Electronic Colloquium on Computational Complexity, Report No. 134 (2018)

Cosystolic Expanders over any Abelian Group

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July 12, 2018

Abstract

In this work we show a general reduction from high dimensional complexes to their links based on the spectral properties of the links. We use this reduction to show that if a certain property is testable in the links, then it is also testable in the complex. In particular, we show that expansion of the links of a complex over any abelian group implies that the complex is an expander *over the same group*.

Previous works studied the expansion properties of complexes from their links, but their proofs were tailored for the field $\mathbb{F}_2$. They showed that the combination of spectral and topological properties of the links of a complex implies its expansion over the field $\mathbb{F}_2$. We show here a general reduction based *only* on the spectral properties of the links.

We then show that all the links of Ramanujan complexes (which are called spherical buildings) are expanders over any abelian group. For this we generalize the result of [LMM16], who showed that spherical buildings are coboundary expanders over $\mathbb{F}_2$. Combined with our reduction, this implies the existence of bounded degree cosystolic expanders over any abelian group.

1 Introduction

In property testing questions, the task is to determine whether an object has a predetermined property or is far from having the property by inspecting only a small part of the object. A property is said to be *testable* if such a test exists with a small probability of failure. When the property is given by a set of constraints, i.e., an object has the property if it satisfies all the constraints, then the natural test picks one constraint at random and accepts if and only if it is satisfied. An equivalent way of saying that the property is testable in this case, is that every object which is far from the property violates many constraints, proportional to its distance from the property.

The main contribution of this paper is a general reduction which implies the testability of a certain property in a high dimensional complex (which is the high dimensional analog of a graph) from its testability in the local views of the complex. In many cases of high dimensional complexes, the local views are easier to understand and analyze than the whole complex. Our reduction allows to study the testability of the whole complex (which is usually harder) from the testability of its local views (which is usually easier).

1.1 High dimensional expanders

A d-dimensional simplicial complex is a $(d+1)$-hypergraph which is closed under taking subsets, namely, for any $(d+1)$-hyperedge in the complex, all of its subsets are also in the complex.

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ISSN 1433-8092

A hyperedge is called a *face* of the complex, and its dimension is one less than its cardinality. For a complex X, we denote by $X(0)$ the set of 0-dimensional faces, which are the vertices, by $X(1)$ the 1-dimensional faces, which are the edges, and so on up to $X(d)$, which are the top dimensional faces. As an example, a 1-dimensional complex is just a graph, and a 2-dimensional complex contains also triangles in addition to vertices and edges. A family of complexes is said to be of *bounded degree* if the number of faces incident to any vertex of any complex in the family is bounded by a constant, independent of the number of vertices in the complex.

In the last decade, a theory of high dimensional expanders has begun to emerge (see [Lub17] for a recent survey). High dimensional expanders possess stronger properties than their one dimensional analogs, which are expander graphs. As such, they are much harder to achieve. In contrast to expander graphs, where a random sparse graph is an expander with high probability, a random sparse high dimensional complex with high probability is not an expander. Moreover, while explicit constructions of expander graphs exist in abundance, only one construction of bounded degree high dimensional expanders is known to exist.

In recent years, several distinct notions of high dimensional expanders have been studied. The notion of *coboundary expansion* has been introduced by Linial and Meshulam [LM06] in their work on homological connectivity of random complexes, and independently by Gromov [Gro10] in his work on the topological overlapping property. Coboundary expansion is an extension of combinatorial expansion of graphs from a cohomological point of view. A relaxation of coboundary expansion, called *cosystolic expansion*, has been introduced in [KKL14, EK16]. Cosystolic expansion has been shown in [KKL14, EK16] to imply the topological overlapping property defined by Gromov [Gro10].

Both properties of expansion are a form of property testing questions. We start by introducing the notion of cosystolic expansion as a generalization of graph expansion to higher dimensions. Then we explain its relation to property testing (for more on the relation between high dimensional expanders and property testing see [KL14]).

Let $X = (V, E)$ be a 1-dimensional complex (i.e., a graph). For any two sets $S, S' \subseteq V$, define the distance between S and S' as $\mathrm{dist}(S, S') = |(S \setminus S') \cup (S' \setminus S)|$. For any subset of vertices $S \subseteq V$, define $\delta(S)$ as the set of edges with one endpoint in S and one endpoint outside of S (also called the coboundary of S). Note that for any connected component S, $\delta(S) = \emptyset$. Moreover, for $S \in \{\emptyset, V\}$, $\delta(S) = \emptyset$ for trivial reasons. The sets for which $\delta(S) = \emptyset$ are called the 0-*cocycles*, and the sets $\{\emptyset, V\} \subseteq \{\text{0-cocycles}\}$ are called the 0-*coboundaries*. We say that X is an (ε, μ)-cosystolic expander for $\varepsilon, \mu > 0$ if for any $S \subseteq V$, $S \notin \{\text{0-coboundaries}\}$:

- If $S \in \{\text{0-cocycles}\}$ then $|S| \geq \mu |V|$.
- Otherwise,
$$\frac{|\delta(S)|}{\mathrm{dist}(S, \{\text{0-cocycles}\})} \geq \varepsilon,$$

where $\mathrm{dist}(S, \{\text{0-cocycles}\}) = \min\{\mathrm{dist}(S, S') \mid S' \in \{\text{0-cocycles}\}\}$. In other words, cosystolic expansion in the case of graphs means that the graph is composed of connected components of size at least μ, where the Cheeger constant of each of them is at least ε.

In order to make the generalization to higher dimensions clearer, let us replace subsets of vertices with functions $f : V \to \mathbb{F}_2$, where we identify a subset of vertices with its characteristic function. The size of a function f is defined as $|f| = |\mathrm{supp}(f)|$, the distance between two functions f, f' is defined by $\mathrm{dist}(f, f') = |f - f'|$, and the coboundary operator is defined by $\delta(f)(\{u, v\}) = f(u) + f(v)$ for any edge $\{u, v\} \in E$.

Now, cosystolic expansion is generalized naturally to higher dimensions: Let $X = (V, E, T)$ be a 2-dimensional complex. For any function $g : E \to \mathbb{F}_2$, define the coboundary of g by

$\delta(g)(\{u,v,w\}) = g(\{u,v\}) + g(\{u,w\}) + g(\{v,w\})$ for any triangle $\{u,v,w\} \in T$. It is an easy exercise to check that for any function on the vertices $f : V \to \mathbb{F}_2$, $\delta(\delta(f)) = 0$, i.e., for $g \in \{\delta(f) \mid f : V \to \mathbb{F}_2\}$, $\delta(g)(\{u,v,w\}) = 0$ for every triangle $\{u,v,w\} \in T$. As in the 1-dimensional case, all functions $g : E \to \mathbb{F}_2$ for which $\delta(g) = 0$ are called the 1-cocycles, and the functions for which $\delta(g) = 0$ for trivial reasons (i.e., all functions $g \in \{\delta(f) \mid f : V \to \mathbb{F}_2\}$) are called the 1-coboundaries. We say that X is an (ε,μ)-cosystolic expander for $\varepsilon,\mu > 0$ if:

1. For any $f : V \to \mathbb{F}_2$, $f \notin \{\text{0-coboundaries}\}$:
   - If $f \in \{\text{0-cocycles}\}$ then $|f| \geq \mu|V|$.
   - Otherwise,
   $$\frac{|\delta(f)|}{\operatorname{dist}(f, \{\text{0-cocycles}\})} \geq \varepsilon.$$
2. For any $g : E \to \mathbb{F}_2$, $g \notin \{\text{1-coboundaries}\}$:
   - If $g \in \{\text{1-cocycles}\}$ then $|g| \geq \mu|E|$.
   - Otherwise,
   $$\frac{|\delta(g)|}{\operatorname{dist}(g, \{\text{1-cocycles}\})} \geq \varepsilon.$$

In a similar way, for any d-dimensional complex X, we say that X is a cosystolic expander if the above conditions hold for any function $f : X(k) \to \mathbb{F}_2$ for all $0 \leq k < d$.

As mentioned, we can view cosystolic expansion as a property testing question. A function $f : X(k) \to \mathbb{F}_2$ which satisfies $\delta(f)(\sigma) = 0$ for every $\sigma \in X(k+1)$ is a k-cocycle. We can view the set of equations $\{\delta(f)(\sigma) = 0\}_{\sigma \in X(k+1)}$ as a set of constraints defining the property of being a k-cocycle. If X is a cosystolic expander, then the property of being a k-cocycle is testable by the natural test of picking a $(k+1)$-face $\sigma \in X(k+1)$ at random and checking whether $\delta(f)(\sigma) = 0$. By the cosystolic expansion criterion, the failure probability of this test is ε-proportional to the distance of f from the k-cocycles. In other words, a complex is a cosystolic expander if any function that is far from the cocycles violates many constraints.

1.2 Our contribution

Before introducing our results, we need to present two important definitions regarding high dimensional complexes:

1. For any $0 \leq k < d$, the *k-skeleton* of X is the complex obtained by taking only faces of dimension $\leq k$ in X. In particular, the 1-skeleton of X is its underlying graph (ignoring the higher dimensional faces).

2. For any face $\sigma \in X$, the *link* of σ is the subcomplex obtained by taking all faces in X which contain σ and removing σ from all of them, formally defined as $X_\sigma = \{\tau \setminus \sigma \mid \sigma \subseteq \tau \in X\}$. Note that X_σ is a subcomplex of dimension $d - |\sigma|$.

Previous works on Ramanujan complexes showed that their $(d-1)$-skeleton is a cosystolic expander over the field $\mathbb{F}_2$ [KKL14, EK16] (i.e., when we consider functions whose target is $\mathbb{F}_2$ as described above). The existence of cosystolic expanders over $\mathbb{F}_2$ has already found great applications as it implies an affirmative answer to Gromov's question regarding the existence of bounded degree complexes with the topological overlapping property [Gro10]. In this work we show that the $(d-1)$-skeleton of a Ramanujan complex is a cosystolic expander over any abelian group (i.e., when the target of the functions is any abelian group).

1.2.1 General reduction from a complex to its links

Our main result is the following theorem:

Theorem 1.1 (Reduction from a complex to its links, informal, for formal see theorem 3.1)**.** *If the underlying graph of every link in X has a good enough spectral gap, then there is a reduction from X to its links. In particular, for any function $f : X(k) \to G$, $0 \le k < d$, G any abelian group, there exist a dimension $0 \le i \le k$ and a constant $c > 0$, such that the coboundary of f is roughly the average of the coboundaries in the links of i-dimensional faces multiplied by c.*

While previous works studied the expansion of a complex from its links, their proofs were tailored for the field $\mathbb{F}_2$. They showed that the combination of spectral gap and coboundary expansion of the links of a complex implies its expansion over the field $\mathbb{F}_2$. We show here a general reduction from the complex to its links, based *only* on the spectral gap of the links. This reduction follows by finding for each function a specific dimension such that the local views in this dimension coincides with the global view of the function. As a corollary, we derive the general result, that expansion of the links of a complex over any abelian group implies that the complex is an expander *over the same group.*

1.2.2 Ramanujan complexes and their links

Ramanujan complexes are the high dimensional analogs of the celebrated LPS graphs [LPS88]. LPS graphs are constructed by taking quotients of the infinite tree, which is the best expander possible. The infinite tree has a high dimensional analog, called the Bruhat-Tits building. This led [LSV05b] to study quotients of it as a generalization of LPS graphs. By taking quotients of the Bruhat-Tits building, [LSV05a] achieve an explicit construction of bounded degree simplicial complexes which locally look like the infinite object. These complexes are called *Ramanujan complexes.* (For more on Ramanujan complexes see [Lub14].)

Every link of a Ramanujan complex is a very symmetric complex called the *spherical building* (more details on the spherical building are presented in §4). The spherical building by itself is of unbounded degree, since the number of faces incident to any vertex grows with the number of vertices in the complex, but the Ramanujan complex is of a bounded degree. In [EK16], the authors showed that the 1-skeleton of the spherical building has an excellent spectral gap (which is controlled by a parameter called the *thickness* of the building), so it is left to show that the spherical building is a coboundary expander over any abelian group.

In [LMM16], the authors show that the spherical building is a coboundary expander over $\mathbb{F}_2$. We generalize their work by taking care of orientations of faces (which was not necessary in their work since they proved it only for $\mathbb{F}_2$). We show that with some modifications which take orientations into account, the proof of [LMM16] can work over any abelian group. We then prove the following theorem.

Theorem 1.2. *The spherical building is a coboundary expander over any abelian group.*

Since we got that the links of (thick enough) Ramanujan complexes have an excellent spectral gap and are coboundary expanders over any abelian group, our reduction implies the following theorem.

Theorem 1.3. *For a thick enough d-dimensional Ramanujan complex, its $(d-1)$-skeleton is a cosystolic expander over any abelian group.*

An important implication of this theorem is that when combined with [Evr] it implies the existence of *bounded degree coboundary expanders.* Coboundary expansion is a stronger notion than cosystolic expansion, and currently the existence of bounded degree coboundary expanders of dimension ≥ 2 is not known. Ramanujan complexes are not coboundary expanders over $\mathbb{F}_2$ because they have non-trivial cocycles (i.e., cocycles which are not coboundaries). Evra [Evr] has shown that over some other finite groups (not $\mathbb{F}_2$), there are no non-trivial cocycles. Combined with theorem 1.3, this implies for the first time the existence of bounded degree coboundary expanders for dimension ≥ 2.

1.3 Discussion and future work

Recent works showed that cosystolic expansion over $\mathbb{F}_2$ implies Gromov's topological overlapping property [EK16]. We believe that cosystolic expansion over general abelian groups should have far reaching applications that are beyond one's imagination. For instance, it could lead to new constructions of locally testable codes.

1.4 Organization

We start with a preliminaries section that contains the basics of cochains with norms in high dimensional complexes. In section 3 we prove theorem 1.1 and show how it implies cosystolic expansion over any abelian group. In section 4 we introduce the links of Ramanujan complexes, which are called spherical buildings, and prove that they are coboundary expanders over any abelian group.

2 Preliminaries

Let X be a d-dimensional simplicial complex. For any $-1 \leq k \leq d$, denote by $X(k)$ the set of k-dimensional faces of X (where $X(-1) = \{\emptyset\}$ contains the only -1-dimensional face, which is the face with 0 vertices). An ordered set $\vec{\sigma} = (v_0, v_1, \ldots, v_k)$ is an *ordered face* of X if the unordered set $\sigma = \{v_0, v_1, \ldots, v_k\}$ is a face of X. Denote by $\vec{X}(k)$ the set of ordered k-dimensional faces of X. The space of k-*cochains* over an abelian group G is defined as

$$C^k = C^k(X; G) = \{f : \vec{X}(k) \to G \mid f \text{ is antisymmetric}\},$$

where f is antisymmetric if for any permutation $\pi \in Sym(k+1)$,

$$f((v_{\pi(0)}, v_{\pi(1)}, \ldots, v_{\pi(k)})) = sgn(\pi) f((v_0, v_1, \ldots, v_k)).$$

Note that for $G = \mathbb{F}_2$, the k-cochains are just subsets of $X(k)$ (where we identify a subset of faces with its characteristic function). In the works of [LMM16] and [EK16], the authors worked only with cochains over $\mathbb{F}_2$ so they did not have to worry about ordered faces and change of signs. We let the cochains to be over any abelian group so we need to take these considerations into account.

We measure the size of a cochain according to its hamming weight with proportion to the top dimension of the complex, as follows. Let $r_d, r_{d-1}, \ldots, r_{-1}$ be a sequence of random faces of X, where r_d is distributed uniformly on $X(d)$, and for any $k < d$, r_k is obtained by removing a uniformly random vertex from r_{k+1}. All the probabilities we measure in this work would be over this distribution of random faces. For any k-cochain $f \in C^k$, we denote its support by $A = \text{supp}(f) = \{\sigma \in X(k) \mid f(\sigma) \neq 0\}$, and define its norm to be $\|f\| = \|A\| = \Pr[r_k \in A]$. (Note that the support of f is a set of unordered faces, and it is well defined even though the cochain is defined on ordered faces, since it does not matter which ordering we take.)

For any $\vec{\sigma} = (v_0, v_1, \ldots, v_k)$ we denote by $\vec{\sigma} \setminus \{v_i\} = (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k)$ the ordered $(k-1)$-face obtained by removing v_i from $\vec{\sigma}$. The k-coboundary operator $\delta = \delta^k : C^k \to C^{k+1}$ is defined as

$$\delta(f)(\vec{\sigma}) = \sum_{i=0}^{k+1} (-1)^i f(\vec{\sigma} \setminus \{v_i\}).$$

Denote by $B^k = \mathrm{Im}(\delta^{k-1}) = \{\delta^{k-1}(f) \mid f \in C^{k-1}\}$ the k-coboundaries of X, and by $Z^k = \ker(\delta^k) = \{f \in C^k \mid \delta^k(f) = 0\}$ the k-cocycles of X. The distance of a k-cochain $f \in C^k$ from the k-coboundaries is defined as $\mathrm{dist}(f, B^k) = \min\{\|f - b\| \mid b \in B^k\}$. Similarly, the distance from the k-cocycles is defined as $\mathrm{dist}(f, Z^k) = \min\{\|f - z\| \mid z \in Z^k\}$.

We can now present the notion of coboundary expansion as was introduced by Linial-Meshulam [LM06] and by Gromov [Gro10].

Definition 2.1 (Coboundary expansion)**.** Let X be a d-dimensional simplicial complex and G an abelian group. X is called an *ε-coboundary expander* over G, if for any k-cochain which is not a k-coboundary $f \in C^k(X;G) \setminus B^k(X;G)$, $0 \leq k \leq d-1$,

$$\frac{\|\delta(f)\|}{\mathrm{dist}(f, B^k(X;G))} \geq \varepsilon.$$

As it turns out, coboundary expansion is a very strong requirement. Currently it is not known whether bounded degree coboundary expanders of dimension ≥ 2 even exist. This leads to the relaxation of coboundary expansion, called cosystolic expansion, which was introduced by [EK16], and is defined as follows.

Definition 2.2 (Cosystolic expansion)**.** Let X be a d-dimensional simplicial complex and G an abelian group. X is called an *(ε, μ)-cosystolic expander* over G, if:

1. For any $f \in C^k(X;G) \setminus Z^k(X;G)$, $0 \leq k \leq d-1$,
$$\frac{\|\delta(f)\|}{\mathrm{dist}(f, Z^k(X;G))} \geq \varepsilon.$$
2. For any $z \in Z^k(X;G) \setminus B^k(X;G)$, $0 \leq k \leq d-1$,
$$\|z\| \geq \mu.$$

Recall that for any $\sigma \in X$, its *link* is the subcomplex obtained by taking all the faces in X which contain σ, and removing σ from all of them. Since the link of σ is a complex by itself, we can talk about cochains and norms in the link. Consider a $(k - |\sigma|)$-cochain in the link of σ, $f \in C^{k-|\sigma|}(X_\sigma; G)$. Its norm in the link is the probability that a random face would fall in $\mathrm{supp}(f)$ when the top face is distributed uniformly over the top faces in X_σ. Thus,

$$\|f\| = \Pr[r_k \setminus \sigma \in \mathrm{supp}(f) \mid r_{|\sigma|-1} = \sigma],$$

where $\|f\|$ is the norm in the link, and $r_k, r_{|\sigma|-1}$ are the random faces chosen in X.

From now on we fix for any face in the complex an arbitrary choice of ordering, so for any $\sigma \in X$ there is one fixed ordered face $\vec{\sigma}$ which corresponds to it. The choice of ordering does not matter, it just has to be consistent. For any k-cochain $f \in C^k$ and any face $\sigma \in X$, we define the *localization* of f to the link of σ, denoted by f_σ, as follows. For any ordered $(k - |\sigma|)$-face $\vec{\tau} \in \vec{X}_\sigma(k - |\sigma|)$, we define $f_\sigma(\vec{\tau}) = f(\vec{\sigma\tau})$, where $\vec{\sigma\tau} \in \vec{X}(k)$ is the ordered k-face obtained by concatenating $\vec{\tau}$ to $\vec{\sigma}$. We say that a cochain $f \in C^k$ is *minimal* if $\|f\| = \mathrm{dist}(f, B^k)$. We say that f is *locally minimal* if its localization to any link is minimal, i.e., if f_σ is minimal in X_σ for any $\emptyset \neq \sigma \in X$.

The following two lemmas regarding minimal and locally minimal cochains will be necessary later.

Lemma 2.3 (Minimal cochains are closed under inclusion)**.** *Let X be a d-dimensional simplicial complex and G an abelian group. For any $f, g \in C^k(X; G)$, $0 \leq k \leq d$, if f is a minimal cochain and $g(\vec{\sigma}) = f(\vec{\sigma})$ for every $\sigma \in supp(g)$, then g is a minimal cochain.*

Proof. Note that for any k-cochain $h \in C^k$,

$$\|f - h\| - \|g - h\| \leq \|f - g\| = \|f\| - \|g\|, \tag{2.1}$$

where the equality follows by the fact that $g(\vec{\sigma}) = f(\vec{\sigma})$ for every $\sigma \in \operatorname{supp}(g)$. Then for any k-coboundary $b \in B^k$,

$$\|g\| = \|g\| + \|f\| - \|f\| \leq \|g\| + \|f - b\| - \|f\| \leq \|g - b\|,$$

where the first inequality follows by the fact that f is a minimal cochain, and the second inequality follows by (2.1). $\square$

Lemma 2.4 (Minimal cochain is also locally minimal)**.** *Let X be a d-dimensional simplicial complex and G an abelian group. For any $f \in C^k(X; G)$, $0 \leq k \leq d$, if f is minimal, then f is also locally minimal.*

Proof. Let $f \in C^k(X; G)$ be a minimal cochain. Assume towards contradiction that f is not locally minimal. So there exists a face $\emptyset \neq \sigma \in X$ and a cochain $h \in C^{k-|\sigma|-1}(X_\sigma; G)$ such that

$$\|f_\sigma - \delta(h)\| < \|f_\sigma\|. \tag{2.2}$$

Define $g \in C^{k-1}(X; G)$ by $g(\vec{\sigma\tau}) = h(\vec{\tau})$ for any $\tau \in X_\sigma$, and for any other face $g(\vec{\tau}) = 0$. Note that $g_\sigma = h$, then by (2.2),

$$\|f - \delta(g)\| < \|f\|,$$

in contradiction to the minimality of f. It follows that f is locally minimal. $\square$

We have one more definition we want to present in this section.

Definition 2.5 (Skeleton expansion)**.** Let X be a d-dimensional simplicial complex. X is called an *α-skeleton expander*, if for any subset of vertices $S \subseteq X(0)$,

$$\|E(S)\| \leq \|S\|^2 + \alpha \|S\|,$$

where $E(S)$ denotes the set of edges with both endpoints in S.

3 Cosystolic expansion

Our aim in this section is to prove the reduction from a complex to its links and to show how it implies cosystolic expansion over any abelian group. Our main result is the following theorem.

Theorem 3.1 (Reduction from a complex to its links, formal, for informal see theorem 1.1)**.** *Let X be a d-dimensional simplicial complex, G an abelian group and $0 \le k \le d-1$. If for every $\sigma \in X$, the link X_σ is an α-skeleton expander, then for any constants $0 = c_{-1} \le c_0 \le \cdots \le c_k \le 1$ and k-cochain $f \in C^k(X;G)$, if f is locally minimal and $\|f\| \le \alpha$, then there exists $0 \le i \le k$ such that*

$$\|\delta(f)\| \ge \Big(\beta_i c_i - (k+1-i)(i+1)c_{i-1} - \alpha^{2^{-d}}(k+1)(k+2)2^{k+2}\Big)\|f\|,$$

where

$$\beta_i = \min\left\{ \frac{\|\delta(g)\|}{dist(g, B^{k-|\sigma|}(X_\sigma;G))} : \sigma \in X(i),\ g \in C^{k-|\sigma|}(X_\sigma;G) \setminus B^{k-|\sigma|}(X_\sigma;G) \right\}.$$

The implication to cosystolic expansion goes through an intermediate notion of small-set expansion, defined as follows.

Definition 3.2 (Small-set expansion)**.** Let X be a d-dimensional simplicial complex and G an abelian group. X is called an (ε,μ)-*small-set expander* over G, if for any $f \in C^k(X;G)$, $0 \le k \le d-1$,

$$f \text{ is locally minimal and } \|f\| \le \mu \quad \Rightarrow \quad \|\delta(f)\| \ge \varepsilon \|f\|.$$

The rest of this section is organized as follows. We first show that when all the links of a complex are coboundary expanders, theorem 3.1 implies that the complex is a small-set expander. Then we show, based on the strategy of [KKL14], that for bounded degree complexes, the $(d-1)$-skeleton of a small-set expander is a cosystolic expander. In the end of this section we prove theorem 3.1.

3.1 When all the links are coboundary expanders, the reduction implies small-set expansion

Theorem 3.3. *Let X be a d-dimensional simplicial complex, G an abelian group and $0 < \beta < 1$. There exist $\varepsilon = \varepsilon(d,\beta)$ and $\alpha = \alpha(d,\beta)$, such that if for every $\sigma \in X$ the link X_σ is an α-skeleton expander and for any $\emptyset \ne \sigma \in X$ the link X_σ is a β-coboundary expander over G, then X is an (ε,α)-small-set expander over G.*

Proof. Fix $0 < \rho < 1$, and let

$$\varepsilon = (1-\rho)\left(1 + \frac{d(d-1)^{2(d-1)}}{\beta^{d-1}(1-\beta)}\right)^{-1}, \qquad \alpha = \left(\frac{\rho}{1-\rho} \cdot \frac{\varepsilon}{d(d+1)2^{d+1}}\right)^{2^d}.$$

Now, let $0 \le k \le d-1$, and define the following constants:

- $c_{-1} = 0$,
- $c_0 = \dfrac{\varepsilon}{(1-\rho)\beta}$,
- $c_i = c_0 + \dfrac{k^2}{\beta} c_{i-1} \qquad \forall i \in \{1,\dots,k-1\}$,
- $c_k = \beta c_0 + (k+1)c_{k-1}$.

Note that

$$c_k = c_0\left(\beta + (k+1)\sum_{i=0}^{k-1}\left(\frac{k^2}{\beta}\right)^i\right) \le c_0\left(\beta + \frac{(k+1)k^{2k}}{\beta^{k-1}(k^2-\beta)}\right) = \frac{\varepsilon}{1-\rho}\left(1 + \frac{(k+1)k^{2k}}{\beta^k(k^2-\beta)}\right) \le 1,$$

so the conditions of theorem 3.1 are satisfied. Let $f \in C^k(X;G)$ be a locally minimal k-cochain with $\|f\| \le \alpha$, and let $0 \le i \le k$ be the good dimension promised by theorem 3.1.

1. If $i = k$, note that for any $\sigma \in \operatorname{supp}(f)$, $\|\delta(f_\sigma)\| \ge \|f_\sigma\|$, so theorem 3.1 yields

$$\|\delta(f)\| \ge \left(c_k - (k+1)c_{k-1} - \frac{\rho}{1-\rho}\varepsilon\right)\|f\| \ge \varepsilon\|f\|.$$

2. Otherwise, by the β-coboundary expansion of the links, theorem 3.1 yields

$$\|\delta(f)\| \ge \left(\beta c_i - k^2 c_{i-1} - \frac{\rho}{1-\rho}\varepsilon\right)\|f\| \ge \varepsilon\|f\|.$$

□

3.2 Small-set expansion implies cosystolic expansion

We show now, based on the strategy of [KKL14], that for bounded degree complexes, the $(d-1)$-skeleton of a small-set expander is a cosystolic expander, where a complex X is said to be *Q-bounded degree* if for any $v \in X(0)$ it holds that $|X_v| \le Q$. We prove the following theorem.

Theorem 3.4 (Small-set expansion implies cosystolic expansion for one dimension less)**.** *Let X be a d-dimensional Q-bounded degree (ε,μ)-small-set expander over an abelian group G. Then the $(d-1)$-skeleton of X is a $(\min\{\mu, Q^{-2}\},\mu)$-cosystolic expander over G.*

The proof of theorem 3.4 follows by the following two propositions.

Proposition 3.5 (Small-set expansion implies cocycle expansion for one dimension less)**.** *Let X be a d-dimensional Q-bounded degree (ε,μ)-small-set expander over an abelian group G. For any $f \in C^k(X;G) \setminus Z^k(X;G)$, $0 \le k \le d-2$, it holds that*

$$\frac{\|\delta(f)\|}{\mathit{dist}(f, Z^k(X;G))} \ge \min\{\mu, Q^{-2}\}.$$

Proposition 3.6 (Small-set expansion implies large non-trivial cocycles)**.** *Let X be a d-dimensional (ε,μ)-small-set expander over an abelian group G. For any $z \in Z^k(X;G) \setminus B^k(X;G)$, $0 \le k \le d-1$, it holds that $\|z\| \ge \mu$.*

Proof of theorem 3.4. Condition 1 of cosystolic expansion (definition 2.2) is proposition 3.5 and condition 2 is proposition 3.6. □

3.2.1 Proof of proposition 3.5

In order to prove proposition 3.5, we need the following lemma.

Lemma 3.7. *Let X be a d-dimensional Q-bounded degree simplicial complex and G an abelian group. For any $f \in C^k(X;G)$, $0 \le k \le d-1$, there exists $g \in C^{k-1}(X;G)$ such that:*

1. $\|g\| \leq Q^2\|f\|$.

2. $f - \delta(g)$ *is locally minimal.*

3. $\|f - \delta(g)\| \leq \|f\|$.

Proof. We prove by induction on $\|f\|$. For the base case, $\|f\| = 0$, the claim holds trivially for $g = 0$. Assume the claim holds for any cochain f' with $\|f'\| < \|f\|$. Now, if f is locally minimal, the claim holds for $g = 0$. Otherwise, there exists some $\sigma \in X$ such that f_σ is not minimal in X_σ. So there exists a cochain in the link of σ, $h \in C^{k-1-|\sigma|}(X_\sigma; G)$, such that $\|f_\sigma - \delta(h)\| < \|f_\sigma\|$. Define $g' \in C^{k-1}(X; G)$ by $g'(\vec{\sigma\tau}) = h(\vec{\tau})$ for any $\tau \in X_\sigma$ and for any other face $g'(\vec{\tau}) = 0$. It follows that $\|f - \delta(g')\| < \|f\|$. By the induction assumption, there exists $g'' \in C^{k-1}(X; G)$ such that:

1. $\|g''\| \leq Q^2\|f - \delta(g')\|$.

2. $f - \delta(g') - \delta(g'') = f - \delta(g' + g'')$ is locally minimal.

3. $\|f - \delta(g') - \delta(g'')\| = \|f - \delta(g' + g'')\| \leq \|f - \delta(g')\| < \|f\|$.

Denote by $g = g' + g''$, and note that conditions 2 and 3 are satisfied. As for condition 1, note that since

$$\|f\| = \sum_{\sigma \in \operatorname{supp}(f)} \Pr[r_k = \sigma] = \sum_{\sigma \in \operatorname{supp}(f)} \sum_{\substack{\tau \in X(d) \\ \tau \supset \sigma}} \Pr[r_d = \tau \wedge r_k = \sigma] = \sum_{\sigma \in \operatorname{supp}(f)} \sum_{\substack{\tau \in X(d) \\ \tau \supset \sigma}} \frac{1}{|X(d)|\binom{d+1}{k+1}},$$

then

$$\left\|f - \delta(g')\right\| < \|f\| \leq \|f\| - \frac{1}{|X(d)|\binom{d+1}{k+1}}. \tag{3.1}$$

Also, since $\operatorname{supp}(g')$ contains only faces which contain σ, then

$$\begin{aligned} \left\|g'\right\| &\leq \sum_{\substack{\tau \in X(k) \\ \tau \supset \sigma}} \Pr[r_k = \tau] = \sum_{\substack{\tau \in X(k) \\ \tau \supset \sigma}} \sum_{\substack{\rho \in X(d) \\ \rho \supset \tau}} \Pr[r_d = \rho \wedge r_k = \tau] \\ &= \sum_{\substack{\tau \in X(k) \\ \tau \supset \sigma}} \sum_{\substack{\rho \in X(d) \\ \rho \supset \tau}} \frac{1}{|X(d)|\binom{d+1}{k+1}} \leq \frac{Q^2}{|X(d)|\binom{d+1}{k+1}}. \end{aligned} \tag{3.2}$$

Combining (3.1), (3.2) and property 1 from the induction assumption yields

$$\|g\| \leq \left\|g'\right\| + \left\|g''\right\| \leq \frac{Q^2}{|X(d)|\binom{d+1}{k+1}} + Q^2\left(\|f\| - \frac{1}{|X(d)|\binom{d+1}{k+1}}\right) = Q^2\|f\|,$$

and condition 1 is satisfied as well. □

We can now prove proposition 3.5.

Proof of proposition 3.5. Let $f \in C^k(X; G) \setminus Z^k(X; G)$, $0 \leq k \leq d-2$. If $\|\delta(f)\| \geq \mu$ we are done, so assume that $\|\delta(f)\| < \mu$. Let $g \in C^k(X; G)$ be the k-cochain promised by lemma 3.7 when applied on $\delta(f)$. By properties 2 and 3 of lemma 3.7, $\delta(f) - \delta(g) = \delta(f - g)$ is a $(k+1)$-cochain which is locally minimal and $\|\delta(f - g)\| \leq \mu$. By the small-set expansion,

$$0 = \|\delta(\delta(f - g))\| \geq \varepsilon\|\delta(f - g)\|,$$

so $\delta(f-g)=0$, which means that $f-g \in Z^k(X;G)$. By property 1 of lemma 3.7 we know that $\|g\| \leq Q^2\|\delta(f)\|$, which yields

$$\|\delta(f)\| \geq Q^{-2}\|g\| = Q^{-2}\|f-(f-g)\| \geq Q^{-2} \cdot \mathrm{dist}(f, Z^k(X;G)).$$

□

3.2.2 Proof of proposition 3.6

Note that it is enough to show that the proposition holds for minimal non-trivial cocycles, since if $z \in Z^k(X;G) \setminus B^k(X;G)$ is not minimal, then there exists a coboundary $b \in B^k(X;G)$ such that $\|z\| \geq \|z-b\|$ and $z-b \in Z^k(X;G) \setminus B^k(X;G)$ is minimal.

Let $z \in Z^k(X;G)$ be a minimal cocycle. We show that if $\|z\| < \mu$, then $z \in B^k(X;G)$. Since z is minimal, by lemma 2.4 z is also locally minimal, so by the small-set expansion, $\|\delta(z)\| \geq \varepsilon\|z\|$. But on the other hand $z \in Z^k(X;G)$ so $\|\delta(z)\| = 0$. It follows that $\|z\| = 0$, so $z \in B^k(X;G)$ as required. □

3.3 Proof of theorem 3.1

The rest of this section is dedicated to proving theorem 3.1. In the following lemma we show that whenever all the information of a cochain is seen in a link, then its local coboundaries coincide with global coboundaries.

Lemma 3.8 (Local-to-global coboundaries)**.** *Let X be a d-dimensional simplicial complex, G an abelian group, $f \in C^k(X;G)$, $0 \leq k \leq d-1$, and $\sigma \in X(i)$, $i < k$. For any $\tau \in supp(\delta(f_\sigma))$, if $\sigma \cup \tau \setminus \{v\} \notin supp(f)$ for every $v \in \sigma$, then $\sigma \cup \tau \in supp(\delta(f))$.*

Proof. Let us denote $\vec{\sigma} = (v_0, \dots, v_i)$ and $\vec{\tau} = (v_{i+1}, \dots, v_{k+1})$ (where $\vec{\sigma}$ and $\vec{\tau}$ are the fixed ordered faces corresponding to σ and τ). Then

$$\begin{aligned}\delta(f)(\vec{\sigma\tau}) &= \sum_{j=0}^{k+1}(-1)^j f(\vec{\sigma\tau} \setminus \{v_j\}) = \sum_{j=i+1}^{k+1}(-1)^j f(\vec{\sigma\tau} \setminus \{v_j\}) \\ &= (-1)^{i+1}\sum_{j=0}^{k-i}(-1)^j f_\sigma(\vec{\tau} \setminus \{v_{j+i+1}\}) = (-1)^{i+1}\delta(f_\sigma)(\vec{\tau}) \neq 0.\end{aligned}$$

□

We now define a machinery of fat faces, which essentially lets us move calculations down the dimensions. Let $\eta > 0$ be a fatness constant. For any subset of k-faces $A \subseteq X(k)$ we define the sets of *fat faces* as follows. The set of fat k-faces is defined as $A_k = A$, and for any $-1 \leq i \leq k-1$ we define the set of fat i-faces $A_i \subseteq X(i)$ by

$$A_i = \{\sigma \in X(i) \mid \Pr[r_{i+1} \in A_{i+1} \mid r_i = \sigma] \geq \eta^{2^{k-i-1}}\}.$$

The following lemma shows that for any $-1 \leq i \leq k-1$, the size of A_i cannot be much larger than the size of A.

Lemma 3.9. *Let X be a d-dimensional simplicial complex and $\eta > 0$ a fatness constant. For any subset of k-faces $A \subseteq X(k)$, $0 \leq k \leq d-1$, and $-1 \leq i \leq k-1$,*

$$\|A_i\| \leq \eta^{1-2^{k-i}}\|A\|.$$

Proof. By laws of probability, for any $-1 \leq j \leq k-1$,

$$\Pr[r_j \in A_j] = \frac{\Pr[r_{j+1} \in A_{j+1} \wedge r_j \in A_j]}{\Pr[r_{j+1} \in A_{j+1} \mid r_j \in A_j]} \leq \eta^{-2^{k-j-1}} \Pr[r_{j+1} \in A_{j+1}]. \tag{3.3}$$

Applying (3.3) iteratively for $j = i, i+1, \ldots, k-1$ finishes the proof. $\square$

For any $\sigma \in X(i)$, $-1 \leq i \leq k$, we denote by $A{\downarrow}\sigma \subseteq A$ the set of faces in A which have a sequence of containments (in [EK16] it is called a ladder) of fat faces down to σ, formally,

$$A{\downarrow}\sigma = \{\tau \in A \mid \exists \tau_{k-1} \in A_{k-1}, \ldots, \tau_{i+1} \in A_{i+1} \text{ s.t. } \tau \supset \tau_{k-1} \supset \cdots \supset \tau_{i+1} \supset \sigma\}.$$

Recall that for a k-cochain $f \in C^k$, we denote its support by $A = \mathrm{supp}(f)$. So we also define $f{\downarrow}\sigma$ to be the restriction of f to $A{\downarrow}\sigma$, formally,

$$(f{\downarrow}\sigma)(\vec{\tau}) = \begin{cases} f(\vec{\tau}) & \tau \in A{\downarrow}\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

A good situation for us is that for any two fat faces which intersect on a codimension 1 face, their intersection is a fat face. This essentially allows us to move calculations down the dimensions. We denote by $\Upsilon \subseteq X(k+1)$ the set of bad $(k+1)$-faces, for which a bad situation exists, formally,

$$\Upsilon = \{\tau \in X(k+1) \mid \exists i \leq k \text{ and } \sigma, \sigma' \subset \tau \text{ s.t. } \sigma, \sigma' \in A_i \text{ and } \sigma \cap \sigma' \in X(i-1) \setminus A_{i-1}\}.$$

In the following proposition we show how we use this machinery of fat faces. The idea is that either we get a lot of expansion from a certain dimension or we can move down one dimension lower.

Proposition 3.10. *Let X be a d-dimensional simplicial complex, G an abelian group and $\eta > 0$ a fatness constant. For any $f \in C^k(X;G)$, $0 \leq k \leq d-1$, and $0 \leq i \leq k$,*

$$\|\delta(f)\| \geq \min_{\sigma \in A_i} \left\{ \frac{\|\delta((f{\downarrow}\sigma)_\sigma)\|}{\|(f{\downarrow}\sigma)_\sigma\|} \right\} \Pr[r_k \in A{\downarrow}r_i \wedge r_i \in A_i] - $$
$$(k+1-i)(i+1)\Pr[r_k \in A{\downarrow}r_{i-1} \wedge r_{i-1} \in A_{i-1}] - \|\Upsilon\|.$$

Proof. By lemma 3.8 we know that every local coboundary of $(f{\downarrow}\sigma)_\sigma$ is also a global coboundary, i.e., $\tau \in \mathrm{supp}(\delta((f{\downarrow}\sigma)_\sigma)) \Rightarrow \sigma \cup \tau \in \mathrm{supp}(\delta(f{\downarrow}\sigma))$. Thus,

$$\|\delta((f{\downarrow}\sigma)_\sigma)\| \leq \|(\delta(f{\downarrow}\sigma))_\sigma\| = \Pr[r_{k+1} \in \mathrm{supp}(\delta(f{\downarrow}\sigma)) \mid r_i = \sigma]. \tag{3.4}$$

Consider a face $\tau \in \mathrm{supp}(\delta(f{\downarrow}\sigma))$. By definition, it contains at least one k-face $\tau^* \subset \tau$, such that $\tau^* \in A{\downarrow}\sigma$. We claim that one of the following cases must occur:

1. τ is a bad face.
2. σ contains a fat $(i-1)$-face $\sigma^* \in A_{i-1}$, such that $\tau^* \in A{\downarrow}\sigma^*$.
3. $\tau \in \mathrm{supp}(\delta(f))$.

If τ is a bad face, the claim holds, so assume that τ is not a bad face. By definition, there exists a sequence of fat faces $\tau_{k-1} \in A_{k-1}, \tau_{k-2} \in A_{k-2}, \ldots, \tau_{i+1} \in A_{i+1}$, such that $\tau \supset \tau^* \supset \tau_{k-1} \supset \cdots \supset \tau_{i+1} \supset \sigma$. Let us denote $\tau = \{v_0, v_1, \ldots, v_{k+1}\}$, $\tau^* = \tau \setminus \{v_{k+1}\}$, $\tau_{k-1} = \tau^* \setminus \{v_k\}$, and so on down to $\sigma = \tau_{i+1} \setminus \{v_{i+1}\}$. Now, if $\tau \setminus \{v_j\} \in A$ for some $j \in \{0, \ldots, i\}$, then $\tau^* \setminus \{v_j\} \in A_{k-1}$ since it is the intersection of two fat k-faces, and then $\tau_{k-1} \setminus \{v_j\} \in A_{k-2}$, and so on down to $\sigma^* = \sigma \setminus \{v_j\} \in A_{i-1}$, and case 2 holds. Otherwise, for any $j \in \{i+1, \ldots, k\}$, a similar argument shows that if $\tau \setminus \{v_j\} \in A$ then $\tau \setminus \{v_j\} \in A{\downarrow}\sigma$. It follows that f and $f{\downarrow}\sigma$ agree on all k-faces that are contained in τ, and case 3 holds. Thus,

$$\tau \in \operatorname{supp}(\delta(f{\downarrow}\sigma)) \quad \Rightarrow \quad (\tau \in \Upsilon) \vee (\tau^* \in A{\downarrow}\sigma^*) \vee (\tau \in \operatorname{supp}(\delta(f))). \tag{3.5}$$

Using (3.5) and summing over all $\tau \in \operatorname{supp}(\delta(f{\downarrow}\sigma))$ yields

$$\begin{aligned} \Pr[r_{k+1} \in \operatorname{supp}(\delta(f{\downarrow}\sigma)) \mid r_i = \sigma] \leq \\ & \Pr[r_{k+1} \in \Upsilon \mid r_i = \sigma] + \\ & (k+1-i)(i+1)\Pr[r_k \in A{\downarrow}r_{i-1} \wedge r_{i-1} = A_{i-1} \mid r_i = \sigma] + \\ & \Pr[r_{k+1} \in \operatorname{supp}(\delta(f)) \mid r_i = \sigma], \end{aligned} \tag{3.6}$$

where the $(k+1-i)(i+1)$ factor is due the probability that $r_k = \tau^*$ and $r_{i-1} = \sigma^*$ given that $r_{k+1} = \tau \supset \tau^*$ and $r_i = \sigma \supset \sigma^*$. Substituting (3.4) in (3.6), and multiplying and dividing by $\|(f{\downarrow}\sigma)_\sigma\| = \Pr[r_k \in A{\downarrow}\sigma \mid r_i = \sigma]$ yields

$$\begin{aligned} \frac{\|\delta((f{\downarrow}\sigma)_\sigma)\|}{\|(f{\downarrow}\sigma)_\sigma\|}\Pr[r_k \in f{\downarrow}\sigma \mid r_i = \sigma] \leq \\ & \Pr[r_{k+1} \in \Upsilon \mid r_i = \sigma] + \\ & (k+1-i)(i+1)\Pr[r_k \in A{\downarrow}r_{i-1} \wedge r_{i-1} = A_{i-1} \mid r_i = \sigma] + \\ & \Pr[r_{k+1} \in \operatorname{supp}(\delta(f)) \mid r_i = \sigma]. \end{aligned} \tag{3.7}$$

Multiplying (3.7) by $\Pr[r_i = \sigma]$, summing over all $\sigma \in A_i$, and applying the law of total probability to the right-hand side yields

$$\begin{aligned} \sum_{\sigma \in A_i} \frac{\|\delta((f{\downarrow}\sigma)_\sigma)\|}{\|(f{\downarrow}\sigma)_\sigma\|}\Pr[r_k \in f{\downarrow}\sigma \wedge r_i = \sigma] \leq \\ & \Pr[r_{k+1} \in \Upsilon] + \\ & (k+1-i)(i+1)\Pr[r_k \in A{\downarrow}r_{i-1} \wedge r_{i-1} = A_{i-1}] + \\ & \Pr[r_{k+1} \in \operatorname{supp}(\delta(f))]. \end{aligned}$$

Taking the minimum over all $\sigma \in A_i$ and rearranging completes the proof. □

It is left to bound the size of the bad faces. The following proposition shows that the size of the bad faces is controlled by the skeleton expansion of the links.

Proposition 3.11 (Skeleton expansion implies small set of bad faces)**.** *Let X be a d-dimensional simplicial complex, $\eta > 0$ a fatness constant and $0 < \alpha \leq \eta^{2^{d-1}}$. If for any $\sigma \in X$, the link X_σ is an α-skeleton expander, then for any subset of k-faces $A \subseteq X(k)$, $0 \leq k \leq d-1$,*

$$\|\Upsilon\| \leq \eta(k+1)(k+2)2^{k+2}\|A\|.$$

Proof. By definition, any bad face $\tau \in \Upsilon$ contains at least one pair of faces $\sigma, \sigma' \subset \tau$ such that $\sigma, \sigma' \in A_i$, $\sigma \cup \sigma' \in X(i+1)$, and $\sigma \cap \sigma' \in X(i-1) \setminus A_{i-1}$ for some $0 \le i \le k$. For any $\tau \in \Upsilon$, choose one such pair $\sigma, \sigma' \subset \tau$ and denote by $\widehat{\tau} = \sigma \cup \sigma'$ and by $\check{\tau} = \sigma \cap \sigma'$. Note that $\widehat{\tau}$ is seen in the link of $\check{\tau}$ as an edge between two fat vertices. Denote by $\Upsilon_i = \{\tau \in \Upsilon \mid \widehat{\tau} \in X(i)\}$, so the set of bad faces can be decomposed to $\Upsilon = \bigsqcup_{i=1}^{k+1} \Upsilon_i$. Now,

$$
\begin{aligned}
\Pr[r_{k+1} \in \Upsilon] &= \sum_{i=1}^{k+1} \sum_{\tau \in \Upsilon_i} \Pr[r_{k+1} = \tau] = \sum_{i=1}^{k+1} \sum_{\tau \in \Upsilon_i} \frac{\Pr[r_{k+1} = \tau \wedge r_i = \widehat{\tau} \wedge r_{i-2} = \check{\tau}]}{\Pr[r_i = \widehat{\tau} \wedge r_{i-2} = \check{\tau} \mid r_{k+1} = \tau]} \\
&\le \sum_{i=1}^{k+1} \sum_{\tau \in \Upsilon_i} \binom{k+2}{i+1} \binom{i+1}{i-1} \Pr[r_i = \widehat{\tau} \wedge r_{i-2} = \check{\tau}] \\
&\le \sum_{i=1}^{k+1} \sum_{\tau \in \Upsilon_i} \binom{k+2}{i+1} \binom{i+1}{i-1} (\eta^{2^{k+1-i}} + \alpha) \Pr[r_{i-1} \in A_{i-1} \wedge r_{i-2} = \check{\tau}] \\
&\le \sum_{i=1}^{k+1} \binom{k+2}{i+1} \binom{i+1}{i-1} \frac{2\eta^{2^{k+1-i}}}{\eta^{2^{k+1-i}-1}} \Pr[r_k \in A] \\
&\le (k+2)(k+1)\eta \Pr[r_k \in A] \sum_{i=1}^{k+1} \binom{k+2}{i+1},
\end{aligned}
$$

where the second inequality follows by the α-skeleton expansion of the links, and the third inequality follows by the law of total probability and by lemma 3.9. $\square$

We can now prove theorem 3.1.

Proof of theorem 3.1. Define the fatness constant $\eta = \alpha^{2^{-d}}$. Now, let $f \in C^k(X; G)$ be a locally minimal k-cochain with $\|f\| \le \alpha \le \eta^{2^{k+1}}$. By lemma 3.9 it follows that

$$\|A_{-1}\| \le \eta^{1-2^{k+1}} \|f\| \le \eta < 1.$$

But since $X(-1)$ contains only one face, i.e., $\|A_{-1}\| \in \{0, 1\}$, then $\|A_{-1}\| = 0$. In other words, the empty-set is not a fat face, thus $\Pr[r_k \in A \downarrow r_{-1} \wedge r_{-1} \in A_{-1}] = 0$. Also note that $\Pr[r_k \in A \downarrow r_k \wedge r_k \in A_k] = \|f\| \ge c_k \|f\|$.

Now, if $\Pr[r_k \in A \downarrow r_i \wedge r_i \in A_i] \ge c_i \|f\|$ for all $0 \le i \le k$, then applying proposition 3.10 on $i = 0$ yields

$$\|\delta(f)\| \ge \min_{\sigma \in A_0} \left\{ \frac{\|\delta((f \downarrow \sigma)_\sigma)\|}{\|(f \downarrow \sigma)_\sigma\|} \right\} c_0 - \|\Upsilon\|. \tag{3.8}$$

Otherwise, let $0 \le j \le k-1$ be the maximal for which $\Pr[r_k \in A \downarrow r_j \wedge r_j \in A_j] < c_j \|f\|$. Applying proposition 3.10 on $i = j+1$ yields

$$\|\delta(f)\| \ge \min_{\sigma \in A_i} \left\{ \frac{\|\delta((f \downarrow \sigma)_\sigma)\|}{\|(f \downarrow \sigma)_\sigma\|} \right\} c_i - (k+1-i)(i+1)c_{i-1} - \|\Upsilon\|. \tag{3.9}$$

Since f is locally minimal, by lemma 2.3, $(f \downarrow \sigma)_\sigma$ is minimal in X_σ for any $\emptyset \ne \sigma \in X$. Thus,

$$\|(f \downarrow \sigma)_\sigma\| = \operatorname{dist}((f \downarrow \sigma)_\sigma, B^{k-|\sigma|}(X_\sigma; G)). \tag{3.10}$$

By proposition 3.11 we know that

$$\|\Upsilon\| \le \alpha^{2^{-d}} (k+1)(k+2) 2^{k+2} \|f\|. \tag{3.11}$$

Substituting (3.10) and (3.11) in (3.8) or (3.9) completes the proof. $\square$

4 Spherical buildings

Spherical buildings are very symmetric complexes with a nice geometric structure. An example for a spherical building is the following complex. Let $d \in \mathbb{N}$ and q a prime power. Denote by $V = \mathbb{F}_q^d$ the d-dimensional vector space over $\mathbb{F}_q$. The vertices of the complex are proper subspaces of V (i.e., not $\{0\}$ and V), and its faces are flags of subspaces. The resulting complex is a $(d-2)$-dimensional spherical building (since maximal flags have $d-1$ vertices). For $d=3$ this is the famous "lines vs. planes" graph which is known to be an excellent expander.

Any d-dimensional spherical building X comes with a collection of d-dimensional subcomplexes, called *apartments*, such that all the apartments are isomorphic to each other and for any two faces in the complex there exists an apartment containing both of them. An important fact is that the size of each apartment is bounded by a constant θ_d which depends only on d (and not on the number of vertices). Also, there exists a group of automorphisms $\Gamma \leq Aut(X)$ which acts transitively on the d-dimensional faces of X, i.e., for any $\sigma, \sigma' \in X(d)$, there exists $g \in \Gamma$ such that $g\sigma = \sigma'$.

In [EK16], the authors showed that the spherical building is an α-skeleton expander for an arbitrary $\alpha > 0$ (it is controlled by a parameter called the *thickness* of the building). In [LMM16], the authors showed that the spherical building is a coboundary expander, but only over the field $\mathbb{F}_2$. This is not enough for us as we need coboundary expansion over $\mathbb{Z}$. We follow their strategy and with some modifications we prove the following theorem.

Theorem 4.1. *The d-dimensional spherical building is a β-coboundary expander over any abelian group for*

$$\beta = \left(2^d \theta_d\right)^{-1}.$$

The proof of theorem 4.1 is essentially composed of two propositions. We use the geometric structure of the spherical building in order to relate the coboundary of a cochain to its distance from the coboundaries. By this relation we over-count each face in the coboundary many times. Then we use its symmetric structure in order to bound these over-counts.

For any $-1 \leq k \leq d-1$, we denote by $\mathcal{F}_k = X(d) \times X(k)$ the set of all pairs of top faces and k-dimensional faces . For any $(\sigma, \tau) \in \mathcal{F}_k$, let $A_{\sigma,\tau}$ be the complex obtained by the intersection of all the apartments in X which contain both σ and τ. Note that if $\tau \subset \tau'$, then $A_{\sigma,\tau} \subset A_{\sigma,\tau'}$.

The following proposition is implied by the geometric structure of the spherical building. Each apartment of the spherical building is a sphere and any piece of it, as we defined above, is either a sphere or it is contractible. It allows us to relate the coboundary of a cochain to its distance from the coboundaries by over-counting each face as the amount of apartments containing it.

Proposition 4.2. *Let X be a d-dimensional spherical building and G an abelian group. For any $f \in C^k(X;G)$, $-1 \leq k \leq d-1$, and $\sigma \in X(d)$,*

$$dist(f, B^k(X;G)) \leq \sum_{\tau \in X(k)} \|\tau\| \cdot |supp(\delta(f)) \cap A_{\sigma,\tau}|.$$

The next proposition is implied by the symmetric structure of the spherical building. Since it possess so many symmetries, all the apartments are spread around it evenly. This implies that any face cannot be contained in many apartments, so we can bound the number of times we over-count each face.

Proposition 4.3. *Let X be a d-dimensional spherical building. For any $-1 \le k \le d-1$ and $\rho \in X$,*

$$\sum_{\substack{(\sigma,\tau)\in\mathcal{F}_k:\\ \rho\in A_{\sigma,\tau}}} \|\tau\| \le \theta_d \cdot |\{\sigma \in X(d) \mid \rho \subseteq \sigma\}|.$$

We show first how theorem 4.1 is implied by the above two propositions.

Proof of theorem 4.1. Let $f \in C^k(X;G)$, $-1 \le k \le d-1$. By proposition 4.2,

$$\begin{aligned} |X(d)| \cdot \mathrm{dist}(f, B^k(X;G)) &= \sum_{\sigma\in X(d)} \mathrm{dist}(f, B^k(X;G)) \\ &\le \sum_{\sigma\in X(d)} \sum_{\tau\in X(k)} \|\tau\| \cdot |\mathrm{supp}(\delta(f)) \cap A_{\sigma,\tau}| \\ &= \sum_{(\sigma,\tau)\in\mathcal{F}_k} \|\tau\| \cdot |\mathrm{supp}(\delta(f)) \cap A_{\sigma,\tau}| \\ &= \sum_{\rho\in\mathrm{supp}(\delta(f))} \sum_{\substack{(\sigma,\tau)\in\mathcal{F}_k:\\ \rho\in A_{\sigma,\tau}}} \|\tau\|. \end{aligned} \tag{4.1}$$

By proposition 4.3,

$$\sum_{\substack{(\sigma,\tau)\in\mathcal{F}_k:\\ \rho\in A_{\sigma,\tau}}} \|\tau\| \le \theta_d \cdot |\{\sigma \in X(d) \mid \rho \subseteq \sigma\}| = \theta_d |X(d)| \binom{d+1}{k+2} \|\rho\| \tag{4.2}$$

Combining (4.1) and (4.2) yields

$$\mathrm{dist}(f, B^k(X;G)) \le \sum_{\rho\in\mathrm{supp}(\delta(f))} \theta_d \binom{d+1}{k+2} \|\rho\| = \theta_d \binom{d+1}{k+2} \|\delta(f)\|,$$

where rearranging completes the proof. $\square$

4.1 Proof of proposition 4.2

We recall some basic definitions of simplicial complexes. Let X be a d-dimensional simplicial complex. For any $-1 \le k \le d$, a *k-chain* is a linear combination of the k-dimensional faces. Denote the group of k-chains by

$$C_k(X) = \left\{ \sum_{\sigma\in X(k)} a_\sigma \cdot \sigma \;\middle|\; \forall \sigma, a_\sigma \in \mathbb{Z} \right\}.$$

We fix some arbitrary orientations of the faces in X, so when considering a face, there is one fixed ordering of its vertices. Then, the boundary of a k-face $(v_0, v_1, \ldots, v_k) \in X(k)$ is

$$\partial((v_0, \ldots, v_k)) = \sum_{i=0}^{k} (-1)^i (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k),$$

and the boundary of a k-chain $c \in C_k(X)$ is

$$\partial(c) = \sum_{\sigma\in X(k)} a_\sigma \cdot \partial(\sigma).$$

For ease of notation, for $\sigma = (v_0, \dots, v_k)$, we denote $\sigma_i = (v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_k)$. Note that the boundary operator commutes with the coboundary operator defined in §2, i.e., for any k-cochain $f \in C^k(X; G)$ and a $(k+1)$-face $\sigma \in X(k+1)$,

$$\delta(f)(\sigma) = \sum_{i=0}^{k} (-1)^i f(\sigma_i) = f(\partial(\sigma)).$$

The following lemma from [LMM16] shows a nice filling property of the complexes $A_{\sigma,\tau}$ defined above.

Lemma 4.4. *[LMM16, Claim 3.5] Let X be a d-dimensional spherical building. For any $(\sigma, \tau) \in \mathcal{F}_k$, $-1 \le k \le d-1$, and an i-chain $c \in C_i(A_{\sigma,\tau})$, $0 \le i \le d-1$, if $\partial(c) = 0$ then there exists an $(i+1)$-chain $c' \in C_{i+1}(A_{\sigma,\tau})$ such that $\partial(c') = c$.*

We use this filling property in order to define a family of chains such that each two consecutive chains are related by the boundary operator.

Lemma 4.5. *Let X be a d-dimensional spherical building. There exists a family of chains*

$$\mathcal{C} = \{c_{\sigma,\tau} \in C_{k+1}(A_{\sigma,\tau}) \mid -1 \le k \le d-1,\ (\sigma, \tau) \in \mathcal{F}_k\},$$

such that

$$\partial(c_{\sigma,\tau}) = (-1)^{k+1}\tau + \sum_{i=0}^{k} (-1)^i c_{\sigma,\tau_i}.$$

Proof. We define $\mathcal{C}$ inductively. For $k = -1$, we have only the empty set $\emptyset \in X(-1)$. For any $\sigma \in X(d)$, choose an arbitrary vertex $v_\sigma \in A_{\sigma,\emptyset}(0)$ and define $c_{\sigma,\emptyset} = v_\sigma$. Then it holds that

$$\partial(c_{\sigma,\emptyset}) = \partial(v_\sigma) = (-1)^0\emptyset,$$

as required. Assume now that $\mathcal{C}$ is defined for any $-1 \le i \le k-1$. For any $(\sigma, \tau) \in \mathcal{F}_k$ define $c_{\sigma,\tau}$ as follows. Consider the k-chain $c = (-1)^{k+1}\tau + \sum_{i=0}^{k} (-1)^i c_{\sigma,\tau_i}$, and note that $\partial(c) = 0$ since

$$\begin{aligned}
\partial(c) &= (-1)^{k+1}\partial(\tau) + \sum_{i=0}^{k} (-1)^i \partial(c_{\sigma,\tau_i}) \\
&= (-1)^{k+1}\partial(\tau) + \sum_{i=0}^{k} (-1)^i \left((-1)^k \tau_i + \sum_{j=0}^{k-1} (-1)^j c_{\sigma,\tau_{ij}} \right) \\
&= (-1)^{k+1}\partial(\tau) + (-1)^k \partial(\tau) + \sum_{i>j} (-1)^{i+j} c_{\sigma,\tau_{ij}} + \sum_{i<j} (-1)^{i+j-1} c_{\sigma,\tau_{ij}} = 0.
\end{aligned}$$

By lemma 4.4 it follows that there exists an $(k+1)$-chain $c' \in C_{k+1}(A_{\sigma,\tau})$ such that $\partial(c') = c$, so define $c_{\sigma,\tau} = c'$. $\square$

For any $\sigma \in X(d)$ and $0 \le k \le d$, we define the contraction operator $\iota_\sigma = \iota_{\sigma,k} : C^k(X; G) \to C^{k-1}(X; G)$ as follows. For any $f \in C^k(X; G)$ and $\tau \in X(k-1)$,

$$\iota_\sigma(f)(\tau) = (-1)^k f(c_{\sigma,\tau}).$$

This contraction operator allows us to relate the coboundary of a cochain to its distance from the coboundaries, as shown in the next lemma.

Lemma 4.6. *Let X be a d-dimensional spherical building and G an abelian group. For any $f \in C^k(X; G)$, $0 \le k \le d-1$, and $\sigma \in X(d)$,*

$$\delta(\iota_\sigma(f)) + \iota_\sigma(\delta(f)) = f.$$

Proof. For any $\tau \in X(k)$,

$$\begin{aligned}
&\delta(\iota_\sigma(f))(\tau) + \iota_\sigma(\delta(f))(\tau) \\
&\qquad = \sum_{i=0}^{k} (-1)^i (\iota_\sigma(f))(\tau_i) + (-1)^{k+1} (\delta(f))(c_{\sigma,\tau}) \\
&\qquad = \sum_{i=0}^{k} (-1)^i (-1)^k f(c_{\sigma,\tau_i}) + (-1)^{k+1} f(\partial(c_{\sigma,\tau})) \\
&\qquad = (-1)^k \sum_{i=0}^{k} (-1)^i f(c_{\sigma,\tau_i}) + (-1)^{k+1} \left((-1)^{k+1} f(\tau) + \sum_{i=0}^{k} (-1)^i f(c_{\sigma,\tau_i}) \right) = f(\tau).
\end{aligned}$$

□

We can now prove proposition 4.2.

Proof of proposition 4.2. Let $f \in C^k(X; G)$, $-1 \le k \le d-1$. By lemma 4.6, for any $\sigma \in X(d)$,

$$\|\iota_\sigma(\delta(f))\| = \|f - \delta(\iota_\sigma(f))\| \ge \mathrm{dist}(f, B^k(X; G)). \tag{4.3}$$

Note that for any $\tau \in X(k)$,

$$\begin{aligned}
\iota_\sigma(\delta(f))(\tau) \ne 0 \quad &\Rightarrow \quad \delta(f)(c_{\sigma,\tau}) \ne 0 \\
&\Rightarrow \quad \exists \rho \in \mathrm{supp}(\delta(f)) \cap \mathrm{supp}(c_{\sigma,\tau}) \\
&\Rightarrow \quad \exists \rho \in \mathrm{supp}(\delta(f)) \cap A_{\sigma,\tau},
\end{aligned}$$

which yields that

$$\|\iota_\sigma(\delta(f))\| = \sum_{\tau \in \mathrm{supp}(\iota_\sigma(\delta(f)))} \|\tau\| \le \sum_{\tau \in X(k)} \|\tau\| \cdot |\mathrm{supp}(\delta(f)) \cap A_{\sigma,\tau}|. \tag{4.4}$$

Combining (4.3) and (4.4) finishes the proof. □

4.2 Proof of proposition 4.3

The key point of the proof is that the spherical building possess so many symmetries, so for any face, only a small portion of the apartments contains it. We show it formally in the following lemma.

Lemma 4.7. *Let X be a d-dimensional spherical building and $\Gamma \le Aut(X)$ the group that acts transitively on $X(d)$. For any $\rho \in X$ and $(\sigma, \tau) \in \mathcal{F}_k$, $-1 \le k \le d-1$,*

$$\frac{|\{g \in \Gamma \mid g\rho \in A_{\sigma,\tau}\}|}{|\Gamma|} \le \theta_d \frac{|\{\sigma \in X(d) \mid \rho \subseteq \sigma\}|}{|X(d)|}.$$

Proof. For any $\rho \in X$, denote by $\Gamma_\rho = \{g \in \Gamma \mid g\rho = \rho\}$ the stabilizer of ρ, and consider the quotient Γ/Γ_ρ. The elements in Γ/Γ_ρ are equivalence classes of the form $g\Gamma_\rho = \{gh \mid h \in \Gamma_\rho\}$. Consider a face $\sigma \in X(d)$ such that $\rho \subseteq \sigma$. The elements in Γ_ρ can move σ only to other d-dimensional faces which contain ρ. It follows that for any equivalence class $g\Gamma_\rho$, the number of d-dimensional faces that the elements in $g\Gamma_\rho$ can move σ is bounded by the number of d-dimensional faces which contain σ. Since Γ is transitive, there must be enough equivalence classes to cover all $X(d)$. Thus,

$$\frac{|\Gamma|}{|\Gamma_\rho|} = \left|{}^{\Gamma}\!/\!{}_{\Gamma_\rho}\right| \geq \frac{|X(d)|}{|\{\sigma \in X(d) \mid \rho \subseteq \sigma\}|}, \tag{4.5}$$

where the equality follows by Lagrange's theorem. Next, note that for any $\rho' \in X$, there are $|\Gamma_\rho|$ elements $g \in \Gamma$ for which $g\rho = \rho'$. Therefore, for any $(\sigma, \tau) \in \mathcal{F}_k$,

$$|\{g \in \Gamma \mid g\rho \in A_{\sigma,\tau}\}| \leq |A_{\sigma,\tau}| \cdot |\Gamma_\rho| \leq \theta_d \frac{|\{\sigma \in X(d) \mid \rho \subseteq \sigma\}|}{|X(d)|}|\Gamma|,$$

where the second inequality follows by (4.5). $\square$

We can now prove proposition 4.3.

Proof of proposition 4.3. Note that for any $\rho \in X$ and $g \in \Gamma$,

$$\sum_{\substack{(\sigma,\tau)\in\mathcal{F}_k:\\ \rho\in A_{\sigma,\tau}}} \|\tau\| = \sum_{\substack{(g\sigma,g\tau)\in\mathcal{F}_k:\\ g\rho\in gA_{\sigma,\tau}}} \|\tau\| = \sum_{\substack{(g\sigma,g\tau)\in\mathcal{F}_k:\\ g\rho\in A_{g\sigma,g\tau}}} \|\tau\| = \sum_{\substack{(\sigma,\tau)\in\mathcal{F}_k:\\ g\rho\in A_{\sigma,\tau}}} \|\tau\|. \tag{4.6}$$

Thus, it is possible to change the order of summation, i.e.,

$$\sum_{g\in\Gamma} \sum_{\substack{(\sigma,\tau)\in\mathcal{F}_k:\\ \rho\in A_{\sigma,\tau}}} \|\tau\| = \sum_{(\sigma,\tau)\in\mathcal{F}_k} \sum_{\substack{g\in\Gamma:\\ g\rho\in A_{\sigma,\tau}}} \|\tau\|.$$

It follows that for any $\rho \in X$,

$$\begin{aligned}
\sum_{\substack{(\sigma,\tau)\in\mathcal{F}_k:\\ \rho\in A_{\sigma,\tau}}} \|\tau\| &= \frac{1}{|\Gamma|}\sum_{g\in\Gamma} \sum_{\substack{(\sigma,\tau)\in\mathcal{F}_k:\\ \rho\in A_{\sigma,\tau}}} \|\tau\| = \frac{1}{|\Gamma|} \sum_{(\sigma,\tau)\in\mathcal{F}_k} \sum_{\substack{g\in\Gamma:\\ g\rho\in A_{\sigma,\tau}}} \|\tau\| \\
&= \sum_{(\sigma,\tau)\in\mathcal{F}_k} \|\tau\| \frac{|\{g \in \Gamma \mid g\rho \in A_{\sigma,\tau}\}|}{|\Gamma|} \\
&\leq \theta_d \frac{|\{\sigma \in X(d) \mid \rho \subseteq \sigma\}|}{|X(d)|} \sum_{(\sigma,\tau)\in\mathcal{F}_k} \|\tau\| \\
&= \theta_d |\{\sigma \in X(d) \mid \rho \subseteq \sigma\}|,
\end{aligned}$$

where the inequality follows by lemma 4.7. $\square$

References

[EK16] S. Evra and T. Kaufman. Bounded degree cosystolic expanders of every dimension. In *Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2016, Cambridge, MA, USA, June 18-21, 2016*, pages 36–48, 2016.

[Evr] S. Evra. Personal communication.

[Gro10] M. Gromov. Singularities, Expanders and Topology of Maps. Part 2: from Combinatorics to Topology Via Algebraic Isoperimetry. *Geometric And Functional Analysis*, 20(2):416–526, 2010.

[KKL14] T. Kaufman, D. Kazhdan, and A. Lubotzky. Ramanujan Complexes and Bounded Degree Topological Expanders. In *Foundations of Computer Science (FOCS), 2014 IEEE 55th Annual Symposium on*, pages 484–493, 2014.

[KL14] T. Kaufman and A. Lubotzky. High dimensional expanders and property testing. In *Innovations in Theoretical Computer Science, ITCS'14, Princeton, NJ, USA, January 12-14, 2014*, pages 501–506, 2014.

[LM06] N. Linial and R. Meshulam. Homological connectivity of random 2-complexes. *Combinatorica*, 26(4):475–487, 2006.

[LMM16] A. Lubotzky, R. Meshulam, and S. Mozes. Expansion of building-like complexes. *Groups, Geometry, and Dynamics*, 10(1):155–175, 2016.

[LPS88] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261–277, 1988.

[LSV05a] A. Lubotzky, B. Samuels, and U. Vishne. Explicit constructions of ramanujan complexes of type $\widetilde{A}_d$. *European Journal of Combinatorics*, 26(6):965–993, 2005.

[LSV05b] A. Lubotzky, B. Samuels, and U. Vishne. Ramanujan complexes of type $\widetilde{A}_d$. *Israel Journal of Mathematics*, 149(1):267–299, 2005.

[Lub14] A. Lubotzky. Ramanujan complexes and high dimensional expanders. *Japanese Journal of Mathematics*, 9(2):137–169, 2014.

[Lub17] A. Lubotzky. High Dimensional Expanders. arXiv:1712.02526, 2017.

ECCC ISSN 1433-8092

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