



Beating Fredman-Komlós for perfect k -hashing*

Venkatesan Guruswami[†] Andrii Riazanov

Computer Science Department
Carnegie Mellon University
Pittsburgh, PA 15213.

{venkatg,riazanov}@cs.cmu.edu

Abstract

We say a subset $C \subseteq \{1, 2, \dots, k\}^n$ is a k -hash code (also called k -separated) if for every subset of k codewords from C , there exists a coordinate where all these codewords have distinct values. Understanding the largest possible rate (in bits), defined as $(\log_2 |C|)/n$, of a k -hash code is a classical problem. It arises in two equivalent contexts: (i) the smallest size possible for a perfect hash family that maps a universe of N elements into $\{1, 2, \dots, k\}$, and (ii) the zero-error capacity for decoding with lists of size less than k for a certain combinatorial channel.

A general upper bound of $k!/k^{k-1}$ on the rate of a k -hash code (in the limit of large n) was obtained by Fredman and Komlós in 1984 for any $k \geq 4$. While better bounds have been obtained for $k = 4$, their original bound has remained the best known for each $k \geq 5$. In this work, we obtain the first improvement to the Fredman-Komlós bound for every $k \geq 5$. While we get explicit (numerical) bounds for $k = 5, 6$, for larger k we only show that the FK bound can be improved by a positive, but unspecified, amount. Under a conjecture on the optimum value of a certain polynomial optimization problem over the simplex, our methods allow an effective bound to be computed for every k .

*Research supported in part by NSF CCF-1422045 and CCF-1563742.

[†]Some of this work was done when the author was visiting the School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore and the Center of Mathematical Sciences and Applications, Harvard University.

1 Introduction

A code of length n over an alphabet of size k is a subset $C \subseteq \{1, 2, \dots, k\}^n$. We say such a code C is a k -hash code (also called k -separated in the literature), if for every subset of k distinct codewords $\{c^{(1)}, c^{(2)}, \dots, c^{(k)}\}$ from C , there exists a coordinate j such that all these codewords differ in this coordinate, i.e. $\{c_j^{(1)}, c_j^{(2)}, \dots, c_j^{(k)}\} = \{1, 2, \dots, k\}$. The rate (in bits) of the code is defined as $R = \frac{\log_2 |C|}{n}$. Then for each fixed integer k , let R_k be the supremum, as $n \rightarrow \infty$, of the rate of the largest k -hash code of length n .

The study of the quantity R_k is a fundamental problem in combinatorics, information theory, and computer science. As the name suggests, k -hash codes have strong connections to the hashing problem. A family of functions mapping a universe of size N to the set $\{1, 2, \dots, k\}$ is called a perfect k -hash family if any k elements of the universe are mapped in one-to-one fashion by at least one hash function from this family. If C is a k -hash code, then a perfect k -hash family for universe C with n functions is just the family of coordinate projections. Therefore, R_k gives the growth rate of the size of universes for which perfect k -hash families of a given size exist. Equivalently, an upper bound on R_k is equivalent to a lower bound on the size of a perfect k -hash family as a function of the universe size.

An equivalent information-theoretic context in which k -hash codes arise concerns zero-error list decoding on certain channels. A channel can be thought of as a bipartite graph (V, W, E) , where V is the set of channel inputs, W is the set of channel outputs, and $(v, w) \in E$ if on input v the channel can output w . The $k/(k-1)$ channel then is the channel with $V = W = \{1, 2, \dots, k\}$, and $(v, w) \in E$ iff $v \neq w$. In this context, R_k is the largest asymptotic rate at which one can communicate using n repeated uses of the channel (as n grows), if we want to ensure that the receiver can identify a subset of at most $k-1$ sequences that is guaranteed to contain the transmitted sequence. See [1, 2] for more details.

Studying the rates of the codes and hashing family sizes in the above settings is a longstanding problem. A probabilistic argument shows the existence of k -hash codes with rate at least $\frac{1}{k-1} \log \frac{1}{1-k!/k^k} - o(1)$ [3, 4], and better bounds are known for some small values of k . Our focus here is on *upper bounds* on R_k , that is limitations on the size of k -hash codes. Here the best-known general upper bound on the rate R_k dates all the way back to the 1984 paper of Fredman and Komlós [3]:

$$R_k \leq \frac{k!}{k^{k-1}} =: \alpha_k . \tag{1}$$

For large k the multiplicative discrepancy between the probabilistic lower bound on R_k and the above Fredman-Komlós upper bound (1) grows approximately as k^2 , so the current bounds on the rate require tightening to obtain better estimations of R_k . There is another trivial upper bound, $R_k \leq \log_2 \left(\frac{k}{k-1} \right)$, that follows from a simple double-counting or first moment method. The above bound (1) is much better than this bound for $k \geq 4$. For $k = 3$ (which is called the trifference problem by Körner), however, $R_3 \leq \log_2(3/2) \approx 0.585$ remains the best upper bound, and improving it (or showing it can be achieved!) is a major combinatorial challenge. For the case $k = 4$, the bound (1) which states $R_4 \leq 0.375$ has been improved, first by Arikan to 0.3512 [5], and recently by Dalai, Guruswami, and Radhakrishnan [2] to $6/19 \leq 0.3158$.

However, the above quantity α_k remained the best known upper bound on R_k for each $k > 4$. Our main result gives the first improvement to the Fredman-Komlós bound (1) for $k \geq 5$, proving that R_k is strictly smaller than α_k for every k .

Theorem 1. *For all $k \geq 4$ there exists β_k such that $R_k \leq \beta_k < \alpha_k$. For $k = 5, 6$, we have the explicit upper bounds $R_5 < \beta_5 = 0.190825 < 0.192 = \alpha_5$, and $R_6 < \beta_6 = 0.0922787 < 0.092\overline{5} = \frac{5}{54} = \alpha_6$.*

Our proof relies on the continuity arguments, which doesn't yield an effective way to find explicit values of β_k . However, we present a technical approach to compute our upper bound β_k , which relies on polynomial optimization over the simplex. Using the tools of numerical optimization in *Mathematica*, we calculate the values of β_5 and β_6 . We also make a conjecture on the optimum value of a certain polynomial optimization problem over the simplex, assuming which our methods allow an effective upper bound that is strictly smaller than α_k to be computed for every k .

Our approach is also applicable to the (b, k) -hashing problem for $b \geq k$, where one considers codes $C \subseteq \{1, 2, \dots, b\}^n$ with the property that for any k distinct codewords $\{c^{(1)}, c^{(2)}, \dots, c^{(k)}\}$ from C there exists a coordinate j such that all these codewords differ in this coordinate. Using exactly the same arguments, we obtain an improvement on the Körner-Marton upper bound [6] on the rate of such codes. When $b = k$, this latter bound is identical to the Fredman-Komlós bound, but can be better than the corresponding bound in [3] when $b > k$. For some small pairs of values (b, k) , with $b > k$, the Körner-Marton bound was further improved by Arikan [5]. In those cases, the bounds we get are probably weaker than Arikan's. For this reason and for the sake of simplicity, in this paper we analyze only the case $b = k$, which corresponds to k -hashing. But all our proofs generalize in a straightforward way for (b, k) -hashing as well. We briefly describe the background on the (b, k) -hashing problem in Appendix A.

2 Background and approach

The previous general upper bounds on the rates of k -hash codes by Fredman and Komlós [3], Körner and Marton [6], and Arikan [5] are all based on information-theoretic inequalities for graph covering, related to the Hansel lemma [7]. Körner [4] cast the Fredman-Komlós proof in the language of graph entropy, which he had introduced in [8] (see [9] for a nice survey on graph entropy). Körner and Marton [6] generalized this approach to the hypergraph case, which led to improvements to the Fredman-Komlós bound for the (b, k) -hashing problem in certain cases when $b > k$, but not for R_k . In this paper we use the following version of the Hansel lemma, which is also proved in [10] via a simple probabilistic argument:

Lemma 1 (Hansel). *Let K_m be a complete graph on m vertices. Let also G_1, G_2, \dots, G_t be bipartite graphs, such that $E(K_m) = \bigcup_{i=1}^t E(G_i)$. Denote by $\tau(G_i)$ the fraction of non-isolated vertices in G_i . Then the following holds:*

$$\log m \leq \sum_{i=1}^t \tau(G_i). \quad (2)$$

To relate this lemma to the context of the paper, consider a k -hash code $C \subseteq [k]^n$. Take a subset of this code $\{x_1, x_2, \dots, x_{k-2}\} \subseteq C$, and define bipartite graphs $G_i^{x_1, \dots, x_{k-2}}$, for $i \in [n]$, as follows:

$$V(G_i^{x_1, \dots, x_{k-2}}) = C \setminus \{x_1, x_2, \dots, x_{k-2}\},$$

$$E(G_i^{x_1, \dots, x_{k-2}}) = \left\{ \{y_1, y_2\} : (y_1)_i, (y_2)_i, (x_1)_i, (x_2)_i, \dots, (x_{k-2})_i \text{ are distinct} \right\}.$$

Note that since C is a k -hash code, for any pair $\{y_1, y_2\} \subseteq C \setminus \{x_1, x_2, \dots, x_{k-2}\}$, there exists some coordinate i , such that all the k codewords $y_1, y_2, x_1, x_2, \dots, x_{k-2}$ differ in the i^{th} coordinate. In other words, $\{y_1, y_2\} \in E(G_i^{x_1, \dots, x_{k-2}})$ for this i . Therefore, it holds $E(K_{|C|-(k-2)}) = \bigcup_{i=1}^n E(G_i^{x_1, \dots, x_{k-2}})$. Then Hansel lemma 1 applies directly, and denoting $\tau_i(x_1, x_2, \dots, x_{k-2}) = \tau(G_i^{x_1, \dots, x_{k-2}})$, we obtain

$$\log(|C| - k + 2) \leq \sum_{i=1}^n \tau_i(x_1, x_2, \dots, x_{k-2}). \quad (3)$$

Taking the expectation over the choice of x_1, x_2, \dots, x_{k-2} , we get

$$\log(|C| - k + 2) \leq \sum_{i=1}^n \mathbb{E}[\tau_i(x_1, x_2, \dots, x_{k-2})]. \quad (4)$$

By bounding the RHS of the above inequality one might obtain an upper bound on $\log|C|$, and thus on the rate of this code. Different strategies to pick the codewords $\{x_1, x_2, \dots, x_{k-2}\}$ from C lead to different approaches to bound the RHS of (4). Here we briefly present the ideas underlying the previous works and then outline our approach.

In the original bound by Fredman and Komlós [3], the codewords x_1, x_2, \dots, x_{k-2} are picked independently at random from the code C . Then one can use symmetry arguments (or Muirhead's inequality) to give an upper bound on the RHS of (4), which will lead to the bound on the rate

$$R_k \leq \frac{k!}{k^{k-1}}. \quad (5)$$

Due to the symmetry arguments involved, this bound is actually tight only in the case when the frequencies of the symbols of the code C in each coordinate are uniform.

Arikan [5, 11] used rate versus distance results from the coding theory to ensure that it is possible to pick x_1, x_2, \dots, x_{k-2} which agree on many coordinates. Note that this already guarantees that many terms in the RHS of (3) equal 0. Together with an argument which allows to modify the code so that it doesn't have any coordinate where the symbols have an overly skewed frequency, Arikan was able to improve the bound (5) for $k = 4$. However, no improvement was gained for larger k .

Dalai, Guruswami, and Radhakrishnan [2] combine aspects of the above two approaches for the case $k = 4$. As in Arikan's work, they pick x_1, x_2 to agree on the first several coordinates. However, instead of a fixed such choice, they pick such a pair at random (from a natural distribution). The technical crux of their argument is a concavity claim for some quadratic form which says that despite conditioning on a common prefix, which might greatly alter the frequency vector of symbols in any coordinate in the suffix, the Fredman-Komlós bound for completely random x_1, x_2 is still valid on those coordinates. (In some sense, only the average frequency vector over all prefixes matters, not the individual ones.) Actually this holds modulo a technical condition that there are no coordinates with very skewed symbol distribution, which can be ensured by some pre-processing of the code similar to [11]. Thus some terms in (4) are equal to 0 and the other are bounded by $3/8$, and balancing these appropriately, a bound of $R_4 \leq \frac{6}{19}$ is obtained in [2].

In this work, we follow the strategy of [2] for general k by picking x_1, x_2, \dots, x_{k-2} randomly so that they all lie in a subcode of C that takes at most $(k - 3)$ values on each coordinate from the

the large set T . This again implies that the coordinates from T contribute 0 to the RHS of (4). In this case, however, the analogous concavity claim seems out of reach, as one has to argue about degree $(k-2)$ polynomials rather than quadratics. We instead take a different approach that works directly with the arbitrary symbol frequencies that may arise upon conditioning within a subcode, avoiding the averaging or concavity step. (This leads to worse bounds, but our main goal is to beat the Fredman-Komlós bound for $k > 4$ by some positive amount at all, doesn't matter how small.) However, another problem arises in that the constraint on the code to have non-skewed frequencies in each coordinate cannot be dealt with using Arikan's argument for large k . To cope with this issue, we differentiate two separate cases: (i) where C has only a few coordinates with skewed distributions of symbols, and (ii) where there are a lot of such coordinates.

- In the first case, we pick the coordinates T (where x_1, x_2, \dots, x_{k-2} are chosen to collide) to include all these skewed coordinates. Note that this is unlike [5, 2] where any choice of T of prescribed size works. Our choice of T ensures that in the remaining coordinates the frequency vector is not too far from uniform, and we apply the approach of [2] directly. We reduce the task of showing that this improves upon the Fredman-Komlós bound to the continuity of certain functions, which we argue via Berge's maximum principle ([12]).
- In the second case, we use the original random strategy of picking x_1, x_2, \dots, x_{k-2} as in [3]. The idea here is that the bound (5) is tight only when all the frequencies of symbols are exactly uniform. Then, in the case when there are a lot of far-from-uniform frequencies, it is possible to improve the bound (5).

By picking the correct way to differentiate between skewed and non-skewed distributions, we then obtain an improvement on the Fredman-Komlós bound (5) for every $k \geq 4$. As mentioned earlier, this is the *first* such improvement for $k \geq 5$. For $k = 5$ and $k = 6$ we use numerical optimization tools to provide new explicit bounds on R_k , presented in section 4.2.

3 Upper bound on the rate of k -hash codes

Let $\Sigma = \{1, 2, \dots, k\} = [k]$, and let $C \subseteq \Sigma^n$ be a k -hash code with rate $R = \frac{\log |C|}{n}$ (all logarithms are to the base 2). Let $f_i \in \mathbb{R}^k$ be the frequency vector of symbols of the code for each coordinate $i \in [n]$, namely:

$$f_i[a] = \frac{1}{|C|} |\{x \in C : x_i = a\}|.$$

Throughout the analysis, we will be interested in two cases: when for most of the coordinates the distribution of codeword symbols is close to uniform (non-skewed), or when this doesn't hold. To define the term "close to uniform" formally, we consider a *threshold* γ , that satisfies $\frac{1}{2k-3} \leq \gamma \leq \frac{1}{k}$, and say that $f \in \mathbb{R}^k$ is close to uniform when for all components of f it holds $f[a] \geq \gamma$. Denote then $P_\gamma = \{i \in [n] : \min_{a \in \Sigma} f_i[a] \geq \gamma\}$ – the set of all the coordinates for which the distribution of

codeword symbols is close to uniform. Denote also $\ell = \left\lfloor \frac{nR - \log n}{\log \left(\frac{k}{k-3}\right)} \right\rfloor$. We then consider two cases:

1. *Unbalanced*: $|P_\gamma| < n - \ell$, so there is a decent fraction of coordinates where the distribution of codeword symbols is skewed. For this case, we apply the random strategy to pick x_1, x_2, \dots, x_{k-2} in (4).

2. *Almost balanced:* $|P_\gamma| \geq n - \ell$, so for almost all coordinates, the distribution of codeword symbols is close to uniform. Then we follow the approach from [2] to pick x_1, x_2, \dots, x_{k-2} which collide on many coordinates.

For both of these cases, we will obtain some bounds on the rate of C , which depend on the threshold γ . It then will remain to choose γ in a manner ensuring that both these bounds beat (5). Then, since for any code C exactly one of the cases holds, we can obtain a general upper bound on the rate.

Before we continue with studying the two cases separately, let's look at how we can estimate $\tau_i(x_1, x_2, \dots, x_{k-2})$. Clearly, the codeword $y \in C$ appears non-isolated in the graph $G_i^{x_1, x_2, \dots, x_{k-2}}$ if and only if all the codewords x_1, x_2, \dots, x_{k-2} and y differ in the i^{th} coordinate. Therefore, the fraction of non-isolated vertices in $G_i^{x_1, x_2, \dots, x_{k-2}}$ is exactly

$$\begin{aligned} & \tau_i(x_1, \dots, x_{k-2}) \\ &= \left(\frac{|C|}{|C| - (k-2)} \right) \left(1 - f_i[x_{1i}] - f_i[x_{2i}] - \dots - f_i[x_{(k-2)i}] \right) \mathbf{1}[x_{1i}, x_{2i}, \dots, x_{(k-2)i} \text{ distinct}], \end{aligned} \quad (6)$$

where $\mathbf{1}(E)$ is the indicator variable for an event/condition E .

3.1 Unbalanced case

We will pick x_1, x_2, \dots, x_{k-2} uniformly at random without replacement from C to obtain an upper bound on the rate of C from (4). Taking the expectations of the both sides in (6), we obtain

$$\begin{aligned} & \mathbb{E}[\tau_i(x_1, \dots, x_{k-2})] \\ &= \frac{|C|}{|C| - k + 2} \sum_{\substack{a_1, \dots, a_{k-2} \in \Sigma \\ \{a_s\} \text{ distinct}}} \left(1 - \sum_{s=1}^{k-2} f_i[a_s] \right) \cdot \mathbb{P}[(x_s)_i = a_s, s = 1, \dots, (k-2)] \\ &= \frac{|C|}{|C| - k + 2} \frac{|C|}{|C|} \frac{|C|}{|C| - 1} \dots \frac{|C|}{|C| - (k-3)} \sum_{\substack{a_1, a_2, \dots, a_{k-2} \in \Sigma \\ \{a_s\} \text{ distinct}}} \left(1 - \sum_{s=1}^{k-2} f_i[a_s] \right) \cdot f_i[a_1] f_i[a_2] \dots f_i[a_{k-2}], \end{aligned} \quad (7)$$

where the coefficients $\frac{|C|}{|C|-j}$, $j = 0, 1, \dots, k-3$ appear because we pick elements from C without replacement. We then define the following function of two probability vectors $g, f \in \mathbb{R}^k$:

$$\phi_k(g, f) = \sum_{\substack{a_1, a_2, \dots, a_{k-2} \in \Sigma \\ \{a_s\} \text{ distinct}}} \prod_{s=1}^{k-2} g[a_s] \left(1 - \sum_{s=1}^{k-2} f[a_s] \right). \quad (8)$$

Then the inequality (7) can be written as follows:

$$\mathbb{E}[\tau_i(x_1, x_2, \dots, x_{k-2})] \leq \phi_k(f_i, f_i)(1 + o(1)). \quad (9)$$

Since $\sum_{a \in \Sigma} f_i[a] = 1$, it is easy to see that $\phi_k(f_i, f_i)$ is a symmetric expression in $f_i[a]$ for all $a \in \Sigma$. Denote by $S_h^t(g)$ the h -th elementary symmetric sum of the first t coordinates of the vector

$g \in \mathbb{R}^k$, i.e. the sum of all products of h distinct elements from $\{g[1], g[2], \dots, g[t]\}$. For example,

$$S_3^4(g) = g[1]g[2]g[3] + g[1]g[2]g[4] + g[1]g[3]g[4] + g[2]g[3]g[4].$$

Then we can write

$$\phi_k(f_i, f_i) = (k-2)! \cdot \binom{k-1}{k-2} S_{k-1}^k(f_i) = (k-1)! \cdot S_{k-1}^k(f_i)$$

It is easy to see that $S_h^k(g)$ for g being a probability vector in \mathbb{R}^k is maximized when g is uniform. Indeed, if there are two non-equal coordinates $g[a] \neq g[b]$, then substituting the values in these coordinates by their arithmetic average strictly increases the value of $S_h^k(g)$. Then let us denote by u the uniform distribution on k elements, i.e. $u[a] = 1/k$ for all $a \in [k]$, and so it holds $S_h^k(g) \leq S_h^k(u)$. Then in (9) we obtain

$$\mathbb{E}[\tau_i(x_1, x_2, \dots, x_{k-2})] \leq (k-1)! \cdot S_{k-1}^k(f_i) \cdot (1 + o(1)) \leq (k-1)! \cdot S_{k-1}^k(u) \cdot (1 + o(1)),$$

where it holds

$$S_{k-1}^k(u) = \binom{k}{k-1} \cdot \left(\frac{1}{k}\right)^{k-1} = \frac{1}{k^{k-2}}.$$

Therefore, we retrieve

$$\mathbb{E}[\tau_i(x_1, x_2, \dots, x_{k-2})] \leq \frac{(k-1)!}{k^{k-2}} \cdot (1 + o(1)) = \frac{k!}{k^{k-1}} \cdot (1 + o(1)). \quad (10)$$

Substituting this inequality into (4), notice that we derive exactly the Fredman-Komlós bound (5). Denote then

$$\alpha_k = \frac{k!}{k^{k-1}},$$

the Fredman-Komlós upper bound on the rate R_k .

Now recall that we are considering the unbalanced case, in which there are a lot of coordinates with frequencies of codeword symbols being far from uniform. Take i to be any of such coordinates, and let for convenience $f = f_i$, so it holds $\min_{a \in \Sigma} f[a] < \gamma$. Without loss of generality, say $f[k] < \gamma$. Notice the following trivial property of symmetric sums:

$$\phi_k(f, f) = (k-1)! \cdot S_{k-1}^k(f) = (k-1)! \left(S_{k-1}^{k-1}(f) + f[k] \cdot S_{k-2}^{k-1}(f) \right).$$

The above expression is symmetric in the first $(k-1)$ coordinates of f . Let's then fix $f[k]$, and do the same averaging operations with all the remaining coordinates of f , making in the end $f'[1] = f'[2] = \dots = f'[k-1] = \frac{1-f[k]}{k-1}$. The value of $\phi_k(f, f)$ only increases after such operations, so

$$\phi_k(f, f) \leq (k-1)! \left(f'[1]f'[2] \cdots f'[k-1] + f[k] \cdot S_{k-2}^{k-1}(f') \right).$$

Denote $y = \frac{1-f[k]}{k-1}$, so $f[k] = 1 - (k-1)y$. Since $0 \leq f[k] < \gamma$ by the assumption above, it holds $\frac{1-\gamma}{k-1} \leq y \leq \frac{1}{k-1}$. Note that we took the threshold $\gamma \leq \frac{1}{k}$, thus $y \geq \frac{1-\gamma}{k-1} \geq \frac{1}{k}$. Then the following inequality holds:

$$\phi_k(f, f) \leq (k-1)! \left(y^{k-1} + (1 - (k-1)y) \cdot (k-1)y^{k-2} \right) = (k-1)! y^{k-2} \left((k-1) - (k^2 - 2k)y \right).$$

Denote $G_k(y) = (k-1)!y^{k-2} \left((k-1) - (k^2 - 2k)y \right)$, so $\phi_k(f, f) \leq G_k(y)$. Note that

$$(G_k(y))' = (k-1)!(k-1)(k-2)y^{k-3}(1-ky),$$

so the derivative of G_k is negative on the interval $\frac{1}{k} \leq \frac{1-\gamma}{k-1} < y \leq \frac{1}{k-1}$, and it is zero at $y = \frac{1}{k}$. Therefore, we finally obtain for any such f :

$$\phi_k(f, f) \leq \max_{y \in [\frac{1-\gamma}{k-1}, \frac{1}{k}]} G_k(y) = G_k\left(\frac{1-\gamma}{k-1}\right). \quad (11)$$

Note that it always holds $G_k\left(\frac{1-\gamma}{k-1}\right) \leq G_k\left(\frac{1}{k}\right) = \alpha_k$ for any $\gamma \leq \frac{1}{k}$, and the strict inequality $G_k\left(\frac{1-\gamma}{k-1}\right) < G_k\left(\frac{1}{k}\right) = \alpha_k$ holds when $\gamma < \frac{1}{k}$.

So if for some coordinate i it holds $\min_{a \in [k]} f_i[a] < \gamma$, we have the bound

$$\mathbb{E}[\tau_i(x_1, x_2, \dots, x_{k-2})] \leq G_k\left(\frac{1-\gamma}{k-1}\right) (1 + o(1)). \quad (12)$$

For now we obtained two bounds for the summands in the RHS of (4): (i) the bound (10) holds for all the coordinates, and (ii) the bound (12) holds for the coordinates with codeword symbol frequencies far from uniform. As we noted above, the second bound is strictly stronger than the first bound when we take the threshold $\gamma < \frac{1}{k}$. Also recall that in the unbalanced case which we now consider, there are a lot of coordinates of the second type, so essentially the bound (12) applies many times. Let's now formalize this argument to obtain an improvement on the Fredman-Komlós bound for the unbalanced case.

Denote for convenience $\xi_k(\gamma) = G_k\left(\frac{1-\gamma}{k-1}\right)$, then we have

$$\xi_k(\gamma) = (k-1)! \frac{(1-\gamma)^{k-2} (k-1)^2 - (k^2 - 2k)(1-\gamma)}{(k-1)^{k-2} (k-1)} = \frac{(k-2)!(1-\gamma)^{k-2} ((k^2 - 2k)\gamma + 1)}{(k-1)^{k-2}}. \quad (13)$$

Let also $\varepsilon_k(\gamma) = \alpha_k - \xi_k(\gamma) \geq 0$. Recall that we denoted by P_γ the set of coordinates i for which it holds $\min_{a \in \Sigma} f_i[a] \geq \gamma$. For such i we directly apply the bound (10). For all the other coordinates $i \in [n] \setminus P_\gamma$ we use the inequality (12). In the unbalanced case $|P_\gamma| < n - \ell$, thus $n - |P_\gamma| > \ell$. Applying all these arguments to (4), we obtain

$$\begin{aligned} \log(|C| - k + 2) &\leq \left(|P_\gamma| \alpha_k + (n - |P_\gamma|) (\alpha_k - \varepsilon_k(\gamma)) \right) (1 + o(1)) \\ &= \left(n \alpha_k - (n - |P_\gamma|) \varepsilon_k(\gamma) \right) (1 + o(1)) \\ &< \left(n \alpha_k - \ell \varepsilon_k(\gamma) \right) (1 + o(1)) \\ &\leq \left(n \alpha_k - \frac{nR}{\log\left(\frac{k}{k-3}\right)} \varepsilon_k(\gamma) + \frac{\log n}{\log\left(\frac{k}{k-3}\right)} \varepsilon_k(\gamma) + \varepsilon_k(\gamma) \right) (1 + o(1)) \\ &= \left(n \alpha_k - \frac{nR}{\log\left(\frac{k}{k-3}\right)} \varepsilon_k(\gamma) + o(n) \right) (1 + o(1)), \end{aligned}$$

where recall $\ell = \left\lceil \frac{nR - \log n}{\log\left(\frac{k}{k-3}\right)} \right\rceil$. Since $|C| = 2^{Rn}$ by definition of the rate R , the above implies for $n \rightarrow \infty$:

$$R \leq \alpha_k - \frac{R\varepsilon_k(\gamma)}{\log\left(\frac{k}{k-3}\right)} + o(1),$$

$$\boxed{R_k^{\text{unbal}}(\gamma) \leq \frac{\alpha_k}{1 + \frac{\alpha_k - \xi_k(\gamma)}{\log\left(\frac{k}{k-3}\right)}}.} \quad (14)$$

Note that if we take $\gamma = \frac{1}{k}$ in the above, we obtain $R_k^{\text{unbal}}(1/k) = \alpha_k$, since $\xi_k(1/k) = G\left(\frac{1-1/k}{k-1}\right) = G\left(\frac{1}{k}\right) = \alpha_k$. Now if we take $\gamma < \frac{1}{k}$, we showed above that $\xi_k(\gamma) < \alpha_k$, and then (14) will give a better bound on $R_k^{\text{unbal}}(\gamma)$. In other words,

$$R_k^{\text{unbal}}\left(\frac{1}{k} - \varepsilon\right) < R_k^{\text{unbal}}\left(\frac{1}{k}\right) \leq \alpha_k \quad (15)$$

for any small $\varepsilon > 0$. So we beat the Fredman-Komlós bound for the unbalanced case for any choice of the threshold $\gamma < \frac{1}{k}$.

3.2 Almost balanced case

For this case we follow the approach used in [2] for 4-hashing. Namely, we will consider a rich subcode of codewords which can take a restricted set of symbols on some fixed set of coordinates, and choose x_1, x_2, \dots, x_{k-2} randomly from this subcode. In the almost balanced case, we are able to ensure that the distributions of codeword symbols in all non-fixed coordinates are close to uniform, which will allow us to use some continuity argument to bound the RHS of (4).

In this case we assume $|P_\gamma| \geq n - \ell$, so there are at most ℓ coordinates where the distribution of codeword symbols is skewed. The set of such coordinates is $\overline{P}_\gamma = [n] \setminus P_\gamma$, $|\overline{P}_\gamma| \leq \ell$. Then take any subset $T \subset [n]$, such that $\overline{P}_\gamma \subseteq T$ and $|T| = \ell$, and denote $S = [n] \setminus T$.

Our goal is to find a subcode of C of sufficient size, such that any $(k-2)$ codewords x_1, x_2, \dots, x_{k-2} from this subcode collide in all the coordinates from T . In other words, for any coordinate $t \in T$ there should exist i, j such that $(x_i)_t = (x_j)_t$. This will ensure that the coordinates from T contribute 0 to the RHS of (4), which will allow us to prove a better bound on the rate of the code C . We will now define the subcodes which satisfy this property.

First, denote by $\binom{\Sigma}{p}$ the family of p -element subsets of the alphabet $\Sigma = \{1, 2, \dots, k\}$. Then define

$$\Omega = \underbrace{\binom{\Sigma}{k-3} \times \binom{\Sigma}{k-3} \times \dots \times \binom{\Sigma}{k-3}}_{\ell}.$$

Now, for any $\omega \in \Omega$ and any string $s \in \Sigma^\ell$, denote $s \vdash \omega$ if $s_1 \in \omega_1, s_2 \in \omega_2, \dots, s_\ell \in \omega_\ell$. Then, for any $\omega \in \Omega$, we define:

$$C_\omega = \{x \in C : x_{\{T\}} \vdash \omega\},$$

where $x_{\{T\}}$ is the projection of the codeword x on the set of coordinates T . Notice that C_ω has the property we discussed above. Indeed, for any pick $x_1, x_2, \dots, x_{k-2} \in C_\omega$ and any $t \in T$, it

holds $(x_1)_t, (x_2)_t, \dots, (x_{k-2})_t \in \omega_t$, but $|\omega_t| = k - 3$, and therefore $(x_1)_t, (x_2)_t, \dots, (x_{k-2})_t$ are not all distinct.

Denote then $M_\omega = |C_\omega|$. Note that for each $x \in C$ there are exactly $\binom{k-1}{k-4}^\ell$ different elements $\omega \in \Omega$ such that $x_{\{T\}} \vdash \omega$. Therefore

$$\sum_{\omega \in \Omega} M_\omega = |C| \cdot \binom{k-1}{k-4}^\ell.$$

It suffices to prove that there exists at least one $\omega \in \Omega$ such that $M_\omega \geq n$ for our arguments further. For the sake of contradiction, suppose then that $M_\omega < n$ for all $\omega \in \Omega$. But then it holds

$$2^{nR} = |C| = \sum_{\omega \in \Omega} M_\omega \frac{1}{\binom{k-1}{k-4}^\ell} < \frac{\binom{k}{k-3}^\ell}{\binom{k-1}{k-4}^\ell} \cdot n = \left(\frac{k}{k-3} \right)^\ell n = 2^{\ell \cdot \log \frac{k}{k-3} + \log n} \leq 2^{nR},$$

where recall $\ell = \left\lfloor \frac{nR - \log n}{\log \left(\frac{k}{k-3} \right)} \right\rfloor$. Since we obtained a contradiction above, there exists $\omega \in \Omega$ such that $M_\omega \geq n$.

We are finally ready to describe the strategy to pick the codewords x_1, x_2, \dots, x_{k-2} in the almost balanced case. We do the following: first, deterministically choose some $\omega \in \Omega$ such that $M_\omega \geq n$, and then pick x_1, x_2, \dots, x_{k-2} uniformly at random (without replacement) from C_ω . Since all the codewords collide on the coordinates from the set T , we obtain in (4):

$$\log(|C| - k + 2) \leq \sum_{m \in [n]} \mathbb{E}[\tau_m(x_1, x_2, \dots, x_{k-2})] = \sum_{m \in S} \mathbb{E}[\tau_m(x_1, x_2, \dots, x_{k-2})]. \quad (16)$$

Now fix some $m \in S$, and let $f_{m|\omega}$ be the frequency vector of the m^{th} coordinate in the subcode C_ω . Taking expectation over the choice of x_1, x_2, \dots, x_{k-2} in (6) with respect to the the random strategy described above, we have

$$\begin{aligned} & \mathbb{E}[\tau_m(x_1, x_2, \dots, x_{k-2})] \\ &= \frac{|C|}{|C| - k + 2} \prod_{j=0}^{k-3} \frac{|C_\omega|}{|C_\omega| - j} \sum_{\substack{a_1, a_2, \dots, a_{k-2} \in \Sigma \\ \{a_s\} \text{ distinct}}} \left(1 - \sum_{s=1}^{k-2} f_m[a_s] \right) \cdot f_{m|\omega}[a_1] f_{m|\omega}[a_2] \cdots f_{m|\omega}[a_{k-2}], \end{aligned} \quad (17)$$

where the coefficients $\frac{|C_\omega|}{|C_\omega| - j}$, $j = 0, 1, \dots, (k-3)$ appear because we pick $(k-2)$ elements from C_ω without replacement. Since we took ω such that $|C_\omega| \geq n$, it holds $\frac{|C_\omega|}{|C_\omega| - j} \leq \frac{n}{n-j}$.

Recall that we defined the following function, which operates on probability vectors $f, g \in \mathbb{R}^k$:

$$\phi_k(g, f) = \sum_{\substack{a_1, a_2, \dots, a_{k-2} \in \Sigma \\ \{a_s\} \text{ distinct}}} \prod_{s=1}^{k-2} g[a_s] \left(1 - \sum_{s=1}^{k-2} f[a_s] \right). \quad (18)$$

So we derive

$$\mathbb{E}[\tau_m(x_1, x_2, \dots, x_{k-2})] \leq \frac{n}{n-k+2} \prod_{j=0}^{k-3} \left(\frac{n}{n-j} \right) \phi_k(f_{m|\omega}, f_m) = \phi_k(f_{m|\omega}, f_m) \cdot (1 + o(1)). \quad (19)$$

Consider the following definition:

$$\boxed{\theta_k(\gamma) = \max_{g,f} \{\phi_k(g, f) : f, g \in \mathbb{R}^k \text{ are probability vectors, } \min_{a \in \Sigma} f[a] \geq \gamma\}. \quad (20)}$$

Let's first consider what bound we obtain using this definition, and then analyze how $\theta_k(\gamma)$ behaves.

Since $\min_{a \in \Sigma} f_m[a] \geq \gamma$ by construction of the set S , it holds $\phi_k(f_m|_\omega, f_m) \leq \theta_k(\gamma)$ for any $m \in S$, so substituting it into (19) gives

$$\mathbb{E}[\tau_m(x_1, x_2, \dots, x_{k-2})] \leq \theta_k(\gamma) \cdot (1 + o(1)).$$

Therefore, in (16) we derive

$$\begin{aligned} \log(|C| - k + 2) &\leq |S| \cdot \theta_k(\gamma)(1 + o(1)) \\ &= (n - \ell) \cdot \theta_k(\gamma)(1 + o(1)) \\ &\leq \left(n - \frac{nR}{\log\left(\frac{k}{k-3}\right)} + \frac{\log n}{\log\left(\frac{k}{k-3}\right)} + 1 \right) \theta_k(\gamma)(1 + o(1)). \end{aligned}$$

Recall that $|C| = 2^{nR}$, thus for $n \rightarrow \infty$ we have

$$R \leq \left(1 - \frac{R}{\log\left(\frac{k}{k-3}\right)} \right) \theta_k(\gamma) + o(1),$$

$$\boxed{R_k^{\text{bal}}(\gamma) \leq \frac{\theta_k(\gamma)}{1 + \frac{\theta_k(\gamma)}{\log\left(\frac{k}{k-3}\right)}}. \quad (21)}$$

It now remains to understand how $\theta_k(\gamma)$, defined in (20), behaves as a function of γ .

Continuity of θ_k . First of all, notice that for $\gamma = \frac{1}{k}$ there exists only one probability vector f such that $\min_{a \in \Sigma} f[a] \geq \gamma$, namely the uniform vector u . But then $\phi_k(g, u)$ is just an elementary symmetric sum of all the coordinates of g , and therefore, using the simple averaging argument similar to the one we used in Section 3.1, we obtain $\phi_k(g, u) \leq \phi_k(u, u) = \alpha_k$, and so $\theta_k(1/k) = \alpha_k$.

Now we use an analysis instrument to prove that $\theta_k(\gamma)$ is continuous at $\gamma = \frac{1}{k}$. Namely, we use Berge's maximum theorem [12], which we state here for completeness:

Theorem 2 (Berge's maximum theorem). *Let X and Γ be metric spaces, and $h : X \times \Gamma \rightarrow \mathbb{R}$ be a function jointly continuous in its two arguments. Let also $D : \Gamma \rightrightarrows X$ be a compact-valued correspondence, i.e. D maps Γ to the compact subsets of X : $D(\gamma) \subseteq X$ and $D(\gamma)$ is a compact for any $\gamma \in \Gamma$. Define for $\gamma \in \Gamma$:*

$$h^*(\gamma) = \max_{x \in D(\gamma)} h(x, \gamma).$$

If D is continuous (both upper and lower hemicontinuous) at γ , then h^ is continuous at γ .*

(The correspondences, or multi-valued functions, is the generalization of functions, and the continuity of correspondences is a generalization of continuity of functions. Refer to [12], Chapter E for details.)

Claim 1. $\theta_k(\gamma)$ is continuous at $\gamma = \frac{1}{k}$.

Proof. This is a direct application of Berge's maximum theorem. Take in the settings of the theorem $X = \mathbb{R}^k \times \mathbb{R}^k$, $\Gamma = \mathbb{R}$, and function $h(x, \gamma) = h((g, f), \gamma) = \phi_k(g, f)$, where the variable x is $(g, f) \in \mathbb{R}^k \times \mathbb{R}^k$, and h doesn't depend on the variable γ . Let us define the correspondence

$$D(\gamma) = \Delta_k \times \Delta_k^{(\gamma)} \subseteq X,$$

where

$$\Delta_k = \{x \in \mathbb{R}_+^k : \langle x, \mathbf{1} \rangle = 1\}$$

is the k -dimensional simplex, and

$$\Delta_k^{(\gamma)} = \left\{x \in \mathbb{R}_+^k : \langle x, \mathbf{1} \rangle = 1; \quad x_i \geq \min\{\gamma, 1/k\}, \quad \forall i \in [k]^1\right\}.$$

It is clear that $D(\gamma)$ is a compact-valued correspondence, and it easily follows from the definition of continuity of correspondences that $D(\gamma)$ is continuous at any γ . By definition (20), for $\gamma \leq 1/k$ we have

$$\theta_k(\gamma) = \max_{g, f} \{\phi_k(g, f) : (g, f) \in \Delta_k \times \Delta_k^{(\gamma)}\} = \max_{x \in D(\gamma)} h(x, \gamma) = h^*(\gamma)$$

in the settings of the theorem. Therefore, we conclude that $\theta_k(\gamma)$ is continuous at any γ , in particular at $\gamma = 1/k$. \square

3.3 Improvement of the Fredman-Komlós bound

Now we prove that it is possible to choose such a threshold γ that both bounds (14) and (21) are stronger than the Fredman-Komlós bound. Since for any code C either the unbalanced or the almost balanced case holds, it will imply the general upper bound on R_k .

For $\gamma = \frac{1}{k}$, in (21) we have

$$R_k^{\text{bal}}\left(\frac{1}{k}\right) \leq \frac{\theta_k(1/k)}{1 + \frac{\theta_k(1/k)}{\log\left(\frac{k}{k-3}\right)}} = \frac{\alpha_k}{1 + \frac{\alpha_k}{\log\left(\frac{k}{k-3}\right)}} < \alpha_k. \quad (22)$$

Clearly, for small enough $\rho > 0$ it holds $\frac{\alpha_k + \rho}{1 + \frac{\alpha_k + \rho}{\log\left(\frac{k}{k-3}\right)}} < \alpha_k$. We proved in Claim 1 that $\theta_k(\gamma)$ is a continuous function, and we also have $\theta_k(1/k) = \alpha_k$. Therefore, there exists such small $\varepsilon > 0$ that $\theta_k(1/k - \varepsilon) < \theta_k(1/k) + \rho = \alpha_k + \rho$, so

$$R_k^{\text{bal}}\left(\frac{1}{k} - \varepsilon\right) \leq \frac{\alpha_k + \rho}{1 + \frac{\alpha_k + \rho}{\log\left(\frac{k}{k-3}\right)}} < \alpha_k.$$

Taking the same ε in (15), we also have

$$R_k^{\text{unbal}}\left(\frac{1}{k} - \varepsilon\right) < \alpha_k,$$

¹We require this inequality rather than just $x \geq \gamma$ because in the latter case it would hold $D(1/k + \varepsilon) = \emptyset$ for any $\varepsilon > 0$, while $D(1/k)$ is a non-empty compact. In such settings $D(\gamma)$ wouldn't be continuous in $\gamma = 1/k$ for the natural reasons (though it would be "one-side continuous" in the settings of correspondences). Since we require $x \geq \min\{\gamma, 1/k\}$ component-wise, it will hold $D(1/k + \varepsilon) = D(1/k)$, and the correspondence D will be continuous.

and therefore we obtain for the general bound

$$R_k \leq \max \left\{ R_k^{\text{bal}}(1/k - \varepsilon); R_k^{\text{unbal}}(1/k - \varepsilon) \right\} < \alpha_k.$$

This shows that we improved the Fredman-Komlós bound for every k , at least by some tiny amount.

The above proof doesn't give an effective method to find an explicit new bound on R_k , since we use the continuity arguments in it. However, in the next section we present quite a different way to address the balanced case, and provide a specific way to obtain explicit bounds on R_k . Unfortunately, this approach leads to parametrized optimization of a large (increasing with k) number of polynomials. Using this method, we present new explicit bounds on R_5 and R_6 in section 4.2. Finally, we present a conjecture, assuming which we are able to find effective upper bound on R_k stronger than the Fredman-Komlós bound, via finding the root of a polynomial of degree $O(k)$ which lies within a specific interval.

4 Optimization approach to the almost balanced case

In this section we introduce the approach to estimate the value of $\theta_k(\gamma)$. Take f to be a probability vector in \mathbb{R}^k , such that $f[a] \geq \gamma$ for $a \in \Sigma$. For convenience, we will use “ f_a ” to refer to the a^{th} coordinate of f rather than “ $f[a]$ ”. Recall:

$$\phi_k(g, f) = \sum_{\substack{a_1, a_2, \dots, a_{k-2} \in \Sigma \\ \{a_i\} \text{ distinct}}} \prod_{i=1}^{k-2} g_{a_i} \left(1 - \sum_{i=1}^{k-2} f_{a_i} \right) = (k-2)! \left[\sum_{a_1 < a_2 < \dots < a_{k-2} \in \Sigma} \prod_{i=1}^{k-2} g_{a_i} \left(1 - \sum_{i=1}^{k-2} f_{a_i} \right) \right].$$

Let $t = k(k-1)/2$ and let P_1, P_2, \dots, P_t be an enumeration of all $(k-2)$ -element subsets of $\Sigma = \{1, 2, \dots, k\}$.

Then we can rewrite

$$\phi_k(g, f) = (k-2)! \left[\sum_{j=1}^t \prod_{a \in P_j} g_a \left(1 - \sum_{a \in P_j} f_a \right) \right].$$

Let $d_j = \prod_{a \in P_j} g_a$, $w_j = \left(1 - \sum_{a \in P_j} f_a \right)$ for $j = 1, 2, \dots, t$, and therefore we simply have

$$\phi_k(g, f) = (k-2)! \sum_{i=1}^t d_i w_i.$$

Since $f_a \geq \gamma$ for any $a \in \Sigma$, it holds $w_i \leq 1 - (k-2)\gamma$. On the other hand, since f is a probability vector, $w_i = f_{a_1} + f_{a_2}$, where $\{a_1, a_2\} = \Sigma \setminus P_i$, and so $w_i \geq 2\gamma$. Denoting $w'_i = w_i - 2\gamma$, we obtain:

$$0 \leq w'_i \leq (1 - k\gamma),$$

$$\phi_k(g, f) = (k-2)! \left[2\gamma \left(\sum_{i=1}^t d_i \right) + \sum_{i=1}^t d_i w'_i \right].$$

Since $w_i = f_{a_1} + f_{a_2}$ for $\{a_1, a_2\} = \Sigma \setminus P_i$, we can argue that $\sum_{i=1}^t w_i$ is symmetric sum of all f_a for $a \in \Sigma$, where each f_a occurs $\binom{k-1}{k-2} = (k-1)$ times. Since $\sum_{a \in \Sigma} f_a = 1$, it then holds

$$\sum_{i=1}^t w_i = (k-1) \sum_{a \in \Sigma} f_a = (k-1) \Rightarrow \sum_{i=1}^t w'_i = (k-1) - 2\gamma t = (k-1)(1 - k\gamma).$$

Then consider the following optimization problem:

$$\begin{aligned} \max_y \quad & \sum_{i=1}^t d_i y_i, \\ \text{s.t.} \quad & 0 \leq y_i \leq (1 - k\gamma), \\ & \sum_{i=1}^t y_i = (k-1)(1 - k\gamma). \end{aligned}$$

Note that the vector $(w'_1, w'_2, \dots, w'_t)$ is feasible for the above program, and let y^* be the optimal solution for this program. Then we have:

$$\phi_k(g, f) = (k-2)! \left[2\gamma \left(\sum_{i=1}^t d_i \right) + \sum_{i=1}^t d_i w'_i \right] \leq (k-2)! \left[2\gamma \left(\sum_{i=1}^t d_i \right) + \sum_{i=1}^t d_i y_i^* \right]. \quad (23)$$

It is straightforward to see that the optimal solution y^* to the above program has $(k-1)$ non-zero coordinates, corresponding to the first $(k-1)$ greatest values among $\{d_1, d_2, \dots, d_t\}$, each equal to $(1 - k\gamma)$, and zeros in the remaining coordinates. In other words, denote $d_{(i)}$ to be the i^{th} ordered statistic of the set $\{d_1, d_2, \dots, d_t\}$, so $d_{(1)}$ is the maximum of this set, and $d_{(t)}$ is the minimum. Then

$$\sum_{i=1}^t d_i y_i^* = (1 - k\gamma) \sum_{j=1}^{k-1} d_{(j)},$$

and therefore in (23) obtain

$$\phi_k(g, f) \leq (k-2)! \left[(1 - (k-2)\gamma) \left(\sum_{j=1}^{k-1} d_{(j)} \right) + 2\gamma \sum_{j=k}^t d_{(j)} \right]. \quad (24)$$

Without loss of generality, assume $g_1 \geq g_2 \geq \dots \geq g_k \geq 0$. Even with this fixed ordering of the coordinates of g , there still could be different cases of ordering within the set $\{d_1, d_2, \dots, d_t\}$, and for each of these cases we obtain some different function of d_i 's in the RHS of (24). Say there can be q_k different possible sets of first $(k-1)$ ordered statistics within the set $\{d_1, d_2, \dots, d_t\}$, then there would be q_k different functionals of d_i 's, and thus of g_i 's, in the RHS of (24), call them $\Theta_k^{(1)}(g, \gamma), \Theta_k^{(2)}(g, \gamma), \dots, \Theta_k^{(q_k)}(g, \gamma)$. Since exactly one ordering is correct for any particular vector g , we obtain

$$\phi_k(g, f) \leq \max_{i=1,2,\dots,q_k} \Theta_k^{(i)}(g, \gamma).$$

Then define

$$\theta_k^{(i)}(\gamma) = \max_x \left\{ \Theta_k^{(i)}(x, \gamma) : \sum_{j=1}^k x_j = 1, x_j \geq 0 \right\}, \quad \text{for } i = 1, 2, \dots, q_k, \quad (25)$$

and so the quantity $\theta_k(\gamma)$ defined in (20) satisfies

$$\theta_k(\gamma) \leq \max_{i=1,2,\dots,q_k} \theta_k^{(i)}(\gamma).$$

So to find an upper bound on $\theta_k(\gamma)$ it suffices to find the maximums of all $\theta_k^{(i)}(\gamma)$ for $i = 1, 2, \dots, q_k$. However, q_k grows rapidly as k increases, so it is not clear how to do this efficiently. We introduce a conjecture below, which suggests that we can determine which of the values $\theta_k^{(i)}(\gamma)$, $i = 1, 2, \dots, q_k$, is the greatest for any k .

Specifically, the conjecture is stated as follows: we assume that the maximum among all the values $\theta_k^{(i)}(\gamma)$, $i = 1, 2, \dots, q_k$, is the greatest for the functional $\Theta_k^{(i)}(x, \gamma)$ corresponding to the case, when the first $(k-1)$ ordered statistics of the set $\{d_1, d_2, \dots, d_t\}$ form the set $\left\{ \frac{\prod_{i=1}^{k-1} g_i}{g_a} \right\}_{a \in [k-1]}$. In other words, $\{d_{(1)}, d_{(2)}, \dots, d_{(k-1)}\}$ correspond to the sets P_j that contain all their $(k-2)$ elements from $\{1, 2, \dots, k-1\}$ (recall $d_j = \prod_{a \in P_j} g_j$). So the first $(k-1)$ ordered statistics are formed as the products of only the first $(k-1)$ coordinates of g , ignoring the coordinate g_k .

Recall that we denote by $S_h^t(g)$ the h -th elementary symmetric sum of a set of t coordinates $\{g_1, g_2, \dots, g_t\}$ (i.e. the sum of all products of h distinct elements from the first t coordinates of g). Then the above-mentioned conjecture can be formalized as follows:

Conjecture 1.

$$\theta_k(\gamma) = \max_x \left\{ (k-2)! \left[\left(1 - (k-2)\gamma \right) S_{k-2}^{k-1}(x) + 2\gamma \cdot x_k \cdot S_{k-3}^{k-1}(x) \right] : \sum_{i=1}^k x_i = 1, x \geq 0 \right\}. \quad (26)$$

Indeed, the function

$$\Theta_k^\gamma(g) = (k-2)! \left[\left(1 - (k-2)\gamma \right) S_{k-2}^{k-1}(g) + 2\gamma \cdot g_k \cdot S_{k-3}^{k-1}(g) \right]$$

just corresponds to the functional in the RHS of (24) in the case we discussed above, since in this case the elements $\{d_{(1)}, d_{(2)}, \dots, d_{(k-1)}\}$ are just the summands from S_{k-2}^{k-1} , while all the other elements $\{d_{(k)}, d_{(k+1)}, \dots, d_{(t)}\}$ are of type $g_k \cdot \prod_{i=1}^{k-3} g_{a_i}$.

Let's now find the RHS of (26). Since $1 - (k-2)\gamma \geq 2\gamma$, it is easy to see that we may consider only vectors x for which x_k is the minimal over all other coordinates. Indeed, in other case, if $x_k > x_i$, switching the values in x_k and x_i will only increase the functional $\Theta_k^\gamma(x)$. So from now on, we consider x_k to be minimal among the coordinates of x .

Now notice that the sums $S_{k-2}^{k-1}(x)$ and $S_{k-3}^{k-1}(x)$ are elementary symmetric sums with respect to x_1, x_2, \dots, x_{k-1} , which means that if some two of x_i, x_j are different for $i, j \leq k-2$, then we can substitute them both by their average, and the functional $\Theta_k(x)$ will not decrease. Therefore, we conclude that the maximum of RHS of (26) is achieved when $x_1 = x_2 = \dots = x_{k-1} = \beta$, and then $x_k = 1 - (k-1)\beta$, so it must hold $\beta \geq 1 - (k-1)\beta \geq 0$, thus $\frac{1}{k} \leq \beta \leq \frac{1}{k-1}$. Denoting $Q_k^\gamma(\beta) = \Theta_k^\gamma(\beta, \beta, \dots, 1 - (k-1)\beta)$, we obtain (assuming Conjecture 1):

$$\theta_k(\gamma) = \max_{\frac{1}{k} \leq \beta \leq \frac{1}{k-1}} Q_k^\gamma(\beta).$$

Then we compute:

$$\begin{aligned}
Q_k^\gamma(\beta) &= (k-2)! \left[\left(1 - (k-2)\gamma\right) (k-1)\beta^{k-2} + 2\gamma \left(1 - (k-1)\beta\right) \frac{(k-1)(k-2)}{2} \beta^{k-3} \right] \\
&= (k-1)! \beta^{k-3} \left(\beta \left(1 - (k^2 - 2k)\gamma\right) + (k-2)\gamma \right); \\
(Q_k^\gamma(\beta))' &= (k-1)! (k-2) \beta^{k-4} \left(\beta \left(1 - (k^2 - 2k)\gamma\right) + (k-3)\gamma \right); \\
(Q_k^\gamma(\beta))'' &= (k-1)! (k-2)(k-3) \beta^{k-5} \left(\beta \left(1 - (k^2 - 2k)\gamma\right) + (k-4)\gamma \right).
\end{aligned}$$

The second derivative of $Q_k^\gamma(\beta)$ is negative whenever $\beta > \frac{(k-4)\gamma}{(k^2-2k)\gamma-1}$, and it is easy to see that $\frac{(k-4)\gamma}{(k^2-2k)\gamma-1} < \frac{1}{k}$ if $\gamma > \frac{1}{2k}$. Since we initially considered $\gamma \geq \frac{1}{2k-3} > \frac{1}{2k}$, it holds $\frac{(k-4)\gamma}{(k^2-2k)\gamma-1} < \frac{1}{k} \leq \beta \leq \frac{1}{k-1}$. So $(Q_k^\gamma(\beta))'' < 0$ for β being in the interval of interest.

Next, since $\gamma \leq \frac{1}{k}$, it is straightforward to see that $(Q_k^\gamma(\frac{1}{k}))' \geq 0$.

Finally, for $\gamma \geq \frac{1}{2k-3}$ it is easy to check that $(Q_k^\gamma(\frac{1}{k-1}))' \leq 0$.

Altogether, we can conclude that for the interval $\frac{1}{2k-3} \leq \gamma \leq \frac{1}{k}$ the maximum of $Q_k^\gamma(\beta)$ can be found by solving the equation $(Q_k^\gamma(\beta))' = 0$, and thus the optimal solution is $\beta^* = \frac{(k-3)\gamma}{(k^2-2k)\gamma-1}$.

Finally:

$$\boxed{\theta_k(\gamma) = Q_k^\gamma(\beta^*) = \frac{(k-1)!(k-3)^{k-3}\gamma^{k-2}}{((k^2-2k)\gamma-1)^{k-3}},} \quad (27)$$

assuming the conjecture 1 holds.

4.1 Optimal threshold (assuming the conjecture)

The optimal threshold γ is such that the bounds (14) and (21) are equal, since we might only claim $R_k \leq \max\{R_k^{\text{bal}}, R_k^{\text{unbal}}\}$, and the first bound becomes weaker as γ increases, while the second bound becomes stronger. Therefore, the optimal threshold is the solution of the following equation:

$$R_k^{\text{unbal}}(\gamma) = \frac{\theta_k(\gamma)}{1 + \frac{\theta_k(\gamma)}{\log(\frac{k}{k-3})}} = \frac{\alpha_k}{1 + \frac{\alpha_k - \xi_k(\gamma)}{\log(\frac{k}{k-3})}} = R_k^{\text{bal}}(\gamma), \quad (28)$$

where $\alpha_k = \frac{k!}{k^{k-1}}$ is the Fredman-Komlós bound, $\theta_k(\gamma)$ can be found using expression (27) (assuming the conjecture holds), and $\xi_k(\gamma)$ is found via (13). Note that both $\theta_k(\gamma)$ and $\xi_k(\gamma)$ are rational functions with degrees $O(k)$, and therefore the above equation is equivalent to finding a root of a polynomial of degree $O(k)$ in γ , that lies in the interval $(\frac{1}{2k-3}, \frac{1}{k})$. Such a solution certainly exists, because $R_k^{\text{bal}}(1/k) = \alpha_k > R_k^{\text{unbal}}(1/k)$, but it is easy to verify that $R_k^{\text{bal}}(\frac{1}{2k-3}) < \alpha_k < R_k^{\text{unbal}}(\frac{1}{2k-3})$. Therefore, there exists a point $\gamma^* \in (\frac{1}{2k-3}, \frac{1}{k})$ where $R_k^{\text{unbal}}(\gamma^*) = R_k^{\text{bal}}(\gamma^*)$, since these functions are continuous.

We can now note that much weaker version of the conjecture would be enough to be able to use these arguments. In fact, we only need for the equation (26) to hold specifically for the

value of the threshold $\gamma = \gamma^*$. So it just suffices to solve all optimization problems (25) for this value γ^* , and check if the conjecture indeed holds (namely, that the maximum of $\Theta_k^{(i)}(g, \gamma)$ is the greatest for the functional $\Theta_k^\gamma(g)$ described in the conjecture). In case it actually holds, we are able to apply inequalities (21) and (14) and obtain an explicit upper bound on R_k , which beats the Fredman-Komlós bound. In the next section we do so for $k = 5$ and $k = 6$.

4.2 New bounds for $k = 5$ and $k = 6$

Applying (28) for $k = 5$, for instance, is just solving the following equation:

$$\frac{96\gamma^3 \log \frac{5}{2}}{(15\gamma - 1)^2 \log \frac{5}{2} + 96\gamma^3} = \frac{32\alpha_5 \log \frac{5}{2}}{32 \log \frac{5}{2} + 32\alpha_5 - 3(1 - \gamma)^3(15\gamma + 1)},$$

where $\frac{1}{7} \leq \gamma \leq \frac{1}{5}$. The only feasible solution to the above is $\gamma^* \approx 0.136163$. We then use the tools of numerical optimization in *Wolfram Mathematica* [13] to verify that the conjecture holds. For this case, we have $q_5 = 2$, so we only need to optimize two functionals over the simplex, and compare the two optimal values. After verifying the conjecture, we obtain the new bound for 5-hashing:

$$R_5 < 0.190825 < 0.192 = \frac{24}{125} = \alpha_5.$$

For $k = 6$, the above approach gives us:

$$R_6 < 0.0922787 < 0.0\overline{925} = \frac{5}{54} = \alpha_6.$$

($0.0\overline{925}$ stands for repeating decimal $0.0925925925\dots$)

A (b, k) -hashing

As we mentioned in the Introduction, the problem for which Fredman and Komlós [3] proved a bound was in fact broader than the k -hashing problem. Namely, for $b \geq k$, say that a code $C \subseteq [b]^n$ is a (b, k) -hash code if for any k distinct codewords from C there exists a coordinate in which all these codewords differ. Then the (b, k) -hashing problem consists in estimating the maximum possible rate $R_{(b,k)}$ of (b, k) -hash codes. This can be equivalently formulated in the context of hash functions.

All the bounds for this generalized version of the problem rely on extended versions of the Hansel lemma. Fredman and Komlós [3] allowed for the graphs G_i in the settings of Lemma 1 to be multipartite rather than just bipartite, and later Körner and Marton [6] also proved the generalization of the lemma for hypergraphs. The generalized version of the lemma was also proven in [10] using probabilistic arguments.

Lemma 2 (Hansel for hypergraphs). *Let $K_m^{(d)}$ be a complete d -uniform hypergraph on m vertices. Let also G_1, G_2, \dots, G_t be c -partite d -uniform hypergraphs, such that $E(K_m^{(d)}) = \bigcup_{i=1}^t E(G_i)$. Denote by $\tau(G_i)$ the fraction of non-isolated vertices in G_i . Then the following holds:*

$$\frac{\log m}{\log(d-1)} \leq \log \frac{c}{d-1} \cdot \sum_{i=1}^t \tau(G_i). \quad (29)$$

Again, to get the bound on the rates of (b, k) -codes, consider some (b, k) -hash code $C \subseteq [b]^n$. Take a subset of this code $\{x_1, x_2, \dots, x_j\} \subseteq C$, where $1 \leq j \leq k - 2$. We now define $(b - j)$ -partite $(k - j)$ -uniform hypergraphs $G_i^{x_1, \dots, x_j}$, for $i \in [n]$, as follows:

$$V(G_i^{x_1, \dots, x_j}) = C \setminus \{x_1, x_2, \dots, x_j\},$$

$$E(G_i^{x_1, \dots, x_j}) = \left\{ \{y_1, y_2, \dots, y_{k-j}\} : (y_1)_i, (y_2)_i, \dots, (y_{k-j})_i, (x_1)_i, (x_2)_i, \dots, (x_j)_i \text{ are distinct} \right\}.$$

Directly applying the the above Hansel lemma for hypergraphs and denoting $\tau_i(x_1, x_2, \dots, x_j) = \tau(G_i^{x_1, \dots, x_j})$, we obtain:

$$\frac{\log(|C| - j)}{\log(k - j - 1)} \leq \log \frac{b - j}{k - j - 1} \sum_{i=1}^n \tau_i(x_1, x_2, \dots, x_j). \quad (30)$$

Similarly, one then might use different ways to pick x_1, x_2, \dots, x_j in order to obtain the upper bound on the rate of C from the above.

In [3] for the usual graph case ($j = k - 2$), and then in [6] for hypergraphs, the codewords x_1, x_2, \dots, x_j are picked independently at random from the code C , and (30) gives the following bound (Körner-Martón bound):

$$R_{(b,k)} \leq \min_{0 \leq j \leq k-2} \frac{b^{j+1}}{b^{j+1}} \log \frac{b - j}{k - j - 1}, \quad (31)$$

where $b^{j+1} = b(b - 1) \dots (b - j)$.

Note that for the case $b = k$ (k -hashing) it can be shown that the above minimum is attained at $j = k - 2$. But in this case the bound (31) turns into the Fredman-Komlós bound (5), so this approach doesn't give any improvement for k -hashing.

In [5] Arikan, using the rate versus distance ideas discussed in the section 2, provides the following bound on the rate $R_{(b,k)}$ for general b and k :

$$R_{(b,k)} \leq \sup_x \{x \leq \alpha_j(x), j = 2, \dots, k - 2\}, \quad (32)$$

where

$$\alpha_j(x) = \frac{b - j}{k - 1} 2^{-x} \left(1 - \frac{x}{\log b}\right) \frac{b^j}{b^j} \log \frac{b - j}{k - 1 - j}$$

for $j = 2, \dots, b - k$, and

$$\alpha_j(x) = \left(1 - \frac{j}{b - k + 1} (1 - 2^{-x})\right) \left(1 - \frac{x}{\log b}\right) \frac{b^j}{b^j} \log \frac{b - j}{k - 1 - j}$$

for $j = b - k + 1, \dots, k - 2$.

Arikan's bound improves the Fredman-Komlós bound (5) for $b = k = 4$, and also beats the Körner-Martón bound (31) for many pairs of (b, k) ; see [5]. However, neither (31) nor (32) beat the bound (5) when $b = k > 4$.

The approach we described in this paper generalizes to the settings of (b, k) -hashing problem in a straightforward way, improving the Körner-Martón bound (31) for any $j = 0, 1, \dots, k - 2$. However, since this bound was already possibly beaten by the Arikan's bound (32) for certain settings of $b \neq k$, and in order to keep the presentation simple, we don't include the proofs for (b, k) -hashing problem in this paper.

References

- [1] P. Elias, “Zero error capacity under list decoding,” *IEEE Trans. Information Theory*, vol. 34, no. 5, pp. 1070–1074, 1988. [Online]. Available: <https://doi.org/10.1109/18.21233>
- [2] M. Dalai, V. Guruswami, and J. Radhakrishnan, “An improved bound on the zero-error list-decoding capacity of the 4/3 channel,” in *2017 IEEE International Symposium on Information Theory (ISIT)*, June 2017, pp. 1658–1662.
- [3] M. L. Fredman and J. Komlós, “On the size of separating systems and families of perfect hash functions,” *SIAM Journal on Algebraic Discrete Methods*, vol. 5, no. 1, pp. 61–68, mar 1984. [Online]. Available: <https://doi.org/10.1137%2F0605009>
- [4] J. Körner, “Fredman–Komlós bounds and information theory,” *SIAM Journal on Algebraic Discrete Methods*, vol. 7, no. 4, pp. 560–570, oct 1986. [Online]. Available: <https://doi.org/10.1137%2F0607062>
- [5] E. Arikan, “A bound on the zero-error list coding capacity,” in *Proceedings. IEEE International Symposium on Information Theory*. IEEE. [Online]. Available: <https://doi.org/10.1109%2Fisit.1993.748467>
- [6] J. Körner and K. Marton, “New bounds for perfect hashing via information theory,” *European Journal of Combinatorics*, vol. 9, no. 6, pp. 523–530, nov 1988. [Online]. Available: <https://doi.org/10.1016%2Fs0195-6698%2888%2980048-9>
- [7] G. Hansel, “Nombre minimal de contacts de fermeture nécessaires pour réaliser une fonction booléenne symétrique de n variables,” *C. R. Acad. Sci. Paris*, pp. 6037–6040, 1964.
- [8] J. Körner, “Coding of an information source having ambiguous alphabet and the entropy of graphs,” *6th Prague Conference on Information Theory*, pp. 411–425, 1973.
- [9] J. Radhakrishnan, “Entropy and counting,” 2001.
- [10] A. Nilli, “Perfect hashing and probability,” *Combinatorics, Probability and Computing*, vol. 3, no. 03, pp. 407–409, sep 1994. [Online]. Available: <https://doi.org/10.1017%2Fs0963548300001280>
- [11] E. Arikan, “An upper bound on the zero-error list-coding capacity,” *IEEE Transactions on Information Theory*, vol. 40, no. 4, pp. 1237–1240, jul 1994. [Online]. Available: <https://doi.org/10.1109%2F18.335947>
- [12] E. Ok, *Real analysis with economic applications*. Princeton University Press, 9 2011.
- [13] W. R. Inc., “Mathematica, Version 11.3,” champaign, IL, 2018.