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Finding forbidden minors in sublinear time: a $O(n^{1/2+o(1)})$-query one-sided tester for minor closed properties on bounded degree graphs

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Abstract

Let G be an undirected, bounded degree graph with n vertices. Fix a finite graph H, and suppose one must remove εn edges from G to make it H-minor free (for some small constant $\varepsilon > 0$). We give an $n^{1/2+o(1)}$-time randomized procedure that, with high probability, finds an H-minor in such a graph. For an example application, suppose one must remove εn edges from a bounded degree graph G to make it planar. This result implies an algorithm, with the same running time, that produces a $K_{3,3}$ or K_5 minor in G. No sublinear time bound was known for this problem, prior to this result.

By the graph minor theorem, we get an analogous result for any minor-closed property. Up to $n^{o(1)}$ factors, this resolves a conjecture of Benjamini-Schramm-Shapira (STOC 2008) on the existence of one-sided property testers for minor-closed properties. Furthermore, our algorithm is nearly optimal, by an $\Omega(\sqrt{n})$ lower bound of Czumaj et al (RSA 2014).

Prior to this work, the only graphs H for which non-trivial property testers were known for H-minor freeness are the following: H being a forest or a cycle (Czumaj et al, RSA 2014), $K_{2,k}$, $(k \times 2)$-grid, and the k-circus (Fichtenberger et al, Arxiv 2017).

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1 Introduction

Deciding if an n-vertex graph G is planar is a classic algorithmic problem, solvable in linear time [HT74]. The Kuratowski-Wagner theorem asserts that any non-planar graph must contain a K_5 or $K_{3,3}$-minor [Kur30, Wag37]. Thus, certifying non-planarity is equivalent to producing such a minor, which can be done in linear time. Can we beat the linear time bound if we knew that G was "sufficiently" non-planar?

Let us assume that we have random access to an adjacency list representation of a bounded degree graph G. Suppose (for some constant $\varepsilon > 0$) one had to remove εn edges from G to make it planar. Can one find a forbidden (K_5 or $K_{3,3}$) minor in $o(n)$ time? It is natural to ask this question for any property expressible through forbidden minors. By the famous Robertson-Seymour graph minor theorem [RS04], any graph property $\mathcal{P}$ that is closed under taking minors can be expressed by a finite list of forbidden minors. We desire sublinear time algorithms to find a forbidden minor in any G that requires εn edge deletions to make it have $\mathcal{P}$.

This problem was first posed by Benjamini-Schramm-Shapira [BSS08] in the context of property testing on bounded degree graphs. We follow the model of property testing on bounded-degree graphs as defined by Goldreich-Ron [GR02]. Fix a degree bound d. Consider $G = (V, E)$, where $V = [n]$, and G is represesnted by an adjacency list. We have random access to the list and can make *neighbor queries*. There is an oracle that, given $v \in V$ and $i \in [d]$, returns the ith neighbor of v (if no neighbor exists, it returns $\perp$).

Given any property $\mathcal{P}$ of bounded-degree graphs, the distance of G to $\mathcal{P}$ is the minimum number of edge additions/removals required to make G have $\mathcal{P}$, divided by the total number of edges, dn. This ensures that the distance is in $[0, 1]$. We say that G is ε-far from $\mathcal{P}$ if the distance to $\mathcal{P}$ is more than ε.

A property tester for $\mathcal{P}$ is a randomized procedure takes as input (query access to) G and a proximity parameter $\varepsilon > 0$. If $G \in \mathcal{P}$, the tester must accept with probability at least $2/3$. If G is ε-far from $\mathcal{P}$, the tester must reject with probability at least $2/3$. A one-sided tester must accept $G \in \mathcal{P}$ with probability 1, and thus provides a certificate of rejection.

We are interested in property $\mathcal{P}$ expressible through *forbidden minors*. Fix a finite graph H. The property $\mathcal{P}_H$ of *H-minor freeness* is exactly the set of graphs that do not contain H as a minor. Observe that one-sided testers for $\mathcal{P}_H$ have a special significance since they must produce an H-minor whenever they reject. One can cast one-sided property testers as sublinear time procedures that find forbidden minors. Our main theorem follows.

Theorem 1.1. *Fix a finite graph H with $|V(H)| = r$ and arbitrarily small $\delta > 0$. Let $\mathcal{P}_H$ be the property of H-minor freeness. There is a randomized algorithm that takes as input a graph G with maximum degree d, and a parameter $\varepsilon > 0$. Its running time is $dn^{1/2+O(\delta r^4)} + d\varepsilon^{-2\exp(2/\delta)/\delta}$. If G is ε-far from $\mathcal{P}_H$, then, with probability $> 2/3$, the algorithm outputs an H-minor in G.*

Equivalently, there exists a one-sided property tester for $\mathcal{P}_H$ with the above running time.

The graph-minor theorem of Robertson and Seymour [RS04] asserts the following. Consider any property $\mathcal{Q}$ that is closed under taking minors. There is a finite list $\boldsymbol{H}$ of graphs such that $G \in \mathcal{Q}$ iff G is H-minor free for all $H \in \boldsymbol{H}$. Hence, if G is ε-far from $\mathcal{Q}$, then G is $\Omega(\varepsilon)$-far from $\mathcal{P}_H$ for some $H \in \boldsymbol{H}$.

Thus, a direct corollary of Theorem 1.1 is the following.

Corollary 1.2. *Let $\mathcal{Q}$ be any minor-closed property of graphs with degree bound d. For any $\delta > 0$, there is a one-sided property tester for $\mathcal{Q}$ with running time $O(dn^{1/2+\delta} + d\varepsilon^{-2\exp(2/\delta)/\delta})$.*

In the following discussion, we suppress dependences on ε and n^δ by $O^*(\cdot)$ (where $\delta > 0$ is arbitrarily small). Previously, the only graphs H for which an analogue of Theorem 1.1 was known are the following: $O^*(1)$ time for H being a forest, $O^*(\sqrt{n})$ for H being a cycle [CGR+14], and $O^*(n^{2/3})$ for H being $K_{2,k}$, the $(k \times 2)$-grid, and the k-circus [FLVW17]. No sublinear time bound was known for planarity.

Corollary 1.2 implies that properties such as planarity, series-parallel graphs, embeddability in bounded genus surfaces, and bounded treewidth are all testable in $O^*(\sqrt{n})$ time.

We note a particularly pleasing application of Theorem 1.1. Suppose bounded degree G has more than $(3+\varepsilon)n$ edges. Then it is guaranteed to be ε-far from being planar, and thus, there is an algorithm to find a forbidden minor in G in $O^*(\sqrt{n})$ time. Since all minor-closed properties have constant average degree bounds, analogous statements can be make for all such properties.

1.1 Related work

Graph minor theory is a deep topic, and we refer the reader to Chapter 12 of Diestel's book [Die10] and Lovász' survey [Lov06]. For our purposes, we use as a black-box polynomial time algorithms that find fixed minors in a graph. A result of Kawarabayashi-Kobayashi-Reed provide an $O(n^2)$ time algorithm [KKR12].

Property testing on graphs is an immensely rich area of study, and we refer the reader to Goldreich's recent textbook for more details [Gol17]. There is a significant difference between the theory of testing for dense graphs and that of bounded-degree graphs. For the former, there is a complete characterization of properties (one-sided, non-adaptive) testable in query complexity independent of graph size. There is a deep connection between property testing and the Szemeredi regularity lemma [AFNS06]. Property testing for bounded degree graphs is much less understood. This study was initiated by Goldreich and Ron, and the first results focused on connectivity properties [GR02]. Czumaj-Sohler-Shapira proved that hereditary properties of non-expanding graphs are testable [CSS09]. A breakthrough result of Benjamini-Schramm-Shapira (henceforth BSS) proved that all minor-closed (more generally, hyperfinite) properties are two-sided testable in constant time. The dependence on ε was subsequently improved by Hassidim et al, using the concept of local partitioning oracles [HKNO09]. A result of Levi-Ron [LR15] significantly simplified and improved this analysis, to get a final query complexity quasi-polynomial in $1/\varepsilon$. Indeed, it is a major open question to get polynomial dependence on $1/\varepsilon$ for two-sided planarity testers. Towards this goal, Ito and Yoshida give such a bound for testing outerplanarity [YI15].

In contrast to dense graph testing, there is a significant jump in complexity for one-sided testers. BSS first raised the question of one-sided testers for minor-closed properties (especially planarity) and conjectured that the bound is $O(\sqrt{n})$. Czumaj et al [CGR+14] made the first step by giving an $\widetilde{O}(\sqrt{n})$ one-sided tester for the property of being C_k-minor free [CGR+14]. For $k = 3$, this is precisely the class of forests. But the main heavy hammer used is a much older result of Goldreich-Ron for one-sided bipartiteness testing for bounded degree graphs [GR99]. (The results in Czumaj et al are obtained by black-box applications of this result.) Czumaj et al adapt the one-sided $\Omega(\sqrt{n})$ lower bound for bipartiteness and show an $\Omega(\sqrt{n})$ lower bound for one-sided testers for H-minor freeness when H has a cycle [CGR+14]. This is complemented with a constant time tester for H-minor freeness when H is a forest.

Recently, Fichtenberger et al give an $\widetilde{O}(n^{2/3})$ tester for H-minor freeness when H is one of the following graphs: $K_{2,k}$, the $(k \times 2)$-grid or the k-circus graph (a wheel where spokes have two edges) [FLVW17]. This subsumes the properties of outerplanarity and cactus graphs. This result uses a different, more combinatorial (as opposed to random walk based) approach than Czumaj et al.

The use of random walks in property testing was pioneered by Goldreich-Ron [GR99] and was then (naturally) used in testing expansion properties and clustering structure [GR00, CS10, KS08, NS07, KPS13, CPS15]. Our approach is directly inspired by the Goldreich-Ron analysis, and we discuss more in the next section. A number of previous results have used random walks for routing in expanders [BFU99, KR96]. We use techniques from Kale-Seshadhri-Peres to analyze random walks on projected Markov Chains [KPS13]. We also employ the local partitioning methods of Spielman-Teng [ST12], in turn derived from the Lovász-Simonovitz analysis technique [LS90].

2 Main Ideas

We give an overview of the proof strategy and discuss the various moving parts of the proof. Our aim is to express as much of the intuition with as little math as possible. For convenience, assume that G is a d-regular graph. It is instructive to understand the method of Goldreich-Ron (henceforth GR) for one-side bipartiteness testing [GR99]. (The cycle minor finding result of Czumaj et al reduce cycle-minor freeness to bipartiteness.) The basic idea to perform $O(\sqrt{n})$ random walks of $\mathrm{poly}(\log n)$ length from a uar vertex s. If two walks end at the same vertex v and the walks induce paths of differing parity (of length), then this gives an odd cycle.

The main insight of Goldreich-Ron is to first analyze this algorithm when G is an expander (and ε-far from bipartite). In this case, the walks from s reach the stationary distribution. One can use a standard collision argument to show that $O(\sqrt{n})$ suffice to hit the same vertex v twice, with different parity paths. The deep insight is that any graph G can be decomposed into pieces where the algorithm works, and each piece P has a small cut to $\overline{P}$. (This has connections with decomposing a graph into expander-like pieces [Tre05, GT12]. Most famously, the Arora-Barak-Steurer algorithm [ABS15] for unique games basically proves such a statement. We note that Goldreich-Ron do *not* decompose into expanders, but rather into pieces where the expander analysis goes through.) So, one might hope to analyze the algorithm by its behavior on each component. Unfortunately, the algorithm cannot produce the decomposition; it can only walk in G and hope that performing random walks in G suffice to simulate the procedure within P. This is extremely challenging, and is precisely what Goldreich-Ron do (it forms the bulk of their analysis). Their main lemma produces a decomposition into such pieces, such that for each piece P, there exists $s \in P$ wherein short random walks (in G) from s reach all vertices in P with sufficient probability. One can think of this a simulation argument: we would like to simulate the random walk algorithm running only on P, through random walks in G.

The challenge of general minors: With planarity in mind, let us focus on finding K_5 minors. It is highly unlikely that random walks from a single vertex will find a such a minor. Intuitively, we would need to find 5 different vertices, launch random walks from all of them, and hope these walks will produce a minor. Thus, we would need to simulate a much more complex procedure than the (odd) cycle finder of GR. Most significantly, we need to understand the random walks behavior from multiple sources within P simultaneously. The GR analysis actually constructs the pieces P by a local partitioning looking at the random walk distribution from a single vertex. There is no guarantee on random walk behavior from other vertices in P.

There is an even bigger challenge from arbitrary minors. The simulation does not say anything about the specific structure of the paths generated. It only deals with the probability of reaching v from s by a random walk in G when v and s are in the same piece. For bipartiteness, as long as we find two paths of differing parity, we are done. They may intersect each other arbitrarily. For finding a K_5 minor, the actual structure is relevant. We would need paths between all pairs of 5 seed vertices to be "disjoint enough" to give a K_5 minor. This appears extremely difficult using the GR analysis. Even if we did understand the random walk behavior (in G) from all vertices in P, we have little control over their behavior when they leave P. They may intersect arbitrarily, and thus destroy any minor structure.

2.1 When do random walks find minors?

Inspired by GR, let us start with an algorithm to find a K_5 minor in an expander G. (Variants of these ideas were present in a result of Kleinberg-Rubinfeld that expanders contain an H-minor for any H with $n/\mathrm{poly}(\log n)$ edges [KR96].) Let ℓ denote the mixing time. Pick u.a.r. a vertex s, and launch 5 random walks each of length ℓ to reach $v_1, v_2, \ldots, v_5$. From each v_i, launch $\sqrt{n}$ random walks each of length ℓ. With high probability, a walk from v_i and a walk from v_j will "collide" (end at the same vertex). We can collect these collisions to get paths between all v_i, v_j, and one can, with some effort, show that these form a K_5-minor.

Our main insight is to show that this algorithm, with minor modifications, works even when random walks have extremely slow mixing properties. When the random walks mix even more slowly than the requisite bound, we can essentially perform local partitioning to pull out very small (n^δ for arbitrarily small $\delta > 0$) pieces that have low conductance cuts. We can simply query all edges in this piece and run a planarity test.

We have a parameter $\delta > 0$ that can be set to an arbitrarily small constant. Let us set our random walk length ℓ to n^δ, and let $\mathbf{p}_{s,\ell}$ be the random walk distribution after ℓ steps from s. Our proof splits into two cases, where $\alpha = c\delta$ for explicit constant $c > 1$:

- Case 1 (the leaky case): For at least εn vertices s, $\|\mathbf{p}_{s,\ell}\|_2^2 \leq 1/n^\alpha$.
- Case 2 (the trapped case): For at least $(1-\varepsilon)n$ vertices s,$\|\mathbf{p}_{s,\ell}\|_2^2 > 1/n^\alpha$.

Note that in the leaky case, random walks are hardly mixing by any standard of convergence. For a random walk of length n^δ, we are merely requiring that the walk (roughly speaking) spreads to a set of size $n^{c\delta}$.

We prove that, in the leaky case, the procedure described in the first paragraph succeeds in finding a K_5 with high probability. We give an outline of this proof strategy.

Let us assume that $\mathbf{p}_{v,\ell/2} = \mathbf{p}_{v,\ell}$ and make a slight modification to the algorithm. We pick $v_1, \ldots, v_5$ as before, with ℓ-length random walks from s. We will perform $O(\sqrt{n})$ $\ell/2$ length random walks from each v_i to produce the K_5 minor. By symmetry of the random walks, the probability that a single walk from v_i and one from v_j collide (to produce a path) is exactly $\mathbf{p}_{v_i,\ell/2} \cdot \mathbf{p}_{v_j,\ell/2}$. Thus, we would like these dot products to be large. By the symmetry of the random walk, the probability of an ℓ-length random walk starting from s and ending at v is $\mathbf{p}_{s,\ell/2} \cdot \mathbf{p}_{v,\ell/2}$. In other words, the entries of $\mathbf{p}_{s,\ell}$ are precisely these dot products, and $\|\mathbf{p}_{s,\ell}\|_2^2 = \sum_{v \in V} (\mathbf{p}_{s,\ell/2} \cdot \mathbf{p}_{v,\ell/2})^2 = \mathbf{E}_{v \sim \mathbf{p}_{s,\ell/2}}[\mathbf{p}_{s,\ell/2} \cdot \mathbf{p}_{v,\ell/2}]$. Since $\mathbf{p}_{s,\ell/2} = \mathbf{p}_{s,\ell}$, we rewrite to get $\mathbf{p}_{s,\ell/2} \cdot \mathbf{p}_{s,\ell/2} = \mathbf{E}_{v \sim \mathbf{p}_{s,\ell/2}}[\mathbf{p}_{s,\ell/2} \cdot \mathbf{p}_{v,\ell/2}]$.

Here is a major insight. Think of the dot products as correlations between distributions. We are saying that the average correlation (over some distribution on vertices) of $\mathbf{p}_{v,\ell/2}$ with $\mathbf{p}_{s,\ell/2}$ is exactly the self-correlation of $\mathbf{p}_{s,\ell/2}$. If the distributions by and large had low ℓ_2-norm (as in the leaky case), we might hope that these distributions are reasonably correlated with each other. Indeed, this is what we prove. Under some conditions, we show that $\mathbf{E}_{v_i, v_j \sim \mathbf{p}_{s,\ell/2}}[\mathbf{p}_{v_i,\ell/2} \cdot \mathbf{p}_{v_j,\ell/2}]$ can be lower bounded. And $\mathbf{p}_{v_i,\ell/2}$ is exactly the distribution the algorithm picks the v_i's from, and so this is evidence that $\ell/2$-length random walks will connect the v_i's through collisions.

There are three difficulties in increasing order of worry. Firstly, we only have a lower bound of the average $\mathbf{p}_{v_i,\ell/2} \cdot \mathbf{p}_{v_j,\ell/2}$. We would need bounds for all (or most) pairs to produce a minor. Secondly, the expected number of collisions between walks from v_i and v_j is controlled by the dot product above, but the variance (which really controls the probability of getting a collision) depends on more refined properties of these distributions. There are instances where the dot product is high, but the collision probability is extremely low. Thirdly, there is no guarantee that these paths will produce a minor since we do not have any obvious constraints on the intermediate vertices in the path.

If one could assume that these distributions were uniform in their support, the second and third problems go away. The first problem is surmounted by a technical trick. It turns out to be cleaner to analyze the probability of getting a biclique minor. So, we perform 50 random walks from s to get sets $A = \{a_1, a_2, \ldots, a_{25}\}$ and an analogous B. We launch $\ell/2$-length random walks from each vertex in $A \cup B$. The average lower bound on the dot product suffices to get a lower bound on the probability of getting a $K_{25,25}$-minor, which contains a K_5-minor.

All in all, under the unreasonable assumptions that the distributions are uniform in their support (call these *support uniform*) and $\mathbf{p}_{s,\ell/2} = \mathbf{p}_{s,\ell}$, the leaky case can be handled.

2.2 R-returning walks

The main technical contribution of our work is in defining R-returning walks. These are walks that periodically return to a given set R of vertices. Fix ℓ as before. Formally, an R-returning walk of length $j\ell$ (for $j \in \mathbb{N}$) is a walk that encounters R at every $i\ell$ step $\forall i \in [j]$. While random walk distributions can be far from support uniform, we can carefully choose R to ensure that the distribution of R-returning walks is approximately support uniform.

In the leaky case, there is some (large) set R, such that $\forall s \in R$, $\|\mathbf{p}_{s,\ell/2}\|_2^2 \leq 1/n^\alpha$. Let $\mathbf{p}_{[R],s,\ell}$ be the random walk distribution restricted to R. Suppose for some $s \in R$, $\|\mathbf{p}_{[R],s,\ell}\|_2^2 \geq 1/n^{\alpha+\delta}$. Observe that each entry in $\mathbf{p}_{[R],s,\ell}$ is $\mathbf{p}_{s,\ell/2} \cdot \mathbf{p}_{v,\ell/2}$, for $s, v \in R$. By Cauchy-Schwartz, this is at most $1/n^\alpha$. For any distribution $\mathbf{v}$, the condition $\|\mathbf{v}\|_2^2 = \|\mathbf{v}\|_\infty$ is equivalent to support uniformity. Thus, $\mathbf{p}_{[R],s,\ell}$ is approximately support uniform, up to n^δ deviations. The math discussed in the previous section goes through for any such s. In other words, if the random walk algorithm started from s, it succeeds in finding a K_5 minor.

Suppose only a negligible fraction of vertices satisfied this condition, and so our algorithm would not actually find such a vertex. Let us remove all these vertices from R (abusing notation, let R be the resulting set). Now, $\forall s \in R$, $\|\mathbf{p}_{[R],s,\ell}\|_2^2 \leq 1/n^{\alpha+\delta}$. So, the bound on the l_2-norm has fallen by an n^δ factor. What does $\mathbf{p}_{[R],s,\ell} \cdot \mathbf{p}_{[R],v,\ell}$ signify? This is the probability of a 2ℓ-length random walk starting from s, ending at v, and encountering R at the ℓth step. This is an R-returning walk of length 2ℓ. Let $\mathbf{q}_{[R],s,2\ell}$ denote the vector of R-returning walk probabilities. Suppose for some s, $\|\mathbf{q}_{[R],s,2\ell}\|_2^2 \geq 1/n^{\alpha+2\delta}$. By Cauchy-Schwartz, $\|\mathbf{q}_{[R],s,2\ell}\|_\infty \leq 1/n^{\alpha+\delta}$, implying that $\mathbf{q}_{[R],s,2\ell}$ is approximately support uniform. Again, the math of the previous section goes through for such an s.

We now remove all vertices that have this property, and end up with R such that $\forall s \in R$, $\|\mathbf{q}_{[R],s,2\ell}\|_2^2 \leq 1/n^{\alpha+2\delta}$. Observe that $\mathbf{q}_{[R],s,2\ell} \cdot \mathbf{q}_{[R],v,2\ell}$ is a probability of a 4ℓ R-returning walk. And we iterate this argument.

In general, this argument goes through phases. In the ith phase, we find $s \in R$ that satisfy

$\|\mathbf{q}_{[R],s,2^i\ell}\|_2^2 \geq 1/n^{\alpha+i\delta}$. We show that the random walk procedure of the previous section (with some modifications) finds a K_5-minor starting from such vertices. We remove all such vertices from R, increment i, and continue the argument. The vertices removed at the ith phase are called the *ith stratum*, and we refer to this entire process as stratification. Intuitively, for vertices in the ith stratum, the R-returning (for the setting of R at that phase) walks roughly form a uniform distribution of support $n^{\alpha+i\delta}$. Thus, for vertices in higher strata, the random walks are spreading to larger sets.

We seem to have ignored a major problem. The $\mathbf{q}$ vectors are *not* distributions, and the vast majority of walks are not R-returning. Indeed, the reduction in norm as we increase strata might simply be an artifact of the lower probability of a longer R-returning walk (note that the walks lengths are increasing exponentially in the phase number). We prove a spectral lemma asserting that this is not the case. As long as R is sufficiently large, the probabilities of R-returning walks are sufficiently high. Unfortunately, these probabilities (must) decrease exponentially in the number of returns. In the ith phase, the walk length is $2^i\ell$ and it must return to R 2^i times. Here is where the n^δ decay in l_2-norm condition saves us. After $1/\delta$ phases, the $\|\mathbf{q}_{[R],s,2^i\ell}\|_2^2$ is basically $1/n$. The spectral lemma tells us that if R is still large, the probability that a $2^{1/\delta}\ell$ length walk is R-returning is sufficiently large. Thus, the norm cannot decrease, and almost all vertices end up in the very next stratum. If R was small, then there is an earlier stratum containing $\Omega(\varepsilon n)$ vertices. Regardless of the case, we deduce that the random walk procedure succeeds in finding a minor for $\Omega(\varepsilon n)$ starting vertices.

2.3 The trapped case: local partitioning to the rescue

In this case, for almost all vertices $\|\mathbf{p}_{s,\ell}\|_2^2 \geq 1/n^\alpha$. The proofs of the (contrapositive of the) Cheeger inequality basically imply the existence of a set of low condutance cut P_s "around" s. By local partitioning methods such as those of Spielman-Teng and Anderson-Chung-Lang [ST12, ACL06], we can actually find P_s in roughly n^α time. We expect our graph to basically decompose into $O(n^\alpha)$ sized components with few edges between them. Our algorithm can simply find these pieces P_s and run a planarity test on them. We refer to this as the *local search* procedure.

While the intuition is mostly correct, the analysis is quite challenging. The main problem is that actual partitioning of the graph (into small components connected by low conductance cuts) is fundamentally iterative. It starts by finding a low conductance set P_{s_1}, then finding a low conductance set P_{s_2} in $\overline{P_{s_1}}$, then P_{s_3} in $\overline{P_{s_1} \cup P_{s_2}}$, and so on. In general, this requires conditions on the random walk behavior inside $\overline{\bigcup_{j<i} P_{s_j}}$. On the other hand, our algorithm and the trapped case condition only refer to random walk behavior in all of G. Furthermore, $\overline{\bigcup_{j<i} P_{s_j}}$ can be as small as $\Theta(\varepsilon n)$, and so we do expect the random walk behavior to be quite different.

The GR bipartiteness analysis surmount this problem and performs such a decomposition, but their parameters do not work for us. Starting from a source vertex s, their analysis discovers P_s such that probabilities of reaching any vertex in P_s (from s) is roughly uniform *and* smaller than $1/\sqrt{n}$. On the other hand, we would like to discover all of P_s in $n^{O(\delta)}$ time so that we can run a full planarity test.

We employ a collection of tools, and crucially use the methods of Kale-Peres-Seshadhri to analyze "projected" Markov Chains [KPS13]. Basically, in the analysis above, we have some set S ($\overline{\bigcup_{j<i} P_{s_j}}$) and want to find a low conductance set P completely contained in S. On the other hand, we wish to discover P using random walks in G. We construct a Markov chain, M_S, with

vertex set S, and include new transitions that correspond to walks in G whose intermediate vertices are not in S. Each such transition has an associated "cost", corresponding to the actual length in G. (GR also have a similar idea, although their Markov chain introduces extra vertices to track the length of the walk in G. This makes the analysis somewhat unwieldy, since low conductance cuts in M_S may include these extra vertices.)

Using bounds on the return time of random walks, we have relationships between the average length of a walk in G whose endpoints are in S and the corresponding length when "projected" to M_S. On average, an ℓ-length walk in G with endpoints in S corresponds to an $\ell|S|/n$-length walk in M_S. Roughly speaking, we hope that for many vertices s, an $\ell|S|/n$-length walk in M_S is trapped in a set of size n^α. Unfortunately, the variance of the walk length can be quite high. But, given our parameters, we can simply use a Markov bound to get statements for length from a specific s.

We employ the Lovász-Simonovitz curve technique to produce a low conductance cut P_s in M_S. We can guarantee that all vertices in P_s are reachable with roughly $n^{-\alpha}$ probability from s through $\ell|S|/n$-length random walks in M_S. Using the average length correspondence between walks in M_S to G, we can make a similar statement in G - albeit with a longer length. We basically iterate over this entire argument to produce the decomposition into low conductance pieces.

3 The algorithm

We are given a bounded degree graph $G = (V, E)$, with max degree d. We assume that $V = [n]$. We follow the standard adjacency list model of Goldreich-Ron for (random) access to the graph. Clearly, an algorithm can sample u.a.r. vertices. An algorithm can also perform *edge queries*. Given a pair $(v, i) \in [n] \times [d]$, the output is the ith neighbor of v (according to the adjacency list ordering). If the degree of v is smaller than i, the output is $\perp$.

In the algorithm, the phrase "random walk" really refers to a lazy random walk on G. Given a current vertex v, with probability 1/2, the walk remains at v. With probability 1/2, the procedure generates u.a.r. $i \in [d]$. It performs the edge query for (v, i). If the output is $\perp$, the walk remains at v, otherwise the walk visits the output vertex. This is a symmetric Markov chain with a uniform stationary distribution.

Our main procedure `FindMinor`(G, ε, H), tries find a H-minor in G. We will prove that it succeeds with high probability if G is ε-far from being H-minor free. There are three subroutines that it uses.

- `LocalSearch`(s): This procedure perform a small number of short random walks to find the piece described in §2.3. This produces a small subgraph of G, where an exact H-minor finding algorithm is used.
- `FindPath`(u, v, k, i): This procedure tries to find a path from u to v. The parameter i decides the length of the walk, and the procedure performs k walks from u and v. If any pair of these walks collide, this path is output.
- `FindBiclique`(s): This is the main procedure, mostly as described in §2.1. It attempts to find a sufficiently large biclique minor. First, it generates seed sets A and B by performing random walks from s. Then, it calls `FindPath` on all pairs in $A \times B$.

We fix a collection of parameters at the outset that make repeated appearances in the analysis.

- δ: An arbitrarily small constant.

- r: The number of vertices in H.
- ℓ: The random walk length. This will be $n^{5\delta}$.
- $\varepsilon_{\mathsf{CUTOFF}}$: $\varepsilon_{\mathsf{CUTOFF}} = n^{\frac{-\delta}{exp(2/\delta)}}$. This is mostly a convenience for analysis, since if $\varepsilon < \varepsilon_{\mathsf{CUTOFF}}$, the algorithm just queries the whole graph.
- $\texttt{KKR}(F, H)$: This refers to an exact H-minor finding process. For concreteness, we use the quadratic time procedure of Kawarabayashi-Kobayashi-Reed [KKR12].

```
FindMinor(G, ε, H)
  1. If ε < ε_CUTOFF, query all of G, and output KKR(G, H)
  2. Else
     (a) Repeat ρ_1 = ε^{-2} n^{35δr^4} times:
           i. Pick uar s ∈ V
          ii. Call LocalSearch(s) and FindBiclique(s).
```

```
LocalSearch(s)
  1. Initialize set B = ∅.
  2. For h = 1, ..., n^{7δr^4}:
     (a) Perform ε^{-1} n^{30δr^4} independent random walks of length h from s. Add all
         vertices reached at the end to B.
  3. Determine G[B], the subgraph induced by B.
  4. Run KKR(G[B], H). If it returns an H-minor, output that and terminate.
```

```
FindBiclique(s)
  1. For i = 5r^4, ..., 1/δ + 4:
     (a) Perform 2r^2 independent random walks of length 2^{i+1}ℓ from s. Let the destina-
         tions of the first r^2 walks be multiset A, and the destinations of the remaining
         walks be B.
     (b) For each a ∈ A, b ∈ B:
           i. Run FindPath(a, b, n^{δ(i+9)/2}, i)
     (c) If all calls to FindPath return a path, then let the collection of paths be
         the subgraph F. Run KKR(F, H). If it returns an H-minor, output that and
         terminate.
```

```
FindPath(u, v, k, i)
  1. Perform k random walks of length 2^i ℓ from u and v.
  2. If a walk from u and v terminate at the same vertex, return these paths. (Otherwise,
     return nothing.)
```

Theorem 3.1. *If G is ε-far from being H-minor free, then* $\texttt{FindMinor}(G, \varepsilon, H)$ *finds an H-minor of G with probability at least $2/3$. Furthermore,* $\texttt{FindMinor}$ *has a running time of $dn^{1/2+O(\delta r^4)} + d\varepsilon^{-2\exp(2/\delta)/\delta}$.*

We note that the query complexity is fairly easy to see. The total queries made in the $\texttt{LocalSearch}$ calls is $dn^{O(\delta r^4)}$. The main work happens in the calls of $\texttt{FindPath}$, within $\texttt{FindBiclique}$. Observe that k is set to $n^{\delta(i+9)/2}$, where $i \leq 1/\delta + 4$. This leads to the $\sqrt{n}$ in the final complexity. (In general, a setting of $\delta < 1/\log(\varepsilon^{-1} \log\log n)$ suffices for an $n^{1/2+o(1)}$ running time.)

Outline: There are a number of moving parts in the proof, which we relegate to their own subsections. We first develop the notion of R-returning walks and the stratification process, given in §4. In §5, we use these techniques to prove that `FindBiclique` discovers a sufficiently large biclique-minor in the leaky case. In §6, we prove a local partitioning lemma that will be used to handle the trapped case. Finally, in §7, we put the tools together to complete the proof of Theorem 3.1.

4 Returning walks and stratification

We introduce the concept of R-returning random walks for any $R \subseteq V$. These definitions are with respect to a fixed length ℓ.

Definition 4.1. *For any set of vertices R, $s \in R$, $u \in R$, and $i \in \mathbb{N}$, we define the* R-returning probability *as follows. We denote by $q^{(i)}_{[R],s}(u)$ the probability that a $2^i\ell$-length random walk from s ends at u, and encounters a vertex in S at every $j\ell^{th}$ step, for all $1 \leq j \leq 2^i$. The R-returning probability vector, denoted by $\boldsymbol{q}^{(i)}_{[R],s}$ is the $|R|$-dimensional vector of returning probabilities.*

Proposition 4.2. $q^{(i+1)}_{[R],s}(u) = \boldsymbol{q}^{(i)}_{[R],s} \cdot \boldsymbol{q}^{(i)}_{[R],u}$

Proof. We use the symmetric nature of (returning) random walks in G.

$$q^{(i+1)}_{[R],s}(u) = \sum_{w \in S} q^{(i)}_{[R],s}(w) q^{(i)}_{[R],w}(u) = \sum_{w \in R} q^{(u)}_{[R],s}(w) q^{(i)}_{[R],u}(w) = \boldsymbol{q}^{(i)}_{[R],s} \cdot \boldsymbol{q}^{(i)}_{[R],u}$$

□

Let M be the transition matrix of the standard random walk on G. Let $\mathbb{P}_R$ be the $n \times |R|$ matrix on R, where each column is the unit vector for some $s \in R$. For any set U, we use $\mathbf{1}_U$ for the indicator vector on U. If no subscript is given, it is the all ones vector, for the appropriate dimension.

Proposition 4.3. $\boldsymbol{q}^{(i)}_{[R],s} = (\mathbb{P}_R^T M^\ell \mathbb{P}_R)^{2^i} \mathbf{1}_s$

Now for a critical lemma. We can lower bound the total probability of an R-returning random walk.

Lemma 4.4. $|R|^{-1} \sum_{s \in R} \|\boldsymbol{q}^{(i)}_{[R],s}\|_1 \geq (|R|/2n)^{2^i+1}$

Proof. We will express $\sum_{s \in R} \|\boldsymbol{q}^{(i)}_{[R],s}\|_1 = \mathbf{1}^T (\mathbb{P}_R^T M^\ell \mathbb{P}_R)^{2^i} \mathbf{1}$. First, let us prove for $i = 0$. Observe that $\sum_{s \in R} \|\boldsymbol{q}^{(0)}_{[R],s}\|_1 = \mathbf{1}_R^T M^\ell \mathbf{1}_R = ((M^T)^{\ell/2} \mathbf{1}_R)^T (M^{\ell/2} \mathbf{1}_R) = \|M^{\ell/2} \mathbf{1}_R\|_2^2$. Since $M^{\ell/2}$ is a stochastic matrix, $\|M^{\ell/2} \mathbf{1}_R\|_1 = \|\mathbf{1}_R\|_1 = |R|$. By a standard norm inequality, $\|M^{\ell/2} \mathbf{1}_R\|_2^2 \geq \|M^{\ell/2} \mathbf{1}_R\|_1^2/n = |R|^2/n$. This completes the proof for $i = 0$.

Let $N = \mathbb{P}_R^T M^\ell \mathbb{P}_R$, which is a symmetric matrix. We have just proven that $\mathbf{1}^T N \mathbf{1} \geq |R|^2/n$. Let the eigenvalues of N be $1 \leq \lambda_1 \leq \lambda_2 \ldots \lambda_s$, with corresponding eigenvectors $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_s}$. We can express $\mathbf{1} = \sum_{k \leq s} \alpha_k \mathbf{u_k}$, where $\sum_k \alpha_k^2 = |R|$. Writing out in the eigenbasis, $\mathbf{1}^T N \mathbf{1} = \sum_k \alpha_k^2 \lambda_k \geq |R|^2/n$. Partition the eigenvalues into the heavy and lights set as follows: $H = \{k | \lambda_k \geq |R|/2n\}$ and $L = \overline{H}$.

$$|R|^2/n \leq \sum_k \alpha_k^2 \lambda_k \leq (|R|/2n) \sum_{k \in L} \alpha_k^2 + \sum_{k \in H} \alpha_k^2 \Longrightarrow \sum_{k \in H} \alpha_k^2 \geq |R|^2/2n$$

Now, we deal with general i.

$$\mathbf{1}^T N^{2^i} \mathbf{1} = \sum_k \alpha_k^2 \lambda_k^{2^i} \geq \sum_{k \in H} \alpha_k^2 \lambda_k^{2^i} \geq (|R|/2n)^{2^i} (|R|^2/2n) = |R| \times (|R|/2n)^{2^i+1}$$

□

4.1 Stratification

We can now define the stratification process. This is a collection of disjoint sets of vertices denoted by $S_0, S_1, \ldots$, each of which are called *strata*. There are corresponding *residue* sets denoted by $R_0, R_1, \ldots$. We begin with $R_0 \subseteq V$ and construct the subsequent residue sets by the recurrence $R_i = R_0 \setminus \bigcup_{j<i} S_j$.

Definition 4.5. *Suppose R_i has been constructed. A vertex $s \in R_i$ is placed in S_i if $\|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_2^2 \geq 1/n^{\delta i}$.*

A consequence of this construction is that the R_i-returning walk vectors for a given stratum cannot be too long.

Claim 4.6. *For all $s \in R_i$ and $j \leq i$, $\|\boldsymbol{q}^{(j)}_{[R_i],s}\|_2^2 \leq 1/n^{\delta(j-1)}$.*

Proof. Suppose $\|\boldsymbol{q}^{(j)}_{[R_i],s}\|_2^2 > 1/n^{\delta(j-1)}$. By assumption, $s \in R_i \subseteq R_{j-1}$. An R_i-returning walk from s is also an R_{j-1}-returning walk. Thus, every entry of $\boldsymbol{q}^{(j)}_{[R_{j-1}],s}$ is at least that of $\boldsymbol{q}^{(j)}_{[R_i],s}$. So $\|\boldsymbol{q}^{(j)}_{[R_{j-1}],s}\|_2^2 \geq \|\boldsymbol{q}^{(j)}_{[R_i],s}\|_2^2 > 1/n^{\delta(j-1)}$. This implies that $s \in R_{j-1}$, contradicting the assumption that $s \in R_i$. □

This allows to get an ℓ_∞ bound on the returning probability vectors. Note that we allow j to be $i+1$ in the following bound.

Claim 4.7. *For all $s \in R_i$ and $j \leq i+1$, $\|\boldsymbol{q}^{(j)}_{[R_i],s}\|_\infty \leq 1/n^{\delta(j-2)}$.*

Proof. By Prop. 4.2, for any $v \in R_i$, $q^{(j)}_{[R_i],s}(v) = \boldsymbol{q}^{(j-1)}_{[R_i],s} \cdot \boldsymbol{q}^{(j-1)}_{[R_i],v}$. Note that $j-1 \leq i$. By Cauchy-Schwartz and Claim 4.6, $q^{(j)}_{[R_i],s}(v) \leq 1/n^{\delta(j-2)}$. □

As a consequence of these bounds, we are able to bound the amount of probability mass retained by R_i-returning walks.

Claim 4.8. *For all $s \in S_i$, $||\boldsymbol{q}^{(i+1)}_{[R_i],s}||_1 \geq n^{-\delta}$.*

Proof. Since $s \in S_i$, we have $||\boldsymbol{q}^{(i+1)}_{[R_i],s}||_2^2 \geq n^{-i\delta}$, and by Claim 4.7, $||\boldsymbol{q}^{(i+1)}_{[R_i],s}||_\infty \leq n^{-i(\delta-1)}$. By Hölder's inequality (or just upper bounding by the maximum) $||\boldsymbol{q}^{(i+1)}_{[R_i],s}||_2^2 \leq ||\boldsymbol{q}^{(i+1)}_{[R_i],s}||_1 ||\boldsymbol{q}^{(i+1)}_{[R_i],s}||_\infty$ and hence $||\boldsymbol{q}^{(i+1)}_{[R_i],s}||_1 \geq n^{-i\delta} n^{\delta(i-1)} = n^{-\delta}$. □

An important consequence of this definition is that most vertices lie in "early" strata.

Lemma 4.9. *Let $\varepsilon \geq \varepsilon_{\mathsf{CUTOFF}}$. Then, at most $\varepsilon n / \log n$ vertices are in $R_{1/\delta+3}$.*

Proof. Suppose for the sake of contradiction that $R_{1/\delta+3}$ has at least $\varepsilon\frac{n}{\log n}$ vertices. By Lemma 4.4,

$$|R_{1/\delta+3}|^{-1}\sum_{s\in R_{1/\delta+3}}\|\boldsymbol{q}^{(1/\delta+3)}_{[R_{1/\delta+3}],s}\|_1 \geq \left(\frac{\varepsilon}{\log n}\right)^{2^{1/\delta}+1}. \tag{1}$$

Thus, the bound holds for some specific $s \in R_{1/\delta+3}$, and thus

$$||\boldsymbol{q}^{(1/\delta+3)}_{[R_{1/\delta+2}],s}\|_2^2 \geq n^{-1}\left(\frac{\varepsilon}{\log n}\right)^{2^{1/\delta+1}+2}. \tag{2}$$

Observe that $\varepsilon \geq \varepsilon_{\mathsf{CUTOFF}} \geq n^{-\delta/\exp(1/\delta)}$. For sufficiently small δ, the latter is greater than $(\log n)n^{-2\delta/(2^{1/\delta+1}+2)}$. Plugging into the RHS of the previous equation, $||\boldsymbol{q}^{(1/\delta+3)}_{[R_{1/\delta+2}],s}\|_2^2 \geq 1/n^{1+2\delta} = 1/n^{\delta(1/\delta+2)}$. This implies that $v \in S_{1/\delta+2}$, which is a contradiction. $\square$

4.2 The correlation lemma

The following lemma is an important tool in our analysis. Roughly speaking, the intuition is as follows. Fix some $s \in S_i$. By Prop. 4.2, the probability $q^{(i+1)}_{[R_i],s}(v)$ is the correlation between the vectors $\boldsymbol{q}^{(i)}_{[R_i],s}$ and $\boldsymbol{q}^{(i)}_{[R_i],v}$. If many of these probabilities are large, then there are many v such that $\boldsymbol{q}^{(i)}_{[R_i],v}$ is correlated with $\boldsymbol{q}^{(i)}_{[R_i],s}$. But if a large set of vectors is correlated with a fixed vector $\boldsymbol{q}^{(i)}_{[R_i],s}$, then we expect that many of these vectors are correlated among themselves.

Definition 4.10. *For $s \in R_i$, the distribution $\mathcal{D}_{s,i}$ has support R_i, and the probability of $u \in R_i$ is $\hat{q}^{(i+1)}_{[R_i],s}(v) = q^{(i+1)}_{[R_i],s}(v)/\|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_1$. (For convenience, we will drop the subscript i in $\mathcal{D}_{s,i}$ when it is apparent.)*

Lemma 4.11. *Fix arbitrary $s \in R_i$.*

$$\mathbf{E}_{u_1,u_2\sim\mathcal{D}_{s,i}}[\boldsymbol{q}^{(i)}_{[R_i],u_1}\cdot\boldsymbol{q}^{(i)}_{[R_i],u_2}] \geq \frac{1}{\|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_1^2}\cdot\frac{\|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_2^4}{\|\boldsymbol{q}^{(i)}_{[R_i],s}\|_2^2}$$

Proof.

$$\mathbf{E}_{u_1,u_2\sim\mathcal{D}_{s,i}}[\boldsymbol{q}^{(i)}_{[R_i],u_1}\cdot\boldsymbol{q}^{(i)}_{[R_i],u_2}] \tag{3}$$

$$=\sum_{u_1,u_2\in R_i}\|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_1^{-2}q^{(i+1)}_{[R_i],s}(u_1)q^{(i+1)}_{[R_i],s}(u_2)\boldsymbol{q}^{(i)}_{[R_i],u_1}\cdot\boldsymbol{q}^{(i)}_{[R_i],u_2} \tag{4}$$

$$=\|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_1^{-2}\sum_{u_1,u_2\in R_i}(\boldsymbol{q}^{(i)}_{[R_i],s}\cdot\boldsymbol{q}^{(i)}_{[R_i],u_1})(\boldsymbol{q}^{(i)}_{[R_i],s}\cdot\boldsymbol{q}^{(i)}_{[R_i],u_2})(\boldsymbol{q}^{(i)}_{[R_i],u_1}\cdot\boldsymbol{q}^{(i)}_{[R_i],u_2}) \quad \text{(Prop. 4.2)} \tag{5}$$

$$=\|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_1^{-2}\sum_{u_1,u_2\in R_i}(\boldsymbol{q}^{(i)}_{[R_i],s}\cdot\boldsymbol{q}^{(i)}_{[R_i],u_1})(\boldsymbol{q}^{(i)}_{[R_i],s}\cdot\boldsymbol{q}^{(i)}_{[R_i],u_2})\sum_{w\in R_i}q^{(i)}_{[R_i],u_1}(w)q^{(i)}_{[R_i],u_2}(w)) \tag{6}$$

$$=\|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_1^{-2}\sum_{w\in R_i}\sum_{u_1,u_2\in R_i}[(\boldsymbol{q}^{(i)}_{[R_i],s}\cdot\boldsymbol{q}^{(i)}_{[R_i],u_1})q^{(i)}_{[R_i],u_1}(w)][(\boldsymbol{q}^{(i)}_{[R_i],s}\cdot\boldsymbol{q}^{(i)}_{[R_i],u_2})q^{(i)}_{[R_i],u_2}(w)] \tag{7}$$

$$=\|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_1^{-2}\sum_{w\in R_i}\left[\sum_{u\in R_i}(\boldsymbol{q}^{(i)}_{[R_i],s}\cdot\boldsymbol{q}^{(i)}_{[R_i],u})q^{(i)}_{[R_i],u}(w)\right]^2 \tag{8}$$

We now write out $\|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_2^2 = \sum_{u \in R_i} q^{(i+1)}_{[R_i],s}(u)^2 = \sum_{u \in R_i} (\boldsymbol{q}^{(i)}_{[R_i],s} \cdot \boldsymbol{q}^{(i)}_{[R_i],u})^2$, by Prop. 4.2. We expand further below. The only inequality is Cauchy-Schwartz.

$$\begin{aligned}
\|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_2^2 &= \sum_{u \in R_i} (\boldsymbol{q}^{(i)}_{[R_i],s} \cdot \boldsymbol{q}^{(i)}_{[R_i],u}) \sum_{w \in R_i} q^{(i)}_{[R_i],s}(w) q^{(i)}_{[R_i],u}(w) && (9) \\
&= \sum_{w \in R_i} q^{(i)}_{[R_i],s}(w) \Big[\sum_{u \in R_i} (\boldsymbol{q}^{(i)}_{[R_i],s} \cdot \boldsymbol{q}^{(i)}_{[R_i],u}) q^{(i)}_{[R_i],u}(w) \Big] && (10) \\
&\leq \sqrt{\sum_{w \in R_i} q^{(i)}_{[R_i],s}(w)^2} \sqrt{\sum_{w \in R_i} \Big[\sum_{u \in R_i} (\boldsymbol{q}^{(i)}_{[R_i],s} \cdot \boldsymbol{q}^{(i)}_{[R_i],u}) q^{(i)}_{[R_i],u}(w) \Big]^2} && (11) \\
&= \|\boldsymbol{q}^{(i)}_{[R_i],s}\|_2 \|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_1 \sqrt{\mathbf{E}_{u_1,u_2 \sim \mathcal{D}_{s_i}} [\boldsymbol{q}^{(i)}_{[R_i],u_1} \cdot \boldsymbol{q}^{(i)}_{[R_i],u_2}]} \quad \text{(by (8))} && (12)
\end{aligned}$$

We rearrange and take squares to complete the proof. $\square$

We can apply previous norm bounds to get an explicit lower bound. To see the significance of the following lemma, note that by Claim 4.6 and Cauchy-Schwartz, $\forall u_1, u_2 \in R_i$, $\boldsymbol{q}^{(i)}_{[R_i],u_1} \cdot \boldsymbol{q}^{(i)}_{[R_i],u_2} \leq 1/n^{\delta(i-1)}$ (fairly close to the lower bound below).

Lemma 4.12. *Fix arbitrary $s \in S_i$.*

$$\mathbf{E}_{u_1,u_2 \sim \mathcal{D}_{s,i}} [\boldsymbol{q}^{(i)}_{[R_i],u_1} \cdot \boldsymbol{q}^{(i)}_{[R_i],u_2}] \geq 1/n^{\delta(i+1)}$$

Proof. By Lemma 4.11, the LHS is at least $\frac{1}{\|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_1^2} \cdot \frac{\|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_2^4}{\|\boldsymbol{q}^{(i)}_{[R_i],s}\|_2^2}$. Note that $\|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_1 \leq 1$. By Definition 4.5, $\|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_2^2 \geq 1/n^{\delta i}$. Since $s \in S_i \subseteq R_i$, by Claim 4.6, $\|\boldsymbol{q}^{(i)}_{[R_i],s}\|_2^2 \leq 1/n^{\delta(i-1)}$. $\square$

5 Analysis of `FindBiclique`

This is arguably the most important theorem in our analysis. It shows that the `FindBiclique`(s) procedure discovers a K_{r^2,r^2} minor with sufficiently high probability, when s is in a sufficiently high stratum.

Theorem 5.1. *Suppose $s \in S_i$, for $5r^4 \leq i \leq 1/\delta + 3$. The probability that the paths discovered in* `FindBiclique`(s) *contain a K_{r^2,r^2} minor is at least $n^{-4\delta r^4}$.*

Theorem 5.1 is proved in §5.5. Towards the proof, we will need multiple tools. In §5.1, we perform a standard calculation to bound the success probability of `FindPath`. In §5.2, we use this bound to show that the sets A and B sampled by `FindBiclique` are successfully connected by paths, as discovered by `FindPath`. In §5.3, we argue that the intersections of these paths is "well-behaved" enough to actually give a K_{r^2,r^2} minor.

We note that the $\sqrt{n}$ in the final running time comes from the calls to `FindPath` in `FindBiclique`.

5.1 The procedure FindPath

We reproduce here for convenience the procedure FindPath. It is a relatively straightforward application of the birthday paradox argument on top of bidirectional search for path finding.

> FindPath(u, v, k, i)
> 1. Perform k random walks of length $2^i \ell$ from u and v.
> 2. If a walk from u and v terminate at the same vertex, return these paths.

Lemma 5.2. *Consider $u, v \in R_i$. Suppose there exist $\alpha \leq \beta$ such that $\max(\|\boldsymbol{q}^{(i)}_{[R_i],u}\|_2^2, \|\boldsymbol{q}^{(i)}_{[R_i],v}\|_2^2) \leq 1/n^\alpha$ and $\boldsymbol{q}^{(i)}_{[R_i],u} \cdot \boldsymbol{q}^{(i)}_{[R_i],v} \geq 1/n^\beta$. Then with $k \geq cn^{\beta/2+4(\beta-\alpha)}$, FindPath$(u, v, k, i)$ returns an R_i-returning path of length $2^{i+1}\ell$ with probability $\geq 2/3$.*

Proof. First, define $W = \{w | q^{(i)}_{[R_i],u}(w)/q^{(i)}_{[R_i],v}(w) \in [1/(4n^{\beta-\alpha}), 4n^{\beta-\alpha}]\}$.

$$\begin{aligned}\sum_{w \notin W} q^{(i)}_{[R_i],u}(w) q^{(i)}_{[R_i],v}(w) &\leq (4n^{\beta-\alpha})^{-1} \sum_{w \notin W} \max(q^{(i)}_{[R_i],u}(w), q^{(i)}_{[R_i],v}(w))^2 \\ &\leq (4n^{\beta-\alpha})^{-1} (\|\boldsymbol{q}^{(i)}_{[R_i],u}\|_2^2 + \|\boldsymbol{q}^{(i)}_{[R_i],v}\|_2^2) \leq 1/2n^\beta\end{aligned}$$

Therefore, $\sum_{w \in W} q^{(i)}_{[R_i],u}(w) q^{(i)}_{[R_i],v}(w) \geq 1/2n^\beta$.

For $a, b \leq k$, let $X_{a,b}$ be the indicator for the following event: the ath $2^i\ell$-length random walk from u is an R_i-returning walk that ends at some $w \in W$, and the bth random walk from v is also R_i-returning, ending at the same w. Let $X = \sum_{a,b \leq k} X_{a,b}$. Observe that the probability that FindPath(u, v, k, i) returns a path is at least $\Pr[X > 0]$.

In what follows, c' is a large constant that depends on c. We can bound $\mathbf{E}[\sum_{a,b \leq k} X_{a,b}] = k^2 \sum_{w \in W} q^{(i)}_{[R_i],u}(w) q^{(i)}_{[R_i],v}(w) \geq k^2/2n^\beta \geq c'n^{4(\beta-\alpha)}$. Let us now bound the variance. First, let us expand out the expected square.

$$\mathbf{E}[(\sum_{a,b} X_{a,b})^2] = \sum_{a,b} \mathbf{E}[X_{a,b}^2] + 2 \sum_{a \neq a', b} \mathbf{E}[X_{a,b} X_{a',b}] + 2 \sum_{a, b \neq b'} \mathbf{E}[X_{a,b} X_{a',b}] + 2 \sum_{a \neq a', b \neq b'} \mathbf{E}[X_{a,b} X_{a',b'}] \quad (13)$$

Observe that $X_{a,b}^2 = X_{a,b}$. Furthermore, for $a \neq a', b \neq b'$, by independence of the walks, $\mathbf{E}[X_{a,b} X_{a',b'}] = \mathbf{E}[X_{a,b}]\mathbf{E}[X_{a',b'}]$. (This term will cancel out in the variance.) By symmetry, $\sum_{a \neq a', b} \mathbf{E}[X_{a,b} X_{a',b}] \leq k^3 \mathbf{E}[X_{1,1} X_{2,1}]$ (and analogously for the third term in (13)). Plugging these in and expanding out the $\mathbf{E}[X]^2$,

$$\mathbf{var}[X] \leq \mathbf{E}[X] + 2k^3 \mathbf{E}[X_{1,1} X_{2,1}] + 2k^3 \mathbf{E}[X_{1,1} X_{1,2}]$$

Note that $X_{1,1} X_{2,1} = 1$ when the first and second walks from u end at the same vertex where the first walk from v ends. Thus, $\mathbf{E}[X_{1,1} X_{2,1}] = \sum_{w \in W} q^{(i)}_{[R_i],u}(w)^2 q^{(i)}_{[R_i],v}(w)$. Since $w \in W$, $q^{(i)}_{[R_i],v}(w) \leq$

$2n^{\beta-\alpha} q^{(i)}_{[R_i],u}(w)$. Plugging this bound in,

$$4k^3\mathbf{E}[X_{1,1}X_{2,1}] \leq 4k^3 n^{\beta-\alpha} \sum_{w\in W} q^{(i)}_{[R_i],u}(w)^3 \leq 4k^3 n^{\beta-\alpha}[\sum_{w\in W} q^{(i)}_{[R_i],u}(w)^2]^{3/2} \tag{14}$$

$$= 4n^{\beta-\alpha}[k^2 \sum_{w\in W} q^{(i)}_{[R_i],u}(w)^2]^{3/2} \tag{15}$$

$$\leq 16n^{2(\beta-\alpha)}[k^2 \sum_{w\in W} q^{(i)}_{[R_i],u}(w) q^{(i)}_{[R_i],v}(w)]^{3/2} \tag{16}$$

$$\leq (\mathbf{E}[X]^{1/2}/c')(\mathbf{E}[X]^{3/2}) = \mathbf{E}[X]^2/c' \tag{17}$$

We get an identical bound for $2k^3\mathbf{E}[X_{1,1}X_{1,2}]$. Putting it all together, we can prove that $\mathbf{var}[X] \leq 4\mathbf{E}[X]^2/c'$. An application of Chebyshev proves that $\Pr[X > 0] > 2/3$. □

5.2 The procedure `FindBiclique`

For convenience, we reproduce `FindBiclique`.

`FindBiclique`(s)

1. For $i = 5r^4, \ldots, 1/\delta + 4$:
   (a) Perform $2r^2$ independent random walks of length $2^{i+1}\ell$ from s. Let the destinations of the first r^2 walks be multiset A, and the destinations of the remaining walks be B.
   (b) For each $a \in A$, $b \in B$:
      i. Run `FindPath`$(a, b, n^{\delta(i+9)/2}, i)$
   (c) If all calls to `FindPath` return a path, then let the collection of paths be the subgraph F. Run `KKR`(F, H). If it returns an H-minor, output that and terminate.

Lemma 5.3. *Suppose $s \in S_i$, for some $i \leq 2/\delta$. Then, with probability $\Omega((2n)^{-2\delta r^4})$ conditioned on the event that $A, B \subseteq R_i$, `FindBiclique`(s) finds a path between every $a \in A, b \in B$.*

Proof. The probability that a $2^{i+1}\ell$-length random walk from s ends at u is at least $q^{(i+1)}_{[S_i],s}(u) = \hat{q}^{(i+1)}_{[R_i],s}(u)\|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_1$. In the rest of the proof, let $t = |A| = |B| = r^2$ denote the common size of the multisets A and B. For any $a, b \in V$, let $\tau_{a,b}$ be the probability that `FindPath`$(a, b, n^{1/2+4\delta}, i)$ succeeds in finding a path between a and b. The probability of success for `FindBiclique`(s, k) given that $A, B \subseteq R_i$ is at least

$$\sum_{A\in R_i^t} \sum_{B\in R_i^t} \prod_{a\in A} \hat{q}^{(i+1)}_{[R_i],s}(a) \prod_{b\in B} \hat{q}^{(i+1)}_{[R_i],s}(b)\tau_{a,b} = \sum_{B\in R_i^t} \prod_{b\in B} \hat{q}^{(i+1)}_{[R_i],s}(b)\Big(\sum_{a\in R_i} \hat{q}^{(i+1)}_{[R_i],s}(a) \prod_{b\in B} \tau_{a,b}\Big)^t \tag{18}$$

Observe that $\prod_{b\in B} \hat{q}^{(i+1)}_{[R_i],s}(b)$ is a probability distribution over B. By Jensen, we lower bound, and manipulate further.

$$
\begin{aligned}
&\sum_{A\in R_i^t}\sum_{B\in R_i^t}\prod_{a\in A}\hat{q}^{(i+1)}_{[R_i],s}(a)\prod_{b\in B}\hat{q}^{(i+1)}_{[R_i],s}(b)\tau_{a,b} \\
\geq\ & \Big[\sum_{B\in R_i^t}\prod_{b\in B}\hat{q}^{(i+1)}_{[R_i],s}(b)\sum_{a\in R_i}\hat{q}^{(i+1)}_{[R_i],s}(a)\prod_{b\in B}\tau_{a,b}\Big]^t && (19)\\
=\ & \Big[\sum_{a\in R_i}\sum_{B\in R_i^t}\hat{q}^{(i+1)}_{[R_i],s}(a)\prod_{b\in B}\hat{q}^{(i+1)}_{[R_i],s}(b)\prod_{b\in B}\tau_{a,b}\Big]^t && (20)\\
=\ & \Big[\sum_{a\in R_i}\hat{q}^{(i+1)}_{[R_i],s}(a)\big(\sum_{b\in R_i}\hat{q}^{(i+1)}_{[R_i],s}(b)\tau_{a,b}\big)^t\Big]^t && (21)\\
\geq\ & \Big[\sum_{a\in R_i}\sum_{b\in R_i}\hat{q}^{(i+1)}_{[R_i],s}(a)\hat{q}^{(i+1)}_{[R_i],s}(b)\tau_{a,b}\Big]^{t^2} \quad \text{(Jensen)} && (22)\\
=\ & \Big[\mathbf{E}_{a,b\sim\mathcal{D}_{s,i}}[\tau_{a,b}]\Big]^{t^2} && (23)
\end{aligned}
$$

By Claim 4.6, for every $a \in R_i$, $\|\boldsymbol{q}^{(i)}_{[R_i],a}\|_2^2 \leq 1/n^{\delta(i-1)}$ (similarly for $b \in R_i$). By Lemma 5.2, if $\boldsymbol{q}^{(i)}_{[R_i],a} \cdot \boldsymbol{q}^{(i)}_{[R_i],b} \geq 1/2n^{\delta(i+1)}$, then $\tau_{a,b} = \Omega(1)$. By Lemma 4.12, $\mathbf{E}_{a,b}[\boldsymbol{q}^{(i)}_{[R_i],a} \cdot \boldsymbol{q}^{(i)}_{[R_i],b}] \geq 1/n^{\delta(i+1)}$. Applying Cauchy-Schwartz, $\boldsymbol{q}^{(i)}_{[R_i],a} \cdot \boldsymbol{q}^{(i)}_{[R_i],b} \leq 1/n^{\delta(i-1)}$. Let p be the probability (over a, b) that $\boldsymbol{q}^{(i)}_{[R_i],a} \cdot \boldsymbol{q}^{(i)}_{[R_i],b} \geq 1/2n^{\delta(i+1)}$.

$$1/n^{\delta(i+1)} \leq \mathbf{E}_{a,b}[\boldsymbol{q}^{(i)}_{[R_i],a} \cdot \boldsymbol{q}^{(i)}_{[R_i],b}] \leq (1-p)/2n^{\delta(i+1)} + p/n^{\delta(i-1)}$$

Thus, $p \geq 1/2n^{2\delta}$, and with this probability $\tau_{a,b} = \Omega(1)$. Plugging into (23), the probability of success is at least $(1/2n^{2\delta})^{r^2}$.

□

5.3 Criteria for `FindBiclique` to reveal a minor

We state some conditions on the paths that `FindBiclique` discovers that will ensure that they reveal a K_{r^2,r^2} minor. For some fixed walk length, let the walk performed by `FindPath` from vertex a to vertex b be denoted as $W_{a,b}$ (and we use $W_{a,b}(t)$ to denote the tth vertex visited in this walk). In order to distinguish intersections between walks that help us find minors from those that do not, consider an integral threshold length $\tau = 2^{i-1}\ell$.

Definition 5.4. *We call an intersection between walks **good** if it meets either of the following criteria:*

1. $W_{a,b}(t_1) = W_{a,c}(t_2)$ *and* $t_1, t_2 \leq \tau$
2. $W_{a,b}(t_1) = W_{b,a}(t_2)$ *and* $\max(t_1, t_2) \geq \tau$.

Definition 5.5. *We call an intersection between walks **bad** if it meets any of the following criteria:*

1. $W_{a,b}(t_1) = W_{a',b'}(t_2)$ *and* a, a', b' *are all distinct.*
2. $W_{a,b}(t_1) = W_{a,b'}(t_2)$ *and* $\max(t_1, t_2) \geq \tau$

3. $W_{a,b}(t_1) = W_{b,a}(t_2)$ *and* $\max t_1, t_2 \leq \tau$.

Moreover, we will refer to bad intersections of type i *depending on which of the above criteria is met.*

Claim 5.6. *Suppose $W_{a,b}(t_1) = W_{a',b'}(t_2)$ for not necessarily distinct a, a', b, b' is not a bad intersection. Then it is a good intersection.*

Proof. Since there are no bad intersections of type 1, the only possible relationships between a, a', b, b' are those where a, a', b' are not distinct, i.e. we have one of two cases:

1. $W_{a,b}(t_1) = W_{a,b'}(t_2)$ - since there are no bad intersections of type 2, it must be that $\max(t_1, t_2) < \tau$ and hence the inequality holds individually for t_1 and t_2, and so this is a type 1 good intersection.
2. $W_{a,b}(t_1) = W_{b,a}(t_2)$ - since there are no type 3 bad intersections, it must be that $\max(t_1, t_2) \geq \tau$ and hence this is a type 2 good intersection.

□

Claim 5.7. *Suppose we are given two disjoints sets of vertices in G, A and B each of size r^2, and for each $a \in A$, $b \in B$ we have a walks $W_{a,b}$ and $W_{b,a}$ such that each such pair of walks end at the same vertex. Moreover, assume that every intersection between walks is a good intersection. Then G contains a K_{r^2,r^2} minor.*

Proof. For a vertex $a \in A \cup B$, call the set of vertices visited in the first τ steps in walks from a to be a's cluster, denoted by $c(a)$. We wish to show that there exists a disjoint path between every cluster for $a \in A$ and $b \in B$, i.e. we wish to construct paths $P_{a,b}$ such that $P_{a,b}$ starts in $c(a)$ and ends in $c(b)$ and does not intersect any other path. We use $W_{a,b}[t_0, t_1]$ to denote the set of vertices visited on walk $W_{a,b}$ between the t_0^{th} step and the t_1^{th} step.

Consider a pair a, b. By assumption, there exists at least one pair of natural numbers t_1, t_2 such that $W_{a,b}(t_1) = W_{b,a}(t_2)$. Consider the pair such that $\min(t_1, t_2)$ is minimized. Without loss of generality, there are three cases to consider:

$t_1 \leq \tau,\ t_2 \leq \tau$ - this case is impossible since every intersection is good

$t_1 \geq \tau,\ t_2 \geq \tau$ - We set $P_{a,b} = W_{a,b}[\tau, t_1] \cup W_{b,a}[\tau, t_2]$.

$t_1 \leq \tau,\ t_2 \geq \tau$ - We set $P_{a,b} = W_{b,a}[\tau, t_2]$.

In either of the above cases if $P_{a,b}$ were to intersect $W_{c,d}$, we would have a type 2 or type 3 bad intersection, which cannot be the case since all intersections are good. We set each $P_{a,b}$ for each pair as above so that we have disjoint paths between each pair. Contracting all the clusters, and then contracting the paths gives us a K_{r^2,r^2} minor □

5.4 Bad intersection probabilities

In this section, we bound the probability of "bad intersections" among the paths discovered by `FindBiclique`.

Definition 5.8. *Let $\sigma_{s,S,t}(v)$ be the probability of a walk from s to t of length t being S-returning. (We allow $\ell \nmid t$, and require that the walk encounters S at every $j\ell$th step, for $j \leq \lfloor t/\ell \rfloor$.)*

We use $\boldsymbol{\sigma}_{s,S,t}$ to denote the vector of these probabilities. More generally, given any distribution vector $\boldsymbol{x}$ on V, we use $\boldsymbol{\sigma}_{\boldsymbol{x},S,t}$ to denote the vector of probabilities at time t for an S-returning walk.

We stress that this is not a conditional probability. Note that if $t = 2^i\ell$, then $\boldsymbol{\sigma}_{s,S,t} = \boldsymbol{q}^{(i)}_{[S],s}$. We show some simple propositions on these vectors. We use $\mathbb{I}_S$ to denote the projection onto the coordinates S; alternately, it preserves all coordinates in S and zeroes out other coordinates.

Proposition 5.9. *The vector $\boldsymbol{\sigma}_{\boldsymbol{x},S,t}$ evolves according to the following recurrence. Firstly, $\boldsymbol{\sigma}_{\boldsymbol{x},S,0} = \boldsymbol{x}$. For $t \geq 1$ such that $\ell \nmid t$, $\boldsymbol{\sigma}_{\boldsymbol{x},S,t} = M\boldsymbol{\sigma}_{\boldsymbol{x},S,t-1}$. For $t \geq 1$ such that $\ell \mid t$, $\boldsymbol{\sigma}_{\boldsymbol{x},S,t} = \mathbb{I}_S M\boldsymbol{\sigma}_{\boldsymbol{x},S,t-1}$*

Proposition 5.10. *For all $\boldsymbol{x}$ and all $t \geq 1$, $\|\boldsymbol{\sigma}_{\boldsymbol{x},S,t}\|_\infty \leq \|\boldsymbol{\sigma}_{\boldsymbol{x},S,t-1}\|_\infty$.*

Proof. Observe that M basically computes the "new" value at a vertex by averaging the values of the neighbors (and itself). This can never increase the maximum value. Furthermore, $\mathbb{I}_S$ only zeroes out some coordinates. This proves the proposition. $\square$

Now, we start bounding the probability of certain types of bad intersections.

Definition 5.11. *Let the random variable denoting the set of vertices encountered in an S-returning walk of length $2^i\ell$ from u be denoted $W_{u,S,i}$. Furthermore, let the tth vertex of the walk be $W_{u,S,i}(t)$.*

The distribution of t-length walks from u is denoted $\mathcal{W}_{u,t}$. (When t is apparent from context, we will drop the subscript.)

Claim 5.12. *Fix any $F \subseteq V$ and $a \in R_i$. The probability that an R_i-returning walk from a of length $2^i\ell$ intersects F at step $t \geq 2^{i-1}\ell$ is at most $2^i\ell|F|/n^{\delta(i-2)}$.*

Proof. By the union bound, the probability is bounded above by

$$\sum_{t\geq 2^{i-1}\ell}\sum_{u\in F}\Pr_{\mathcal{W}_a}[W_{a,R_i,i}(t) = u] \leq \sum_{t\geq 2^{i-1}\ell}\sum_{u\in F}\|\boldsymbol{\sigma}_{a,R_i,t}\|_\infty \tag{24}$$

By Prop. 5.10, the infinity norm is bounded above by $\|\boldsymbol{\sigma}_{a,R_i,2^{i-1}\ell}\|_\infty = \|\boldsymbol{q}^{(i-1)}_{[R_i],a}\|_\infty$. By Claim 4.7, the latter is at most $1/n^{\delta(i-2)}$. Plugging in (24), we get an upper bound of $2^{i-1}\ell|F|/n^{\delta(i-2)}$. $\square$

Claim 5.13. *Fix any $s \in S_i$. For any $F \subseteq V$,*

$$\Pr_{a\sim\mathcal{D}_s,\mathcal{W}_a}[W_{a,R_i,i} \cap F \neq \emptyset] \leq 2^i\ell|F|/(n^{\delta(i-1)}\|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_1).$$

Proof. Let $\boldsymbol{x}$ be the probability vector corresponding to $\mathcal{D}_s$. By Prop. 5.10, $\forall t \geq 1$, $\|\boldsymbol{\sigma}_{\boldsymbol{x},R_i,t}\|_\infty \leq \|\boldsymbol{x}\|_\infty$. By Definition 4.10, $\|\boldsymbol{x}\|_\infty = \|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_\infty/\|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_1$. Using Claim 4.7, this is at most $1/(n^{\delta(i-1)}\|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_1)$.

$$\begin{aligned}\Pr_{a\sim\mathcal{D}_s,\mathcal{W}_a}[W_{a,R_i,i} \cap F \neq \emptyset] &\leq \sum_{t\leq 2^i\ell}\sum_{v\in F}\Pr_{a\sim\mathcal{D}_s,\mathcal{W}_a}[W_{a,R_i,t} = v] \\ &\leq \sum_{t\leq 2^i\ell}\sum_{v\in F}\|\boldsymbol{x}\|_\infty \leq 2^i\ell|F|/(n^{\delta(i-1)}\|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_1)\end{aligned}$$

$\square$

Claim 5.14. *Fix $s \in S_i$ and $a \in R_i$.*

$$\begin{aligned}&\Pr_{b\sim\mathcal{D}_s,\mathcal{W}_a,\mathcal{W}_b}[\exists t_a, t_b, \min(t_a,t_b) \leq 2^{i-1}\ell | W_{a,R_i,i}(t_a) = W_{b,R_i,i}(t_b) \wedge W_{a,R_i,i}(2^i\ell) = W_{b,R_i,i}(2^i\ell)] \\ \leq\ & 2^{2i}\ell^2/(n^{\delta(2i-2)}\|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_1)\end{aligned}$$

Proof. Let us write out the main event in English. We fix an arbitrary a, and pick $b \sim \mathcal{D}_s$. We perform R_i-returning walks of length $2^i\ell$ from both a and b. We are bounding the probability that the "initial half" (less than $2^{i-1}\ell$ steps) of one of the walks intersects with the other, and subsequently, both walks end at the same vertex.

To that end, let us define two vertices w_1, w_2. We want to bound the probability of that both walks first encounter w_1, and then end at w_2. It is be very useful to treat the latter part simply as two walks from w_1, where one of them is at least of length $2^{i-1}\ell$. Note that w_1 may not be in R_i.

Since we only focus on returning walks over R_i, we drop the subscript of R_i. Let $Z_{a,t}$ be the random variable denoting the tth vertex of an R_i-returning walk from a. Let us also define R_i-returning walks with an offset g, starting from w. Basically, such a walk starts from w (that may not be in R_i) and performs g steps to end up in R_i. Subsequently, it behaves as an R_i-returning walk. Observe that the second parts of the walks are R_i-returning walks from w_1, with offsets of $\ell - t_a (\text{mod } \ell)$, $\ell - t_b (\text{mod } \ell)$. Let $Y_{w,t}$ be the random variable denoting the tth vertex of an R_i-returning walk from w, with the offset $\ell - t (\text{mod } \ell)$. We use primed versions for independent such variables.

Let us fix values for t_a, t_b such that $\min(t_a, t_b) \leq 2^{i-1}\ell$. (We will eventually union bound over all such values.) The probability we wish to bound is the following. We use independence of the walks to split the probabilities.

$$
\begin{aligned}
&\sum_{w_1 \in V} \sum_{w_2 \in V} \Pr_{b \sim \mathcal{D}_s, \mathcal{W}_a, \mathcal{W}_b, \mathcal{W}_{w_1}} [Z_{a,t_a} = w_1 \wedge Z_{b,t_b} = w_1 \wedge Y_{w_1, 2^i\ell - t_a} = w_2 \wedge Y'_{w_1, 2^i\ell - t_b} = w_2] \\
= &\sum_{w_1 \in V} \sum_{w_2 \in V} \Pr_{\mathcal{W}_a}[Z_{a,t_a} = w_1] \Pr_{b \sim \mathcal{D}_s, \mathcal{W}_b}[Z_{b,t_b} = w_1] \Pr_{\mathcal{W}_{w_1}}[Y_{w_1, 2^i\ell - t_a} = w_2] \Pr_{\mathcal{W}_{w_1}}[Y_{w_1, 2^i\ell - t_b} = w_2]
\end{aligned}
$$

Consider $\Pr_{b \sim \mathcal{D}_s, \mathcal{W}_b}[Z_{b,t_b} = w_1]$. This is exactly the w_1th entry in $\boldsymbol{\sigma}_{\boldsymbol{x}, \mathbb{R}_i, t_b}$ where $\boldsymbol{x}$ is the distribution given by $\mathcal{D}_s$. By Prop. 5.10, this is at most $\|\boldsymbol{x}\|_\infty$, which is at most $1/(n^{\delta(i-1)} \|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_1)$ (as argued in the proof of Claim 5.13).

One of $2^i\ell - t_a, 2^i\ell - t_b$ is at least $2^{i-1}\ell$. Thus, one of $\Pr_{\mathcal{W}_{w_1}}[Y_{w_1, 2^i\ell - t_a} = w_2]$ or $\Pr_{\mathcal{W}_{w_1}}[Y_{w_1, 2^i\ell - t_b} = w_2]$ refers to a walk of length at least $2^{i-1}\ell$. Let us bound $\Pr_{\mathcal{W}_{w_1}}[Y_{w_1,t} = w_2]$ for $t \geq 2^i\ell$. We can break such a walk into two parts: the first $\ell - t (\text{mod } \ell)$ steps lead to some $v \in R_i$, and the second part is an R_i-returning walk of length at least $2^i\ell$ from v to w. Recall that $p_{x,d}(y)$ is the standard random walk probability of starting from x and ending at y after d steps. For some $t' \geq 2^i\ell$,

$$
\begin{aligned}
\Pr_{\mathcal{W}_{w_1}}[Y_{w_1,t} = w_2] &= \sum_{v \in R_i} p_{w_1, \ell - t (\text{mod } \ell)}(v) \sigma_{v, R_i, t'}(w_2) \leq \sum_{v \in R_i} p_{w_1, \ell - t (\text{mod } \ell)}(v) \|\boldsymbol{\sigma}_{v, R_i, t'}\|_\infty \\
&\leq \sum_{v \in R_i} p_{w_1, \ell - t (\text{mod } \ell)}(v) \|\boldsymbol{q}^{(i)}_{[R_i],v}\|_\infty \leq \sum_{v \in R_i} p_{w_1, \ell - t (\text{mod } \ell)}(v) n^{-\delta(i-1)} = n^{-\delta(i-1)}
\end{aligned}
$$

Plugging these bounds in (25), for fixed t_a, t_b, there exists $t \in \{2^i\ell - t_a, 2^i\ell - t_b\}$ such that the probability of the main event is at most

$$
\begin{aligned}
&(1/n^{\delta(i-1)} \|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_1) \cdot (1/n^{\delta(i-1)}) \sum_{w_1 \in V} \sum_{w_2 \in V} \Pr_{\mathcal{W}_a}[Z_{a,t_a} = w_1] \Pr_{\mathcal{W}_{w_1}}[Y_{w_1,t} = w_2] \\
\leq \;& 1/(n^{\delta(2i-2)} \|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_1) \sum_{w_1 \in V} \Pr_{\mathcal{W}_a}[Z_{a,t_a} = w_1] \sum_{w_2 \in V} \Pr_{\mathcal{W}_{w_1}}[Y_{w_1,t} = w_2] = 1/(n^{\delta(2i-2)} \|\boldsymbol{q}^{(i+1)}_{[R_i],s}\|_1)
\end{aligned}
$$

A union bound over all pairs of t_a, t_b completes the proof. ∎

Claim 5.15. *Given an $s \in S_i$, the probability that* `FindBiclique`*(s) returns a bad intersection of type 2 is at most*

$$\frac{2r^6(2^i\ell)^2 n^{13\delta/2}}{n^{\delta i/2}}$$

conditioned on $A, B \subseteq R_i$.

Proof. Suppose that the subsets A and B in `FindBiclique` are both subsets of R_i. Fix $a \in A \cup B$, and set $F = \bigcup_{b \in B} W_{a,b}$. By Claim 5.12, the probability that a single walk of length $2^i\ell$ from a intersects F after 2^{i-1} steps is at most

$$\frac{2^i\ell|F|}{n^{\delta(i-2)}} = \frac{2^i\ell r^2 \cdot 2^i\ell}{n^{\delta(i-1)}},$$

and union bounding over all $r^2 n^{\delta(i+1)/2+4\delta}$ walks performed from a in all calls to FindPath, we get

$$\Pr_{W_i \sim \mathcal{W}_a}[\exists i, \exists t \leq 2^{i-1}\ell \text{ such that } W_i(t) \text{ is bad type } 2 | A \cup B \subseteq R_i] \leq \frac{r^4(2^i\ell)^2 n^{13\delta/2}}{n^{\delta i/2}}. \tag{25}$$

By a union bound over all $a \in A \cup B$, the above expression becomes at most $\frac{2r^6(2^i\ell)^2 n^{13\delta/2}}{n^{\delta i/2}}$. □

Claim 5.16. *For an $s \in S_i$, the probability that* `FindBiclique`*(s) returns a bad intersection of type 1 is at most*

$$\frac{2r^8(2^i\ell)^2 n^{13\delta/2}}{n^{\delta i/2}||\boldsymbol{q}^{(i+1)}_{[R_i],s}||_1}$$

Proof. Fix some $a' \in A$ and $b' \in B$, and suppose that `FindBiclique`(s) has found a path between a' and b', and denote the set of vertices in this path by F. Over the randomness of the choice of a, by Claim 5.13, the probability that a single walk from a intersects F is at most $2^i\ell(2^{i+1}\ell)/(n^{\delta(i-1)}||\boldsymbol{q}^{(i+1)}_{[R_i],s}||_1)$. Since there are $r^2 n^{\delta(i+1)/2+4\delta}$ walks performed from any particular a, the probability that a single a has a type 1 bad intersection with the a' to b' path is at most

$$r^2 n^{\delta(i+1)/2+4\delta} \frac{2^i\ell(2^{i+1}\ell)}{n^{\delta(i-1)}||\boldsymbol{q}^{(i+1)}_{[R_i],s}||_1} = \frac{2r^2(2^i\ell)^2 n^{13\delta/2}}{n^{\delta i/2}||\boldsymbol{q}^{(i+1)}_{[R_i],s}||_1} \tag{26}$$

and union bounding this over all r^6 triples a, a', b' completes the proof. □

Claim 5.17. *For an $s \in S_i$, the probability that* `FindBiclique`*(s) returns a bad intersection of type 3 is at most*

$$\frac{r^4(2^i\ell)^2 n^{11\delta}}{||\boldsymbol{q}^{(i+1)}_{[R_i],s}||_1 n^{\delta i}}$$

conditioned on the event that $A \cup B \subseteq R_i$.

Proof. Consider some vertex $a \in R_i$. By Claim 5.14 and a union bound over all $n^{\delta(i+9)}$ pairs of walks performed by `FindPath`, the probability that walks between a and some other vertex b engage in type 3 bad intersections in one call to `FindPath` is bounded above by $(2^i\ell)^2 n^{11\delta}/(||\boldsymbol{q}^{(i+1)}_{[R_i],s}||_1 n^{\delta i})$. Given that all vertices in A and B are in R_i, the probability that there is a single bad intersection of type 3 is given by union bounding over all r^4 pairs. □

5.5 Proof of Theorem 5.1

Proof. By Lemma 5.3, the probability that `FindBiclique` finds two multisets A and B with paths between every $a \in A$ and $b \in B$ conditioned on the event that $A, B \subseteq R_i$ is at least $(2n)^{-2\delta r^4}$. By Claim 5.6 and Claim 5.7, we need only show that the probabilities of bad intersections returned by `FindBiclique` is low. Each of the bounds on bad intersections conditioned on $A, B \subseteq R_i$ (Claim 5.15, Claim 5.16 and Claim 5.17) are at most

$$\frac{2r^8(2^{2i})n^{21\delta}}{||\boldsymbol{q}^{(i+1)}_{[R_i],s}||_1 n^{\delta i/2}} = \frac{2r^8 n^{21\delta}}{||\boldsymbol{q}^{(i+1)}_{[R_i],s}||_1 (n^{\delta/2}/4)^i}, \tag{27}$$

which if we set $i = 5r^4$, and union bound over the three bad intersection types gives us a total probability of at most

$$\frac{6r^8 n^{13\delta}}{||\boldsymbol{q}^{(i+1)}_{[R_i],s}||_1 (n^{\delta/2}/4)^{5r^4}} \leq \frac{6r^8(4^{5r^4})}{||\boldsymbol{q}^{(i+1)}_{[R_i],s}||_1 n^{2\delta(r^4+1)}}. \tag{28}$$

Let X denote the event that `FindBiclique` returns paths between all $a \in A$ and $b \in B$, Y denote the probability that `FindBiclique` returns a bad intersection and Z denote the event that $A, B \subseteq R_i$. We lower bound $\Pr[X \cap \neg Y | Z]$ by $\Pr[X|Z] - \Pr[Y|Z]$.

$$\Pr[X|Z] - \Pr[Y|Z] \geq \frac{1}{(2n)^{2\delta r^4}} - \frac{6r^8(4^{5r^4})}{||\boldsymbol{q}^{(i+1)}_{[R_i],s}||_1 n^{\delta(2+2r^4)}} \tag{29}$$

$$= \frac{2^{-2\delta r^4}||\boldsymbol{q}^{(i+1)}_{[R_i],s}||_1 - 6r^8 4^{5r^4} n^{-2\delta}}{n^{2\delta r^4}||\boldsymbol{q}^{(i+1)}_{[R_i],s}||_1} \tag{30}$$

$$\tag{31}$$

By Claim 4.8, the numerator above for sufficiently large n is at least

$$2^{-2\delta r^4} n^{-\delta} - 6r^8 4^{5r^4} n^{-2\delta} \geq 2^{-2\delta r^4 - 1} n^{-\delta} \tag{32}$$

and so

$$\Pr[X \cap \neg Y | Z] \geq \frac{1}{2^{2\delta r^4+1} n^{2\delta r^4+\delta}} \tag{33}$$

We know that $\Pr[Z] \geq ||\boldsymbol{q}^{(i+1)}_{[R_i],s}||_1^{r^4}$ and multiplying the above expression by this and applying Claim 4.8 gives the desired result. □

6 Local partitioning in the trapped case

Theorem 5.1 tells us that if there are $\Omega(n^{1-\delta})$ vertices in strata numbered $5r^4$ and above, then `FindMinor` finds a biclique minor with high probability. We now build the machinery to deal with case when most vertices lie in low strata. Equivalently, random walks from most vertices are trapped in a very small subset.

We basically argue that that (almost all) vertices in low strata can be partitioned into "pieces", such that each piece is a low conductance cut, and (a superset of) each piece can be found by performing randoming walks in G. The application of this lemma is along the lines of the GR bipartiteness analysis. If `FindMinor` fails to find a minor, this lemma can be iteratively applied to make G H-minor free by removing few edges. (This argument is give in §7.)

We use $p_{s,t}(v)$ to denote the probability that at t length random walk from s ends at v.

Lemma 6.1. *Consider some subset $S \subseteq V$ and $i \in \mathbb{N}$ such that $\forall s \in S, \|\boldsymbol{q}^{(i)}_{[S],s}\|_2^2 \leq 1/n^{\delta(i-1)}$. Define $S' \subseteq S$ to be $\{s | s \in S$ and $\|\boldsymbol{q}^{(i+1)}_{[S],s}\|_2^2 \geq 1/n^{\delta i}\}$.*

Suppose $|S'| \geq \alpha n$. Then, there is a subset $\widetilde{S} \subseteq S'$, $|\widetilde{S}| \geq \alpha n/8$ such that for $\forall s \in \tilde{S}$: there exists a subset $P_s \subseteq S$ where

- *$E(P_s, S \setminus P_s) \leq 2n^{-\delta/4} d|P_s|/\alpha$*
- *$\forall v \in P_s$, $\exists t \leq 160n^{\delta(i+7)}/\alpha$ such that $p_{s,t}(v) \geq \alpha/n^{\delta(2i+14)}$.*

The aim of this section is to prove this lemma. Henceforth, we will assume that S, S' are as defined in the lemma.

The main use of the norm bounds for vertices in S' is encapsulated in the following lemma. It essentially says that every vertex $s \in S'$ there is a "favorite set" of vertices which a random walk of length $2^{i+1}\ell$ from s has a high probability of occupying. This is the only property we require of $s \in S'$.

Claim 6.2. *For every $s \in S'$, there exists a set $U_s \subseteq S$, $|U_s| \geq n^{\delta(i-2)}$ such that $\forall u \in U_s$, $p_{s,2^{i+1}\ell}(u) \geq 1/2n^{\delta i}$.*

Proof. By Prop. 4.2, for any $u \in U$, $q^{(i+1)}_{[S],s}(u) = \boldsymbol{q}^{(i)}_{[S],s} \cdot \boldsymbol{q}^{(i)}_{[S],u}$. By the property of S and Cauchy-Schwartz, $q^{(i+1)}_{[S],s}(u) \leq 1/n^{\delta(i-1)}$.

Since $s \in S'$, $\sum_{u \in S} q^{(i+1)}_{[S],s}(u)^2 \geq 1/n^{\delta i}$. Let us simply define U_s to be $\{u | u \in S, q^{(i+1)}_{[S],s}(u) \geq 1/2n^{\delta i}\}$. Note that $p_{s,2^{i+1}\ell}(u) \geq q^{(i+1)}_{[S],s}(u)$.

$$1/n^{\delta i} \leq \sum_{u \in S} q^{(i+1)}_{[S],s}(u)^2 = \sum_{u \in U_s} q^{(i+1)}_{[S],s}(u)^2 + \sum_{u \notin U_s} q^{(i+1)}_{[S],s}(u)^2 \leq |U_s|/n^{2\delta(i-1)} + 1/2n^{\delta i}$$

We rearrange to bound the size of U_s. □

6.1 Local partitioning on the projected Markov chain

We define the "projection" of the random walk onto the set S. This uses a construction of [KPS13]. We define a Markov chain M_S over the set S. We retain all transitions from the original random walk on G that are within S, and we denote these by $e^{(1)}_{u,v}$ for every u to v transition in the random walk on G. Additionally, for every $u, v \in S$ and $t \geq 2$, we add a transition $e^{(t)}_{u,v}$. The probability of this transition is equal to the total probability of t-length walks in G from u to v, where all internal vertices in the walk lie outside S.

M_S forms a doubly stochastic Markov chain. Since G is irreducible and the stationary mass on S is non-zero, all walks eventually reach S. Thus the outgoing transition probabilities from each v in M_S sum to 1, and hence M_S is a valid Markov chain. Since for any t length walk from u to v, all vertices on which fall outside S, we added a transition $e^{(t)}_{u,v}$, it can be seen that the transition probabilities for all edges which end at v in M_S sum to 1. Therefore the transition matrix of M_S remains doubly stochastic, and the stationary distribution is uniform on S.

Observe that every transition in M_S has a corresponding "length" in G. For a transition $e^{(t)}_{u,v}$ in M_S, we define the length of this transition to be t. For clarity, we use "hops" to denote the length of a walk in M_S, and retain "length" for walks in G. The length of an h hop random walk in M_S is defined to be the sum of the lengths of the transitions it takes. We note that these ideas come from the work of Kale-Peres-Seshadhri to analyze random walks in noisy expanders [KPS13].

We use $\boldsymbol{\tau}_{s,h}$ to denote the distribution of the h-hop walk from s, and $\tau_{s,h}(v)$ to denote the corresponding probability of reaching v. We use $\mathcal{W}_h$ to denote the distribution of h-hop walks starting from the uniform distribution.

We state Kac's formula (Corollary 24 in Chapter 2 of [AF02], restated).

Lemma 6.3. *(Kac's formula) The expected return time (in G) to S of a random walk starting from S is the fractional stationary mass of S, ie $|S|/n$.*

The following is a direct corollary.

Lemma 6.4. $\mathbf{E}_{W\sim\mathcal{W}_h}$[*length of* W] $= hn/|S|$

Proof. Since the walk starts at the stationary distribution, it remains in this distribution at all hops. Thus, it suffices to get the expected length for just the first step (and multiply with h). Observe that this is precisely the expected return time to S, if we performed random walks in G. By Kac's fromula above, the expected return time to S equals the reciprocal of the stationary mass of S, which is just $n/|S|$. ☐

The next lemma shows that for many of the vertices in M_S we are able to associate a small set of vertices in M_S which are reached by short random walks with high probability. Contrast this with Claim 6.2 where we showed a similar statement for random walks on the original graph.

Lemma 6.5. *Assume $\ell = n^{5\delta}$. There exists a subset $S'' \subseteq S$, $|S''| \geq |S'|/2$, such that $\forall s \in S''$, $\|\boldsymbol{\tau}_{s,n^\delta}\|_\infty \geq 1/n^{\delta(i+6)}$.*

Proof. Let us define event $\mathcal{E}_{s,v,h}$. We fix the starting vertex s, and perform an h-hop random walk in M_S. The event $\mathcal{E}_{s,v,h}$ occurs when the walk ends at v in a $2^{i+1}\ell$-length walk. We define U_s as given in Claim 6.2. For any h,

$$\frac{1}{|S|}\sum_{s\in S'}\sum_{v\in U_s}\Pr[\mathcal{E}_{s,v,h}](2^{i+1}\ell) \leq \mathbf{E}_{W\sim\mathcal{W}_h}[\text{length of } W] = hn/|S|$$

Suppose $h \leq 2^{i+1}\ell/n^{4\delta}$. (This is true for all $h \leq n^\delta$). Then $\sum_{s\in S'}\sum_{v\in U_s}\Pr[\mathcal{E}_{s,v,h}] \leq n^{1-4\delta}$, and $\sum_{h\leq n^\delta}\sum_{s\in S'}\sum_{v\in U_s}\Pr[\mathcal{E}_{s,v,h}] \leq n^{1-3\delta}$.

Let us define

$$\texttt{trap}_{h,\ell}(s) = \Pr[\text{an at most } h\text{-hop walk from } s \text{ ends in } U_s \text{ after exactly } 2^{i+1}\ell \text{ steps }].$$

so that we have

$$\sum_{s\in S'}\sum_{v\in U_s}\sum_{h\leq n^\delta}\Pr[\mathcal{E}_{s,v,h}] = \sum_{s\in S'}\texttt{trap}_{n^\delta,\ell}(s) \leq n^{1-3\delta}.$$

Let $S'' \subset S$ be the set of vertices, v, where $\texttt{trap}_{n^\delta,\ell}(v) \leq 2\mathbf{E}_{s\sim S'}[\texttt{trap}_{n^\delta,\ell}(s)] = 2n^{1-3\delta}/|S'|$ $\leq 2/(\alpha n^{3\delta})$. By the Markov bound, $|S''| \geq |S'|/2$. By averaging over U_s, we can assert the following. For every $s \in S''$, there exists some $v \in U_s$ such that $\sum_{h \leq n^\delta} \Pr[\mathcal{E}_{s,v,h}] \leq 2/(\alpha n^{3\delta}|U_s|)$. Claim 6.2 bounds $|U_s| \geq n^{\delta(i-2)}$, and thus

$$\sum_{h \leq n^\delta} \Pr[\mathcal{E}_{s,v,h}] \leq 2/(\alpha n^{\delta(i+1)}). \tag{34}$$

Observe that $p_{s,2^{i+1}\ell}(v) = \sum_{h \leq 2^{i+1}\ell} \Pr[\mathcal{E}_{s,v,h}]$ (because the number of hops is always at most the length). Since $v \in U_s$, by Claim 6.2, $p_{s,2^{i+1}\ell}(v) \geq 1/2n^{\delta i}$. Subtracting the bound on (34) and using the fact that $\varepsilon > \varepsilon_{\mathsf{CUTOFF}}$ and $\alpha = \Omega(\varepsilon/\log n)$, $\sum_{h \in [n^\delta, 2^{i+1}\ell]} \Pr[\mathcal{E}_{s,v,h}] \geq 1/4n^{\delta i}$. Observe that $\boldsymbol{\tau}_{s,v}(h) \geq \Pr[\mathcal{E}_{s,v,h}]$. By averaging, for some $h \in [n^\delta, 2^{i+1}\ell]$, $\tau_{s,h}(v) \geq 1/(2^{i+3}n^{\delta i}\ell) \geq 1/n^{\delta(i+6)}$. So, $\|\boldsymbol{\tau}_{s,h}\|_\infty \geq 1/n^{\delta(i+6)}$. Since $\boldsymbol{\tau}_{s,h}$ is a random walk vector, the infinity norm is non-increasing in h. This completes the proof. $\square$

We now perform local partitioning on M_S, starting with an arbitrary $s \in S''$. We apply the Lovász-Simonovitz curve technique. (The definitions are originally from [LS90]. Refer to Lecture 7 of Spielman's notes [Spi] as well as Section 2 in Spielman-Teng [ST12].) This will require a series of definitions.

- Ordering of states at time t: At time t, let us order the vertices in M_S as $v_1^{(t)}, v_2^{(t)}, \ldots$ such that $\tau_{s,t}(v_1^{(t)}) \geq \tau_{s,t}(v_2^{(t)}) \ldots$, breaking ties by vertex id.
- The LS curve h_t: We define a function $h_t : [0, |S|] \to [0, 1]$ as follows. For every $k \in [|S|]$, set $h_t(k) = \sum_{j \leq k}[\tau_{s,t}(v_j^{(t)}) - 1/n]$. (Set $h_t(0) = 0$.) For every $x \in (k, k+1)$, we linearly interpolate to construct $h(x)$. Alternately, $h_t(x) = \max_{\vec{w} \in [0,1]^{|S|}, \|\vec{w}\|_1 = x} \sum_{v \in S}[\tau_{s,t}(v) - 1/n]w_i$.
- Level sets: For $k \in [0, |S|]$, we define the (k, t)-level set, $L_{k,t}$ to be $\{v_1^{(t)}, v_2^{(t)}, \ldots, v_k^{(t)}\}$. The *minimum probability* of $L_{k,t}$ denotes $\tau_{s,t}(v^{(t)})$.
- Conductance: for some $T \subseteq S$ we define the conductance of T in M_S to be

$$\Phi(T) = \frac{\sum_{\substack{u \in T \\ v \in S \setminus T}} \tau_{u,1}(v)}{\min\{|S \setminus T|, |T|\}}$$

The main lemma of Lovász-Simonovitz is the following (Lemma 1.4 of [LS90], also refer to Theorem 7.3.3 of Lecture 7 in [Spi]).

Lemma 6.6. *For all k and all t,*

$$h_t(k) \leq \frac{1}{2}[h_{t-1}(k - 2\min(k, n-k)\Phi(L_{k,t})) + h_{t-1}(k + 2\min(k, n-k)\Phi(L_{k,t}))]$$

We employ this lemma to prove a condition of the level set conductances.

Lemma 6.7. *Suppose for all $t' \leq t$, if $L_{k,t}$ has a minimum probability of at least $1/10n^{\delta(i+6)}$, and $\Phi(L_{k,t}) \geq n^{-\delta/4}$. Then for all $x \in [0, n]$, $h_t(x) \leq \sqrt{x}(1 - n^{-\delta/2}/4)^t + x/10n^{\delta(i+6)}$.*

Proof. We assume the conductance bound for all sets with minimum probability $\leq 1/10n^{\delta(i+6)}$. Note that h_t is a concave, piecewise linear curve, while the RHS is a concave curve. Thus, it suffices to prove this for the extremal points of h_t, namely, for integral x. Thus, we will assume that $x = k$, for $k \in [|S|]$.

We will prove by induction over t. If $k \geq n^{\delta(i+6)}$, then the RHS is at least 1. Thus, the bound is trivially true. Let us assume that $k < n^{\delta(i+6)} < n/2$. We now split the proof into two cases. First, let us consider the case where $\Phi(L_{k,t}) \geq n^{-\delta/4}$ (the standard LS calculation). By Lemma 6.6 and concavity of h,

$$\begin{aligned} h_t(k) &\leq \frac{1}{2}\left(h_{t-1}(k(1-2n^{-\delta/4})) + h_{t-1}(k(1+2n^{-\delta/4}))\right) && (35)\\ &\leq \frac{1}{2}\left(\sqrt{k(1-2n^{-\delta/4}}(1-n^{-\delta/2}/4)^{t-1} + \sqrt{k(1+2n^{-\delta/4})}(1-n^{-\delta/2}/4)^{t-1} + \frac{2k}{10n^{\delta(i+6)}}\right) && (36)\\ &\leq \frac{1}{2}\left(\sqrt{k}(1-2n^{-\delta/4})^{t-1}(\sqrt{1-2n^{-\delta/4}} + \sqrt{1+2n^{-\delta/4}}) + \frac{2k}{10n^{\delta(i+6)}}\right) && (37)\\ &\leq \sqrt{k}(1-n^{\delta/2}/2)^t + k/n^{\delta(i+6)} && (38) \end{aligned}$$

For the last inequality, we use the bound $(\sqrt{1+z}+\sqrt{1-z})/2 \leq 1 - z^2/8$.

□

6.2 Proof of Lemma 6.1

We can now prove our main partitioning lemma.

Proof. Define S'' as given in Lemma 6.5. For any $s \in S''$, $\|\boldsymbol{\tau}_{s,n^\delta}\|_\infty \geq 1/n^{\delta(i+6)}$. By the definition of the LS curve, $h_{n^\delta}(1) \geq 1/n^{\delta(i+6)}$. Suppose (for contradiction's sake) all level sets for $t \leq n^\delta$ with minimum probability at least $1/10n^{\delta(i+6)}$ have conductance at least $n^{-\delta/4}$. By Lemma 6.7, $h_{n^\delta}(1) \leq (1-n^{-\delta/2}/4)^{n^\delta} + 1/10n^{\delta(i+6)} < 1/n^{\delta(i+6)}$. This contradicts the bound obtained by Lemma 6.5.

Thus, for every $s \in S''$, there exists some level set for $t_s \leq n^\delta$ with minimum probability at least $1/10n^{\delta(i+6)}$ and conductance $< n^{-\delta/4}$. Let us call this level set P_s. By the construction of M_S, the volume of P_s, $\texttt{vol}(P_s)$, is at most $2d|P_S|n/|S| \leq 2d|P_S|/\alpha$. Hence,

$$\Phi(P_s) \geq \frac{E(P_s, S \setminus P_s)}{\texttt{vol}(P_s)} \geq \frac{\alpha E(P_s, S \setminus P_s)}{2d|P_s|}$$

Rearranging, we get $E(P_s, S \setminus P_s) \leq n^{-\delta/4}(2d|P_s|)/\alpha$.

For all $s \in S''$ and $v \in P_s$, $\tau_{s,n^\delta}(v) \geq 1/10n^{\delta(i+6)}$. Set $L = 160n^{\delta(i+7)}/\alpha$. Let $\widetilde{S}$ be the subset of S'' such that $\forall v \in P_s$, $\sum_{l \leq L} p_{s,l}(v) \geq 1/20n^{\delta(i+6)}$. By averaging, $\exists l \leq L$ such that $p_{s,l}(v) \geq \alpha/n^{\delta(2i+14)}$.

We have seen that $\widetilde{S}$ satisfies the two desired properties: for all $s \in \widetilde{S}$ $E(P_s, S \setminus P_s) \leq 2n^{-\delta/4}d|P_s|/\alpha$ and for all $v \in P_s$, $\exists t \leq 160n^{\delta(i+7)}$ such that $p_{s,t}(v) \geq \alpha/n^{\delta(2i+14)}$. It only remains to prove a lower bound on size. We basically use a Markov bound to achieve that.

Consider any $s \in S'' \setminus \widetilde{S}$. There exists some $v_s \in P_s$ such that $\tau_{s,n^\delta}(v_s) \geq 1/10n^{\delta(i+6)}$ but $\sum_{l \leq L} p_{s,l}(v_s) < 1/20n^{\delta(i+6)}$. Let us use $\hat{p}_{s,l}(v_s)$ to denote the probability of reaching v_s from s in an l-length walk that makes n^δ hops. Observe that

$$\begin{aligned}
&\tau_{s,n^\delta}(v_s) && (39)\\
&= \sum_{l \geq n^\delta} \hat{p}_{s,l}(v_s) && (40)\\
&= \sum_{l=n^\delta}^{L} \hat{p}_{s,l}(v_s) + \sum_{l>L} \hat{p}_{s,l}(v_s) && (41)\\
&\leq \sum_{l=n^\delta}^{L} p_{s,l}(v_s) + \sum_{l>L} \hat{p}_{s,l}(v_s) < 1/20n^{\delta(i+6)} + \sum_{l>L} \hat{p}_{s,l}(v_s) && (42)
\end{aligned}$$

Since $\tau_{s,n^\delta}(v_s) \geq 1/10n^{\delta(i+6)}$, $\sum_{l>L} \hat{p}_{s,l}(v_s) > 1/20n^{\delta(i+6)}$. Thus,

$$\frac{1}{|S|} \sum_{s \in S'' \setminus \widetilde{S}} \sum_{l>L} \hat{p}_{s,l}(v_s) L > \frac{L \cdot |S'' \setminus \widetilde{S}|}{20|S|n^{\delta(i+6)}} = \frac{160\alpha^{-1} n^{\delta(i+7)} \cdot |S'' \setminus \widetilde{S}|}{20|S|n^{\delta(i+6)}} = \frac{8n^\delta |S'' \setminus \widetilde{S}|}{\alpha |S|} \tag{43}$$

By Lemma 6.4,

$$\frac{1}{|S|} \sum_{s \in S'' \setminus \widetilde{S}} \sum_{l>L} \hat{p}_{s,l}(v_s) L \leq \mathbf{E}_{W \sim \mathcal{W}_n^\delta}[\text{length of } W] = \frac{n^{1+\delta}}{|S|} \tag{44}$$

Combining the above, $|S'' \setminus \widetilde{S}| \leq \alpha n/8$. By Lemma 6.5, $|S''| \geq |S'|/2 \geq \alpha n/2$, yielding the bound $|\widetilde{S}| \geq \alpha n/4$.

$\square$

7 Wrapping it all up: the proof of Theorem 3.1

We now have all the tools required to complete the proof of Theorem 3.1. Our aim is to show that if $\texttt{FindMinor}(G, \varepsilon, H)$ outputs an H-minor with probability $< 2/3$, then G is ε-close to being H-minor free. Henceforth in this section, we will simply assume the "if" condition.

The proof will be constructive and prescribes the edges to remove, based on the behavior of the various subroutines of FindMinor. We define a decomposition procedure that is used by the proof to achieve this construction.

We set parameter $\alpha = \varepsilon/(40r^4 \log n)$.

Decompose(G)

1. Initialize $S = V$ and $\mathcal{P} = \emptyset$.
2. For $i = 1, \ldots, 4r^4$:
   (a) Assign $S' := \left\{ s \in S : ||\boldsymbol{q}^{(i+1)}_{[S],s}||_2^2 \geq 1/n^{\delta i} \right\}$
   (b) While $|S'| \geq \alpha n$:
      i. Choose arbitrary $s \in S''$, and let P_s be as in Lemma 6.1.
      ii. Add P_s to $\mathcal{P}$ and assign $S := S \setminus P_s$
      iii. Assign $S' := \left\{ s \in S : ||\boldsymbol{q}^{(i+1)}_{[S],s}||_2^2 \geq 1/n^{\delta i} \right\}$
   (c) Assign $S := S \setminus S'$
   (d) Assign $X_i := S'$
3. Let $X = \bigcup_i X_i$.
4. Output the partition $\mathcal{P}, X, S$

The output of `Decompose` is a collection of sets: those in $\mathcal{P}$, X, and S. Let us take a moment to understand the `Decompose` algorithm. It takes as input the vertex set V of vertices which it is supposed to partition. If a huge fraction of vertices occupy the lower strata, then it is clear from §4.1 that the union of first $4r^4$ strata contains almost all the vertices. The `Decompose` procedure, powered by Lemma 6.1, builds up these strata iteratively one stratum at a time in step 2. The union of all of these strata goes inside the collection $\mathcal{P}$. In the i^{th} iteration of step $2(b)$, `Decompose` procedure explores "on the fly" whether it is possible to add more vertices to the i^{th} stratum (which when fully discovered, is added to $\mathcal{P}$ in step $2.(b).(iii)$). As long as the set S' is large, Lemma 6.1 implies it is possible to keep building the i^{th} stratum. When the set S' becomes fairly small, the `while` loop started in step $2(b)$ terminates and X_i is set equal to the current "noisy set" S' which was not decomposed in the i^{th} step. At the end of all the iterations, S is remaining $r$esidual set – vertices in which never got pulled in any of the $4r^4$ strata or any of the previous noisy X_i's.

We will also define the *ball* around a vertex s, denoted B_s.

$$B_s = \left\{ v \in V : \exists t \leq 160 n^{6\delta r^4}/\alpha \text{ such that } p_{s,t}(v) \geq \frac{\alpha}{n^{11\delta r^4}} \right\}.$$

Combining the various theorems in the previous sections, we can prove the following decomposition statement.

Lemma 7.1. *Assume $\varepsilon > \varepsilon_{\mathsf{CUTOFF}}$. Suppose* `FindMinor`$(G, \varepsilon, H)$ *outputs an H-minor with probability $< 2/3$. Then, the output of* `Decompose` *satisfies the following conditions.*

- $|X| \leq \varepsilon n/10$.
- $|S| \leq \varepsilon n/10$.
- *For all $P_s \in \mathcal{P}$, $P_s \subseteq B_s$*
- *There are at most $\varepsilon n/10$ edges that go between different P_s sets.*

Proof. Except for the bound on S, all other conditions follow from Lemma 6.1 and the parameter settings.

First, consider the X_i's formed by `Decompose`. Each of these has size at most αn, and there are at most $4r^4$ of these. Clearly, their union has size at most $\varepsilon n/10$.

The third condition holds directly from Lemma 6.1. Consider the number of edges that go between P_s and the rest of S, when P_s was constructed (in `Decompose`). By Lemma 6.1 again, the number of these edges is at most $2n^{-\delta/4}d|P_s|/\alpha = 40r^4(\log n)\varepsilon^{-1}n^{-\delta/4}d|P_s|$. Note that $\varepsilon > \varepsilon_{\mathsf{CUTOFF}}$. For sufficiently small constant δ, the number of edges between P_s and $S \setminus P_s$ (at the time of removal) is at most $\varepsilon|P_s|/10$. The total number of such edges is at most $\varepsilon n/10$ (since P_s are all disjoint). Observe that any edge that goes betweeo different P_s sets must be such a cut edge, proving the fourth condition.

The second condition on $|S|$ is where we require the behavior of `FindMinor`. Suppose, for contradiction's sake, that $|S| > \varepsilon n/10$. Consider the stratification process with $R_0 = S$. By construction of S, $\forall s \in S$, $||\boldsymbol{q}^{(4r^4+1)}_{[S],s}|| \leq 1/n^{4\delta r^4}$. Thus, all of these vertices will lie in strata numbered $4r^4$ or above. Since $\varepsilon > \varepsilon_{\mathsf{CUTOFF}}$, by Lemma 4.9, at most $\varepsilon n/\log n$ vertices are in strata numbered more than $1/\delta + 3$. By Theorem 5.1, for at least $\varepsilon n/10 - \varepsilon n/\log n \geq n^{1-\delta}$ vertices, the probability that the paths discovered by `FindBiclique`(s) contain a K_{r^2,r^2} minor is at least $n^{-4\delta r^4}$. Since a K_{r^2,r_2} minor contains an H minor, the algorithm (in this situation) will succeed in finding an H minor.

All in all, this implies that the probability that a single call to `FindBiclique` finds an H minor is at least $n^{-5\delta r^4}$. The probability that $n^{20\delta r^4}$ calls do not find such a minor is $< 1/6$, a contradiction. Thus, $|S| \leq \varepsilon n/10$.

□

Claim 7.2. *Assume $\varepsilon > \varepsilon_{\mathsf{CUTOFF}}$. Suppose* `FindMinor`$(G, \varepsilon, H)$ *outputs an H-minor with probability $< 2/3$. Then, for at most $n^{1-30\delta r^4}$ vertices s, B_s induces an H-minor.*

Proof. Suppose not. Observe that `LocalSearch` is called on $n^{35\delta r^4}$ vertices, so with probability at least $1 - (1 - n^{-30\delta r^4})^{n^{35\delta r^4}} > 5/6$, the algorithm samples vertex s where B_s induces an H-minor. There are at most $160\alpha^{-1}n^{6\delta r^4} \times \alpha^{-1}n^{11\delta r^4} \leq n^{18\delta r^4}$ vertices in B_s. The probability of ending at such a vertex v (from s) is at least $\alpha n^{-11\delta r^4} \geq n^{-12\delta r^4}$. Since `LocalSearch`$(s)$ performs $n^{30\delta r^4}$ random walks from s, the probability of not adding v to B is at most $\exp(n^{15\delta r^4})$. Taking a union bound over all of B_s, with probability at least $5/6$, the set B discovered by `LocalSearch`(s) is a superset of B_s. In this situation, `LocalSearch`(s) finds an H-minor. Thus, the overall probability of outputting an H-minor is at least $(5/6)^2 > 2/3$, a contradiction. □

And now, we can prove the correctness guarantee of `FindMinor`.

Claim 7.3. *Suppose* `FindMinor`(G, ε, H) *outputs an H-minor with probability $< 2/3$. Then G is ε-close to being H-minor free.*

Proof. If $\varepsilon \leq \varepsilon_{\mathsf{CUTOFF}}$, then `FindMinor` runs an exact procedure. So the claim is clearly true. Henceforth, assume $\varepsilon > \varepsilon_{\mathsf{CUTOFF}}$. Apply Lemma 7.1 to partition V into $\mathcal{P}, X, S$ as given.

Call $P_s \in \mathcal{P}$ bad, if P_s induces an H-minor. Similarly, call B_s bad if it induces an H-minor. Observe that for all $P_s \in \mathcal{P}$, $P_s \subseteq B_s$. Thus, the union of bad P_s sets is contained in the union on bad B_s sets. The size of any B_s is at most $160\alpha^{-1}n^{6\delta r^4} \times \alpha^{-1}n^{11\delta r^4} \leq n^{18\delta r^4}$. There are at most $n^{1-30\delta r^4}$ bad B_s sets, by Claim 7.2. So, the total union is at most $n^{1-12\delta r^4} \leq \varepsilon n/10$.

We can make G H-minor free by deleting all edges incident to X, all edges incident to S, all edges incident to vertices in any bad P_s sets, and all edges between P_s sets. By Lemma 7.1 and the bound given above, the total number of edges deleted is at most $4\varepsilon dn/10 < \varepsilon dn$. □

Finally, we bound the running time.

Claim 7.4. *The running time of* `FindMinor`(G, ε, H) *is $dn^{1/2+O(\delta r^4)} + d\varepsilon^{-2\exp(2/\delta)/\delta}$.*

Proof. If $\varepsilon < \varepsilon_{\mathsf{CUTOFF}}$, then the running time is simply $O(n^2)$. Since $\varepsilon < n^{-\delta/\exp(2/\delta)}$, this can be expressed as $\varepsilon^{-2\exp(2/\delta)/\delta}$.

Assume $\varepsilon \geq \varepsilon_{\mathsf{CUTOFF}}$. The total number of vertices encountered by all the `LocalSearch` calls is $n^{O(\delta r^4)}$. There is an extra d factor to determine all incident edges, through vertex queries. Thus, the total running time is $dn^{O(\delta r^4)}$, because of the quadratic overhead of `KKR`. Consider a single iteration for the main loop of `FindBiclique`. First, `FindBiclique` performs $2r^2$ random walks of length $2^{i+1}n^{5\delta}$, and then for each of these, `FindPath` performs $n^{\delta(i+1)/2+4\delta}$ walks of length $2^i n^{5\delta}$. Hence, the total steps (and thus, queries) in all walks performed by a single call to `FindBiclique` is

$$\sum_{i=5r^4}^{1/\delta+3} \left(2r^2 2^{i+1} n^{5\delta} + 2r^2 n^{\delta(i+1)/2+4\delta} 2^i n^{5\delta}\right) = r^2 n^{1/2+O(\delta)}. \tag{45}$$

While this is the total number of vertices encountered, we note that the calls made to $\mathtt{KKR}(F, H)$ are for much smaller graphs. The output of find path has size $O(2^{1/\delta} n^{5\delta})$, and the subgraph F constructed has at most $O(2^{1/\delta} n^{5\delta})$ vertices. We incur an extra d factor to determine the induced subgraph, through vertex queries. Thus, the time for each call to $\mathtt{KKR}(F, H)$ is $n^{O(\delta)}$. There are $n^{O(\delta r^4)}$ calls to `FindBiclique`, and we can bound the total running time by $dn^{1/2+O(\delta r^4)}$. □

References

[ABS15] Sanjeev Arora, Boaz Barak, and David Steurer. Subexponential algorithms for unique games and related problems. *Journal of the ACM*, 62(5):42:1–42:25, 2015. 3

[ACL06] R. Andersen, F. R. K. Chung, and K. Lang. Local graph partitioning using pagerank vectors. *Foundations of Computer Science (FOCS)*, pages 475–486, 2006. 6

[AF02] David Aldous and James Allen Fill. Reversible markov chains and random walks on graphs, 2002. Unfinished monograph, recompiled 2014, available at `http://www.stat.berkeley.edu/~aldous/RWG/book.html`. 22

[AFNS06] N. Alon, E. Fischer, I. Newman, and A. Shapira. A combinatorial characterization of the testable graph properties : it's all about regularity. *Symposium on the Theory of Computing (STOC)*, pages 251–260, 2006. 2

[BFU99] Andrei Z. Broder, Alan M. Frieze, and Eli Upfal. Static and dynamic path selection on expander graphs: A random walk approach. *Random Struct. Algorithms*, 14(1):87–109, 1999. 3

[BSS08] I. Benjamini, O. Schramm, and A. Shapira. Every minor-closed property of sparse graphs is testable. In *Symposium on the Theory of Computing (STOC)*, pages 393–402, 2008. 1

[CGR+14] Artur Czumaj, Oded Goldreich, Dana Ron, C Seshadhri, Asaf Shapira, and Christian Sohler. Finding cycles and trees in sublinear time. *Random Structures & Algorithms*, 45(2):139–184, 2014. 2

[CPS15] Artur Czumaj, Pan Peng, and Christian Sohler. Testing cluster structure of graphs. In *Symposium on the Theory of Computing (STOC)*, pages 723–732, 2015. 3

[CS10] Artur Czumaj and Christian Sohler. Testing expansion in bounded-degree graphs. *Combinatorics, Probability & Computing*, 19(5-6):693–709, 2010. 3

[CSS09] Artur Czumaj, Asaf Shapira, and Christian Sohler. Testing hereditary properties of nonexpanding bounded-degree graphs. *SIAM Journal on Computing*, 38(6):2499–2510, 2009. 2

[Die10] Reinhard Diestel. *Graph Theory, Fourth Edition.* Springer, 2010. 2

[FLVW17] Hendrik Fichtenberger, Reut Levi, Yadu Vasudev, and Maximilian Wötzel. On testing minor-freeness in bounded degree graphs with one-sided error. *CoRR*, abs/1707.06126, 2017. 2

[Gol17] O. Goldreich. *Introduction to Property Testing.* Cambridge University Press, 2017. 2

[GR99] O. Goldreich and D. Ron. A sublinear bipartite tester for bounded degree graphs. *Combinatorica*, 19(3):335–373, 1999. 2, 3

[GR00] O. Goldreich and D. Ron. On testing expansion in bounded-degree graphs. *ECCC*, TR00-020, 2000. 3

[GR02] O. Goldreich and D. Ron. Property testing in bounded degree graphs. *Algorithmica*, 32(2):302–343, 2002. 1, 2

[GT12] Shayan Oveis Gharan and Luca Trevisan. Approximating the expansion profile and almost optimal local graph clustering. In *Foundations of Computer Science (FOCS)*, pages 187–196. IEEE Computer Society, 2012. 3

[HKNO09] A. Hassidim, J. Kelner, H. Nguyen, and K. Onak. Local graph partitions for approximation and testing. In *Foundations of Computer Science (FOCS)*, pages 22–31, 2009. 2

[HT74] John Hopcroft and Robert Tarjan. Efficient planarity testing. *Journal of the ACM (JACM)*, 21(4):549–568, 1974. 1

[KKR12] Ken-ichi Kawarabayashi, Yusuke Kobayashi, and Bruce A. Reed. The disjoint paths problem in quadratic time. *J. Comb. Theory, Ser. B*, 102(2):424–435, 2012. 2, 8

[KPS13] Satyen Kale, Yuval Peres, and C. Seshadhri. Noise tolerance of expanders and sublinear expansion reconstruction. *SIAM J. Comput.*, 42(1):305–323, 2013. 3, 6, 21, 22

[KR96] Jon M. Kleinberg and Ronitt Rubinfeld. Short paths in expander graphs. In *Foundations of Computer Science (FOCS)*, pages 86–95. IEEE Computer Society, 1996. 3, 4

[KS08] S. Kale and C. Seshadhri. Testing expansion in bounded degree graphs. *Proc. 35th ICALP*, pages 527–538, 2008. 3

[Kur30] K. Kuratowski. Sur le problème des courbes gauches en topologie. *Fundamenta Mathematica*, 15:271–283, 1930. 1

[Lov06] L. Lovász. Graph minor theory. *Bulletin of the American Mathematical Society*, 43(1):75–86, 2006. 2

[LR15] Reut Levi and Dana Ron. A quasi-polynomial time partition oracle for graphs with an excluded minor. *ACM Transactions on Algorithms (TALG)*, 11(3):24, 2015. 2

[LS90] László Lovász and Miklós Simonovits. The mixing rate of markov chains, an isoperimetric inequality, and computing the volume. In *Foundations of Computer Science (FOCS)*, pages 346–354, 1990. 3, 23

[NS07] A. Nachmias and A. Shapira. Testing the expansion of a graph. *ECCC*, TR07-118, 2007. 3

[RS04] N. Robertson and P. D. Seymour. Graph minors. XX. Wagner's conjecture. *Journal of Combinatorial Theory Series B*, 92(1):325–357, 2004. 1

[Spi] D. Spielman. Lecture notes on spectral graph theory. http://www.cs.yale.edu/homes/spielman/eigs/. 23

[ST12] D. Spielman and S.-H. Teng. A local clustering algorithm for massive graphs and its application to nearly-linear time graph partitioning. *SIAM Journal on Computing*, 42(1):1–26, 2012. 3, 6, 23

[Tre05] Luca Trevisan. Approximation algorithms for unique games. In *Foundations of Computer Science (FOCS)*, pages 197–205. IEEE, 2005. 3

[Wag37] K. Wagner. Über eine eigenschaft der ebenen komplexe. *Mathematische Annalen*, 114:570–590, 1937. 1

[YI15] Yuichi Yoshida and Hiro Ito. Testing outerplanarity of bounded degree graphs. *Algorithmica*, 73(1):1–20, 2015. 2

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