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Quantum Lovász Local Lemma: Shearer's Bound is Tight

Kun He*, Qian Li *, Xiaoming Sun* and Jiapeng Zhang †

Abstract

Lovász Local Lemma (LLL) is a very powerful tool in combinatorics and probability theory to show the possibility of avoiding all "bad" events under some "weakly dependent" condition. Over the last decades, the algorithmic aspect of LLL has also attracted lots of attention in theoretical computer science [15, 19, 24]. A tight criterion under which the *abstract* version LLL holds was given by Shearer [31]. It turns out that Shearer's bound is generally not tight for *variable* version LLL (VLLL) [16]. Recently, Ambainis et al. [3] introduced a quantum version LLL (QLLL), which was then shown to be powerful for quantum satisfiability problem.

In this paper, we prove that Shearer's bound is tight for QLLL, affirming a conjecture proposed by Sattath et. al. [28]. Our result shows the tightness of Gilyén and Sattath's algorithm [11], and implies that the lattice gas partition function fully characterizes quantum satisfiability for almost all Hamiltonians with large enough qudits [28].

Commuting LLL (CLLL), LLL for commuting local Hamiltonians which are widely studied in literature, is also investigated here. We prove that the tight regions of CLLL and QLLL are generally different. Thus, the efficient region of algorithms for CLLL can go beyond shearer's bound. Our proof is by first bridging CLLL and VLLL on a family of interaction bipartite graphs and then applying the tools of VLLL, e.g., the gapless/gapful results, to CLLL. We also provide a sufficient and necessary condition for deciding whether the tight regions of QLLL and CLLL are the same for a given interaction bipartite graph.

*Institute of Computing Technology, Chinese Academy of Sciences. University of Chinese Academy of Sciences. Beijing, China. Email:hekun, liqian, sunxiaoming@ict.ac.cn

†University of California, San Diego. Email:jpeng.zhang@gmail.com

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1 Introduction

Classical Lovász Local Lemma The famous *Lovász Local Lemma* (or LLL) is a very powerful tool in combinatorics and probability theory to show the possibility of avoiding all "bad" events under some "weakly dependent" condition, and has numerous applications. Formally, given a set $\mathcal{A}$ of bad events in a probability space, LLL provides the condition under which $\mathbb{P}(\cap_{A\in\mathcal{A}}\overline{A}) > 0$. The dependency among events is usually characterized by dependency graph. A dependency graph is an undirected graph $G_D = ([m], E_D)$ such that for any vertex i, A_i is independent of $\{A_j : j \notin \Gamma_i \cup \{i\}\}$, where Γ_i stands for the neighborhood of i in G_D. In this setting, finding the conditions under which $\mathbb{P}(\cap_{A\in\mathcal{A}}\overline{A}) > 0$ is reduced to the following problem: given a graph G_D, determine its abstract interior $\mathcal{I}(G_D)$ which is the set of vectors $\mathbf{p}$ such that $\mathbb{P}\left(\cap_{A\in\mathcal{A}}\overline{A}\right) > 0$ for any event set $\mathcal{A}$ with dependency graph G_D and probability vector $\mathbf{p}$. *Local* solutions to this problem, including the first LLL proved in 1975 by Erdős and Lovász [8], are referred as abstract-LLL.

The most frequently used abstract-LLL is as follows:

Theorem 1.1 ([32]). *Given a dependency graph $G_D = ([m], E_D)$ and a probability vector $\mathbf{p} \in (0,1)^n$, if there exist real numbers $x_1, ..., x_n \in (0,1)$ such that $p_i \leq x_i \prod_{j\in\Gamma_i}(1-x_j)$ for any $i \in [m]$, then $\mathbf{p} \in \mathcal{I}(G_D)$.*

Shearer [31] provided the exact characterization of $\mathcal{I}(G_D)$ with the independence polynomial defined as follows.

Definition 1.1 (Multivariate independence polynomial). Let $G_D = (V, E)$, $\mathbf{x} = (x_v : v \in V)$ and let $Ind(G_D)$ be the set of all independent sets of G_D. Then we call $I(G_D, \mathbf{x}) = \sum_{S\in Ind(G_D)}(-1)^{|S|}\prod_{v\in S} x_v$ the *multivariate independence polynomial*.

Definition 1.2. Probability vector $\mathbf{p} = (p_v : v \in V) \in \mathbb{R}^{|V|}$ is called *above Shearer's bound* for a dependency graph G_D if there is a vertex set $V' \subseteq V$ such that for the corresponding induced subgraph $G_D[V'] := (V', E') : I(G'_D, (p_v : v \in V')) \leq 0$. Otherwise we say $\mathbf{p}$ is *b*elow Shearer's bound.

The tight criterion under which *abstract* version LLL holds provided by Shearer is as follows.

Theorem 1.2 ([31]). *For a dependency graph $G_D = (V, E)$ and probabilities $\mathbf{p} \in \mathbb{R}^{|V|}$ the following conditions are equivalent:*

1. *$\mathbf{p}$ is below Shearer's bound for G_D.*
2. *for any probability space Ω and $\{A_v \subseteq \Omega : v \in V\}$ events having G_D as dependency graph and satisfying $\mathbb{P}(A_v) \leq p_v$, we have $\mathbb{P}(\overline{\cup_{v\in V} A_v}) \geq I(G_D, \mathbf{p}) > 0$.*

In other words, $\mathbf{p} \in \mathcal{I}(G_D)$ if and only if $\mathbf{p}$ is below Shearer's bound for G_D.

Another important version of LLL, *variable version Lovász Local Lemma* (or VLLL), which exploits richer dependency structures of the events, has also been studied [16, 19]. In this setting, each event A_i can be fully determined by some subset $\mathcal{X}_i$ of a set of mutually independent random variables $\mathcal{X} = (X_1, \cdots, X_n)$. Thus, the dependency can be naturally characterized by the event-variable graph defined as follows. An event-variable graph is a bipartite graph $G_B = ([m], [n], E)$ such that for any $X_j \in \mathcal{X}_i$, there is an edge $(i, j) \in [m] \times [n]$. Similar to the abstract-LLL, the VLLL is for solving the following problem: given a bipartite graph G_B, determine its variable interior $\mathcal{VI}(G_B)$ which is the set of vectors

$\mathbf{p}$ such that $\mathbb{P}\left(\cap_{A\in\mathcal{A}}\overline{A}\right) > 0$ for any variable-generated event system $\mathcal{A}$ with event-variable graph G_B and probability vector $\mathbf{p}$.

The VLLL is important because many problems in which LLL has applications naturally conform with the variable setting, including hypergraph coloring [22], satisfiability [9, 10], counting solutions to CNF formulas [23], acyclic edge coloring [12], etc. Moreover, most of recent progresses on the algorithmic aspects of LLL are based on the variable model [19, 24, 26].

A key problem around the VLLL is whether Shearers bound is tight for variable-LLL [19]. Formally, given a bipartite graph $G_B = (U, V, E)$, its *base graph* is defined as the graph $G_D(G_B) = (U, E')$ such that for any two nodes $u_i, u_j \in U$, there is an edge $(u_i, u_j) \in E'$ if and only if u_i and u_j share some common neighbor in G_B. That is to say, $G_D(G_B)$ is a dependency graph of the variable-generated event system with event-variable graph G_B. Thus, we have $\mathcal{I}(G_D(G_B)) \subseteq \mathcal{VI}(G_B)$ immediately. If $\mathcal{I}(G_D(G_B)) \neq \mathcal{VI}(G_B)$, we say that Shearer's bound is not tight for G_B, or G_B has a gap. The first example of gap existence is a bipartite graph whose base graph is a cycle of length 4 [19]. Recently, He et al. [16] have shown that Shearer's bound is generally not tight for variable-LLL.

Quantum Satisfiability and Quantum Lovasz Local Lemma Most systems of physical interest can be described by local Hamiltonians $H = \sum_i H_i$ where each k-local term H_i acts nontrivially only on at most k qudits. We say H is frustration free if the ground state $|\phi\rangle$ of H is also the ground state of every H_i. Let Π_i be the projection operator on the excited states of H_i and $\Pi = \sum \Pi_i$, it is easy to see the frustration freeness of H and Π are the same. Henceforth, we only care about the Hamiltonians which are projectors. Determining whether a given Π is frustration free (or satisfiable, in computer science language), known as the quantum satisfiability problem, is a central pillar in quantum complexity theory, and has many applications in quantum many body physics.

Unfortunately, quantum satisfiability problem has been shown to be QMA$_1$-complete [4], which is widely believed to be intractable in general even for quantum computing. This makes it highly desirable to search for efficient heuristics and algorithms in order to, at least, partially answer this question. In the seminal paper, by generalizing the notations of probability and independence as described in the following table, Ambainis et al. [3] introduced a quantum version LLL (or QLLL) respect to the dependency graph, i.e., a sufficient condition under which the Hamiltonian is guaranteed to be frustration free. With QLLL, they greatly improved the known critical density for random k-QSAT from $\Omega(1)$ [20] to $\Omega(2^k/k^2)$, almost meet the best known upper bound $O(2^k)$ [20].

Probability space Ω	$\rightarrow$	Vector space: V
Event A	$\rightarrow$	Subspace $A \subseteq V$
Complement $\overline{A} = \Omega \backslash A$	$\rightarrow$	Orthogonal subspace $A^{\perp}$
Probability $\mathbb{P}(A)$	$\rightarrow$	Relative dimension $R(A) := \frac{\dim(A)}{\dim(V)}$
Disjunction $A \vee B$	$\rightarrow$	$A + B = \{a + b \mid a \in A, b \in B\}$
Conjunction $A \wedge B$	$\rightarrow$	$A \cap B$
Independence $\mathbb{P}(A \wedge B)$	$\rightarrow$	$R(A \cap B) = R(A) \cdot R(B)$
Conditioning $\mathbb{P}(A\|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}$	$\rightarrow$	$R(A\|B) := \frac{R(A\cap B)}{R(B)}$

Recently, Sattath et al. [28] generalized Shearer's theorem to QLLL respect to the interaction bipartite graph, which can be viewed as the quantum analogue of classical event-variable graph, and showed that Shearer's bound is still a sufficient condition here. Remarkably, the probability threshold of Shearer's bound turns out to be the first negative fugacity of the hardcore lattice gas partition function, which has been extensively studied in classical statistical mechanism. Utilizing the tools in classical statistical mechanism, they concretely apply QLLL to evaluating some critical threshold to local Hamiltonians on various regular lattices. In contrast to VLLL [16] which goes beyond Shearer's bound generally, they conjectured that Shearer's bound is tight for QLLL, which, if true, would have important physical significance and several striking consequence [28].

In the past few years, as a special case of quantum satisfiability problem, the commuting local Hamiltonian problem (CLH), where $[\Pi_i, \Pi_j] = 0$ for all i and j, has attracted considerable attention [1, 2, 5, 13, 29]. Commuting Hamiltonians are somewhat "halfway" between classical and quantum, and capable of exhibiting intriguing multi-particle entanglement phenomena, such as the famous toric code [18]. CLH interests people not only because the commutation restriction is natural and often made in physics, but also it may help us to understand the centrality of non-commutation in quantum mechanics. CLH can be viewed as a generalization of the classical SAT, thus CLH is at least NP-hard, and as a sufficient condition, the commuting version LLL (or CLLL) is desirable and would has various applications.

The QLLLs provide sufficient conditions for frustration freeness. A natural question is whether there is an efficient way to prepare a frustration-free state under the conditions of QLLL. A series of results showed that the answer is positive if all local Hamiltonians commute [7, 27, 30]. Recently, Gilyén and Sattath improved on the previous constructive results by designing an algorithm that works efficiently under Shearer's bound for non-commuting terms as well [11].

Therefore, the following two closely related problems beg answers:

1. Tight region for QLLL: complete characterization of the interior of QLLL for a given interaction bipartite graph G_B. Here the interior is the set of vectors $\mathbf{r}$ such that any local Hamiltonians with relative dimensions $\mathbf{r}$ and interaction bipartite graph G_B are frustration free. As Shearer's bound has been shown to be a sufficient condition for QLLL [28], a fundamental question here is whether Shearer's bound is tight. If it is really tight, there are several striking consequences. Firstly, the tightness implies Gilyén and Sattath's algorithm [11] converges up to the tight region. Moreover, the geometrization theorem [21] says that given the interaction bipartite graph, dimensions of qudits, and dimensions of local Hamiltonians, either all such Hamiltonian are frustration free, or almost all such Hamiltonians are not. If Shearer's bound is indeed tight for QLLL, by geometrization theorem we have that the quantum satisfiability for almost all Hamiltonians with large enough qudits are completely characterized by the lattice gas partition function. Meanwhile, the lattice gas critical exponents can be directly applied to the counting of the ground state entropy of almost all quantum Hamiltonians in the frustration free regime. Thus, the tightness means a lot for transferring insights from classical statistical mechanics into the quantum complexity domain [28].

2. Tight region for CLLL: complete characterization of the interior of CLLL for a given interaction bipartite graph G_B which is the set of vectors $\mathbf{r}$ such that any *commuting* Hamiltonians with relative dimensions $\mathbf{r}$ and interaction bipartite graph G_B are frustration free. It is immediate that the interior of QLLL is a subset of the interior of CLLL for any G_B. An interesting question is whether the containment is proper. There are a series of results on the algorithms for preparing a frustration-free state for commuting Hamiltonians under the conditions of QLLL [7, 27, 30]. A natural question is whether the algorithm designed for commuting Hamiltonians can still be efficient beyond the conditions of QLLL, e.g., Shearer's bound. These questions beg answers not only because the

various applications in CLH, but also it may help us to understand the role of non-commutation plays in the quantum world.

1.1 Results and Discussion

In this paper, we mainly concentrate on the following three problems: the tight region for QLLL, the tight region for CLLL, and whether the tight regions for QLLL and CLLL are the same for a given interaction bipartite graph. We provide the complete answer for the first problem and partial answers for other two problems. Our results show that Shearer's bound, which is tight for abstract-LLL, is also tight for QLLL. The CLLL behaves very different from QLLL, i.e., the interior of CLLL goes beyond Shearer's bound generally. And we also show that the tight regions for CLLL and VLLL are the same for a family of Hamiltonians. The main results of this paper are listed and discussed as follows.

1.1.1 Tight region for QLLL

Shearer's bound is tight for QLLL In this paper, we first prove the tightness of Shearer's bound for QLLL, which affirms the conjecture in [28]. More precisely,

Theorem 1.3 (Shearer's bound is tight for QLLL)**.** *Given an interaction bipartite graph G_B and and relative projector ranks $r_i = R(Im\Pi_i)$ for all i,*

- *If $(G_B, \mathbf{r})$ is below Shearer's bound, then all such Hamiltonians are frustration free [28].*
- *Otherwise, there is a set of local Hamiltonians with interaction bipartite graph G_B and relative projector rank $\mathbf{r}$ which is not frustration free.*

In contrast to the VLLL which goes beyond Shearer's bound generally, QLLL is another example exhibiting the difference between the classical world and the quantum world. As mentioned above, Theorem 1.3 means that the position of the first negative fugacity zero of partition function is exactly the critical threshold of quantum satisfiability for almost all Hamiltonians with large enough qudits. The above theorem also shows the tightness of Gilyén and Sattath's algorithm [11], which prepares a frustration free state under Shearer's bound.

1.1.2 Tools for CLLL

Connection between CLLL and VLLL For CLLL, we first prove that if any two Hamiltonians share at most one qudit, then the frustration freeness region of commuting Hamiltonians and that of classical Hamiltonians are the same.

Theorem 1.4. *Given an interaction bipartite graph where any pair of Hamiltonians share at most one qudit, the dimensions of qudits and the relative dimensions of Hamiltonians, if there is such a commuting local Hamiltonian which is frustration free, there is also such a frustration free classical local Hamiltonian.*

The following is a direct corollary.

Corollary 1.5. *Given any interaction bipartite graph G_B where any pair of Hamiltonians share at most one qudit, the tight regions of CLLL and VLLL are the same.*

Many local Hamiltonians studied in literature can be regarded as this type, for example, the Hamiltonians of which the lattice is chain [25], tree [6, 28], cycle, triangular, square [33], and so on. Actually, for lattices where the vertices are qudits and edges are Hamiltonians or vice versa, any two Hamiltonians share at most one qudit. Thus, Corollary 1.5 means that the VLLL can be applied to many important local Hamiltonians in the literature.

A sufficient and necessary condition for gap existence Since the Hamiltonians are restricted to be commuting for CLLL, it is naturally expected that the tight regions of QLLL and CLLL are different. We propose a necessary and sufficient condition to decide whether such a difference exist. Our condition is a nontrivial extension of the classical condition for VLLL [16] to the commuting case. Because we have proved that Shearer's bound is tight for QLLL, to ask whether the difference between QLLL and CLLL exists is equivalent to ask whether Shearer's bound is not tight for CLLL.

For any interaction bipartite graph G_B and any vector $\mathbf{r}$ of positive reals, there is a critical threshold λ_1 such that any Hamiltonians with relative dimensions less than $\lambda_1\mathbf{r}$ are frustration-free but the Hamiltonians with relative dimensions larger than $\lambda_1\mathbf{r}$ can be not. Similarly, for commuting Hamiltonians, we have another critical threshold λ_2. We say there is a gap between QLLL and CLLL for G_B in direction $\mathbf{r}$ if and only if $\lambda_1 \neq \lambda_2$. We say there is a gap between QLLL and CLLL for G_B if there is a direction $\mathbf{r}$ with a gap. For conciseness of presentation, we also say G_B is gapful if it has a gap, and gapless otherwise. The following theorem gives a sufficient and necessary condition for gap existence.

Theorem 1.6. *Given any interaction bipartite graph G_B and a vector $\mathbf{r}$ of positive reals, the following two conditions are equivalent:*

1. *For any rational $\lambda\mathbf{r}$ where λ is less than the critical threshold of CLLL in direction $\mathbf{r}$, there is a commuting Hamiltonian set with interaction bipartite graph G_B and relative dimension vector $\lambda\mathbf{r}$ such that any Hamiltonians sharing qudits are exclusive.*

2. *G_B is gapless for CLLL in the direction of $\mathbf{r}$.*

Here the qualifier "exclusive" means that the images of Hamiltonians are orthogonal. By this theorem, one can prove the existence of a gap just by proving the non-existence of commuting Hamiltonian set, without computing the critical threshold of QLLL or CLLL.

Meanwhile, we also have the following corollary.

Corollary 1.7. *Given any interaction bipartite graph G_B and a rational $\lambda\mathbf{r}$ where λ is the critical threshold of CLLL in direction $\mathbf{r}$, if there is a set of commuting Hamiltonians with interaction bipartite graph G_B and relative dimension vector $\lambda\mathbf{r}$ such that any Hamiltonians sharing qudits are exclusive, then G_B is gapless in the direction of $\mathbf{r}$.*

Reduction method He et. al. proposed five operations for VLLL which transforms a bipartite graph without changing the existence or nonexisence of a gap to discover more instances that have or have no gaps [16]. Here, we extend the reduction rules about these operations to CLLL. Meanwhile, we also propose another operation preserving both gapful and gapless. By these reduction rules, we can prove two interesting results.

Theorem 1.8. *An interaction bipartite graph G_B is gapless for CLLL if G_B is a tree.*

Here the interaction bipartite graph which is a tree includes two interesting families of interaction bipartite graphs, the treelike bipartite graphs [16] and the regular trees [6, 17, 28]. As an application, the critical threshold $\lambda_c = \frac{1}{t} \cdot \frac{(k-1)^{(k-1)}}{k^k}$ for QLLL [28] on infinite (t, k)-regular tree also applies for CLLL and VLLL as well.

We say a graph $G_D = ([m], E_D)$ is strongly a-gapful, if any interaction bipartite graph satisfying $G_D(G_B) = G_D$ is gapful, otherwise we call it is strongly a-gapless. In 2011, Kolipaka et al. [19] proposed to characterize strongly a-gapful graphs for VLLL, and this problem remains open until last year [16]. Here we can give a complete characterization of strongly a-gapful graphs for CLLL by our new reduction rule.

Theorem 1.9. *A dependency graph is strongly a-gapless for CLLL if and only if it is chordal.*

1.1.3 CLLL: beyond Shearer's bound

Tight region for trees Besides proving that all interaction bipartite graphs which are trees are gapless, we also give the tight region for CLLL on trees. Actually, our result provides the tight condition of frustration freeness of commuting Hamiltonians where the interaction bipartite graph is a tree and the dimensions of qudits are given. Our theorem extends the classical result on treelike bipartite graphs [16] to the commuting case on a larger family of graphs even if the dimensions of qudits are given. Without loss of generality, we can assume that the leaves of all trees are qudits, since we can add some 1-dimension qudits if necessary.

Theorem 1.10. *Given an interaction bipartite graph $G_B = ([m], [n], E_B)$ which is a tree and dimensions of qudits $\mathbf{d}$, appoint the qudit n as the root. For ranks $\mathbf{r} \in \mathbb{Z}^m$, define $\mathbf{q} = (q_1, ..., q_n) \in \mathbb{Z}^n$ to be*

$$q_i = \begin{cases} 0 & \text{if vertex } i \text{ is a leaf of } G_B, \\ \sum_{j \in \mathcal{C}_i} \lfloor r_j \cdot \prod_{k \in \mathcal{C}_j} \frac{1}{d_k - q_k} \rfloor & \text{otherwise}. \end{cases} \tag{1}$$

Here $\mathcal{C}_i$ is the set of children of i. Then there is such a commuting instance with rank $\mathbf{r}$ spanning the whole space if and only if there is some $q_i \geq d_i$.

The above theorem also implies the tight region for VLLL, CLLL and QLLL on trees, ignoring the dimensions of qudits.

Corollary 1.11. *Given an interaction bipartite graph $G_B = ([m], [n], E_B)$ which is a tree, appoint the qudit n as the root. For $\mathbf{r} \in (0, 1)^m$, define $\mathbf{q} = (q_1, ..., q_n) \in [0, 1]^n$ to be*

$$q_i = \begin{cases} 0 & \text{if vertex } i \text{ is a leaf of } G_B, \\ \sum_{j \in \mathcal{C}_i} r_j \cdot \prod_{k \in \mathcal{C}_j} \frac{1}{1 - q_k} & \text{otherwise}. \end{cases} \tag{2}$$

Here $\mathcal{C}_i$ is the set of children of i. Then $\mathbf{r} \in \mathcal{VI}(G_B) = \mathcal{CI}(G_B) = \mathcal{I}(G_B)$ if and only if $\forall i \in [n]$, $q_i < 1$.

CLLL is generally different from QLLL By coupling our tools for CLLL with the results about VLLL [16], we could obtain a list of gapless/gapful results for CLLL. We have proved that trees are gapless. On the other side, we can prove the following theorem.

Theorem 1.12. *Any interaction bipartite graph that some Hamiltonians form a cycle is gapful.*

Here, we say more than three Hamiltonians form a cycle if and only if the induced subgraph containing exact these Hamiltonians and their qudits can be transformed to a cycle by deleting dummy qudits.

The above theorem shows that CLLL is very different from QLLL, i.e., the interior of CLLL can go beyond Shearer's bound. Thus, it is possible to design more specialized algorithm for CLLL which is efficient beyond Shearer's bound.

By Theorems 1.8 and 1.12, we can prove the following corollary, which almost gives a complete characterization of gapful/gapless for CLLL except when the base graph has only 3-cliques.

Corollary 1.13. *An interaction bipartite graph is gapless for CLLL if its base graph is tree, is gapful if its base graph has an induced cycle of length at least 4.*

Local Hamiltonians on regular lattices As mentioned above, many local Hamiltonians in studies can be regarded as that any two Hamiltonians share at most one qudit, including various regular lattices in research. For these Hamiltonians, we further have the following results.

Theorem 1.14. *Let $G_B = ([m], [n], E_B)$ be a interaction bipartite graph where any two Hamiltonians share at most one qudit, then G_B is gapful if and only if it is a tree. Moreover, if G_B is gapful, it is gapful in all directions.*

This theorem indicates that the critical thresholds of frustration freeness to commuting Hamiltonians on many regular lattices are different from that to general Hamiltonians.

Finally, we wonder whether CLLL and VLLL are the same.

Conjecture 1.1. For any interaction bipartite graph, the tight regions of CLLL and VLLL are the same.

It should be noticed that QLLL and CLLL only exploit the information of the interaction bipartite graph and relative ranks of Hamiltonians, ignoring the dimensions of qudits. The frustration free conditions for Hamiltonians with specified dimensions of qudits, as another fundamental question, deserves the effort.

The organization of this paper is as follows. Section 2 provides the definitions and notations. In Section 3, we prove that Shearer's bound is tight for QLLL. Section 4 provides the tools for CLLL. We show that the tight region of CLLL is generally different from that of QLLL in Section 5. Section 6 gives the hardness result.

2 Definitions and Notations

In this paper, the tight region of QLLL (CLLL) for a given interaction bipartite graph $G_B = ([m], [n], E_B)$ is defined as the set of vectors $\mathbf{r}$ such that any projectors (commuting projectors) with relative dimensions $\mathbf{r}$ and interaction bipartite graph G_B are frustration free. Without loss of generality, we assume that G_B is connected and $\mathbf{r} = (r_1, r_2, \cdots, r_m) \in (0, 1]^m$. For any $\mathbf{r}$ and $\mathbf{r}'$, we say $\mathbf{r} \geq \mathbf{r}'$ if $\mathbf{r} \geq \mathbf{r}'$ holds for any $i \in [m]$. We say $\mathbf{r} > \mathbf{r}'$ if $\mathbf{r} \geq \mathbf{r}'$ and $\mathbf{r}_i > \mathbf{r}'_i$ holds for some $i \in [m]$. The dependency graphs in this paper are also assumed to be connected. And all vector spaces are of finite dimensions and over $\mathbb{C}$. We will use $\mathbf{r}, \mathbf{p}, \mathbf{q}, \mathbf{d}$ to denote vectors.

Definition 2.1 (Hilbert space of the qudits)**.** Let n be the number of qudits, so the Hilbert space of the quantum system is a nth-order tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$. The $\{\mathcal{H}_i\}$ are of dimension

$\mathbf{d} = (d_1, d_2, \cdots, d_n)$ respectively and labelled with elements from $[n]$. For $A \subseteq [n]$ let $\mathcal{H}_A := \bigotimes_{i\in A} \mathcal{H}_i$ denote the Hilbert space of the qudits in A.

Definition 2.2 (Projectors, subspaces and relative dimensions)**.** Given a subspace $V \subset \mathcal{H}$, let Π_V be the orthogonal projector to V. The relative dimension of Π_V is defined as $R(\Pi_V) := \frac{\mathrm{tr}(\Pi_V)}{\dim(\mathcal{H})} = \frac{\dim(V)}{\dim(\mathcal{H})}$. Easy to see that $R(\Pi_V)$ is a rational number. We say two subspaces V and V' are commuting if Π_V and $\Pi_{V'}$ are commuting. Π_V is called a classical Hamiltonian if Π_V is diagonal with respect to the computational basis. In this paper, the two terms "subspaces" and "projectors" will be used interchangeably.

Definition 2.3 (Interaction bipartite graphs and dependency graphs)**.** Given a bipartite graph $G_B = ([m], [n], E_B)$, we say a set of local Hamiltonians $\mathcal{V} = (V_1, \cdots, V_m)$ conforms with G_B, denoted by $\mathcal{V} \sim G_B$, if for any $i \in [m]$, Π_V acts trivially on qubits $[n] \setminus \mathcal{N}(i)$. Thus we can write $V_i = V_i^{loc} \otimes \mathcal{H}_{[n]\setminus\mathcal{N}(i)}$, where $V^{loc} \subseteq \mathcal{H}_{\mathcal{N}(i)}$. In this paper, for convenience of presentation, we use $\mathcal{N}_{G_B}(i)$ (or $\mathcal{N}(i)$ if G_B is implicit) to denote the neighbors of vertex i in G_B, if which side this vertex belongs to is clear from the context. Here, we usually call G_B the interaction bipartite graph.

The corresponding dependency graph of G_B is defined as $G_D(G_B) = ([m], E_D)$, where $(i, j) \in E_D$ if and only if $\mathcal{N}(i) \cap \mathcal{N}(j) \neq \emptyset$. For $i \in [m]$ let $\Gamma_i := \{j \in [m] : (i, j) \in E_D\}$ and $\Gamma_i^+ := \Gamma_i \cup \{i\}$. We define the multivariate independence polynomial $I(G_B, \mathbf{x})$ (or $I(G_B)$ if $\mathbf{x}$ is implicit) of G_B as that of G_D, i.e., $I(G_B, \mathbf{x}) := I(G_D(G_B), \mathbf{x})$.

For $A \subseteq [m]$, let $G_B[A]$ ($G_D[A]$ resp.) be the induced subgraph of G_B (G_D resp.) discarding $[m]\setminus A$.

Definition 2.4 (Random subspaces)**.** When we say randomly picking a subspace, we always use Haar measure. Whenever it comes to random V, we always mean V_i^{loc} is a random subspace of $\mathcal{H}_{\mathcal{N}(i)}$ according to the Haar measure, except the case specified.

3 QLLL: Shearer's Bound is Tight

The section aims at proving that Shearer's bound is tight for QLLL. As the Shearer's bound has been shown to be a sufficient condition for QLLL [28], it remains to show there exists a set of local Hamiltonians which is not frustration free for any relative dimension vector above Shearer's bound.

Our proof is an induction on the number of Hamiltonians m. Roughly speaking, suppose $V_m, \cdots, V_{m-t+1}$ depend on qudit n, we decompose $\mathcal{H}_n$ to t orthogonal subspaces $\mathcal{H}_n = \bigoplus_{i=1}^t \mathcal{H}_n^i$. If $i \in [m-t+1, m]$, let $V_i^{loc} = \mathcal{H}_n^{m-i+1} \otimes V_i^{loc,-n}$ where $V_i^{loc,-n} \subseteq \mathcal{H}_{\mathcal{N}(i)\setminus\{n\}}$ is a random subspace, otherwise, let V_i^{loc} be randomly picked in $\mathcal{H}_{\mathcal{N}(i)}$. By the induction hypothesis, each $\mathcal{H}_n^i \otimes \mathcal{H}_{[n-1]}$ can be spanned by these V_i's with high probability, so can $\mathcal{H}_{[n]}$. The geometrization theorem shown by Laumann et al. [21] plays an important role here*.

Theorem 3.1 (The geometrization theorem, adapted from [21])**.** *Fix the interaction bipartite graph $G_B = ([m], [n], E_B)$, dimensions of qudits $\mathbf{d} = (d_1, \cdots, d_n)$ and relative dimensions of local Hamiltonians $\mathbf{r} = (r_1, \cdots, r_m)$, if there exist such Hamiltonians $\{V_i^*\}_{i\in[m]}$ satisfying $R(\bigoplus_{i=1}^m V_i^*) = 1$, then*

$$\mathbb{P}_{V_1,\ldots,V_m}[R(\bigoplus_{i=1}^m V_i) = 1] = 1.$$

The following is a direct corollary of the geometrization theorem.

*Though they only showed the case of qubits, the proof applies for the general qudits as well

Corollary 3.2. *Fix the interaction bipartite graph $G_B = ([m], [n], E_B)$, dimensions of qudits $\mathbf{d} = (d_1, \cdots, d_n)$ and relative dimensions of local Hamiltonians $\mathbf{r} = (r_1, \cdots, r_m)$, if there exist such Hamiltonians $\{V_i^*\}_{i\in[m]}$ satisfying $R(\bigoplus_{i=1}^m V_i^*) = 1$, then for any $\mathbf{d}' = (k_1 d_1, \cdots, k_n d_n)$ where k_i's are positive integers, we have*

$$\mathbb{P}_{V_1,\dots,V_m}[R(\bigoplus_{i=1}^{m} V_i) = 1] = 1.$$

Armed with the geometrization theorem, we are going to give the whole proof.

Proposition 3.3. *For any interaction bipartite graph $G_B = ([m], [n], E_B)$ and nonnegative $\mathbf{r} = (r_1, \cdots, r_m)$, if $I(G_B, \mathbf{r}) \le 0$ and $\forall A \subsetneq [m]$, $I(G_B[A], \mathbf{r}) > 0$, then*

1. *$\forall A \subseteq [m]$, $I(G_B[A]) \le 1$,*
2. *$\forall i \in [m]$, $r_i > 0$,*
3. *$G_D(G_B)$ is connected.*

Proof. 1. If $i \notin A$, then $I(G_B[A \cup \{i\}]) = I(G_B[A]) - r_i \cdot I(G_B[A \backslash \Gamma_i])$, which means $I(G_B[A])$ is non-increasing as A grows up, thus $I(G_B[A]) \le I(G_B[\emptyset]) = 1$.

2. Suppose $r_i = 0$, then $I(G_B[[m]\backslash\{i\}]) = I(G_B) \le 0$, a contradiction.

3. Suppose there exists $\emptyset \subsetneq S \subsetneq [m]$, s.t., there is no edge between S and $[m]\backslash S$ in G_D, then $I(G_B) = I(G_B[S]) I(G_B[[m]\backslash S]) \le 0$, which implies $I(G_B[S]) \le 0$ or $I(G_B[[m]\backslash S]) \le 0$, a contradiction.

□

Theorem 1.3 (restated). *For any interaction bipartite graph $G_B = ([m], [n], E_B)$ and rational $\mathbf{r} = (r_1, \cdots, r_m)$ above Shearer's bound (i.e., $\exists A \subseteq [m]$ s.t. $I(G_B[A], \mathbf{r}) \le 0$), then there is a appropriate $\mathbf{d}$, such that*

$$\mathbb{P}_{V_1,\dots,V_m}[R(\bigoplus_{i=1}^{m} V_i) = 1] = 1.$$

Proof. We prove this theorem by induction on m.
Basic: $m \le 2$ holds obviously.
Induction: We assume this theorem has already been proven for small cases. Let $A \subseteq [m]$ be of the minimal size such that $I(G_B[A], \mathbf{r}) \le 0$. If $A \subsetneq [m]$, then by the induction hypothesis,

$$\mathbb{P}_{V_1,\dots,V_m}[R(\bigoplus_{i\in A} V_i) = 1] = 1.$$

In the following we assume $A = [m]$, thus $I(G_B[A], \mathbf{r}) > 0$ for any $A \subsetneq [m]$. By Proposition 3.3, G_D is connected, so there must exist a qudit s.t. at least two local Hamiltonians acts on it. Without loss of generality, we assume this qudit is $\mathcal{H}_n$, and the Hamiltonians acting on $\mathcal{H}_n$ are $V_m, \cdots, V_{m-t+1}$, where $t \ge 2$.

According to Theorem 3.1, it suffices to show that there exists $V_1, \cdots, V_m$ with given relative dimension $\mathbf{r}$ such that $R(\bigoplus_{i=1}^m V_i) = 1$. Now we are going to show the existence. First, we decompose $\mathcal{H}_n$ into t orthogonal subspaces $\mathcal{H}_n^1, \cdots, \mathcal{H}_n^t$ where $\dim(\mathcal{H}_n^i) := \frac{r_{m-i+1}\cdot(1-I(G_B[[m]\backslash\Gamma^+_{m-i+1}],\mathbf{r}))}{\sum_{i=1}^t r_{m-i+1}\cdot(1-I(G_B[[m]\backslash\Gamma^+_{m-i+1}],\mathbf{r}))} \cdot \dim(\mathcal{H}_n)$. This is well-defined, since

1. $\sum_{i=1}^{t} \dim(\mathcal{H}_n^i) = \dim(\mathcal{H}_n)$.
2. $\forall i \le t$, $r_{m-i+1}(1 - I(G_B[[m]\backslash\Gamma^+_{m-i+1}]) > 0$ due to Proposition 3.3.
3. $\forall i \le t$, $r_{m-i+1}(1 - I(G_B[[m]\backslash\Gamma^+_{m-i+1}])$ is a rational number, so a appropriate $d_n = \dim(\mathcal{H}_n)$ can make all $\dim(\mathcal{H}_n^i)$ positive integer.

Then, we choose $V_1, \ldots, V_m$ randomly as follows,

- For $i \ge m-t+1$, let $V_i^{loc} = V_i^{loc,-n} \otimes \mathcal{H}_n^{m-i+1}$, where $V_i^{loc,-n}$ is randomly picked in $\mathcal{H}_{\mathcal{N}(i)\backslash\{n\}}$ with $R(V_i) = \frac{\dim(V_i^{loc})}{\dim(\mathcal{H}_{\mathcal{N}(i)})} = \frac{\dim(\mathcal{H}_n^{m-i+1})\cdot\dim(V_i^{loc,-n})}{\dim(\mathcal{H}_n)\dim(\mathcal{H}_{\mathcal{N}(i)\backslash\{n\}})} = r_i$. Thus, we have
$$\frac{\dim(V_i^{loc,-n})}{\dim(\mathcal{H}_{\mathcal{N}(i)\backslash\{n\}})} = r_i \cdot \frac{\sum_{i=1}^{t} r_i \cdot (1 - I(G_B[[m]\backslash\Gamma_i^+], \mathbf{r}))}{r_i \cdot (1 - I(G_B[[m]\backslash\Gamma_i^+], \mathbf{r}))}.$$
- For $i \le m-t$, V_i^{loc} is randomly picked in $\mathcal{H}_{\mathcal{N}(i)}$ with $R(V_i) = \frac{\dim(V_i^{loc})}{\dim(\mathcal{H}_{\mathcal{N}(i)})} = r_i$.

Now, we are going to show the following claim, which implies the existence by using the union bound.

Claim. For appropriate $(d_1, \cdots, d_{n-1})$, and $V_1, \cdots, V_m$ are randomly picked using the above method, then for any $i \in [t]$,
$$\mathbb{P}_{V_1,\ldots,V_m}[\mathcal{H}_{[n-1]} \otimes \mathcal{H}_n^i \subseteq \bigoplus_{i=1}^{m} V_i] = 1.$$

Proof. We only show the case $i = 1$, and the other cases follows the same argument. Let $V_m' = V_m^{loc,-n} \otimes \mathcal{H}_{[n-1]\backslash\mathcal{N}(m)}$ and $V_i' = V_i^{loc} \otimes \mathcal{H}_{[n-1]\backslash\mathcal{N}(i)}$ for $i \le m-t$. Note that for $i \le m-t$ or $i = m$, $V_i' \otimes \mathcal{H}_n^1 \subseteq V_i$, so it suffices to show
$$\mathbb{P}_{V_1\cdots,V_m}[(\bigoplus_{i=1}^{m-t} V_i') \oplus V_m' = \mathcal{H}_{[n-1]}] = 1.$$

Note that the induced subgraph of G_B on $([m-t] \cup \{m\}, [n-1])$ is the interaction bipartite graph of these V_i''s, denoted by G_B'. In addition, $R(V_m') = \frac{\dim(V_m')}{\dim(\mathcal{H}_{[n-1]})} = \frac{\dim(V_m^{loc,-n})}{\dim(\mathcal{H}_{\mathcal{N}(m)\backslash\{n\}})} = \frac{\sum_{i=1}^{t} r_i \cdot (1 - I(G_B[[m]\backslash\Gamma_i^+], \mathbf{r}))}{1 - I(G_B[[m]\backslash\Gamma_m^+], \mathbf{r})}$, and for $i \le m-k$, $R(V_i') = \frac{\dim(V_i^{loc})}{\dim(\mathcal{H}_{\mathcal{N}(i)})} = R(V_i)$. Let $\mathbf{r}' = (R(V_1'), \cdots, R(V_t'), R(V_m'))$, thus
$$\begin{aligned} I(G_B', \mathbf{r}') &= I(G_B'[[m-k]], \mathbf{r}') + R(V_m')(1 - I(G_B'[[m-k]\backslash\Gamma_m]), \mathbf{r}') \\ &= I(G_B[[m-k]], \mathbf{r}) + R(V_m')(1 - I(G_B[[m-k]\backslash\Gamma_m], \mathbf{r})) \\ &= I(G_B[[m-k]], \mathbf{r}) + \sum_{i=1}^{t} r_{m-i+1} \cdot (1 - I(G[[m]\backslash\Gamma^+_{m-i+1}], \mathbf{r})) \\ &= I(G_B, \mathbf{r}) \le 0. \end{aligned} \tag{3}$$

The second equality is because $G_D(G_B'[A]) = G_D[G_B[A]]$ if $A \subseteq [m-k]$ and $R(V_i') = r_i$. Now, by induction hypothesis, we get the conclusion. □ □

As an application, it is not difficult to see that the region of frustration freeness approaches to $\mathcal{I}(G_B)$, as the dimension of qudits goes to infinite.

Corollary 3.4. *For any interaction bipartite graph $G_B = ([m],[n],E_B)$ and any $\epsilon > 0$, there is $\mathbf{D} = (D_1, \cdots, D_n)$, such that for all $\mathbf{d} \geq \mathbf{D}$ and $\mathbf{r} = (r_1, \cdots, r_m)$ above Shearer's bound, there is a $\mathbf{r}'$ with $\|r' - r\|_1 \leq \epsilon$, s.t. randomly picked $V_1, \cdots, V_m$ with $R(V_i) = r'_i$ span the whole space.*

4 Tools for Commuting LLL

In this section, we focus on the tight region of commuting LLL. The CLLL lies between VLLL and QLLL, and appears to be much more similar with VLLL than QLLL.

4.1 Properties of Relative Dimension

Additional to the properties for general subspaces proved in [3], we prove some additional properties of the *relative dimension* only holding for the commuting case. These additional properties will be used in the following proofs implicitly.

Lemma 4.1. *For any commuting subspaces V, W, V_i the following hold*

(i) Mutual independence for orthogonal complementary space: let $V^\perp$ be the orthogonal complement of V, then $R(V|W) + R(V^\perp|W) = 1$. Thus if $R(V \cap W) = R(V) \cdot R(W)$, then $R(V^\perp \cap W) = R(V^\perp) \cdot R(W)$

(ii) Inclusion-exclusion principle:

$$R\left(\bigoplus_{i=1}^{n} V_i\right) = \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{1 \leq i_1 < \cdots < i_k \leq n} R(V_{i_1} \cap \cdots \cap V_{i_k})\right).$$

Proof. Because subspaces V, W, V_i are commuting, all the terms can be diagonalized simultaneously with respect to an orthonormal basis. Without loss of generality, we assume the orthonormal basis is $\{|e_1\rangle, \cdots, |e_m\rangle\}$. Then we can define a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as follows. Let $\Omega = \{|e_1\rangle, \cdots, |e_m\rangle\}$, $\mathbb{P}(|e_i\rangle) = 1/m$ for each i and $\mathcal{F} = 2^\Omega$. Let $A_V = \{|e_i\rangle : |e_i\rangle \in V\}$, then $R(V) = \mathbb{P}(A_V)$, $A_{V\cap W} = A_V \cap A_W$ and $A_{V \oplus W} = A_V \cup A_W$. Thus it is easy to see the above properties holds according to the analogue properties of probability. □

4.2 Connection between CLLL and VLLL

In this section, we prove that the tight regions for CLLL and VLLL are the same for interaction bipartite graphs where any two Hamiltonians share at most one qudit.

A key tool used in our proof is Bravyi and Vyalyi's *Structure Lemma* [5], which dissects the structure of commuting local Hamiltonians and turns out to be a powerful tool in studying commuting local Hamiltonian problems.

Lemma 4.2 (Structure Lemma, adapted from [5])**.** *Suppose $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ are complex Euclidean spaces, Π_V and Π_W are projection operators acting on $\mathcal{X} \otimes \mathcal{Y}$ and $\mathcal{Y} \otimes \mathcal{Z}$ respectively. If $[\Pi_V, \Pi_W] = 0$, then $\mathcal{Y}$ can be decomposed to some orthogonal subspaces $\mathcal{Y} = \bigoplus_i \mathcal{Y}_i = \bigoplus \mathcal{Y}_{i1} \otimes \mathcal{Y}_{i2}$ such that for any i:*

1. *Π_V and Π_W preserve $\mathcal{Y}_i$.*

2. *Restricted to $\mathcal{Y}_i$, Π_V and Π_W act non-trivially only on $\mathcal{Y}_{i1}$ and $\mathcal{Y}_{i2}$, respectively.*

In other words, V can be decomposed to some orthogonal subspace $V = \bigoplus_i V|_{\mathcal{Y}_{i1}} \otimes \mathcal{Y}_{i2}$, where $V|_{\mathcal{Y}_{i1}}$ is a subspace of $\mathcal{X}\otimes\mathcal{Y}_{i1}$, and similarly, W can be written as $W = \bigoplus_i W|_{\mathcal{Y}_{i2}}\otimes\mathcal{Y}_{i1}$, where $W|_{\mathcal{Y}_{i2}} \subseteq \mathcal{Y}_{i2}\otimes\mathcal{Z}$.

The following theorem is a direct application of the structure lemma.

Theorem 1.4. *Given an interaction bipartite graph where any pair of Hamiltonians share at most one qudit, the dimensions of qudits and the relative dimensions of Hamiltonians, if there is such a commuting local Hamiltonian which is frustration free, there is also such a frustration free classical local Hamiltonian.*

Proof. Let $G_B = ([m], [n], E_B)$ be the interaction bipartite graph, $\mathbf{d}$ be the dimension of qudits. W.l.o.g., assume $\mathcal{N}(\Pi_1) = [k]$. For each $i \in [k]$, let $\Pi^i := \sum_{j\neq 1: i\in\mathcal{N}(\Pi_j)} \Pi_j$ be the sum of all local Hamiltonians acting on $\mathcal{H}_i$ except Π_1. Since each Π_i other than Π_1 intersects Π_1 on at most one qudit, so does each Π^i. By applying the structure lemma simultaneously for all the first k qudits, we have that for each $i \in [k]$, $\mathcal{H}_i$ can be decomposed to some orthogonal subspaces $\mathcal{H}_i = \bigoplus_j \mathcal{H}_{i,j} = \bigoplus_j \mathcal{H}^0_{i,j} \otimes \mathcal{H}^1_{i,j}$, such that for each i, j

1. $\mathcal{H}_{i,j}$ is preserved by Π^i, so does each Π_k.
2. Restricted to $\mathcal{H}_{i,j}$, Π_1 (and Π^i resp.) act non-trivially only on $\mathcal{H}^0_{i,j}$ (and $\mathcal{H}^1_{i,j}$ resp.)

Thus, the first k qudits can be sliced, i.e.,

$$\mathcal{H}_{[k]} = \bigoplus_{j_1,\cdots,j_k} \mathcal{H}_{1,j_1} \otimes \cdots \otimes \mathcal{H}_{k,j_k},$$

and each slice $\mathcal{H}_{1,j_1} \otimes\cdots\otimes \mathcal{H}_{k,j_k}$ is preserved by all Π_i's as well as Π, so Π is unsatisfiable if and only if the restricted Π on each slice is unsatisfiable. Note that in each slice, the restricted Π_1 and each restricted Π^i act on disjoint sets of subqudits, thus properly rotating each of the first k qubits can make Π_i classical, without changing the satisfiability of Π. Repeating this procedure can make all Π be classical. □

By Theorem 1.4 we have the following corollary immediately, which provides a connection between CLLL and VLLL.

Corollary 1.5. *Given any interaction bipartite graph G_B where any pair of Hamiltonians share at most one qudit, the tight regions of CLLL and VLLL are the same.*

As mentioned earlier, many systems of physical interest are of this kind. Thus, Corollary 1.5 means that the VLLL can be applied to many important local Hamiltonians in literature.

The following is a simple corollary of Lemma 4.2, which will be used in the proof of Theorem 1.14.

Corollary 4.3. *In the setting of the Structure Lemma, if $V \perp W = \emptyset$, then one can write $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$, such that $V^{loc} \in \mathcal{X} \otimes \mathcal{Y}_1$ and $V^{loc} \in \mathcal{X} \otimes \mathcal{Y}_2$.*

4.3 A Theorem for Gap Decision

In this subsection, we study whether or when Shearer's bound is tight for CLLL. Here, our main result, namely Theorem 1.6, is a sufficient and necessary condition for deciding whether Shearer's bound remains tight for CLLL on a given interaction bipartite graph, which is an extension of [16, Theorem 5] for VLLL.

Firstly, we give the definitions of interior, boundary and gap.

Definition 4.1 (Commuting Interior)**.** The *commuting interior* of an interaction bipartite graph $G_B = ([m],[n],$
$E_B)$, denoted by $\mathcal{CI}(G_B)$, is the set $\{\mathbf{r} \in (0,1)^m$: there is a rational vector $\mathbf{r}' \geq \mathbf{r}$ such that $R\big(\bigoplus_{V\in\mathcal{V}} V\big)$ < 1 for any commuting subspace set $\mathcal{V} \sim G_B$ with $R(\mathcal{V}) = \mathbf{r}'\}$.

The *commuting interior* is well-defined, since

Lemma 4.4 (Monotonicity Lemma)**.** *Suppose there is a commuting subspace set $\mathcal{V} \sim G_B$ with $R(\mathcal{V}) = \mathbf{r}$ such that $R\big(\bigoplus_{V\in\mathcal{V}} V\big) = 1$. Then for any rational relative dimension vector $\mathbf{r}' \geq \mathbf{r}$, there is a commuting subspace set $\mathcal{V}' \sim G_B$ with $R(\mathcal{V}') = \mathbf{r}'$ such that $R\big(\bigoplus_{V'\in\mathcal{V}'} V'\big) = 1$.*

The monotonicity is obvious for VLLL and QLLL, and becomes less trivial for CLLL due to the commutation restriction. Here, we add a new qudit to get around this problem.

Proof. Without loss of generality, we assume that $\mathbf{r}' = (r_1 + \epsilon, r_2, \cdots, r_m)$ and V_1 is related to $\mathcal{H}_1$. Since $\mathbf{r}'$ and $\mathbf{r}$ are both rational vectors, $\frac{\epsilon}{1-r_1}$ is rational as well. Suppose $\frac{\epsilon}{1-r_1} = a/b$ where a and b are integers. Let $\mathcal{H}'_1 = \mathcal{H}_1 \otimes \mathcal{H}^c_1$ where $\dim(\mathcal{H}^c_1) = b$, and $\mathcal{H}'_i = \mathcal{H}_i$ for any $i \geq 2$. Thus the whole vector space is $\mathcal{H}' = \bigotimes_{i=1}^m \mathcal{H}'_i = \bigotimes_{i=1}^m \mathcal{H}_i \otimes \mathcal{H}^c_1$.

We construct the subspace set $\mathcal{V}'$ as follows. Let $V'_1 = (V_1 \otimes \mathcal{H}^c_1) \oplus (W \otimes \bigotimes_{i=1}^m \mathcal{H}'_i)$ where W can be any subspace of $\mathcal{H}^c_1$ with dimension a. For each $i \geq 2$, let $V'_i = V_i \bigotimes \mathcal{H}^c_1$. It is not difficult to verify that $\mathcal{V}'$ satisfying the conditions. □

Thus, by contradiction, we have:

Corollary 4.5. *Given an interaction bipartite graph $G_B = ([m],[n],E_B)$ and a rational $\mathbf{r} \in \mathcal{CI}(G_B)$, $R\big(\bigoplus_{V\in\mathcal{V}}$
$V\big) < 1$ holds for any commuting subspace set $\mathcal{V} \sim G_B$ with $R(\mathcal{V}) = \mathbf{r}$.*

So, $\mathcal{CI}(G_B)$ consists of two sets: one is the set of *rational* vectors $\mathbf{r}$ such that $R\big(\bigoplus_{V\in\mathcal{V}} V\big) < 1$ for any such commuting subspace set $\mathcal{V}$, and the other is the set of *irrational* vectors $\mathbf{r}$ to make $\mathcal{CI}(G_B)$ continuous.

Definition 4.2 (Commuting Boundary)**.** The *commuting boundary* of an interaction bipartite graph G_B, denoted by $\mathcal{C}\partial(G_B)$, is the set of vectors $\mathbf{r}$ on $(0,1]$ such that $(1-\epsilon)\mathbf{r} \in \mathcal{CI}(G_B)$ and $(1+\epsilon)\mathbf{r} \notin \mathcal{CI}(G_B)$ for any $\epsilon \in (0,1)$. Any $\mathbf{r} \in \mathcal{C}\partial(G_B)$ is called a *commuting boundary vector* of G_B.

According to the definition, the following proposition is obvious.

Proposition 4.6. *Given an interaction bipartite graph $G_B = ([m],[n],E_B)$, for any $\mathbf{r} \in (0,1]^n$, there exists a unique $\lambda > 0$ such that $\lambda\mathbf{r} \in \mathcal{C}\partial(G_B)$.*

Similar to the classic case [16], the idea of exclusiveness is the key of many proofs for CLLL.

Definition 4.3 (Commuting Exclusiveness)**.** A commuting subspace set $\mathcal{V}$ is called *exclusive* with respect to an interaction bipartite graph G_B, if $\mathcal{V}$ conforms with G_B and $R(V_i \cap V_j) = 0$ (or $V_i \perp V_j$) for any i, j such that $i \in \Gamma_j$. We do not mention "with respect to G_B" if it is clear from context. An exclusive subspace set is implicitly commuting.

Recall that $\mathcal{I}(G_D)$ is the classical abstract interior of the dependency graph G_D, which is exactly the set of probability vectors below Shearer's bound. Let $\partial(G_D)$ be the set of critical probability vector, that is

Definition 4.4 (Shearer's Boundary)**.** The *Shearer's boundary* of a graph $G_D = ([m], E_D)$, denoted by $\partial(G_D)$, is the set $\{\mathbf{r} \in (0,1]^m : (1-\epsilon)\mathbf{r} \in \mathcal{I}(G_D)$ and $(1+\epsilon)\mathbf{r} \notin \mathcal{I}(G_D)$ for any $\epsilon \in (0,1)\}$. Any $\mathbf{r} \in \partial(G_D)$ is called an *Shearer's boundary vector* of G_D.

$\mathcal{I}(G_D)$ is a open set, i.e., $\mathcal{I}(G_D) \cap \partial(G_D) = \emptyset$. For simplicity, let $\mathcal{I}(G_B) := \mathcal{I}(G_D(G_B))$. Gilyén and Sattath [11] have shown $\mathcal{I}(G_B) \subseteq \mathcal{CI}(G_B)$ for any interaction bipartite graph G_B. Here, we care about whether or when the boundaries $\partial(G_B)$ and $\mathcal{C}\partial(G_B)$ are same.

Definition 4.5 (Gap)**.** An interaction bipartite graph G_B is called *gapful for CLLL in the direction* of $\mathbf{r} \in (0,1)^m$, if there is a gap between $\partial(G_B)$ and $\mathcal{C}\partial(G_B)$ in this direction, i.e., $\lambda > 0$ such that $\lambda\mathbf{r} \in (\mathcal{CI}(G_B) \cup \mathcal{C}\partial(G_B)) \setminus (\mathcal{I}(G_B) \cup \partial(G_B))$, otherwise it is called gapless in this direction. G_B is said to be *gapful* for CLLL if it is gapful in some direction, otherwise it is gapless. Similarly, we can also define *gapful/gapless* for VLLL. We do not mention "for CLLL" or "for VLLL" if it is clear from context.

Remark. Another natural definition of gapful/gapless is as follows: An interaction bipartite graph G_B is called *gapful for CLLL in the direction* of $\mathbf{r} \in (0,1)^m$, if there is a $\lambda > 0$ such that $\lambda\mathbf{r} \in \mathcal{CI}(G_B) \backslash \mathcal{I}(G_B)$, otherwise it is called gapless in this direction. The only difference of these two definitions appears in the case there is a $\lambda_0 > 0$ such that $\lambda_0\mathbf{r} \in \mathcal{C}\partial(G_B)$, $\lambda_0\mathbf{r} \in \partial(G_B)$ but $\lambda_0\mathbf{r} \notin \mathcal{CI}(G_B)$, $\lambda_0\mathbf{r} \in \mathcal{I}(G_B)$. Informally, the boundaries in direction $\mathbf{r}$ are the same, but the interiors are different. We use the above definition because this case should be regarded as gapless for the same boundaries.

The main result of this section, namely Theorem 1.6, is a necessary and sufficient condition for deciding whether an interaction bipartite graph is gapful. It bridges gaplessness and exclusiveness in the interior. Though this theorem seems similar to Theorem 5 in [16], the proof is very different. The proof in [16] relies on the exclusive cylinder set on the boundary, which connects gaplessness with exclusiveness naturally. The existence of such exclusive cylinder set is ensured by Theorem 3 [16], the key idea of which is that the discreteness degree of each variable is bounded by the number of events related to this variable. However, for CLLL there is no such subspace set on the boundary if the relative dimension on boundary is irrational. Even if the relative dimensions are rational, it is still very difficult to bound the discreteness degree of subspaces because of the possible entanglement. Thus, we need new techniques to connect gaplessness with exclusiveness. Roughly speaking, in our proof, we first get a commuting subspace set, the relative dimensions of which exceeds the boundary. Then we adapt it to be exclusive by slicing the subspaces and discarding some slices. The main techniques used are a probability tool shown in Lemma 4.9 and the structure lemma.

Here are some properties of classical exclusive event sets, which will be used.

Lemma 4.7 (Theorem 1 in [31])**.** *Given G_D and $\mathbf{p} \in \mathcal{I}(G_D) \cup \partial(G_D)$. Among all event sets $\mathcal{A} \sim G_D$ with $\mathbb{P}(\mathcal{A}) = \mathbf{p}$, there is an exclusive one such that $\mathbb{P}(\cup_{A \in \mathcal{A}} A)$ is maximized.*

Lemma 4.8 (Lemma 29 in [16]). *Suppose that G_D is a dependency graph of event sets $\mathcal{A}$ and $\mathcal{B}$, $\mathbb{P}(\mathcal{A}) = \mathbb{P}(\mathcal{B})$, and $\mathcal{B}$ is exclusive. Then $\mathbb{P}(\cup_{A\in\mathcal{A}} A) \le \mathbb{P}(\cup_{B\in\mathcal{B}} B)$, and the equality holds if and only if $\mathcal{A}$ is exclusive.*

By Lemma 4.7 and Lemma 4.8, we have that for any event set $\mathcal{A} \sim G_D$ where $\mathbb{P}(\mathcal{A}) \in \mathcal{I}(G_D) \cup \partial(G_D)$ and any i, j where $i \in \Gamma_j$, if $\mathbb{P}(A_i \cap A_j) > 0$, then $\mathbb{P}\big(\cap_{A\in\mathcal{A}} \overline{A}\big) > 0$. The following lemma, namely Lemma 4.9, further shows that $\mathbb{P}\big(\cap_{A\in\mathcal{A}} \overline{A}\big)$ can be lower bounded, and can be viewed as a quantitative version of Lemma 4.8. The proof of this lemma is presented in the appendix.

Define $I(G_D, \mathbf{p}, k) := \min\{I(G_D[V'], (p_v : v \in V')) : |V'| = k\}$. Let p_{min} be the minimum element in $\mathbf{p} = (p_1, p_2, \cdots, p_m)$ If $\mathbf{p} \in \mathcal{I}(G_D) \cup \partial(G_D)$ and $t \le m-2$, let

$$\mathbb{F}(G_D, \mathbf{p}, t) =: \begin{cases} p_{min}^t \prod_{k=1}^t \frac{I(G_D,\mathbf{p},k)}{(m-1-k)} & \text{if } p \in \partial(G_D), \\ I(G_D, \mathbf{p}) & \text{if } p \in \mathcal{I}(G_D). \end{cases} \tag{4}$$

It is not hard to see $\mathbb{F}(G_D, \mathbf{p}, t) > 0$, $\mathbb{F}(G, \mathbf{p}, t') \le \mathbb{F}(G, \mathbf{p}, t)$ for $t' \ge t$, and $\mathbb{F}(G_D, \mathbf{p}', t) \le \mathbb{F}(G_D, \mathbf{p}, t)$ for any $\mathbf{p}' \ge \mathbf{p}$.

Lemma 4.9. *Given a dependency graph $G_D = ([m], E_D)$, a vector $\mathbf{p} \in \mathcal{I}(G_D) \cup \partial(G_D)$. For any event set $\mathcal{A} \sim G_D$ where $\mathbb{P}(\mathcal{A}) = \mathbf{p}$ and any i, j where $i \in \Gamma_j$, we have $\mathbb{P}\big(\cap_{A\in\mathcal{A}} \overline{A}\big) \ge \mathbb{P}(A_i \cap A_j)\mathbb{F}(G, \mathbf{p}, m-2)$.*

The following property of exclusive subspace sets will also be used.

Lemma 4.10. *Given an interaction bipartite graph $G_B = ([m], [n], E_B)$ and a rational vector $\mathbf{r}$ on $(0, 1]$, if there is an exclusive subspace set $\mathcal{V} \sim G_B$ with $R(\mathcal{V}) = \mathbf{r}$, then for any rational $\mathbf{r}' < \mathbf{r}$, there is an exclusive subspace set $\mathcal{V}' \sim G_B$ with $R(\mathcal{V}') = \mathbf{r}'$.*

Proof. W.l.o.g, we can assume that $\mathbf{r}' = (r_1 - \epsilon, r_2, \cdots, r_m)$ and $\mathcal{H}_1$ is related to V_1. Since $\mathbf{r}'$ and $\mathbf{r}$ are both rational vectors, $\frac{\epsilon}{r_1}$ is rational as well. Suppose $\frac{\epsilon}{r_1} = a/b$ where a and b are integers. Let $\mathcal{H}_1' = \mathcal{H}_1 \otimes \mathcal{H}_1^c$ where $\dim(\mathcal{H}_1^c) = b$, and $\mathcal{H}_i' = \mathcal{H}_i$ for any $i \ge 2$. Thus the whole vector space is $\mathcal{H}' = \bigotimes_{i=1}^m \mathcal{H}_i' = \bigotimes_{i=1}^m \mathcal{H}_i \otimes \mathcal{H}_1^c$.

We construct the subspace set $\mathcal{V}'$ as follows. Let $V_1' = V_1 \otimes W$, where W can be any subspace of $\mathcal{H}_1^c$ with dimension $b - a$. For each $i \ge 2$, let $V_i' = V_i \bigotimes \mathcal{H}_1^c$. It is not difficult to verify that $\mathcal{V}' \sim G_B$, $R(\mathcal{V}') = \mathbf{r}'$, $\mathcal{V}'$ is commuting and exclusive. □

Now we are ready to prove the main result of this section. The following lemma gives the necessary condition of gapless.

Lemma 4.11. *Given an interaction bipartite graph $G_B = ([m], [n], E_B)$. For any rational vector $\mathbf{r} \in \mathcal{CI}(G_B) \setminus \mathcal{C}\partial(G_B)$ on $(0, 1]$ such that G_B is gapless in direction $\mathbf{r}$, there is an exclusive subspace set $\mathcal{V} \sim G_B$ with $R(\mathcal{V}) = \mathbf{r}$.*

Proof. Let $\mathbf{q} = \lambda\mathbf{r} \in \partial(G_B)$. For any $0 < \epsilon \le 1$ where $(1-\epsilon)\mathbf{q}$ is rational, the construction of the exclusive subspace set $\mathcal{V} \sim G_B$ with $R(\mathcal{V}) = (1-\epsilon)\mathbf{q}$ are as follows. We can assume $m \ge 2$, since the case $m = 1$ is trivial.

Let $\epsilon_1 = \frac{(\epsilon q_{min})^2}{(3m)^2\|\mathbf{q}\|_1}\mathbb{F}(G, \mathbf{q}, m-2)$. Let $\mathbf{q}', \mathbf{q}''$ be rational vectors such that $\mathbf{q} < \mathbf{q}' \le \phi((1+\epsilon_1)\mathbf{q})$ and $(1-\epsilon_1)\mathbf{q} \le \mathbf{q}'' \le \mathbf{q}$. Here, $\phi(\mathbf{p}) \in (0, 1]^m$ is the vector whose i-th entry is $\min\{1, p_i\}$ for any i. Note that if $m \ge 2$, then $\mathbf{q} < \mathbf{1}$, thus such $\mathbf{q}'$ always exist.

According to the definition of $\mathcal{C}\partial(G_B)$, there is a commuting subspace set $\mathcal{V}^{(1)} \sim G_B$ with $R(\mathcal{V}^{(1)}) = \mathbf{q}'$ and $R\big(\bigoplus_{V_i^{(1)} \in \mathcal{V}^{(1)}} V_i^{(1)}\big) = 1$, and let $\mathcal{H}^{(1)} = (\mathcal{H}_1^{(1)}, \cdots, \mathcal{H}_m^{(1)})$ denote the corresponding qudits.

We construct a commuting subspace set $\mathcal{V}^{(2)} \sim H$ with $R(\mathcal{V}^{(2)}) = \mathbf{q}''$ from $\mathcal{V}^{(1)}$ as follows. Suppose $q_i''/q_i' = a_i/b_i$ where a_i and b_i are integers for $i \in [m]$. Let $\mathcal{H}_i^{(2)} = \mathcal{H}_i^{(1)} \bigotimes \mathcal{H}_i^c$ where $\dim(\mathcal{H}_i^c) = b_i$, and $V_i^{(2)} = V_i^{(1)} \bigotimes W_i$ where W_i can be any subspace of $\mathcal{H}_i^c$ with dimension a_i. Let $\mathcal{V}^{(2)} = \{V_1^{(2)}, \cdots, V_m^{(2)}\}$. It is not difficult to verify that $\mathcal{V}^{(2)} \sim H$, $\mathbb{P}(\mathcal{V}^{(2)}) = \mathbf{q}''$ and $\mathcal{V}^{(2)}$ is commuting.

Meanwhile, denote the orthogonal complement of W_i in space $\mathcal{H}_i^c$ by $\overline{W_i}$, we have

$$\begin{aligned}
1 - R\big(\textstyle\bigoplus_{V\in\mathcal{V}^{(2)}} V\big) &= R\big(\textstyle\bigoplus_{V\in\mathcal{V}^{(1)}} V\big) - R\big(\textstyle\bigoplus_{V\in\mathcal{V}^{(2)}} V\big) \\
&= R\big(\textstyle\bigoplus_{V\in\mathcal{V}^{(1)}} V \otimes \mathcal{H}_i^c\big) - R\big(\textstyle\bigoplus_{V\in\mathcal{V}^{(2)}} V\big) \\
&\le R\big(\textstyle\bigoplus_{i\in[m]} V_i^{(1)} \bigotimes \overline{W_i}\big) \le \textstyle\sum_{i\in[m]} R(V_i^{(1)} \bigotimes \overline{W_i}) \\
&= \textstyle\sum_{i\in[m]} R(V_i^{(1)}) R(\overline{W_i}) \le \textstyle\sum_{i\in[m]} q_i'(b_i - a_i)/b_i \\
&= \textstyle\sum_{i\in[m]} (q_i' - q_i'') = \|\mathbf{q}'\|_1 - \|\mathbf{q}''\|_1 \le 2\epsilon_1 \|\mathbf{q}\|_1 .
\end{aligned} \tag{5}$$

According to the gaplessness in direction $\mathbf{r}$, $\mathbf{q}'' \in \mathcal{I}_a(H) \cup \partial_a(G)$. From formula (5) and Lemma 4.9, we have for any i, j where $i \in \Gamma_j$,

$$\begin{aligned}
R(V_i^{(2)} \cap V_j^{(2)}) &= \mathbb{P}(A_{V_i^{(2)}} \cap A_{V_j^{(2)}}) \\
&\le \mathbb{P}\big(\textstyle\bigcap_{V_i^{(2)} \in \mathcal{V}^{(2)}} \overline{A_{V_i^{(2)}}}\big) / \mathbb{F}(G, \mathbf{q}'', n-2) \\
&\le \mathbb{P}\big(\textstyle\bigcap_{V_i^{(2)} \in \mathcal{V}^{(2)}} \overline{A_{V_i^{(2)}}}\big) / \mathbb{F}(G, \mathbf{q}, n-2) \\
&= (1 - R\big(\textstyle\bigoplus_{V_i^{(2)} \in \mathcal{V}^{(2)}} V_i^{(2)}\big)) / \mathbb{F}(G, \mathbf{q}, n-2) \\
&\le 2\epsilon_1 \|\mathbf{q}\|_1 / \mathbb{F}(G, \mathbf{q}, n-2) = 2(\tfrac{\epsilon q_{min}}{3n})^2 .
\end{aligned} \tag{6}$$

Recall that A_V is the corresponding classical event of V defined in the proof of Lemma 4.1, and $\mathbb{F}(G, \mathbf{p}, n-2)$ monotonically decreases as $\mathbf{p}$ increases.

Now, we are going to construct an exclusive subspace set $\mathcal{V}^{(3)} \sim G_B$ with $R(\mathcal{V}^{(3)}) \ge (1-\epsilon)\mathbf{q}$ from $\mathcal{V}^{(2)}$, which concludes the proof coupled with Lemma 4.10. For simplicity, we use $\mathcal{H}_i$ and V_i to represent $\mathcal{H}_i^{(2)}$ and $V_i^{(2)}$ respectively.

For any i, j where $i \in \Gamma_j$, according to the structure lemma, $\mathcal{H}_{\mathcal{N}(i)\cap\mathcal{N}(j)}$ can be decomposed to some orthogonal subspaces $\mathcal{H}_{\mathcal{N}(i)\cap\mathcal{N}(j)} = \bigoplus_k W_k = \bigoplus_k W_{k1} \otimes W_{k2}$ s.t.

1. $V_i^{loc} = \bigoplus_k V_i|_{W_{k1}} \otimes W_{k2}$, where $V_i|_{W_{k1}} \subseteq \mathcal{H}_{\mathcal{N}(i)\setminus\mathcal{N}(j)} \otimes W_{k1}$.
2. $V_j^{loc} = \bigoplus_k V_j|_{W_{k2}} \otimes W_{k1}$, where $V_j|_{W_{k2}} \subseteq \mathcal{H}_{\mathcal{N}(j)\setminus\mathcal{N}(i)} \otimes W_{k2}$.

For simplicity, let $X_{ijk} = V_i|_{W_{k1}} \otimes W_{k2} \otimes \mathcal{H}_{[n]\setminus\mathcal{N}(i)}$, $X_{jik} = V_j|_{W_{k2}} \otimes W_{k1} \otimes V_{[n]\setminus\mathcal{N}(j)}$, and $Y_{ijk} = W_k \otimes \mathcal{H}_{[n]\setminus(\mathcal{N}(i)\cap\mathcal{N}(j))}$. Thus $\bigoplus_k X_{ijk} = V_i$, $\bigoplus_k X_{jik} = V_j$ and $\bigoplus_k Y_{ijk} = \mathcal{H}$. For any $i \in [m]$, we define $V_i^{(3)}$ as the orthogonal complement of $\bigoplus_{j: j\in\Gamma_i, k: R(X_{ijk}) \le R(X_{jik})} X_{ijk}$ in V_i. Since X_{ijk} only depends on $\mathcal{H}_{\mathcal{N}(i)}$, so does $V_i^{(3)}$. Therefore, $\mathcal{V}^{(3)} \sim G_B$.

$\mathcal{V}^{(3)}$ is commuting and exclusive: for any given i, j where $i \in \Gamma_j$, it is not hard to see $V_i^{(3)}$ is a subspace of $\bigoplus_{k: R(X_{ijk}) > R(X_{jik})} Y_{ijk}$ and $V_j^{(3)}$ is a subspace of $\bigoplus_{k: R(X_{jik}) > R(X_{ijk})} Y_{jik}$, thus $V_i^{(3)}$ and $V_j^{(3)}$ are orthogonal, therefore commuting and exclusive.

$R(\mathcal{V}^{(3)}) \geq (1-\epsilon)\mathbf{q}$: since V_{ijk} and V_{jik} are R-independent in Y_{ijk}, we have $R\big(X_{ijk} \cap X_{jik}\big) = R\big(X_{ijk} \cap X_{jik} | Y_{ijk}\big) \cdot R(Y_{ijk}) = R(X_{ijk}|Y_{ijk})R(X_{jik}|Y_{ijk})R(Y_{ijk})$, thus,

$$\begin{aligned} R\big(V_i \cap V_j\big) &= R(\textstyle\bigoplus_k X_{ijk} \cap X_{jik}) = \sum_k R(X_{ijk} \cap X_{jik}) \\ &= \textstyle\sum_k R(X_{ijk}|Y_{ijk})R(X_{jik}|Y_{ijk})R(Y_{ijk}) \\ &\geq \textstyle\sum_k \left(\min\{R(X_{ijk}|Y_{ijk}), R(X_{jik}|Y_{ijk})\}\right)^2 R(Y_{ijk}) \\ &= \mathbb{E}_k[(\min\{R(X_{ijk}|Y_{ijk}), R(X_{jik}|Y_{ijk})\})^2]. \end{aligned} \tag{7}$$

Meanwhile, we also have

$$\begin{aligned} \textstyle\sum_k \min\{R(X_{ijk}), R(X_{jik})\} &= \textstyle\sum_k \min\{R(X_{ijk}|Y_{ijk}), R(X_{jik}|Y_{ijk})\} R(Y_{ijk}) \\ &= \mathbb{E}_k(\min\{R(X_{ijk}|Y_{ijk}), R(X_{jik}|Y_{ijk})\}). \end{aligned}$$

By Jensen's inequality, we have

$$\begin{aligned} R\big(V_i \cap V_j\big) &\geq \mathbb{E}_k[(\min\{R(X_{ijk}|Y_{ijk}), R(X_{jik}|Y_{ijk})\})^2] \\ &\geq [\mathbb{E}_k(\min\{R(X_{ijk}|Y_{ijk}), R(X_{jik}|Y_{ijk})\})]^2 \\ &= (\textstyle\sum_k \min\{R(X_{ijk}), R(X_{jik})\})^2. \end{aligned} \tag{8}$$

Thus, for any i,

$$\begin{aligned} &\mathbb{R}\big(\textstyle\sum_{j:j\in\Gamma_i} \bigoplus_{k:R(X_{ijk})\leq R(X_{jik})} X_{ijk}\big) \\ \leq &\textstyle\sum_{j:j\in\Gamma_i} \sum_{k:R(X_{ijk})\leq R(X_{jik})} R(X_{ijk})) \\ \leq &\textstyle\sum_{j:j\in\Gamma_i} \sum_k \min\{R(X_{ijk}), R(X_{jik})\} \\ \leq &\textstyle\sum_{j:j\in\Gamma_i} \big(R\big(V_i \cap V_j\big)\big)^{1/2} && \text{(by (8))} \\ \leq &\textstyle\sum_{j:j\in\Gamma_i} \big(2(\frac{\epsilon q_{min}}{3n})^2\big)^{1/2} && \text{(by (6))} \\ \leq &n\big(2(\tfrac{\epsilon q_{min}}{3n})^2\big)^{1/2} < 2\epsilon q_{min}/3. \end{aligned}$$

Recall $V_i^{(3)}$ is the orthogonal complement of $\sum_{j:j\in\Gamma_i} \bigoplus_{k:R(X_{ijk})\leq R(X_{jik})} X_{ijk}$ in space V_i, we have

$$\begin{aligned} R(V_i^{(3)}) &= R\big(V_i\big) - R\big(\textstyle\sum_{j:j\in\Gamma_i} \bigoplus_{k:R(X_{ijk}\leq R(X_{jik})} X_{ijk})\big) \\ &\geq (1-\epsilon_1)q_i - 2\epsilon q_{min}/3 \geq (1-\epsilon/3)q_i - 2\epsilon q_{min}/3 \geq (1-\epsilon)q_i. \end{aligned}$$

Therefore, we have $R(\mathcal{V}^{(3)}) \geq (1-\epsilon)\mathbf{q}$. □

Theorem 1.6 (restated). *Given an interaction bipartite graph G_B and a vector $\mathbf{r}$ of positive reals, the following two conditions are equivalent:*

1. *For rational $\lambda\mathbf{r} \in \mathcal{CI}(G_B) \setminus \mathcal{C}\partial(G_B)$, there is an exclusive subspace set with interaction bipartite graph G_B and probability vector $\lambda\mathbf{r}$.*
2. *G_B is gapless for CLLL in the direction of $\mathbf{r}$.*

Proof. ($1 \Rightarrow 2$): Arbitrarily fix $\lambda > 0$ such that $\mathbf{q} \triangleq \lambda\mathbf{r} \in \mathbb{I}(G_B)$ and $\mathbf{q}$ is rational. Let $\mathcal{V} \sim G_B$ be an exclusive subspace set such that $R(\mathcal{V}) = \mathbf{q}$ and $R(\bigoplus_{V\in\mathcal{V}} V) < 1$. Recall in the proof of Lemma 4.1, A_V is the classical event corresponding to V, and $A_V \sim G_D(G_B)$, $\mathbb{P}(A_{V_i}) = q_i$, $\mathbb{P}(\cup_i A_{V_i}) = R(\bigoplus_i V_i) < 1$. In addition, A_{V_i}'s are exclusive, thus according to Lemma 4.8, $\mathbf{q} \in \mathcal{I}(G_B)$, which means H is gapless in the direction of $\mathbf{r}$.

(2 $\Rightarrow$ 1): It is immediate by Lemma 4.11. $\square$

By Theorem 1.6, one can prove the existence of a gap just by proving non-existence of exclusive subspace set, without computing the critical threshold of QLLL or CLLL.

The following corollary is immediate by Theorem 1.6 and Lemma 4.10. By this corollary, one can prove gaplessness just by constructing an commuting subspace set, without computing the critical threshold of QLLL or CLLL.

Corollary 1.7 (restated). *Given an interaction bipartite graph G_B and a rational vector $\mathbf{r} \in \mathcal{C}\partial(G_B)$, if there is an exclusive subspace set with interaction bipartite graph G_B and probability vector $\mathbf{r}$, then G_B is gapless in the direction of $\mathbf{r}$.*

4.4 Reduction Rules

To infer gap existence for VLLL of a bipartite graph from known ones, a set of reduction rules are established for VLLL [16]. With these rules, various bipartite graphs, in particular combinatorial ones, are shown to be gapful/gapless. In this subsection, we show these reduction rules apply for CLLL as well. Meanwhile, we introduce another operation (the sixth one) which preserves both gapful and gapless. With these operation, the interaction bipartite graph which is a tree can be shown to be gapless immediately and a complete characterization of strongly a-gapful bipartite graphs is provided.

Given an interaction bipartite graph $G_B = ([m], [n], E_B)$, we consider the following 6 types of operations on G_B are:

1. Delete-R-Leaf: Delete a vertex $j \in [n]$ on the right side with $|\mathcal{N}(j)| \leq 1$, and remove the incident edge if any.
2. Duplicate-L-Vertex: Given a vertex $i \in [m]$ on the left side, add a vertex i' to the left side, and add edges incident to i' so that $\mathcal{N}(i') = \mathcal{N}(i)$.
3. Duplicate-R-Vertex: Given a vertex $j \in [n]$ on the right side, add a vertex j' to the right side, and add some edges incident to j' so that $\mathcal{N}(j') \subseteq \mathcal{N}(j)$.
4. Delete-Edge: Delete an edge from E_B provided that the base graph G_D remains unchanged.
5. Delete-L-Vertex: Delete a vertex $i \in [m]$ on the left side, and remove all the incident edges.
6. Delete-L-Leaf: Delete a vertex $i \in [m]$ on the left hand with $|\mathcal{N}(i)| \leq 1$, and remove the incident edge if any.

We also define the inverses of the above operations. The inverse of an operation O is the operation O' such that for any G_B, $O'(O(G_B)) = O(O'(G_B)) = G_B$.

The next theorems show how these operations influence the existence of gaps for CLLL.

Theorem 4.12. *A gapless interaction bipartite graph remains gapless for CLLL after applying Delete-L-Vertex or the inverse of Delete-Edge.*

Theorem 4.13. *A gapful interaction bipartite graph remains gapful for CLLL after applying Delete-Edge or the inverse of Delete-L-Vertex.*

These two theorems are trivial and the proofs are omitted.

Theorem 4.14. *An interaction bipartite graph $G_B = ([m], [n], E_B)$ is gapful for CLLL, if and only if it is gapful after applying Delete-L-Leaf, Delete-R-Leaf, Duplicate-L-Vertex, Duplicate-R-Vertex, or their inverse operations.*

Proof. (Duplicate-L-Vertex, Duplicate-R-Vertex): The proofs of Duplicate-L-Vertex and Duplicate-R-Vertex are similar to that in [16]. Moreover, let G'_B be the resulting graph by applying Duplicate-R-Vertex to G_B, we have $\mathcal{C}\partial(G'_B) = \mathcal{C}\partial(G_B)$.

(Delete-R-Leaf): Suppose vertex $n+1$ is added to the right side, if $\mathcal{N}(n+1)$ is empty, it's trivial, otherwise assume $\mathcal{N}(n+1) = \{m\}$ and $G'_B = ([m], [n+1], E'_B\})$ is the resulting bipartite graph. Note that the base graph G_D remains unchanged, it suffices to prove $\mathcal{C}\partial(G_B) = \mathcal{C}\partial(G'_B)$. It is easy to see $\mathcal{C}\partial(G'_B) \subseteq \mathcal{CI}(G_B) \cup \mathcal{C}\partial(G_B)$, so it remains to show $\mathcal{C}\partial(G_B) \subseteq \mathcal{CI}(G'_B) \cup \mathcal{C}\partial(G'_B)$.

Consider another interaction bipartite graph $G''_B = ([m], [n+1], E''_B)$ obtained by applying the inverse operation of Delete-Edge to G'_B: $\forall i \in \mathcal{N}(n)$, we add the edge $(i, n+1)$. On one hand, it is easy to see $\mathcal{C}\partial(G''_B) \subseteq \mathcal{CI}(G'_B) \cup \mathcal{C}\partial(G'_B)$. On the other hand, note that $\mathcal{N}(n) = \mathcal{N}(n+1)$, thus G''_B can be viewed as the resulting bipartite graph by applying Duplicate-R-Vertex to G_B, we have $\mathcal{C}\partial(G''_B) = \mathcal{C}\partial(G_B)$.

(Delete-L-Leaf): Suppose vertex $m+1$ is added to the left side, if $\mathcal{N}(m+1)$ is empty, it's trivial, otherwise assume $\mathcal{N}(m+1) = \{n\}$ and $G'_B = ([m], [n+1], E'_B\})$ is the resulting bipartite graph. In addition, assume $\mathcal{N}(m) = \{1, 2, \cdots, k, n+1\}$.

G_B is gapless $\Rightarrow$ G'_B is gapless: By Theorem 1.6, if suffices to show for any rational $\mathbf{r}' \triangleq (r'_1, \cdots, r'_{m+1}) \in \mathcal{CI}(G'_B) \setminus \mathcal{C}\partial(G'_B)$, there is such an exclusive subspace set $\mathcal{V}'$. Let $\mathbf{r} = \big(\frac{r'_1}{1-r'_{m+1}}, \frac{r'_2}{1-r'_{m+1}}, \cdots, \frac{r'_k}{1-r'_{m+1}}, r'_{k+1}, \cdots, r'_m\big)$. First, we claim that $\mathbf{r} \in \mathcal{CI}(G_B) \setminus \mathcal{C}\partial(G_B)$.

This is because otherwise, by the definition of $\mathcal{C}\partial(G_B)$, there is a commuting subspace set $\mathcal{V} \sim G_B$ with $R(\mathcal{V}) = (1+\epsilon)\mathbf{r}$ satisfying $R\big(\bigoplus_{V\in\mathcal{V}} V\big) = 1$ for any rational $\epsilon > 0$. We construct a commuting subspace set $\mathcal{V}' \sim G'_B$ as follows: suppose $1 - r'_{n+1} = a/b$ where a and b are integers. Let $\mathcal{H}'_n = \mathcal{H}_n \otimes \mathcal{H}^c_n$ where $\dim(\mathcal{H}^c_n) = b$, and $\mathcal{H}'_i = \mathcal{H}_i$ for any $i < n$. Then let

- If $i \in [k]$, $V'_i = V_i \bigotimes Y$, where Y can be any subspace of $\mathcal{H}^c_n$ with dimension a.
- If $k < i \leq n$, $V'_i = V_i \bigotimes \mathcal{H}^c_n$.
- If $i = n+1$, $V'_i = Y^\perp \otimes \bigotimes_{i=1}^n \mathcal{H}_i$

It is easy to verify that $\mathcal{V}'$ is commuting, $\mathcal{V}' \sim G'_B$, $R\big(\bigoplus_{V'\in\mathcal{V}'} V'\big) = 1$ and $R(\mathcal{V}') = ((1+\epsilon)r'_1, \cdots, (1+\epsilon)r'_n, r'_{n+1}) \leq (1+\epsilon)\mathbf{r}'$. Thus, by Lemma 4.4 there is also a commuting subspace set $\mathcal{V}'' \sim G'_B$ with relative dimensions $(1+\epsilon)\mathbf{r}'$ such that $R\big(\bigoplus_{V''\in\mathcal{V}''} V''\big) = 1$, a contradiction.

since $\mathbf{r} \in \mathcal{CI}(G_B) \setminus \mathcal{C}\partial(G_B)$ and G_B is gapless, by Thereom 1.6 we have there is an exclusive subspace set $\mathcal{V}$ with interaction bipartite graph G_B and relative dimensions $\mathbf{r}$. Then it is easy to verify that the commuting subspace set $\mathcal{V}'$ constructed above is also exclusive, $\mathcal{V}' \sim G'_B$, and $R(\mathcal{V}') = \mathbf{r}'$.

G'_B is gapless $\Rightarrow$ G_B is gapless: By Theorem 1.6, it suffices to show there is an exclusive $\mathcal{V} \sim G_B$ for any $\mathbf{r} \in \mathcal{CI}(G_B) \setminus \mathcal{C}\partial(G_B)$. Let $(1+\lambda)\mathbf{r} \in \mathcal{C}\partial(G_B)$ be the vector on the boundary, and $\mathbf{r}' = (\mathbf{r}, \epsilon)$ where $\epsilon < \lambda r_1$. It is not hard to see that $\mathbf{r}' \in \mathcal{CI}(G'_B) \setminus \mathcal{C}\partial(G'_B)$. And then, by Theorem 1.6, there is an exclusive subspace set $\mathcal{V}' = \{V'_1, \cdots, V'_{m+1}\}$ with interaction bipartite graph G'_B and relative dimensions $\mathbf{r}'$. Let $\mathcal{V} = \{V'_1, V'_2, \cdots, V'_m\}$, it is easy to see that $\mathcal{V}$ is exclusive, $\mathcal{V} \sim G_B$ and $R(\mathcal{V}) = \mathbf{r}$. $\square$

Remark. Following the above proof or that of [16, Lemma 46], it can be proved that Delete-L-Leaf also applies for VLLL. That is

Theorem 4.15. *An interaction bipartite graph G_B is gapful for VLLL, if and only if it remains gapful after applying Delete-L-Leaf or its inverse operation.*

With these reduction rules, it is easy to see all trees are gapless, which includes two interesting families of interaction bipartite graphs, the treelike bipartite graphs [16] and the regular trees [6, 17, 28].

Theorem 1.8. *An interaction bipartite graph G_B is gapless for CLLL if G_B is a tree.*

Proof. Applying Delete-L-Leaf or Delete-R-Leaf on G_B repeatedly results in an interaction bipartite graph $G'_B = ([1], [1], \{(1, 1)\})$. Obviously, G'_B is gapless, which implies G_B is gapless as well by Theorems 4.14. □

Another application of these reduction rules is the characterization of strongly a-gapful graphs [16, 19]. We say a graph $G_D = ([m], E_D)$ is strongly a-gapful, if any interaction bipartite graph satisfying $G_D(G_B) = G_D$ is gapful, otherwise we call it is strongly a-gapless. The canonical bipartite graph, as defined below, is the key to studying strongly a-gapless.

Definition 4.6 (Canonical bipartite graph [16])**.** Given a dependency graph $G_D = ([m], E_D)$ with $Cliq(G_D) = \{C_1, ..., C_n\}$, its canonical bipartite graph, denoted by $H(G_D)$, is the bipartite graph $([m], [n], E_H)$ where $E_H = \{(i, j) \in [m] \times [n] : i \in C_j\}$. Here, $Cliq(G_D)$ be the set of maximal cliques of G_D.

Informally, $H(G_D)$ is the interaction bipartite graph where each maximal clique has a distinct subsystem and an subspace depends on a subsystem if it is in the corresponding maximal clique. $H(G_D)$ can be shown to have the minimum interior for CLLL among the bipartite graphs whose base graph is G_D, thus G_D is strongly a-gapful for CLLL if and only if $H(G_D)$ is gapful. The proof is very similar with the classical case [16], and we omit it here.

Lemma 4.16. *Given a dependency graph G_D, for any interaction bipartite graph G_B with $G_D(G_B) = G_D$, we have $\mathcal{CI}(G_B) \supseteq \mathcal{CI}(H(G_D))$.*

The following theorem extends the classic result of strongly a-gapful graphs for VLLL to commuting Hamiltonians. A chordal graph is one with no induced cycle of length greater than three. A well known property of chordal graphs is that it has a vertex which lies in exactly one maximal clique.

Theorem 1.9. *A dependency graph is strongly a-gapless for CLLL if and only if it is chordal.*

Proof. $\Longrightarrow$: Suppose G_D is not chordal, then there must exist an induced cycle of length at least four. By Theorem 1.12, $H(G)$ is gapful, which means G_D is strongly a-gapful.

$\Longleftarrow$: The proof of this direction is similar to the proof of [16, Lemma 46], but we still give the proof here for completeness. The proof is by induction on m. The case $m = 1$ is trivial. Suppose we have already shown the small cases, and now we would like to show the case $G_D = ([m], E_D)$. Let $H = H(G_D) = ([m], [n], E_H)$. W.l.o.g, assume that the vertex m of G_D lies in exactly one maximal clique $S = \{m-k+1, ..., m\}$. That is, the vertex m has only one neighbor, say n, in H, and $\mathcal{N}_H(n) = S$. Let H' be the resulting bipartite graph by deleting the vertex m. By Theorem 4.14, the gaplessness of H are H' are same.

Let G'_D be the chordal graph obtained by deleting the vertex m from G_D. If $S \setminus \{m\}$ remains a maximal clique in G'_D, then obviously $H' = H(G'_D)$; Otherwise, $H(G'_D)$ can be obtained by applying the inverse operation of Duplicate-R-Vertex to H'. Hence, we always have that H' is gapless if and only if so is $H(G'_D)$, which implies the conclusion by the induction hypothesis.

□

Remark. Theorem 1.8 is an immediate corollary of Theorem 1.9. The proof is retained since it is concise and a good example to show the power of our reduction rules.

5 CLLL: beyond Shearer's bound

5.1 Tight region for trees

In the above section, we have proved that trees are gapless. In this section, we give the tight region for CLLL on trees. Moreover, our results also applies for the case where the dimensions of qudits are specified. The interaction bipartite graph which is a tree include two interesting families of bipartite graphs, the treelike bipartite graphs defined in [16] and the regular trees defined in [6, 17, 28]. He et al. [16] have already calculated the tight region of treelike bipartite graphs for VLLL. Here we extends the classical result to the commuting case on a larger family of graphs even if the dimensions of qudits are given.

Without loss of generality, we can assume that the leaves of all trees are qudits, since we can add some 1-dimension qudits if necessary.

Theorem 1.10. *Given an interaction bipartite graph $G_B = ([m], [n], E_B)$ which is a tree and dimensions of qudits $\mathbf{d}$, appoint the qudit n as the root. For ranks $\mathbf{r} \in \mathbb{Z}^m$, define $\mathbf{q} = (q_1, ..., q_n) \in \mathbb{Z}^n$ to be*

$$q_i = \begin{cases} 0 & \textit{if vertex } i \textit{ is a leaf of } G_B, \\ \sum_{j\in\mathcal{C}_i} \lfloor r_j \cdot \prod_{k\in\mathcal{C}_j} \frac{1}{d_k - q_k} \rfloor & \textit{otherwise}. \end{cases} \tag{1}$$

Here $\mathcal{C}_i$ is the set of children of i. Then there is such a commuting instance with rank $\mathbf{r}$ spanning the whole space if and only if there is some $q_i \geq d_i$.

Proof. $\Longleftarrow$: Suppose there is $i \in [n]$ such that $q_i \geq d_i$. Fix such an i each of whose descendant k satisfies $q_k < d_k$. Let T_i be the subtree rooted at i. We will construct classical events, i.e., whose basis are computational basis, and show those events in T_i suffice to span the whole space. The constructs are as following:

- For each qudit j in T_i except i, let $\mathcal{H}_j^a := \text{span}\{|1\rangle, \cdots, |q_j\rangle\}$ and $\mathcal{H}_j^b := \text{span}\{|q_j+1\rangle, \cdots, |d_j\rangle\}$.
- For each event j in T_i except $\mathcal{C}_i$, let $V_j := \bigotimes_{k\in\mathcal{C}_j} \mathcal{H}_k^b \otimes \mathcal{H}_{\mathcal{F}_j}^j$. Here $\mathcal{F}_j$ is the father of j, $\dim(\mathcal{H}_{\mathcal{F}_j}^j) = \lfloor r_j \cdot \prod_{k\in\mathcal{C}_j} \frac{1}{d_k - q_k} \rfloor$, and the $\mathcal{H}_{\mathcal{F}_j}^j \subseteq \mathcal{H}_{\mathcal{F}_j}^a$ are mutually orthogonal for j's with the same father $\mathcal{F}_j$, thus these $\mathcal{H}_{\mathcal{F}_j}^j$ span $\mathcal{H}_{\mathcal{F}_j}^a$
- For each event $j \in \mathcal{C}_i$, let V_j be $\bigotimes_{k\in\mathcal{C}_j} \mathcal{H}_k^b \otimes \mathcal{H}_i^k$. Here $\dim(\mathcal{H}_i^k) = \lfloor r_j \cdot \prod_{k\in\mathcal{C}_j} \frac{1}{d_k - q_k} \rfloor$, and $\mathcal{H}_i^k$'s satisfy $\bigoplus_{k\in\mathcal{C}_i} \mathcal{H}_i^k = \mathcal{H}_i$.

Note that rank$(V_j) = \prod_{k\in\mathcal{C}_j}(d_k - p_k)\cdot\lfloor r_j \cdot \prod_{k\in\mathcal{C}_j}\frac{1}{d_k-q_k}\rfloor \leq r_j$. Now we will prove that these subspaces span the whole space, which concludes this direction of the proof. Arbitrarily fix a computational basis $|s\rangle = |s_1, s_2, \cdots, s_n\rangle$.

Let $l = i$. Then, if there are events $j \in \mathcal{C}_l$ such that $|s_l\rangle \in \mathcal{H}^j_{\mathcal{F}_j}$, pick such a j arbitrarily, if $\exists k \in \mathcal{C}_j$ s.t., $|s_k\rangle \in \mathcal{H}^a_k$, let l be such a k arbitrarily. Iterate this process and finally one of the following two cases must be reached.

Case 1: $\mathcal{C}(l) = \emptyset$, namely l is a leaf.

Case 2: $\mathcal{C}(l) \neq \emptyset$ and for any $t \in \mathcal{C}_k$, $|s_t\rangle \in \mathcal{H}^b_t$.

Let the final l be l_0, it is easy to see that Case 1 cannot happen, so we have $|s\rangle \in V_k$, which implies these these subspaces span the whole space.

$\Longrightarrow$: By induction on m. The case $m = 1$ is trivial. Now suppose it holds for $m-1$, and we are going to show it also holds for $G_B = ([m], [n], E_B)$. Let t be any such a event where $\mathcal{C}_t$ are all leaves, G'_B be the interaction bipartite graph by deleting t and its children, $\mathbf{r}' \in \mathbb{Z}^{n-1}$ be the induced $\mathbf{r}$ on G'_B, and $\mathbf{d}'$ be induced $\mathbf{d}$ on G'_B except $d'_{\mathcal{F}_t} = d_{\mathcal{F}_t} - \lfloor r_t \cdot \prod_{k\in\mathcal{C}_t}\frac{1}{d_k-q_k}\rfloor = d_{\mathcal{F}_t} - \lfloor r_t \cdot \prod_{k\in\mathcal{C}_t}\frac{1}{d_k}\rfloor$. Thus $\mathcal{H}'_{\mathcal{F}_t}$ can be viewed as $\mathcal{H}^b_{\mathcal{F}_t}$. Note that $\mathbf{q}'$ is induced $\mathbf{q}'$ on G'_B except $q'_{\mathcal{F}_t} = q_{\mathcal{F}_t} - \lfloor r_t \cdot \prod_{k\in\mathcal{C}_t}\frac{1}{d_k}\rfloor$. So the following claim concludes the proof.

Claim. The case $(H, \mathbf{r}, \mathbf{d})$ has an unsatisfying instance if and only if the case $(H', \mathbf{r}', \mathbf{d}')$ has an unsatisfying one.

Proof. $\Leftarrow$: Suppose the satisfying instance of $(G'_B, \mathbf{r}', \mathbf{d}')$ is $\mathcal{V}' = \{V'_1, \cdots, V'_{t-1}, V'_{t+1}, \cdots, V'_m\}$, which can span $\mathcal{H}^b_{\mathcal{F}_t} \otimes \mathcal{H}_{[n]\setminus\mathcal{F}_t}$ by definition. Let $V_t = \mathcal{H}^a_{\mathcal{F}_t} \otimes \mathcal{H}_{\mathcal{C}_t}$. Then it is easy to see that $\mathcal{V} = \{\mathcal{V}', V_t\}$ is a satisfying instance of $(H, \mathbf{p}, \mathbf{d})$.

$\Leftarrow$: Suppose the satisfying instance of $(H, \mathbf{r}, \mathbf{d})$ is $\mathcal{V} = \{V_1, \cdots, V_m\}$. Let $\mathcal{V}' = \mathcal{V} \setminus \{V_t\}$ Define $\mathcal{H}^g_{\mathcal{F}} = \text{span}(\{|i\rangle : |i\rangle \otimes \mathcal{H}_{[n]\setminus\mathcal{F}_t} \subseteq V_t\})$. It is easy to see that $\dim(\mathcal{H}^g_{\mathcal{F}_t}) \leq \lfloor r_t \cdot \prod_{k\in\mathcal{C}_t}\frac{1}{d_k}\rfloor$, and $(\mathcal{H}_{\mathcal{F}_t} - \mathcal{H}^g_{\mathcal{F}_t}) \otimes \mathcal{H}_{[n]\setminus\mathcal{F}_t} \subseteq \bigoplus_{V\in\mathcal{V}'} V$. Thus $\mathcal{V}'$ is an unsatisfiable instance of $(H', \mathbf{r}', \mathbf{d}')$. □

□

The above theorem also implies the tight region for VLLL, CLLL and QLLL on tree, ignoring the dimensions of qudits.

Corollary 1.11. *Given an interaction bipartite graph $G_B = ([m], [n], E_B)$ which is a tree, appoint the qudit n as the root. For $\mathbf{r} \in (0,1)^m$, define $\mathbf{q} = (q_1, ..., q_n) \in [0,1]^n$ to be*

$$q_i = \begin{cases} 0 & \text{if vertex } i \text{ is a leaf of } G_B, \\ \sum_{j\in\mathcal{C}_i} r_j \cdot \prod_{k\in\mathcal{C}_j}\frac{1}{1-q_k} & \text{otherwise.} \end{cases} \tag{2}$$

Here $\mathcal{C}_i$ is the set of children of i. Then $\mathbf{r} \in \mathcal{VI}(G_B) = \mathcal{CI}(G_B) = \mathcal{I}(G_B)$ if and only if $\forall i \in [n]$, $q_i < 1$.

5.2 CLLL is generally different from QLLL

In this section, we show that many interaction bipartite graphs are gapful for CLLL. An easy observation is that an interaction bipartite graph G_B is gapless for CLLL if it is gapless for VLLL. Thus, the combinatorial interaction bipartite graph gapless for VLLL defined in [16] are also gapless for CLLL.

Definition 5.1 (Combinatorial interaction bipartite graph [16]). Given two positive integers $m < n$, let $G_{n,m} = ([\binom{n}{m}], [n], E_{n,m})$ where $(i,j) \in E_{n,m}$ if and only if j is in the m-sized subset of $[n]$ represented by i. $G_{n,m}$ is called the (n,m)-combinatorial interaction bipartite graph.

Corollary 5.1. *For $n \geq 4$, $G_{n,n-1}$ is gapless.*

Corollary 5.2. *For any constant m, when n is large enough, $G_{n,n-m}$ is gapless.*

On the other side, it can be proved that all cyclic graphs are gapful. An interaction bipartite graph is called n-cyclic, or cyclic for short, if its base graph is a cycle of length n. If $n = 3$, we additionally requires the three Hamiltonians sharing no common qudit. Note that any cyclic interaction bipartite graph satisfies that any pair of Hamiltonians sharing at most one qudit, by Corollary 1.5 we have the tight regions of CLLL and VLLL are the same for cyclic graphs. Thus, we can obtain the tight region of CLLL for cyclic interaction bipartite graph from the tight region of VLLL [16]. Meanwhile, because it has been proved that cyclic graphs are gapful for VLLL, we immediately have the following corollary.

Corollary 5.3. *Cyclic interaction bipartite graphs are gapful.*

By Corollary 5.3, we can get a large class of gapful interaction bipartite graphs.

Definition 5.2 (Containing [16])**.** We say that an interaction bipartite graphs $G_B = ([m], [n], E_B)$ contains another interaction bipartite graphs $G'_B = ([m'], [n'], E'_B)$, if there are injections $\pi_L : [m'] \to [m]$ and $\pi_R : [n'] \to [n]$ such that the following two conditions hold simultaneously:

1. For any $i \in [m']$ and $j \in [n']$, $\pi_R(j) \in \mathcal{N}_{G_B}(\pi_L(i))$ if and only if $j \in \mathcal{N}_{G'_B}(i)$.
2. For any $j \in [n] \setminus \pi_R([n'])$, $j \notin \mathcal{N}_{G_B}(\pi_L(i)) \cap \mathcal{N}_{G_B}(\pi_L(k))$ for any $i, k \in [m']$.

By Theorem 4.14 and Theorem 4.12, an interaction bipartite graph G_B is gapful if it contains a gapful one. According to Theorem 5.3, we obtain the following result.

Theorem 1.12 (restated). *Any interaction bipartite graphs containing a cyclic one is gapful.*

It is easy to verify that any interaction bipartite graph contains a cyclic one if its base graph has an induced cycle of length at least four. Thus, by Theorems 1.8 and 1.12, we have the following corollary, which almost gives a complete characterization of gapful/gapless for CLLL except when the base graph has only 3-cliques.

Corollary 1.13. *An interaction bipartite graph is gapless for CLLL if its base graph is tree, is gapful if its base graph has an induced cycle of length at least 4.*

5.3 Local Hamiltonians on regular lattices

In this subsection, we focus on the interaction bipartite graph where any pair of Hamiltonians share at most one qudit, which includes many system of physical interest, such as lattice[6, 25, 28, 33]. For this special kind of interaction bipartite graphs, we have already known CLLL is equal to VLLL by Corollary 1.5, and here we will completely solve the gapful/gapless problem. Moreover, we find that if a G_B of this kind is gapful, then it is gapful in all directions, here, we should mention that this doesn't hold for general G_B.

Example. Let $G_B = ([4], [3], E_B)$ with $E_B = \{(1,1), (1,2), (2,2), (2,3), (3,3), (3,1), (4,1), (4,2), (4,3)\}$. By Corollary 1.12, we have G_B is gapful because it contains a cyclic interaction bipartite graph. However, G_B is gapless in the direction of $\mathbf{r} = (1/4, 1/4, 1/4, 1/4)$. To see this, firstly, we decompose each $\mathcal{H}_i$ to two orthogonal subspace $\mathcal{H}_i = \mathcal{H}_i^a \oplus \mathcal{H}_i^b$ with $\dim(\mathcal{H}_i^a) = \dim(\mathcal{H}_i^b)$, then let $V_1^{loc} = \mathcal{H}_1^a \otimes \mathcal{H}_2^b$, $V_2^{loc} = \mathcal{H}_2^a \otimes \mathcal{H}_3^b$, $V_3^{loc} = \mathcal{H}_3^a \otimes \mathcal{H}_1^b$, and V_4 be the orthogonal complementary space of $V_1 \oplus V_2 \oplus V_3$, easy to see they are orthogonal and span the whole space, thus by Corollary 1.7, G_B is gapless in this direction.

Theorem 1.14 (restated). *Let $G_B = ([m], [n], E_B)$ be an interaction bipartite graph where any two Hamiltonians share at most one qudit, then*

(a) *G_B is gapless for CLLL (or VLLL) if and only if G_B is a tree.*

(b) *If G_B is not a tree, G_B is gapful for CLLL (or VLLL) in all directions.*

Therefore, on many regular lattices, the critical threshold of commuting Hamiltonians is strictly larger than the general non-commutative one.

Proof. *Part(a).* If G_B is not a tree, it is easy to see that G_B contains a cyclic interaction bipartite graph G'_B. Hence, Theorem 1.8 and Theorem 1.12 implies this part directly.

Part(b). W.l.o.g., assume $G'_B = ([m'], [n'], E'_B)$, where $m' = n'$ and $E_{m'} = \{(i, i), (i, (i+1)(\overline{\text{mod}}\, n')) : i \in [m']\}$. Here the value $k(\overline{\text{mod}}\, n')$ is defined to be $(k-1)(\text{mod}\, n') + 1$. To simplify notation, the operator "$(\overline{\text{mod}}\, n')$" will be omitted whenever clear from context.

By contradiction, assume G_B is gapless in some direction $\mathbf{r}$, and let $\lambda\mathbf{r}$ be the vector on $\mathcal{C}\partial(G_B)$, so by Theorem 1.6, for a sufficiently small $\epsilon > 0$, there is an exclusive subspace set $\mathcal{V} \sim G_B$ with $R(\mathcal{V}) = (1-\epsilon)\lambda\mathbf{r}$. Moreover, according to Corollary 4.3, each $\mathcal{H}_i$, where $i \in [n']$, can be decomposed into three orthogonal subspaces $\mathcal{H}_i = \mathcal{H}_i^0 \oplus \mathcal{H}_i^1 \oplus \mathcal{H}_i^c$, such that for any $i \in [m']$, $V_i \subset \mathcal{H}_i^0 \otimes \mathcal{H}_{i+1}^1 \otimes \mathcal{H}_{[n]\setminus\{i,i+1\}}$, and for any other $i \notin [m']$ which depends on some qudit in $[n']$, $V_i \subset \mathcal{H}_{[n]\setminus[n']} \otimes \bigotimes_{j\in[n']} \mathcal{H}_j^c$. For simplicity, let $\mathcal{H}^0 := \bigotimes_{i\in[n']} \mathcal{H}_i^0$, $\mathcal{H}^1 := \bigotimes_{i\in[n']} \mathcal{H}_i^1$.

W.l.o.g., assume $R(\mathcal{H}^0) \geq R(\mathcal{H}^1)$. Note that

$$R(\mathcal{H}^0) \cdot R(\mathcal{H}^1) = \prod_{i\in[m']} R(\mathcal{H}_i^0) \cdot \prod_{i\in[m']} R(\mathcal{H}_i^1) = \prod_{i\in[m']} R(\mathcal{H}_i^0 \otimes \mathcal{H}_{i+1}^1) \geq \prod_{i\in[m']} r_i,$$

thus $R(\mathcal{H}^0) \geq \sqrt{\prod_{i\in[m']} r_i}$. Moreover, let $\mathcal{D} := \{j \in [m'+1, m] : \mathcal{N}(j) \cap [n'] = \emptyset\}$ be the collection of V_i's which doesn't depend on any qudit in $[n']$, it is easy to see that

$$R(\mathcal{H}^0 \cap \bigoplus_{V\in\mathcal{V}} V) = R(\mathcal{H}^0 \cap \bigoplus_{i\in\mathcal{D}} V_i) = R(\mathcal{H}^0) \cdot R(\bigoplus_{i\in\mathcal{D}} V_i) = R(\mathcal{H}^0) \cdot I(G_D[\mathcal{D}]),$$

which means

$$1 - R(\bigoplus_{V\in\mathcal{V}} V) \geq R(\mathcal{H}^0) - R(\mathcal{H}^0 \cap \bigoplus_{V\in\mathcal{V}} V) \geq \sqrt{\prod_{i\in[m']} r_i} \cdot (1 - I(G_D[\mathcal{D}])) = \theta,$$

where $\theta > 0$ is independent with ϵ.

On the other hand, for a sufficiently small $\delta > 0$ satisfying $(\epsilon + \delta)\lambda\|\mathbf{r}\|_1 \leq \theta$, there is a subspace set $\mathcal{V}' \sim G_B$ with $R(\mathcal{V}) = (1+\delta)\lambda\mathbf{r}$ spanning the whole space. However, by Lemma 4.7 and union bound, we have $R(\bigoplus_{V\in\mathcal{V}'} V) \leq R(\bigoplus_{V\in\mathcal{V}} V) + (\epsilon+\delta)\lambda\|\mathbf{r}\|_1 < 1$, a contradiction. □

6 Hardness Results

It is immediate that testing membership in quantum interior is #P-hard because testing membership in Shearer's region is #P-hard [14]. We show that testing membership in commuting interior is also #P-hard.

Definition 6.1 (CINT Problem)**.** Given an interaction bipartite graph G_B and a rational vector $\mathbf{r}$ on $(0, 1)$, decide whether $\mathbf{r} \in \mathcal{CI}(G_B)$.

Theorem 6.1. *CINT is #P-hard.*

Proof. Given a $(3, 2)$-regular bipartite graph $G_B = ([m], [n], E_B)$, let $G'_B := ([m + 1], [n], E'_B)$ where $E'_B = E_B \cup \{(m+1, 1), (m+1, 2), \ldots, (m+1, n)\}$, and $\mathbf{r}^{(\lambda)} = (\frac{1}{8}, \frac{1}{8}, \ldots, \frac{1}{8}, \lambda)$ with rational $\lambda \in [0, 1]$. To decide whether $\mathbf{r}^{(\lambda)} \in \mathcal{VI}(H')$ has shown to be #P-hard [16]. Moreover, for such G'_B, we have $\mathcal{I}(G'_B) = \mathcal{VI}(G'_B)$ [16]. Since $\mathcal{CI}(G'_B)$ lies between $\mathcal{I}(G'_B)$ and $\mathcal{VI}(G'_B)$, thus $\mathcal{I}(G'_B) = \mathcal{CI}(G'_B) = \mathcal{VI}(G'_B)$, which implies CINT is also #P-hard.

□

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A Proof of Lemma 4.9

Let $\mathcal{A} = \{A_1, ..., A_m\}$ be a set of events conforms with G_D, and $\mathbb{P}(\mathcal{A}) = \mathbf{p} = (p_1, ..., p_m)$. From $\mathbf{p} \in \mathcal{I}(G_D) \cup \partial(G_D)$ and Lemma 4.7, there exists an exclusive event set $\mathcal{B} \sim G_D$ where $\mathbb{P}(\mathcal{B}) = \mathbf{p}$. We assume $\mathbb{P}(A_i \cap A_j) > 0$, since the case $\mathbb{P}(A_i \cap A_j) = 0$ holds trivially. Furthur, we can assume $\mathbf{p} \in \partial(G_D)$, since otherwise $\mathbf{p} \in \mathcal{I}(G_D)$, then due to Theorem 1.2, we have $\mathbb{P}\big(\cap_{A \in \mathcal{A}} \overline{A}\big) \geq I(G, \mathbf{p}) = \mathbb{F}(G, \mathbf{p}, m-2) \geq \mathbb{P}(A_i \cap A_j)\mathbb{F}(G, \mathbf{p}, m-2)$.

Let's borrow the notation from the proof of [31, Theorem 1]. For any $S \subseteq [n]$, define $\alpha(S) = \mathbb{P}(\cap_{i \in S} \overline{A_i})$ and $\beta(S) = \mathbb{P}(\cap_{i \in S} \overline{B_i})$. We first review some useful properties of $\alpha(S)$ and $\beta(S)$. Note that $\alpha(S)/\beta(S)$ monotonically increases as $|S|$ increases provided $\beta(S) \neq 0$. This can be proved by induction on $|S|$. The base cases holds since $\alpha(\emptyset) = \beta(\emptyset)$ and $\alpha(S) = \beta(S)$ for any singleton S. For induction, given $S_1 \subset [m]$ and $j \in [m] \setminus S_1$, let $S_2 = S_1 \cup \{j\}$, $T_2 = S_1 \cap \Gamma_j$, and $T_1 = S_1 \setminus T_2$. We have

$$\tfrac{\alpha(S_2)}{\beta(S_2)} - \tfrac{\alpha(S_1)}{\beta(S_1)} \geq \tfrac{\alpha(S_1) - p_j \alpha(T_1)}{\beta(S_1) - p_j \beta(T_1)} - \tfrac{\alpha(S_1)}{\beta(S_1)} = \tfrac{p_j \beta(T_1)}{\beta(S_1) - p_j \beta(T_1)} \left[\tfrac{\alpha(S_1)}{\beta(S_1)} - \tfrac{\alpha(T_1)}{\beta(T_1)} \right] \geq 0. \tag{9}$$

The last inequality is by induction. The first inequality holds because

$$\begin{aligned} \alpha(S_2) = \ & \mathbb{P}(\cap_{i \in S_2} \overline{A_i}) = \mathbb{P}(\cap_{i \in S_1} \overline{A_i}) - \mathbb{P}(\cap_{i \in S_1} \overline{A_i} \cap A_j) \\ = \ & \mathbb{P}(\cap_{i \in S_1} \overline{A_i}) - \mathbb{P}(\cap_{i \in T_1} \overline{A_i} \cap A_j) + \mathbb{P}(\cap_{i \in T_1} \overline{A_i} \cap A_j \cap (\cup_{i \in T_2} A_i)) \\ \geq \ & \mathbb{P}(\cap_{i \in S_1} \overline{A_i}) - \mathbb{P}(\cap_{i \in T_1} \overline{A_i} \cap A_j) = \alpha(S_1) - p_j \alpha(T_1). \end{aligned} \tag{10}$$

and by a similarly formula we also have $\beta(S_2) = \beta(S_1) - p_j \beta(T_1)$ because $\mathcal{B}$ is exclusive and then $\mathbb{P}(\cap_{i \in T_1} \overline{B_i} \cap B_j \cap (\cup_{i \in T_2} B_i)) = 0$. Hence, $\alpha(S)/\beta(S)$ is increasing.

Now we return to the proof of the lemma. Let $S_2 = [m]$, $S_1 = S_2 \setminus \{j\}$, $T_1 = S_1 \setminus \Gamma_j$, $T_2 = S_1 \setminus T_1 = \Gamma_j$. Since $\beta(S_1) > 0$ and $\beta(S_2) = 0$, $\alpha(S_2) = \alpha(S_2) - \frac{\alpha(S_1)}{\beta(S_1)}\beta(S_2)$. Moreover, since G_D is connected, $|T_1| \le m-2$, we have $\mathbb{P}(A_j \cap (\cup_{i\in T_2} A_i))\mathbb{F}(G_D, \mathbf{p}, |T_1|) \ge \mathbb{P}(A_i \cap A_j)\mathbb{F}(G_D, \mathbf{p}, m-2)$. Thererore, to prove $\mathbb{P}\big(\cap_{A\in\mathcal{A}}\overline{A}\big) \ge \mathbb{P}(A_i \cap A_j)\mathbb{F}(G_D, \mathbf{p}, m-2)$, it suffices to show $\alpha(S_2) - \frac{\alpha(S_1)}{\beta(S_1)}\beta(S_2) \ge \mathbb{P}(A_j \cap (\cup_{i\in T_2} A_i))\mathbb{F}(G_D, \mathbf{p}, |T_1|)$.

Claim. For any $S_2 \subseteq [m]$ and $j \in S_2$, let $S_1 = S_2 \setminus \{j\}, T_1 = S_1 \setminus \Gamma_j, T_2 = S_1 \setminus T_1$. Then $\alpha(S_2) - \frac{\alpha(S_1)}{\beta(S_1)}\beta(S_2) \ge \mathbb{P}(A_j \cap (\cup_{i\in T_2} A_i))\mathbb{F}(G_D, \mathbf{p}, |T_1|)$.

Proof. The following form of $\alpha(S_2) - \frac{\alpha(S_1)}{\beta(S_1)}\beta(S_2)$ will be used:

$$\begin{aligned}&\alpha(S_2) - \tfrac{\alpha(S_1)}{\beta(S_1)}\beta(S_2) = \alpha(S_2) - \tfrac{\alpha(S_1)}{\beta(S_1)}(\beta(S_1) - p_j\beta(T_1))\\ &= \alpha(S_2) - \alpha(S_1) + p_j\alpha(T_1) + p_j\beta(T_1)\left(\tfrac{\alpha(S_1)}{\beta(S_1)} - \tfrac{\alpha(T_1)}{\beta(T_1)}\right).\end{aligned} \tag{11}$$

The proof of this claim is by induction on $|T_1|$.

Basis: $T_1 = \emptyset$. We have

$$\begin{aligned}\alpha(S_2) - \tfrac{\alpha(S_1)}{\beta(S_1)}\beta(S_2) \quad &\ge \alpha(S_2) - \alpha(S_1) + p_j\alpha(T_1)\\ &= \mathbb{P}(\cap_{i\in T_1}\overline{A_i} \cap A_j \cap (\cup_{i\in T_2} A_i)) = \mathbb{P}(A_j \cap (\cup_{i\in T_2} A_i)).\end{aligned}$$

The first inequality is due to formula (11) and the monotonicity of $\alpha(S)/\beta(S)$, the first equality is due to formula (10), and the second equality is due to $T_1 = \emptyset$. Note that $\mathbb{F}(G_D, \mathbf{p}, 0) = 1$, the claim holds for this case.

Hypothesis: The claim holds if $|T_1| < t$.

Induction: Suppose $|T_1| = t$. Since the case $T_2 = \emptyset$ is trivial, we assume $T_2 \ne \emptyset$. By the union bound there is $j' \in T_2$ such that $\mathbb{P}(A_j \cap A_{j'}) \ge \mathbb{P}(A_j \cap (\cup_{i\in T_2} A_i))/|T_2| \ge \mathbb{P}(A_j \cap (\cup_{i\in T_2} A_i))/(m-1-|T_1|)$, the last inequality is becasue $|T_2| \le m-1-|T_1|$.

If $T_1 \cap \Gamma_{j'} = \emptyset$, then

$$\begin{aligned}\alpha(S_2) - \tfrac{\alpha(S_1)}{\beta(S_1)}\beta(S_2) \quad &\ge \alpha(S_2) - \alpha(S_1) + p_j\alpha(T_1) && \text{by (11)}\\ &= \mathbb{P}(\cap_{i\in T_1}\overline{A_i} \cap A_j \cap (\cup_{i\in T_2} A_i)) && \text{by (10)}\\ &\ge \mathbb{P}(\cap_{i\in T_1}\overline{A_i} \cap A_j \cap A_{j'})\\ &= \mathbb{P}(\cap_{i\in T_1}\overline{A_i})\mathbb{P}(A_j \cap A_{j'}) && \text{by } T_1 \cap \Gamma_{j'} = \emptyset\\ &\ge I(G_D, \mathbf{p}, |T_1|)\tfrac{\mathbb{P}(A_j \cap (\cup_{i\in T_2} A_i))}{n-1-|T_1|} && \text{by def. of } I(G_D, \mathbf{p}, t)\\ &\ge \mathbb{F}(G_D, \mathbf{p}, |T_1|)\mathbb{P}(A_j \cap (\cup_{i\in T_2} A_i)). && \text{by def. of } \mathbb{F}(G_D, \mathbf{p}, t)\end{aligned} \tag{12}$$

Otherwise, $T_1 \cap \Gamma_{j'} \ne \emptyset$, then

$$\begin{aligned}&\mathbb{P}(A_{j'} \cap (\cup_{i\in T_1\cap\Gamma_{j'}} A_i)) + \mathbb{P}(\cap_{i\in T_1}\overline{A_i} \cap A_j \cap (\cup_{i\in T_2} A_i))\\ \ge\ &\mathbb{P}(A_{j'} \cap (\cup_{i\in T_1\cap\Gamma_{j'}} A_i)) + \mathbb{P}(\cap_{i\in T_1}\overline{A_i} \cap A_j \cap A_{j'})\\ \ge\ &\mathbb{P}(A_{j'} \cap (\cup_{i\in T_1\cap\Gamma_{j'}} A_i) \cap A_j \cap_{k\in T_1\setminus\Gamma j'} \overline{A_k}) + \mathbb{P}(\cap_{i\in T_1}\overline{A_i} \cap A_j \cap A_{j'})\\ =\ &\mathbb{P}(A_j \cap A_{j'} \cap_{k\in T_1\setminus\Gamma_{j'}} \overline{A_k})\\ =\ &\mathbb{P}(A_j \cap A_{j'})\mathbb{P}(\cap_{k\in T_1\setminus\Gamma_{j'}}\overline{A_k}),\end{aligned} \tag{13}$$

we have

$$
\begin{aligned}
&\mathbb{P}(A_{j'} \cap (\cup_{i \in T_1 \cap \Gamma_{j'}} A_i)) \\
\geq\ &\mathbb{P}(A_j \cap A_{j'})\mathbb{P}(\cap_{k \in T_1 \setminus \Gamma_{j'}} \overline{A_k}) - \mathbb{P}(\cap_{i \in T_1} \overline{A_i} \cap A_j \cap (\cup_{i \in T_2} A_i)) \\
=\ &\mathbb{P}(A_j \cap A_{j'})\mathbb{P}(\cap_{k \in T_1 \setminus \Gamma_{j'}} \overline{A_k}) - (\alpha(S_2) - \alpha(S_1) + p_j \alpha(T_1)).
\end{aligned} \tag{14}
$$

The last equality is due to formula (10). Let $S_2' \triangleq T_1 \cup \{j'\}, S_1' \triangleq T_1, T_1' \triangleq S_1' \setminus \Gamma_{j'}, T_2' \triangleq S_1' \setminus T_1' = T_1 \cap \Gamma_{j'}$. Thus,

$$
\begin{aligned}
\frac{\alpha(S_1)}{\beta(S_1)} - \frac{\alpha(T_1)}{\beta(T_1)} \quad &\geq \frac{\alpha(S_2')}{\beta(S_2')} - \frac{\alpha(S_1')}{\beta(S_1')} = \frac{1}{\beta(S_2')}\left(\alpha(S_2') - \frac{\beta(S_2')\alpha(S_1')}{\beta(S_1')}\right) \\
&\geq \frac{1}{\beta(S_2')}\mathbb{P}(A_{j'} \cap (\cup_{i \in T_2'} A_i))\mathbb{F}(G_D, \mathbf{p}, |T_1'|),
\end{aligned} \tag{15}
$$

the first inequality is since $S_1' = T_1$, $S_2' \subseteq S_1$ and the monotonicity of $\frac{\alpha(S)}{\beta(S)}$. The last inequality is by applying the induction hypothesis to T_1'. Therefore, we have

$$
\begin{aligned}
&\alpha(S_2) - \frac{\alpha(S_1)}{\beta(S_1)}\beta(S_2) \\
=\ &\alpha(S_2) - \alpha(S_1) + p_j\alpha(T_1) + p_j\beta(T_1)\left(\frac{\alpha(S_1)}{\beta(S_1)} - \frac{\alpha(T_1)}{\beta(T_1)}\right) && \text{by (11)} \\
\geq\ &\alpha(S_2) - \alpha(S_1) + p_j\alpha(T_1) + p_j\beta(S_2')\left(\frac{\alpha(S_1)}{\beta(S_1)} - \frac{\alpha(T_1)}{\beta(T_1)}\right) && \text{by } T_1 \subset S_2' \\
\geq\ &\alpha(S_2) - \alpha(S_1) + p_j\alpha(T_1) + p_j\mathbb{P}(A_{j'} \cap (\cup_{i \in T_2'} A_i))\mathbb{F}(G_D, \mathbf{p}, |T_1'|) && \text{by (15)} \\
\geq\ &p_j(\mathbb{P}(A_j \cap A_{j'})\mathbb{P}(\cap_{k \in T_1 \setminus \Gamma_{j'}} \overline{A_k}))\mathbb{F}(G_D, \mathbf{p}, |T_1'|) && \text{by (14)} \\
\geq\ &p_{min} I(G, \mathbf{p}, |T_1|)\mathbb{F}(G_D, \mathbf{p}, |T_1'|)\frac{\mathbb{P}(A_j \cap (\cup_{i \in T_2} A_i))}{(m-1-|T_1|)} \\
\geq\ &\mathbb{P}(A_j \cap (\cup_{i \in T_2} A_i))\mathbb{F}(G_D, \mathbf{p}, |T_1|).
\end{aligned}
$$

□

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