by whether the CNF gets satisfied. We study variants of this game in which the variables may be played in any order, and each turn consists of picking a remaining variable and a truth value for it.

- For the version where the set of variables is partitioned into two halves and each player may only pick variables from his/her half, we prove that the problem is PSPACE-complete for 5-CNFs and in P for 2-CNFs. Previously, it was known to be PSPACE-complete for unbounded-width CNFs (Schaefer, STOC 1976).
- For the general unordered version (where each variable can be picked by either player), we also prove that the problem is PSPACE-complete for 5-CNFs and in P for 2-CNFs. Previously, it was known to be PSPACE-complete for 6-CNFs (Ahroth and Orponen, MFCS 2012) and PSPACE-complete for positive 11-CNFs (Schaefer, STOC 1976).

## 1 Introduction

Conjunctive normal form formulas (CNFs) are among the most prevalent representations of boolean functions. All sorts of computational problems concerning CNFs—such as satisfying them, minimizing them, learning them, refuting them, fooling them, and playing games on them—play central roles in complexity theory. A CNF is a conjunction of clauses, where each clause is a disjunction of literals; a  $w$ -CNF has at most  $w$  literals per clause. The *width*  $w$  is often the most important parameter governing the complexity of problems concerning CNFs. The following are three classical games played on a CNF  $\varphi(x_1, \dots, x_n)$ :

- In the *ordered* game, player 1 assigns a bit value for  $x_1$ , then player 2 assigns  $x_2$ , then player 1 assigns  $x_3$ , and so on, and the winner is determined by whether  $\varphi$  gets satisfied. Note that the variables must be played in the prescribed order  $x_1, x_2, x_3, \dots$ . Deciding who has a winning strategy—better known as TQBF or QSAT—is PSPACE-complete for 3-CNFs [SM73] and in P for 2-CNFs [APT79, Cal08]. Many PSPACE-completeness results have been shown by reducing from the ordered 3-CNF game.
- In the *unordered* game, each player is allowed to pick which remaining variable to play next (as well as which bit value to assign it), and again the winner is determined by whether  $\varphi$  gets satisfied. Deciding who has a winning strategy is PSPACE-complete for 6-CNFs [AO12] and

for 11-CNFs with only positive literals [Sch76, Sch78]. The unordered game on positive CNFs is also known as the maker–breaker game, and a simplified proof of PSPACE-completeness for unbounded-width positive CNFs appears in [Bys04]. Many PSPACE-completeness results have been proven by reducing from the unordered positive CNF game [FG87, Sla00, Sla02, AS03, Bys04, Hea09, TDU11, vV13, FGM<sup>+</sup>15, BDK<sup>+</sup>16]. For the general unordered CNF game, nothing was known for width  $< 6$ ; in particular, the complexity of the unordered 2-CNF game was not studied in the literature before. An experimental evaluation of heuristics for the unordered CNF game appears in [ZM04].

- In the *partitioned* game, the set of variables is partitioned into two halves and each player may only pick variables from his/her half. This is, in a sense, intermediate between ordered and unordered: the ordered game restricts the set of variables available to each player *and* the order they must be played; the unordered game restricts neither; the partitioned game restricts only the former. Deciding who has a winning strategy was shown to be PSPACE-complete for unbounded-width CNFs in [Sch76, Sch78], where it was explicitly posed as an open problem to show PSPACE-completeness with any constant bound on the width. This game has been used for PSPACE-completeness reductions [BDG<sup>+</sup>15], and a variant with a matching between the two players’ variables has also been studied [BI97]. The partitioned 2-CNF game was not studied in the literature before.

We prove that the unordered and partitioned games are both PSPACE-complete for 5-CNFs; the former improves the width 6 bound from [AO12], and the latter resolves the 42-year-old open problem from [Sch76, Sch78]. We also prove that the unordered and partitioned games are both in P for 2-CNFs. The complexity for width 3 and 4 remains open. In the following section we give the precise definitions and theorem statements.

## 1.1 Statement of results

The *unordered CNF game* is defined as follows. There are two players, denoted T (for “true”) and F (for “false”). The input consists of a CNF  $\varphi$ , a set of variables  $X = \{x_1, \dots, x_n\}$  containing all the variables that appear in  $\varphi$  (and possibly more), and a specification of which player goes first. The players alternate turns, and each turn consists of picking a remaining variable from  $X$  and assigning it a value 0 or 1. Once all variables have been assigned, the game ends and T wins if  $\varphi$  is satisfied, and F wins if it is not. We let G (for “game”) denote the problem of deciding which player has a winning strategy, given  $\varphi$ ,  $X$ , and who goes first.

The *partitioned CNF game* is similar to the unordered CNF game, except that  $X$  is partitioned into two halves  $X_T$  and  $X_F$ , and each player may only pick variables from his/her half. If  $n$  is even we require  $|X_T| = |X_F|$ , and if  $n$  is odd we require  $|X_T| = |X_F| + 1$  if T goes first, and  $|X_F| = |X_T| + 1$  if F goes first. We let  $G^\%$  denote the problem of deciding which player has a winning strategy, given  $\varphi$ , the partition  $X = X_T \cup X_F$ , and who goes first.

We let  $G_w$  and  $G_w^\%$  denote the restrictions of G and  $G^\%$ , respectively, to instances where  $\varphi$  has width  $w$ , i.e., each clause has at most  $w$  literals. Now, we state our results as the following theorems:

**Theorem 1.**  $G_5$  is PSPACE-complete.

**Theorem 2.**  $G_5^\%$  is PSPACE-complete.

**Theorem 3.**  $G_2$  is in P, in fact, in Linear Time.

**Theorem 4.**  $G_2^{\%}$  is in P, in fact, in Linear Time.

We prove Theorem 1 and Theorem 2 in Section 2 by showing reductions from the PSPACE-complete games  $G$  and  $G^{\%}$  respectively. For Theorem 3 and Theorem 4 in Section 3 we prove characterizations in terms of the graph representation from the classical 2-SAT algorithm—who has a winning strategy in terms of certain graph properties—and we design linear time algorithms to check these properties.<sup>1</sup>

In the proofs, it is helpful to distinguish four patterns for ‘who goes first’ and ‘who goes last’, we introduce new subscripts. For  $a, b \in \{T, F\}$ , the subscript  $a \cdots b$  means player  $a$  goes first and player  $b$  goes last,  $a \cdots$  means  $a$  goes first, and  $\cdots b$  means  $b$  goes last. These may be combined with the width  $w$  subscript. For example,  $G_{T \cdots F}^{\%}$  (which was denoted  $G_{\% \text{ free}}(\text{CNF})$  in [Sch76, Sch78], by the way) corresponds to the partitioned game where T goes first and F goes last (so  $n = |X|$  must be even), and  $G_{5 \cdots T}$  corresponds to the unordered game with width 5 where T goes last (so either  $n$  is even and F goes first, or  $n$  is odd and T goes first).

## 2 5-CNF

We prove Theorem 1 in Section 2.1 and Theorem 2 in Section 2.2.

### 2.1 $G_5$

In this section we prove Theorem 1. It is trivial to argue that  $G_5 \in \text{PSPACE}$ . We prove PSPACE-hardness by showing a reduction  $G_{T \cdots F} \leq G_{5, T \cdots F}$  in Section 2.1.2.  $G_{T \cdots F}$  is already known to be PSPACE-complete [Sch76, Sch78, Bys04, AO12]. We will talk about the other three patterns  $G_{F \cdots F}$ ,  $G_{T \cdots T}$ ,  $G_{F \cdots T}$  in Section 2.1.3. Before the formal proof we develop the intuition in Section 2.1.1.

#### 2.1.1 Intuition

In NP-completeness, recall the following simple reduction from SAT with unbounded width to 3-SAT. Suppose a SAT instance is given by  $\varphi$  over set of variables  $X$ . If  $(\ell_1 \vee \ell_2 \vee \ell_3 \vee \cdots \vee \ell_k)$  is a clause in  $\varphi$  with width  $k > 3$ , then the reduction introduces fresh variables  $z_1, z_2, \dots, z_{k-1}$  and generates a chain of clauses in  $\varphi'$  as follows:

$$(\ell_1 \vee z_1) \wedge (\bar{z}_1 \vee \ell_2 \vee z_2) \wedge \cdots \wedge (\bar{z}_{i-1} \vee \ell_i \vee z_i) \wedge \cdots \wedge (\bar{z}_{k-2} \vee \ell_{k-1} \vee z_{k-1}) \wedge (\bar{z}_{k-1} \vee \ell_k)$$

Each clause of  $\varphi$  gets a separate set of fresh variables for its chain, and we let  $Z = \{z_1, z_2, \dots\}$  be the set of all fresh variables for all chains. The reduction claims that  $\varphi$  is satisfiable if and only if  $\varphi'$  is satisfiable. We are going to have a stronger property in Claim 1.

**Claim 1.** *For every assignment  $x$  to  $X$ :  $\varphi(x)$  is satisfied iff there exists an assignment  $z$  to  $Z$  such that  $\varphi'(x, z)$  is satisfied.*

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<sup>1</sup>We remark that it is not automatic that two-player games on 2-CNFs are solvable in polynomial time; e.g., the game played on a 2-CNF with only negative literals in which players alternate turns assigning variables of their choice to 0 and where the loser is the first to falsify the 2-CNF, as well as the partitioned variant of this game, are PSPACE-complete [Sch76, Sch78].

*Proof.* Suppose  $x$  satisfies  $\varphi$ . If  $x$  satisfies  $(\ell_1 \vee \ell_2 \vee \ell_3 \vee \dots \vee \ell_k)$  in  $\varphi$  by  $\ell_i = 1$ , then in the corresponding chain of clauses in  $\varphi'$ , the clause having  $\ell_i$  also gets satisfied by  $\ell_i = 1$  and the rest of the clauses in that chain can get satisfied by assigning all  $z$ 's on the left side of  $\ell_i$  as 1 and right side of  $\ell_i$  as 0.

Now suppose  $x$  does not satisfy  $\varphi$ . Then at least one of the clauses of  $\varphi$  has all literals assigned as 0. The corresponding chain of clauses in  $\varphi'$  essentially becomes:

$$(z_1) \wedge (\bar{z}_1 \vee z_2) \wedge \dots \wedge (\bar{z}_{i-1} \vee z_i) \wedge \dots \wedge (\bar{z}_{k-2} \vee z_{k-1}) \wedge (\bar{z}_{k-1})$$

In order to satisfy the above chain,  $z_1 = 1$  and  $z_{k-1} = 0$ . It also introduces the following chain of implications:  $z_1 \Rightarrow z_2 \Rightarrow z_3 \Rightarrow \dots \Rightarrow z_{k-1}$ . Following the chain we get  $(z_1 \Rightarrow z_{k-1}) = (1 \Rightarrow 0)$ . Therefore, we conclude that  $\varphi'(x, z)$  cannot be satisfied for any assignment  $z$ .  $\square$

Now this reduction does not show  $G_{T..F} \leq G_{3,T..F}$  since the games on  $\varphi$  and  $\varphi'$  are not equivalent. We show a simple example to make our point. Consider the following  $G_{T..F}$  game over variables  $\{x_0, x_1, \dots, x_k\}$ .

$$\varphi = x_0 \wedge (x_1 \vee x_2 \vee x_3 \vee \dots \vee x_k), \text{ where } k > 1$$

In the above  $G_{T..F}$  game, T has a winning strategy: On the first move T plays  $x_0 = 1$ . Then whatever F plays, T plays one of the  $k - 1$  many unassigned  $x_i$  from  $\{x_1, x_2, \dots, x_k\}$  as 1. T wins.

But if we introduce fresh variables  $\{z_1, z_2, z_3, \dots\}$  as in the NP-completeness reduction then we get a game over variables  $\{x_0, x_1, x_2, \dots, x_k\} \cup \{z_1, \dots, z_{k-1}\}$ :

$$\varphi' = x_0 \wedge (x_1 \vee z_1) \wedge \dots \wedge (\bar{z}_{i-1} \vee x_i \vee z_i) \wedge \dots \wedge (\bar{z}_{k-1} \vee x_k)$$

In the above  $G_{3,T..F}$  game, F has a winning strategy: On the first move T must play  $x_0 = 1$ , otherwise F wins by  $x_0 = 0$ . Then F plays  $x_1 = 0$  and T must reply by  $z_1 = 1$ , otherwise F wins by  $z_1 = 0$ . Then F plays  $x_2 = 0$  and T must reply by  $z_2 = 1$ , otherwise F wins by  $z_2 = 0$ . The strategy goes on like this until the last clause and F wins by  $x_k = 0$ .

The  $G_{3,T..F}$  game is disadvantageous for T compared to the  $G_{T..F}$  game. The disadvantage arises from F having the beginning move in a fresh chain of clauses.

Now the intuition is to design a game version of the NP-completeness reduction by fixing the imbalance. We design  $\psi$  in such a way that the games on  $\varphi$  and  $\psi$  stay equivalent. In order to counter the unfairness for T due to fresh variables  $\{z_1, z_2, z_3, \dots\}$ , we replace  $z_i$  by a pair of variables  $(a_i, b_i)$  which gives T more opportunities to satisfy the clauses. The construction of a chain of clauses in  $\psi$  from a clause  $(\ell_1 \vee \ell_2 \vee \ell_3 \vee \dots \vee \ell_k)$  in  $\varphi$  goes as follows:

$$(\ell_1 \vee a_1 \vee b_1) \wedge \dots \wedge (\bar{a}_{i-1} \vee \bar{b}_{i-1} \vee \ell_i \vee a_i \vee b_i) \wedge \dots \wedge (\bar{a}_{k-1} \vee \bar{b}_{k-1} \vee \ell_k)$$

We just constructed a 5-CNF  $\psi$ . Let us consider the  $G_{5,T..F}$  game on  $\psi$ . In an optimal gameplay, no player should play  $a$ 's or  $b$ 's before playing  $x$ 's. Intuitively, this is because, if F plays any  $a_i$  or  $b_i$ , then T can reply by making  $a_i \neq b_i$  and both clauses involving  $a_i$  and  $b_i$  will be satisfied, which benefits T. If T plays any  $a_i$  or  $b_i$ , F can reply by making  $a_i = b_i$ , which satisfies one clause involving  $a_i$  and  $b_i$  but the other clause gets two 0 literals. Since only one of the two clauses gets satisfied by  $a_i, b_i$ , T would like to wait for more information before deciding which one to satisfy with  $a_i, b_i$ : it depends on whether they are on the right side or left side of a satisfied  $\ell_i$  in a chain, which in turn depends on the assignment  $x$ .

So, an optimal gameplay consists of two phases. In the first phase, players should play only  $x$ 's. Whoever deviates from this optimal strategy does not have the upper hand. The second phase begins when all the  $x$ 's have been played and someone must start playing  $a$ 's and  $b$ 's. Since the number of fresh variables is even ( $2|Z|$ ) and F plays last, T must be the one to start the second phase, which is essential since if F started the second phase then T could satisfy all the clauses regardless of what happened in the first phase. This observation also allows us to show PSPACE-completeness of  $G_{5,F..F}$ , discussed in Section 2.1.3.

In the second phase, after T plays any  $a_i$  or  $b_i$ , it is optimal for F to reply by making  $a_i = b_i$ . Assuming this optimal gameplay by F, we can consider a pair  $(a_i, b_i)$  as a single variable  $z_i$  which can be assigned only by T. Effectively, the second phase just consists of T choosing an assignment  $z$  to  $\varphi'$  from the NP-completeness reduction. Thus  $\psi(x, a, b)$  is satisfied iff  $\varphi'(x, z)$  is satisfied, which by Claim 1 is possible iff  $\varphi(x)$  is satisfied, where  $x$  is the assignment from the first phase.

### 2.1.2 Formal Proof

We show  $G_{T..F} \leq G_{5,T..F}$ . Suppose an instance of  $G_{T..F}$  is given by  $(\varphi, X)$  where  $\varphi$  is a CNF with unbounded width over set of variables  $X$ . We show how to construct an instance  $(\psi, Y)$  for  $G_{5,T..F}$  where  $\psi$  is a 5-CNF over set of variables  $Y$ . Suppose  $(\ell_1 \vee \ell_2 \vee \ell_3 \vee \dots \vee \ell_k)$  is a clause in  $\varphi$ . If  $k \leq 3$ , the same clause remains in  $\psi$ . If  $k > 3$ , we show how to construct a chain of clauses in  $\psi$ . We introduce two sets of fresh variables  $\{a_1, a_2, a_3, \dots, a_{k-1}\}$  and  $\{b_1, b_2, b_3, \dots, b_{k-1}\}$  as follows:

$$(\ell_1 \vee a_1 \vee b_1) \wedge \dots \wedge (\bar{a}_{i-1} \vee \bar{b}_{i-1} \vee \ell_i \vee a_i \vee b_i) \wedge \dots \wedge (\bar{a}_{k-1} \vee \bar{b}_{k-1} \vee \ell_k)$$

Each clause of  $\varphi$  gets separate sets of fresh variables for its chain, and we let  $A = \{a_1, a_2, a_3, \dots\}$  and  $B = \{b_1, b_2, b_3, \dots\}$  be the sets of all fresh variables for all chains. Finally we get a 5-CNF  $\psi$  over set of variables  $Y = X \cup A \cup B$ .

We claim that T has a winning strategy in  $(\varphi, X)$  iff T has a winning strategy in  $(\psi, Y)$ .

Suppose T has a winning strategy in  $(\varphi, X)$ . We describe T's winning strategy in  $(\psi, Y)$  as Algorithm 1. To see that the strategy works, note that the winning strategy in  $(\varphi, X)$  ensures that  $\varphi(x)$  is satisfied by the assignment  $x$  to  $X$  in the first phase, so according to Claim 1, there is an assignment  $z$  to  $Z$  such that  $\varphi'(x, z)$  is satisfied. T can ensure that for each  $i$ , either  $a_i = z_i$  or  $b_i = z_i$  (since  $a_i = z_i$  or  $b_i = z_i$  due to line 8, or  $a_i \neq b_i$  due to line 4 or line 7) and thus  $\psi(x, a, b)$  gets satisfied, since  $\varphi'(x, z)$  is satisfied and each clause of  $\psi$  is identical to a clause from  $\varphi'$  but with each  $z_i$  replaced with  $a_i \vee b_i$  and  $\bar{z}_i$  replaced with  $\bar{a}_i \vee \bar{b}_i$ .

Suppose F has a winning strategy in  $(\varphi, X)$ . We describe F's winning strategy in  $(\psi, Y)$  as Algorithm 2. To see that the strategy works, note that the winning strategy in  $(\varphi, X)$  ensures that  $\varphi(x)$  is unsatisfied by the assignment  $x$  to  $X$ , so according to Claim 1, for all assignments  $z$  to  $Z$ ,  $\varphi'(x, z)$  is unsatisfied. F can ensure that for each  $i$ ,  $a_i = b_i$ ; let us call this common value  $z_i$ . Thus  $\psi(x, a, b)$  is unsatisfied, since  $\varphi'(x, z)$  is unsatisfied and  $\psi(x, a, b) = \varphi'(x, z)$ .

### 2.1.3 $G_{F..F}$ , $G_{T..T}$ , $G_{F..T}$

**Corollary 1.**  $G_{5,F..F}$  is PSPACE-complete.

*Proof.* The reduction is  $G_{T..F} \leq G_{F..F} \leq G_{5,F..F}$ . First we show  $G_{T..F} \leq G_{F..F}$ . Suppose  $\varphi = c_1 \wedge c_2 \wedge c_3 \wedge \dots \wedge c_m$  over set of variables  $X$  is an instance of  $G_{T..F}$ . We introduce a fresh variable  $z$  and construct  $\psi = (c_1 \vee z) \wedge (c_2 \vee z) \wedge (c_3 \vee z) \wedge \dots \wedge (c_m \vee z)$  over set of variables

**Algorithm 1:** T's winning strategy in  $(\psi, Y)$  when T has a winning strategy in  $(\varphi, X)$

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1 while there is a remaining  $X$ -variable do
2   if (first move) or (F played an  $X$ -variable in the previous move) then
3      $\lfloor$  play according to the same winning strategy as in  $(\varphi, X)$ 
4   else if F played  $a_i$  or  $b_i$  in the previous move then play the other one to make  $a_i \neq b_i$ 
5 while there is a remaining  $A$ -variable or  $B$ -variable do
6   if (F played  $a_i$  or  $b_i$  in the previous move) and (one of  $a_i$  or  $b_i$  remains unplayed) then
7      $\lfloor$  play the other one to make  $a_i \neq b_i$ 
8   else pick a remaining  $a_i$  or  $b_i$  and assign it  $z_i$ 's value from Claim 1

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**Algorithm 2:** F's winning strategy in  $(\psi, Y)$  when F has a winning strategy in  $(\varphi, X)$

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1 while there is a remaining variable do
2   if T played an  $X$ -variable in the previous move then
3      $\lfloor$  play according to the same winning strategy as in  $(\varphi, X)$ 
4   else if T played  $a_i$  or  $b_i$  in the previous move then play the other one to make  $a_i = b_i$ 

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$Y = X \cup \{z\}$ . Now in the  $G_{F\dots F}$  game on  $(\psi, Y)$ , F's first move must be  $z = 0$  otherwise T wins by  $z = 1$  as the first move. Then the rest of the winning strategy for T or F is the same as in  $(\varphi, X)$ . This completes the reduction  $G_{T\dots F} \leq G_{F\dots F}$ .

Now the reduction  $G_{F\dots F} \leq G_{5,F\dots F}$  is identical to Section 2.1.2 except it is F's move first.  $\square$

**Corollary 2.**  $G_{11,T\dots T}$  is PSPACE-complete.

*Proof.* The reduction is  $G_{11,T\dots F}^+ \leq G_{11,T\dots T}$ , where  $G_{11,T\dots F}^+$  is the restriction of  $G_{11,T\dots F}$  to instances with only positive literals (which is known to be PSPACE-complete [Sch76, Sch78]). Given a positive 11-CNF  $\varphi^+$  over set of variables  $X$ , we simply introduce a dummy variable  $z$  that does not appear in  $\varphi^+$  and use  $Y = X \cup \{z\}$ . As  $\varphi^+$  is a positive CNF, no player would want to play  $z$  until the end. Playing  $z$  is like having a pass move. In a positive CNF, T would never want to skip the chance to play a literal as 1 and F would never want to skip the chance to play a literal as 0. So in  $(\varphi^+, Y)$  there is a winning strategy for T or F which is the same as in  $(\varphi^+, X)$  until  $z$  is played at the end. This completes the reduction.  $\square$

**Corollary 3.**  $G_{12,F\dots T}$  is PSPACE-complete.

*Proof.* The reduction is  $G_{11,T\dots T} \leq G_{12,F\dots T}$  (similar to  $G_{T\dots F} \leq G_{F\dots F}$  in Corollary 1): Introduce a fresh variable  $z$  to every clause in  $\varphi$ . Then F must play  $z = 0$  as the first move otherwise T wins by  $z = 1$  as the first move.  $\square$

## 2.2 $G_5^{\%}$

In this section we prove Theorem 2. It is trivial to argue that  $G_5^{\%} \in \text{PSPACE}$ . We prove PSPACE-hardness by showing a reduction  $G_{T\dots F}^{\%} \leq G_{5,T\dots F}^{\%}$  in Section 2.2.2.  $G_{T\dots F}^{\%}$  is already known to be

PSPACE-complete [Sch76, Sch78]. We will talk about the other three patterns  $G_{F\dots F}^{\%}$ ,  $G_{T\dots T}^{\%}$ ,  $G_{F\dots T}^{\%}$  in Section 2.2.3. Before the formal proof we develop the intuition in Section 2.2.1.

### 2.2.1 Intuition

This intuition is a continuation of Section 2.1.1. The reduction is the same as  $G_{T\dots F} \leq G_{5,T\dots F}$  reduction except giving  $A$ -variables to  $T$  and  $B$ -variables to  $F$ . In the general unordered game if any player plays  $a_i$  or  $b_i$ , then the other player can immediately play the other one from  $a_i, b_i$  in a certain advantageous way. In the partitioned version they can do the same thing if  $a_i$  belongs to  $T$  and  $b_i$  belongs to  $F$ .

### 2.2.2 Formal Proof

We show  $G_{T\dots F}^{\%} \leq G_{5,T\dots F}^{\%}$ . Suppose an instance of  $G_{T\dots F}^{\%}$  is given by  $(\varphi, X_T, X_F)$  where  $\varphi$  is a CNF with unbounded width over sets of variables  $X_T$  and  $X_F$ . We show how to construct an instance  $(\psi, Y_T, Y_F)$  for  $G_{5,T\dots F}^{\%}$  where  $\psi$  is a 5-CNF over sets of variables  $Y_T$  and  $Y_F$ . Suppose  $(\ell_1 \vee \ell_2 \vee \ell_3 \vee \dots \vee \ell_k)$  is a clause in  $\varphi$ . If  $k \leq 3$ , the same clause remains in  $\psi$ . If  $k > 3$ , we show how to construct a chain of clauses in  $\psi$ . We introduce two sets of fresh variables  $\{a_1, a_2, a_3, \dots, a_{k-1}\}$  for  $T$  and  $\{b_1, b_2, b_3, \dots, b_{k-1}\}$  for  $F$  as follows:

$$(\ell_1 \vee a_1 \vee b_1) \wedge \dots \wedge (\bar{a}_{i-1} \vee \bar{b}_{i-1} \vee \ell_i \vee a_i \vee b_i) \wedge \dots \wedge (\bar{a}_{k-1} \vee \bar{b}_{k-1} \vee \ell_k)$$

Each clause of  $\varphi$  gets separate sets of fresh variables for its chain, and we let  $A = \{a_1, a_2, a_3, \dots\}$  for  $T$  and  $B = \{b_1, b_2, b_3, \dots\}$  for  $F$  be the sets of all fresh variables for all chains. Finally we get a 5-CNF  $\psi$  over sets of variables  $Y_T = X_T \cup A$  and  $Y_F = X_F \cup B$ .

We claim that  $T$  has a winning strategy in  $(\varphi, X_T, X_F)$  iff  $T$  has a winning strategy in  $(\psi, Y_T, Y_F)$ .

Suppose  $T$  has a winning strategy in  $(\varphi, X_T, X_F)$ . We describe  $T$ 's winning strategy in  $(\psi, Y_T, Y_F)$  as Algorithm 3. To see that the strategy works, note that the winning strategy in  $(\varphi, X_T, X_F)$  ensures that  $\varphi(x)$  is satisfied by the assignment  $x$  to  $X_T \cup X_F$  in the first phase, so according to Claim 1, there is an assignment  $z$  to  $Z$  such that  $\varphi'(x, z)$  is satisfied.  $T$  can ensure that for each  $i$ , either  $a_i = z_i$  or  $b_i = z_i$  (since  $a_i = z_i$  due to line 8, or  $a_i \neq b_i$  due to line 4 or line 7) and thus  $\psi(x, a, b)$  gets satisfied, since  $\varphi'(x, z)$  is satisfied and each clause of  $\psi$  is identical to a clause from  $\varphi'$  but with each  $z_i$  replaced with  $a_i \vee b_i$  and  $\bar{z}_i$  replaced with  $\bar{a}_i \vee \bar{b}_i$ .

Suppose  $F$  has a winning strategy in  $(\varphi, X_T, X_F)$ . We describe  $F$ 's winning strategy in  $(\psi, Y_T, Y_F)$  as Algorithm 4. To see that the strategy works, note that the winning strategy in  $(\varphi, X_T, X_F)$  ensures that  $\varphi(x)$  is unsatisfied by the assignment  $x$  to  $X_T \cup X_F$ , so according to Claim 1, for all assignments  $z$  to  $Z$ ,  $\varphi'(x, z)$  is unsatisfied.  $F$  can ensure that for each  $i$ ,  $a_i = b_i$ ; let us call this common value  $z_i$ . Thus  $\psi(x, a, b)$  is unsatisfied, since  $\varphi'(x, z)$  is unsatisfied and  $\psi(x, a, b) = \varphi'(x, z)$ .

### 2.2.3 $G_{F\dots F}^{\%}$ , $G_{T\dots T}^{\%}$ , $G_{F\dots T}^{\%}$

**Corollary 4.**  $G_{5,F\dots F}^{\%}$  is PSPACE-complete.

*Proof.* The reduction is  $G_{T\dots F}^{\%} \leq G_{F\dots F}^{\%} \leq G_{5,F\dots F}^{\%}$ . First we show  $G_{T\dots F}^{\%} \leq G_{F\dots F}^{\%}$ . Suppose  $(\varphi, X_T, X_F)$  is an instance of  $G_{T\dots F}^{\%}$ . We introduce a dummy variable  $z$  that does not appear in  $\varphi$  and give it to  $F$ :  $Y_T = X_T$ ,  $Y_F = X_F \cup \{z\}$ . Thus  $(\varphi, Y_T, Y_F)$  is an instance of  $G_{F\dots F}^{\%}$ . The reduction works since when  $F$  has a winning strategy in  $(\varphi, X_T, X_F)$ ,  $F$  can play  $z$  as the first

**Algorithm 3:** T's winning strategy in  $(\psi, Y_T, Y_F)$  when T has a winning strategy in  $(\varphi, X_T, X_F)$

```

1 while there is a remaining  $X_T$ -variable do
2   if (first move) or (F played an  $X_F$ -variable in the previous move) then
3      $\lfloor$  play according to the same winning strategy as in  $(\varphi, X_T, X_F)$ 
4   else if F played  $b_i$  in the previous move then play  $a_i$  to make  $a_i \neq b_i$ 
5 while there is a remaining  $A$ -variable do
6   if (F played  $b_i$  in the previous move) and ( $a_i$  remains unplayed) then
7      $\lfloor$  play  $a_i$  to make  $a_i \neq b_i$ 
8   else pick a remaining  $a_i$  and assign it  $z_i$ 's value from Claim 1

```

**Algorithm 4:** F's winning strategy in  $(\psi, Y_T, Y_F)$  when F has a winning strategy in  $(\varphi, X_T, X_F)$

```

1 while there is a remaining variable do
2   if T played an  $X_T$ -variable in the previous move then
3      $\lfloor$  play according to the same winning strategy as in  $(\varphi, X_T, X_F)$ 
4   else if T played  $a_i$  in the previous move then play  $b_i$  to make  $a_i = b_i$ 

```

move then continue the winning strategy as in  $(\varphi, X_T, X_F)$ , and conversely, when T has a winning strategy in  $(\varphi, X_T, X_F)$ , T can use the same strategy in  $(\varphi, Y_T, Y_F)$ —if F does not play  $z$  at the beginning, then this only provides T more information than otherwise, making it easier for T to win. This completes the reduction  $G_{T\dots F}^{\%} \leq G_{F\dots F}^{\%}$ .

Now the reduction  $G_{F\dots F}^{\%} \leq G_{5,F\dots F}^{\%}$  is identical to Section 2.2.2 except it is F's move first.  $\square$

**Observation 1.**  $G_{3,T\dots F}^{\%}$ ,  $G_{3,F\dots F}^{\%}$ ,  $G_{3,T\dots T}^{\%}$ ,  $G_{3,F\dots T}^{\%}$  are NP-hard.

*Proof.* First we show that  $3\text{-SAT} \leq G_{3,T\dots F}^{\%}$ . Suppose  $(\varphi, X)$  is an instance of 3-SAT. We construct the instance  $(\varphi, Y_T, Y_F)$  of  $G_{3,T\dots F}^{\%}$  where  $Y_T = X$  and  $Y_F$  is a new set of fresh variables such that  $|Y_F| = |X|$ . F's moves do not matter. If  $\varphi$  is satisfiable then T can play a satisfying assignments, otherwise T cannot satisfy  $\varphi$ .

The reductions for the other patterns are similar.  $\square$

### 3 2-CNF

In order to analyze the complexity of the games  $G_2$  and  $G_2^{\%}$ , we construct a directed graph  $g(\varphi, X)$  by the classical technique for 2-SAT:

- For each variable  $x_i \in X$ , form two nodes  $x_i$  and  $\bar{x}_i$ . Let  $\ell_i$  refer to either  $x_i$  or  $\bar{x}_i$ .<sup>2</sup>

<sup>2</sup>In Section 2,  $\ell_i$  represented an arbitrary literal; in Section 3,  $\ell_i$  always represents either  $x_i$  or  $\bar{x}_i$ .



- For each clause  $(\ell_i \vee \ell_j)$ , add two directed edges  $\bar{\ell}_i \rightarrow \ell_j$  and  $\ell_i \leftarrow \bar{\ell}_j$ . In case of a single variable clause  $(\ell_i)$ , consider the clause as  $(\ell_i \vee \ell_i)$  and add one directed edge  $\bar{\ell}_i \rightarrow \ell_i$ .

In the graph, every path  $\ell_i \rightsquigarrow \ell_j$  has a mirror path  $\bar{\ell}_i \leftarrow \bar{\ell}_j$ . If there exist two paths  $\ell_i \rightsquigarrow \ell_j$  and  $\ell_i \leftarrow \bar{\ell}_j$ , we express this as  $\ell_i \leftrightarrow \bar{\ell}_j$ . We are interested in strongly connected components, which we call strong components for short.

The 2-CNF game analogy on this graph is, if any variable  $x_i$  is assigned a bit value in  $\varphi$ , then in the graph both nodes  $x_i$  and  $\bar{x}_i$  are assigned. Conversely, if say a player assigns a bit value to a node  $\ell_i$ , then the complement node  $\bar{\ell}_i$  simultaneously gets assigned the opposite value. If  $\ell_i$  refers to  $x_i$ , then  $x_i$  gets assigned the same value as  $\ell_i$  and similarly for  $\ell_i$  referring to  $\bar{x}_i$ . Thus we can describe strategies as assigning bit values to nodes in the graph.

In a satisfying assignment for  $\varphi$ , there must not exist any false implication edge  $(1 \rightarrow 0)$  in the graph. In fact, the graph must not have any path  $(1 \rightsquigarrow 0)$  since the path will contain at least one  $(1 \rightarrow 0)$  edge. Player F's goal is to create a false implication and player T will try to make all implications true.

We prove [Theorem 3](#) in [Section 3.1](#) and [Theorem 4](#) in [Section 3.2](#).

### 3.1 $G_2$

$G_2$  is the unordered analogue of the 2-TQBF game. We prove [Theorem 3](#) by separately considering the cases  $G_{2,F\dots F}$  in [Section 3.1.1](#),  $G_{2,F\dots T}$  in [Section 3.1.2](#), and  $G_{2,T\dots}$  in [Section 3.1.3](#).

#### 3.1.1 $G_{2,F\dots F} \in$ Linear Time

**Lemma 1.** *F has a winning strategy in  $G_{2,F\dots F}$  iff at least one of the following statements holds in the graph  $g(\varphi, X)$ :*

- (1) *There exists a node  $\ell_i$  such that  $\bar{\ell}_i \rightsquigarrow \ell_i$ .*
- (2) *There exist three nodes  $\ell_i, \ell_j, \ell_k$  such that  $\ell_j \rightsquigarrow \ell_i \leftarrow \ell_k$ .*
- (3) *There exist two nodes  $\ell_i, \ell_j$  such that  $\ell_i \leftrightarrow \ell_j$ .*

*Proof.* Suppose at least one of the statements holds.

If statement (1) holds, F can win by  $\ell_i = 0$  as the very first move.

If statement (2) holds but statement (1) does not, there can be two cases:

- In the first case,  $\ell_i, \ell_j, \ell_k$  represent three distinct variables. At the beginning, F can play  $\ell_i = 0$ , then whatever T plays, F still has at least one of  $\ell_j$  or  $\ell_k$  to play. F can assign  $\ell_j$  or  $\ell_k$  as 1 and wins.
- In the second case,  $\ell_i, \ell_j, \ell_k$  do not represent three distinct variables. The only possibility is that  $\ell_k$  is  $\bar{\ell}_j$ , i.e.,  $\ell_j \rightsquigarrow \ell_i \leftarrow \bar{\ell}_j$ . F can play  $\ell_i = 0$ , then whatever the value of  $\ell_j$ , F wins.

If statement (3) holds but statement (1) does not, F can wait by playing variables other than  $x_i, x_j$  with arbitrary values until T plays  $x_i$  or  $x_j$ . Then F can immediately respond by making  $\ell_i \neq \ell_j$  and win. As F moves last, he/she can always wait for that opportunity.

Conversely, suppose none of the statements hold. Then we claim the graph has no two edges that share an endpoint. Otherwise, two edges that share an endpoint would cause statement (2) or statement (3) to be satisfied. We show this by considering all possible ways of two edges sharing an endpoint:

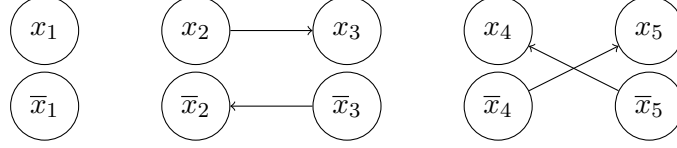


Figure 1: T has a winning strategy in  $G_{2,F\dots F}$  for  $(\bar{x}_2 \vee x_3) \wedge (x_4 \vee x_5)$

**Algorithm 5:** Linear Time Algorithm for  $G_{2,F\dots F}$

```

1 construct  $g(\varphi, X)$ 
2 foreach  $x_i \in X$  do
3   if  $(x_i \rightarrow \bar{x}_i)$  or  $(x_i \leftarrow \bar{x}_i)$  or  $(x_i$  has at least two incident edges) then output F
4 output T

```

- $l_i \leftrightarrow l_j$ : Satisfies statement (3).
- $l_j \rightarrow l_i \leftarrow l_k$  or its mirror  $\bar{l}_j \leftarrow \bar{l}_i \rightarrow \bar{l}_k$ : Satisfies statement (2).
- $l_k \rightarrow l_j \rightarrow l_i$ : Satisfies statement (2).

So, the graph can only have some isolated nodes and isolated edges. Since statement (1) does not hold, there are no edges between complementary nodes. An example of such a graph looks like Figure 1. Conversely, in any such graph (like Figure 1) none of statements (1), (2), (3) holds.

Now, we describe a winning strategy for T on such a graph. If F plays  $l_i$  or  $l_j$  of any fresh (both endpoints unassigned) edge  $l_i \rightarrow l_j$ , T plays in the same edge by the same bit value for the other node, i.e., making  $l_i = l_j$ . Otherwise, T picks any remaining node  $l_i$ . If  $l_i$  is isolated, T assigns any arbitrary bit value. If  $l_i$  has an incoming edge, T plays  $l_i = 1$ . If  $l_i$  has an outgoing edge, T plays  $l_i = 0$ .

The strategy works, since all the edges  $l_i \rightarrow l_j$  will be satisfied, by either  $l_i = l_j$  or  $l_i = 0$  or  $l_j = 1$ .  $\square$

The characterization of such a graph in the proof of Lemma 1 can be verified in linear time, and that yields a Linear Time algorithm for  $G_{2,F\dots F}$ . Details of the idea have been described as Algorithm 5.

### 3.1.2 $G_{2,F\dots T} \in$ Linear Time

The characterization is the same as for  $G_{2,F\dots F}$  but without statement (3).

**Lemma 2.** F has a winning strategy in  $G_{2,F\dots T}$  iff at least one of the following statements holds in the graph  $g(\varphi, X)$ :

- (1) There exists a node  $l_i$  such that  $\bar{l}_i \rightsquigarrow l_i$ .
- (2) There exist three nodes  $l_i, l_j, l_k$  such that  $l_j \rightsquigarrow l_i \leftarrow l_k$ .

*Proof.* Suppose one of the statements holds. In Lemma 1, we have already seen that statement (1) and statement (2) allow player F to win at the beginning.

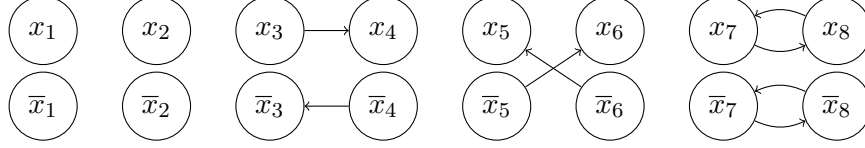


Figure 2: T has a winning strategy in  $G_{2,F...T}$  for  $(\bar{x}_3 \vee x_4) \wedge (x_5 \vee x_6) \wedge (\bar{x}_7 \vee x_8) \wedge (x_7 \vee \bar{x}_8)$

**Algorithm 6:** Linear Time Algorithm for  $G_{2,F...T}$

```

1 construct  $g(\varphi, X)$ 
2 foreach  $x_i \in X$  do
3   if  $(x_i \rightarrow \bar{x}_i)$  or  $(x_i \leftarrow \bar{x}_i)$  or  $(x_i$  has at least two neighbors) then output F
4 output T

```

Conversely, suppose none of the statements hold. The graph can have strong components of size 2. Other than that, there are no two edges sharing an endpoint because statement (2) does not hold. So, the graph can only have some isolated nodes, isolated edges, and isolated strong components of size 2. Since statement (1) does not hold, there are no edges between complementary nodes. An example of such a graph looks like Figure 2. Conversely, in any such graph (like Figure 2) none of statements (1), (2) holds.

Now, we describe a winning strategy for T on such a graph. If F plays  $\ell_i$  or  $\ell_j$  of any fresh (both endpoints unassigned) edge  $\ell_i \rightarrow \ell_j$  or strong component  $\ell_i \leftrightarrow \ell_j$ , T plays in the same edge or strong component by the same bit value for the other node, i.e., making  $\ell_i = \ell_j$ . Otherwise, T picks any remaining isolated node and gives it any arbitrary bit value. Since  $|X|$  is even, T can always play such a node.

The strategy works, since all the edges  $\ell_i \rightarrow \ell_j$  will be satisfied by  $\ell_i = \ell_j$ . □

The characterization of such a graph in the proof of Lemma 2 can be verified in linear time, and that yields a Linear Time algorithm for  $G_{2,F...T}$ . Details of the idea have been described as Algorithm 6.

### 3.1.3 $G_{2,T...} \in$ Linear Time

In order to win  $G_{2,T...}$ , at the beginning T must locate a node  $\ell_i$  such that after playing it, the game is reduced to a  $G_{2,F...}$  game such that T still has a winning strategy in it. So, T's success depends on finding such a node  $\ell_i$ . On the other hand, F's success depends on there not existing such a node  $\ell_i$ .

**Lemma 3.** T has a winning strategy in  $G_{2,T...}$  iff there exists an  $\ell_i$  with no outgoing edges such that after deleting  $\ell_i$ ,  $\bar{\ell}_i$  and their incident edges, in the rest of the graph T has a winning strategy in  $G_{2,F...}$ .

*Proof.* Suppose T has a winning strategy in  $G_{2,T...}$ . Let T's first move in the winning strategy be  $\ell_i = 1$  (or  $\bar{\ell}_i = 0$ ). Then  $\ell_i$  must not have any outgoing edge, otherwise either that edge goes to  $\bar{\ell}_i$  or F could play the other endpoint node of that edge as 0 and win.

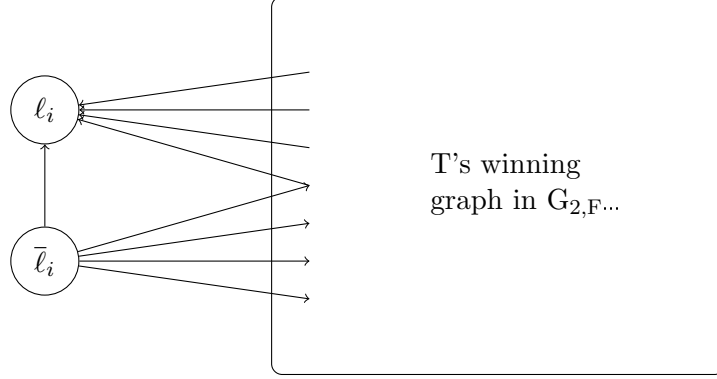


Figure 3: T's winning graph in  $G_{2,T\dots}$  (all edges incident to  $l_i$  or  $\bar{l}_i$  are optional)

Conversely, suppose there exists such an  $l_i$ . At the beginning, T can play  $l_i = 1$ , and all the incoming edges to  $l_i$  and outgoing edges from  $\bar{l}_i$  get satisfied. Then T can continue the game according to the winning strategy in  $G_{2,F\dots}$  for the rest of the graph and win. For example, in Figure 3, T's winning strategy is to play  $l_i = 1$  at the beginning then continue the winning strategy for  $G_{2,F\dots}$ .  $\square$

We define  $L$  as the set of all nodes that have no outgoing edges. If  $|L| = 0$ , then according to Lemma 3, T has no winning strategy in  $G_{2,T\dots}$ . If  $|L| > 0$ , then the trivial algorithm for  $G_{2,T\dots}$  is, checking for each node  $l_i \in L$ , whether or not after playing  $l_i = 1$  the rest of the graph becomes a winning graph for T in  $G_{2,F\dots}$ , i.e., running Algorithm 5 or Algorithm 6 for  $O(|L|)$  times, which is a quadratic time algorithm. We argue that we can do better than that.

We filter the possibilities in  $L$  and show that there are only three cases to consider:

- There exists a node  $l_i \in L$  such that statement (1) from Lemma 1 and Lemma 2 holds. We consider this case in Claim 2.
- There exists a node  $l_i \in L$  such that statement (2) from Lemma 1 and Lemma 2 holds. We consider this case in Claim 3.
- There exists no node  $l_i \in L$  such that statement (1) or statement (2) from Lemma 1 and Lemma 2 holds. We consider this case in Claim 4.

Then in Claim 5 and Claim 6 we analyze the efficiency of this approach.

**Claim 2.** *If there exists  $l_i \in L$  such that  $\bar{l}_i \rightsquigarrow l_i$  and T has a winning strategy in  $G_{2,T\dots}$ , then T's first move must be  $l_i = 1$ .*

*Proof.* Suppose T's first move is not  $l_i = 1$ . If T's first move assigns 1 to a node with an outgoing edge, then T loses as in Lemma 3. Otherwise, T's first move must not involve any variable on the path  $\bar{l}_i \rightsquigarrow l_i$  (since if it assigns 1 to a node on the path other than  $l_i$  then that node has an outgoing edge, and if it assigns 0 to a node on the path other than  $\bar{l}_i$  then that node's complement has an outgoing edge). In this case, in the rest of the game T loses by statement (1) from Lemma 1 and Lemma 2.  $\square$

**Claim 3.** *If there exists  $l_i \in L$  such that  $l_j \rightsquigarrow l_i \leftarrow l_k$  for two other nodes  $l_j, l_k$  and T has a winning strategy in  $G_{2,T\dots}$ , then T's first move must be  $l_i = 1$  or  $\bar{l}_j = 1$  or  $\bar{l}_k = 1$ .*

*Proof.* Suppose T's first move is not  $\ell_i = 1$  or  $\bar{\ell}_j = 1$  or  $\bar{\ell}_k = 1$ . If T's first move assigns 1 to a node with an outgoing edge, then T loses as in Lemma 3. Otherwise, T's first move must not involve any variable on the paths  $\ell_j \rightsquigarrow \ell_i \leftarrow \ell_k$  (since if it assigns 1 to a node on the paths other than  $\ell_i$  then that node has an outgoing edge, and if it assigns 0 to a node on the paths other than  $\ell_j$  or  $\ell_k$  then that node's complement has an outgoing edge). In this case, in the rest of the game T loses by statement (2) from Lemma 1 and Lemma 2.  $\square$

**Claim 4.** *If there exists no  $\ell_i \in L$  such that  $\bar{\ell}_i \rightsquigarrow \ell_i$  or  $\ell_j \rightsquigarrow \ell_i \leftarrow \ell_k$  for two other nodes  $\ell_j, \ell_k$  and T has a winning strategy in  $G_{2,T,\dots}$ , then for all  $\ell_i \in L$ , T has a winning strategy in  $G_{2,T,\dots}$  beginning with  $\ell_i = 1$ .*

*Proof.* For all nodes  $\ell_i \in L$ , statement (1) and statement (2) from Lemma 1 and Lemma 2 do not hold. So all nodes  $\ell_i \in L$  are either isolated single nodes or have only one isolated incoming edge, from another variable's node outside  $L$ . (The argument is similar to the situation when statement (1) and statement (2) do not hold in Lemma 1 and Lemma 2.) If T plays any  $\ell_i \in L$  as  $\ell_i = 1$ , then it does not affect whether or not statements (1), (2), (3) from Lemma 1 and Lemma 2 hold on the rest of the graph. So, if T indeed has a winning strategy then it does not matter which  $\ell_i \in L$  is assigned as 1 as the first move.  $\square$

The overall idea is: If we can find an  $\ell_i$  for which statement (1) or statement (2) from Lemma 1 and Lemma 2 holds, then Claim 2 and Claim 3 allow us to narrow down T's first move to  $O(1)$  possibilities. If we cannot find such an  $\ell_i$ , then Claim 4 allows T to play any arbitrary  $\ell_i \in L$  as the first move because all of them are equivalent as the first move. We define  $L^*$  as the  $O(1)$  possibilities in  $L$ . Then we can run Algorithm 5 or Algorithm 6 for  $|L^*| = O(1)$  times.

In the following two claims, we show how we can efficiently verify whether or not there exists such an  $\ell_i$  for which statement (1) or statement (2) from Lemma 1 and Lemma 2 holds.

**Claim 5.** *There exists a constant-time algorithm for: given  $\ell_i$ , find two other nodes  $\ell_j, \ell_k$  such that  $\ell_j \rightsquigarrow \ell_i \leftarrow \ell_k$  or determine they do not exist.*

*Proof.* It is sufficient to check three cases:

- $\ell_i$  has indegree  $> 1$ : Then we can find  $\ell_j \rightarrow \ell_i \leftarrow \ell_k$ .
- $\ell_i$  has indegree  $= 1$ : There exists  $\ell_j$  with  $\ell_j \rightarrow \ell_i$ . Then look for  $\ell_k$  with  $\ell_k \rightarrow \ell_j$ .
- $\ell_i$  has indegree  $< 1$ : Such  $\ell_j, \ell_k$  do not exist.  $\square$

**Claim 6.** *There exists a constant-time algorithm for: given  $\ell_i$  for which there are no  $\ell_j, \ell_k$  as in Claim 5, decide whether there exists a path  $\bar{\ell}_i \rightsquigarrow \ell_i$ .*

*Proof.* Since  $\ell_j \rightsquigarrow \ell_i \leftarrow \ell_k$  does not hold,  $\ell_i$  has indegree  $\leq 1$  and any incoming neighbor has indegree 0. It is sufficient to check two cases:

- $\ell_i$  has indegree  $= 1$ : Then check if  $\bar{\ell}_i \rightarrow \ell_i$ .
- $\ell_i$  has indegree  $< 1$ : Such a path does not exist.  $\square$

Now combining the whole idea from Claim 2 to Claim 6 we can develop an algorithm for  $G_{2,T,\dots}$ . Details of the idea have been described as Algorithm 7.

**Algorithm 7:** Linear Time Algorithm for  $G_{2,T\dots}$ 

```

1 construct  $g(\varphi, X)$ 
2 let  $L = \{\}, L^* = \{\}$ 
3 foreach node  $\ell_i$  do
4   if  $\ell_i$  has no outgoing edges then  $L = L \cup \{\ell_i\}$ 
5 if  $|L| = 0$  then output F
6 foreach  $\ell_i \in L$  do
7   if  $\ell_j \rightsquigarrow \ell_i \leftarrow \ell_k$  for two other nodes  $\ell_j, \ell_k$  (using Claim 5) then
8      $L^* = L \cap \{\ell_i, \bar{\ell}_j, \bar{\ell}_k\}$  (Claim 3), break loop
9   else if  $\bar{\ell}_i \rightsquigarrow \ell_i$  (using Claim 6) then  $L^* = \{\ell_i\}$  (Claim 2), break loop
10 if  $|L^*| = 0$  then  $L^* = \{\ell_i\}$  for an arbitrary  $\ell_i \in L$  (Claim 4)
11 foreach  $\ell_i \in L^*$  do
12   form graph  $g'$  from  $g(\varphi, X)$  by deleting nodes  $\ell_i, \bar{\ell}_i$  and their incident edges
13   run Algorithm 5 or Algorithm 6 on  $g'$  as the  $G_{2,F\dots}$  game
14   if T has a winning strategy in  $G_{2,F\dots}$  then output T
15 output F

```

### 3.2 $G_2^\%$

In this section we prove Theorem 4 by separately considering the cases  $G_{2,\dots F}^\%$  in Section 3.2.1 and  $G_{2,\dots T}^\%$  in Section 3.2.2. We let  $V_T$  and  $V_F$  be the sets of nodes created from  $X_T$  and  $X_F$  respectively. Also, let  $V = V_T \cup V_F$  be the set of all nodes.

#### 3.2.1 $G_{2,\dots F}^\% \in$ Linear Time

**Lemma 4.** F has a winning strategy in  $G_{2,\dots F}^\%$  iff at least one of the following statements holds in the graph  $g(\varphi, X)$ :

- (1) There exists a node  $\ell_i \in V$  such that  $\bar{\ell}_i \rightsquigarrow \ell_i$ .
- (2) There exist two nodes  $\ell_i, \ell_j \in V_F$  such that  $\ell_i \rightsquigarrow \ell_j$ .
- (3) There exist two nodes  $\ell_i \in V_F$  and  $\ell_j \in V_T$  such that  $\ell_i \rightsquigarrow \ell_j$ .

*Proof.* Suppose at least one of the statements holds.

If statement (1) holds, F can win by any strategy since  $\varphi$  is unsatisfiable.

If statement (2) holds, F can play  $\ell_i = 1$ , and either  $\ell_j$  is  $\bar{\ell}_i = 0$  or F can play  $\ell_j = 0$  and win.

If statement (3) holds, F can wait by playing variables other than  $x_i$  with arbitrary values until T plays  $x_j$ . Then F can respond by making  $\ell_i \neq \ell_j$  and win. As F moves last, he/she can always wait for that opportunity.

Conversely, suppose none of the statements hold. Since statement (1) does not hold, the graph has a satisfying assignment [APT79]. Since statement (2) does not hold, there is no edge or path between any two nodes of  $V_F$ . Since statement (3) also does not hold, there is no node in  $V_F$  that belongs to a strong component of size  $> 1$ . Intuitively, if  $V_F$  is reachable from a node  $\ell_i \in V_T$  then F can force T to assign  $\ell_i = 0$  by assigning the other endpoint of the path as 0, and similarly

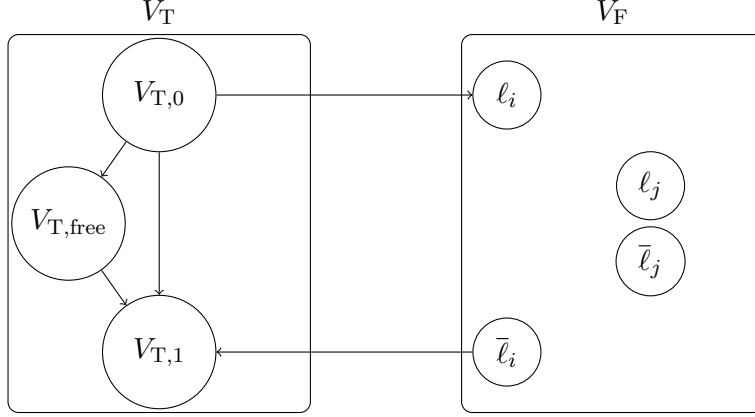


Figure 4: T has a winning strategy in  $G_{2,\dots,F}^{\%}$

if  $l_i \in V_T$  is reachable from  $V_F$  then F can force T to assign  $l_i = 1$ . All other nodes in  $V_T$  are intuitively free from the influence of F's strategy, meaning T is free to assign any bit value he/she likes. This motivates us to partition  $V_T$  into three sets  $V_{T,0}$ ,  $V_{T,1}$ ,  $V_{T,free}$ , defined as follows:

- $V_{T,0} = \{l_j \in V_T : l_j \rightsquigarrow l_i \text{ for some } l_i \in V_F\}$
- $V_{T,1} = \{l_j \in V_T : l_j \leftarrow l_i \text{ for some } l_i \in V_F\}$
- $V_{T,free} = V_T \setminus (V_{T,0} \cup V_{T,1})$

This is indeed a partition: there must not be any common node that is in both  $V_{T,0}$  and  $V_{T,1}$ , because this would create either a path between two nodes of  $V_F$  (satisfying statement (2)) or a cycle touching both  $V_T$  and  $V_F$  (satisfying statement (3)). Note that there cannot be any edge entering  $V_{T,0}$ , leaving  $V_{T,1}$ , or between  $V_{T,free}$  and  $V_F$ . In general  $V_F$  may have many isolated nodes. A general case of the graph looks like Figure 4.

Now we describe a winning strategy for T on such a graph. Whatever F plays, T picks any remaining node to play. If the node is in  $V_{T,0}$ , T assigns it 0. If the node is in  $V_{T,1}$ , T assigns it 1. If the node is in  $V_{T,free}$ , T assigns it according to a satisfying assignment that exists since statement (1) does not hold.

The strategy works since each edge  $l_i \rightarrow l_j$  has either  $l_i \in V_{T,0}$  in which case it gets satisfied by  $l_i = 0$ , or  $l_j \in V_{T,1}$  in which case it gets satisfied by  $l_j = 1$ , or  $l_i, l_j \in V_{T,free}$  in which case it gets satisfied by the satisfying assignment.  $\square$

Now, we develop a linear time algorithm to check statements (1), (2), (3) in Lemma 4. We start by creating a topologically sorted DAG of strong components for the whole graph. The DAG construction can be done in linear time [Tar72]. We can check statements (1), (3) by directly inspecting the strong components. In order to check statement (2) we do dynamic programming over the topological order of strong components to see whether any strong component containing a node in  $V_F$  is reachable from any other such strong component. The idea has been described as Algorithm 8.

### 3.2.2 $G_{2,\dots,T}^{\%} \in$ Linear Time

The characterization is the same as for  $G_{2,\dots,F}^{\%}$  except statement (3).

**Algorithm 8:** Linear Time algorithm for  $G_{2,\dots,F}^{\%$ 

```
1 construct  $g(\varphi, X)$ 
2 construct  $g^*$  as the DAG of strong components from  $g(\varphi, X)$ 
3 let  $S$  = set of all strong components of  $g(\varphi, X)$  (nodes of  $g^*$ )
4 foreach  $\ell_i \in V$  do
5   | let  $s = \ell_i$ 's strong component
6   | if  $(\bar{\ell}_i \in s)$  or  $(\ell_i \in V_F$  and  $|s| > 1)$  then output F
7 let  $S_F$  = set of strong components containing nodes from  $V_F$ 
8 let  $S_T$  = set of strong components containing nodes from  $V_T$ 
9 mark all  $s \in S_F$  as “reachable from  $S_F$ ”
10 topologically order  $s_1, s_2, s_3, \dots \in S$  so edges of  $g^*$  go from lower to higher indices
11 foreach  $i = 1, 2, 3, \dots, |S|$  do
12   | if  $\exists j < i$  such that  $s_j \rightarrow s_i$  and  $s_j$  is marked then
13     |   | if  $s_i \in S_T$  then mark  $s_i$  as “reachable from  $S_F$ ”
14     |   | else output F
15 output T
```

**Lemma 5.** F has a winning strategy in  $G_{2,\dots,T}^{\%$  iff at least one of the following statements holds in the graph  $g(\varphi, X)$ :

- (1) There exists a node  $\ell_i \in V$  such that  $\bar{\ell}_i \rightsquigarrow \ell_i$ .
- (2) There exist two nodes  $\ell_i, \ell_j \in V_F$  such that  $\ell_i \rightsquigarrow \ell_j$ .
- (3) There exist three nodes  $\ell_i \in V_F$  and  $\ell_j, \ell_k \in V_T$  such that  $\ell_j \rightsquigarrow \ell_i \rightsquigarrow \ell_k$ .

*Proof.* Suppose at least one of the statements holds. In Lemma 4, we have already seen that statement (1) and statement (2) allow player F to win.

If statement (3) holds, F can wait by playing variables other than  $x_i$  with arbitrary values until T plays  $x_j$  or  $x_k$ . Then F can respond by making  $\ell_i \neq \ell_j$  or  $\ell_i \neq \ell_k$  and win.

Conversely, suppose none of the statements hold. The graph structure remains the same as we had for  $G_{2,\dots,F}^{\%$ , except it is allowed to have shared strong components of size 2 which form a matching between some nodes of  $V_T$  and  $V_F$ . Intuitively, F can force T to assign  $V_{T,sc}$  nodes as any bit values he/she likes, by assigning the corresponding matching endpoints, and T must wait to find out what those values are. We partition  $V_T$  into four sets  $V_{T,sc}, V_{T,0}, V_{T,1}, V_{T,free}$ , defined as follows:

- $V_{T,sc} = \{\ell_j \in V_T : \ell_j \leftrightarrow \ell_i \text{ for some } \ell_i \in V_F\}$
- $V_{T,0} = \{\ell_j \in V_T : \ell_j \rightsquigarrow \ell_i \text{ for some } \ell_i \in V_F\} \setminus V_{T,sc}$
- $V_{T,1} = \{\ell_j \in V_T : \ell_j \rightsquigarrow \ell_i \text{ for some } \ell_i \in V_F\} \setminus V_{T,sc}$
- $V_{T,free} = V_T \setminus (V_{T,sc} \cup V_{T,0} \cup V_{T,1})$

This is indeed a partition: there must not be any common node that is in both  $V_{T,0}$  and  $V_{T,1}$ , because this would create either a path between two nodes of  $V_F$  (satisfying statement (2)) or a



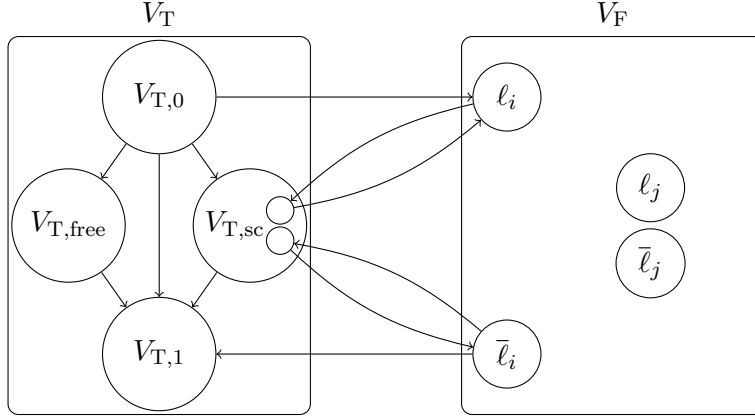


Figure 5: T has a winning strategy in  $G_{2,\dots,T}^{\%}$

cycle of length  $> 2$  that touches both  $V_T$  and  $V_F$  (satisfying statement (2) or statement (3)). Note that there cannot be any edge entering  $V_{T,0}$ , leaving  $V_{T,1}$ , between  $V_{T,free}$  and  $V_{T,sc} \cup V_F$ , between nodes of  $V_{T,sc}$ , or between  $V_{T,sc}$  and  $V_F$  except the matching edges. In general  $V_F$  may have many isolated nodes. A general case of the graph looks like Figure 5.

Now we describe a winning strategy for T on such a graph. If F's previous move was in a shared strong component  $l_j \leftrightarrow l_i$  then make  $l_j = l_i$ . Otherwise, T picks any remaining node not in  $V_{T,sc}$ . If the node is in  $V_{T,0}$ , T assigns it 0. If the node is in  $V_{T,1}$ , T assigns it 1. If the node is in  $V_{T,free}$ , T assigns it according to a satisfying assignment that exists since statement (1) does not hold.

The strategy works since T has the last move, so T will always be able to respond when F plays in a shared strong component to ensure these edges gets satisfied. Each other edge  $l_i \rightarrow l_j$  has either  $l_i \in V_{T,0}$  in which case it gets satisfied by  $l_i = 0$ , or  $l_j \in V_{T,1}$  in which case it gets satisfied by  $l_j = 1$ , or  $l_i, l_j \in V_{T,free}$  in which case it gets satisfied by the satisfying assignment.  $\square$

The algorithm for checking the characterization of such a graph is almost identical to Algorithm 8, except in line 6 it is necessary to check for the size of strong components being greater than 2 instead of 1. The idea has been described as Algorithm 9.

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**Algorithm 9:** Linear Time algorithm for  $G_{2,\dots,T}^{\%}$ 

```
1 construct  $g(\varphi, X)$ 
2 construct  $g^*$  as the DAG of strong components from  $g(\varphi, X)$ 
3 let  $S$  = set of all strong components of  $g(\varphi, X)$  (nodes of  $g^*$ )
4 foreach  $\ell_i \in V$  do
5   | let  $s = \ell_i$ 's strong component
6   | if  $(\bar{\ell}_i \in s)$  or  $(\ell_i \in V_F$  and  $|s| > 2)$  then output F
7 let  $S_F$  = set of strong components containing at least one node from  $V_F$ 
8 let  $S_T$  = set of strong components containing only nodes from  $V_T$ 
9 mark all  $s \in S_F$  as “reachable from  $S_F$ ”
10 topologically order  $s_1, s_2, s_3, \dots \in S$  so edges of  $g^*$  go from lower to higher indices
11 foreach  $i = 1, 2, 3, \dots, |S|$  do
12   | if  $\exists j < i$  such that  $s_j \rightarrow s_i$  and  $s_j$  is marked then
13     |   | if  $s_i \in S_T$  then mark  $s_i$  as “reachable from  $S_F$ ”
14     |   | else output F
15 output T
```

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