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The Diptych of Communication Complexity Classes in the Best-partition Model and the Fixed-partition Model

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Abstract

Most of the research in communication complexity theory is focused on the fixed-partition model (in this model the partition of the input between Alice and Bob is fixed). Nonetheless, the best-partition model (the model that allows Alice and Bob to choose the partition) has a lot of applications in studying stream algorithms and complexity of branching programs.

In the paper, we show how to transform separations between communication complexity classes from the fixed-partition to the best-partition models. Using these and previously known methods, we give an answer to the open question asked by Göös, Pitassi, and Watson by providing an almost complete picture of the relations between best-communication complexity classes between P_{op} and $\mathsf{PSPACE}_{\mathrm{op}}$.

1 Introduction

In the communication complexity theory the following model is studied. Let $f : \{0,1\}^n \times \{0,1\}^m \to \{0,1\}$ be a Boolean function. Alice and Bob want to compute $f(x, y)$ but Alice knows only bits of x and Bob only bits of y. In order to compute the function they communicate via a two-sided channel and the communication complexity of the function f is the number of bits they sent (this model is also known as a fixed-partition model, since we fix a partition of the input between Alice and Bob). If Alice and Bob are deterministic, we call this complexity deterministic communication complexity and denote it $\mathsf{P}(f)$; but like in computational complexity theory it is possible to define bounded error probabilistic communication complexity ($\mathsf{BPP}(f)$), one-side error probabilistic communication complexity ($\mathsf{RP}(f)$), nondeterministic communication complexity ($\mathsf{NP}(f)$), and many others.

Studying different complexity classes form a core of complexity theory, and its principal goal is to give a complete picture of relations between them. Similar research in communication complexity theory was initiated by Babai et al. [2], where they considered classes of functions with polylogarithmic communication complexity. For example, in their paper, they proved that $\mathsf{BPP} \not\subseteq \mathsf{NP}$ (here, and in the sequel we denote by P, NP etc. classes of functions with polylogarithmic communication complexity in the corresponding model). The well-known result of Aho et al. [1] may be reformulated as an equality of communication complexity classes; it says that $\mathsf{P} = \mathsf{NP} \cap \mathsf{coNP}$ for total functions. However, later Klauck [11] showed that for partial functions $\mathsf{NP} \cap \mathsf{coNP}$ is not a subset of BPP. Consequently Buhrman et al. [3] showed that $\mathsf{PP} \not\subseteq \mathsf{P}^{\mathsf{NP}}$ and Papakonstantinou

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et al. [15] showed that $\oplus\mathsf{P}$ is not a subset of P^{NP}. Finally, Göös et al. [6] proved separations for almost all the classes between P and PSPACE.

Nonetheless, that there were a lot of papers about the fixed-partition model of communication complexity, only few works were devoted to the optimal-partition model (or best-communication complexity). In this model Alice and Bob would like to compute the value of a function $f : \{0,1\}^n \to \{0,1\}$ and they can choose a balanced partition of the input bits in advance; we denote the deterministic communication complexity of the function f by $\mathsf{P}_{\mathrm{op}}(f)$, bounded error probabilistic by $\mathsf{BPP}_{\mathrm{op}}(f)$, etc. In 1984 JaJa et al. [8] showed that P_{op} does not contain $\mathsf{RP}_{\mathrm{op}}$ (here and after we denote by P_{op}, $\mathsf{NP}_{\mathrm{op}}$ etc. the class of functions with polylogarithmic communication complexity in the corresponding optimal-partition model), $\mathsf{RP}_{\mathrm{op}}$ does not contain $\mathsf{NP}_{\mathrm{op}}$, and $\mathsf{NP}_{\mathrm{op}}$ is incomparable with $\mathsf{BPP}_{\mathrm{op}}$. Later, Jukna [9] proved that for total functions P_{op} does not even contain intersection of $\mathsf{RP}_{\mathrm{op}}$ and $\mathsf{coRP}_{\mathrm{op}}$ (in contrast with the fixed-partition model).

In Section 3 we give an answer to the open question proposed by Göös et. al [6] and show how to translate almost all separations between communication complexity classes in the fixed-partition model into separations between communication complexity classes in the best-partition model. In order to do it we consider two different transformations.

Shift transformation. Results in [8] were proven using a transformation of Boolean functions, such that it transforms some hard functions for the fixed-partition model into hard functions for the best-partition model; i.e., they considered “shifted” versions of an *equality function* and a *disjointness function*. Later Lam and Ruzzo [14] generalized this technique and proved that for any paddable function f with deterministic (bounded error, unbounded error, zero error, and nondeterministic) communication complexity C, the “shifted” version of f has best-communication complexity C in the same model. Theorem 3.2 generalizes this result and prove this “lifting” for almost all known communication complexity measures.

Segerlinds transformation. Nevertheless, many of the functions separating other classes are not paddable. In order to solve this issue, we introduce the second transformation that is based on the ideas of Segerlind [17]. We extend the result of Knop [13] and construct a transformation $\mathcal{S}_g$, parameterized by a “good” function $g : \{0,1\}^k \to \{0,1\}$, such that if a function $f : \{0,1\}^n \to \{0,1\}$ has a communication complexity C in some model for **some** partition of inputs, then $\mathcal{S}_g(f) : \{0,1\}^l \to \{0,1\}$ has a communication complexity at least C in this model for **every** balanced partition and $l = \mathrm{poly}(n,m)$. Moreover, we prove that if a function $f \circ g$ has a communication complexity at most C in some model for **every** balanced partition of inputs, then $\mathcal{S}_g(f) : \{0,1\}^l \to \{0,1\}$ has a communication complexity at most $C + 2\log n$ in this model for **some** balanced partition of inputs.

As the beforementioned separation between $\mathsf{NP}_{\mathrm{op}} \cap \mathsf{coNP}_{\mathrm{op}}$ and P_{op} shows that some equalities of communication complexity classes in the fixed-partition can not be lifted to the corresponding equalities of communication complexity classes in the best-partition model. In Section 4.1 we give another example of such a phenomenon, we show that there is a total function $f : \{0,1\}^n \to \{0,1\}$, such that $\mathsf{DP}_{\mathrm{op}}(f) = O(\log n)$, $\mathsf{coDP}_{\mathrm{op}}(f) = O(\log n)$, and $\mathsf{P}_{\mathrm{op}}^{\mathsf{NP}[1]}(f) = \Omega(\sqrt{n})$, but in the fixed-partition model such a result is not known for total functions.

1.1 Cartography

All the known inclusions and non-inclusions are drawn on Figure 1. The diagram does not show any unnecessary arrows (arrows that follow from the drawn one) e.g. if there are arrows $A \longrightarrow B$

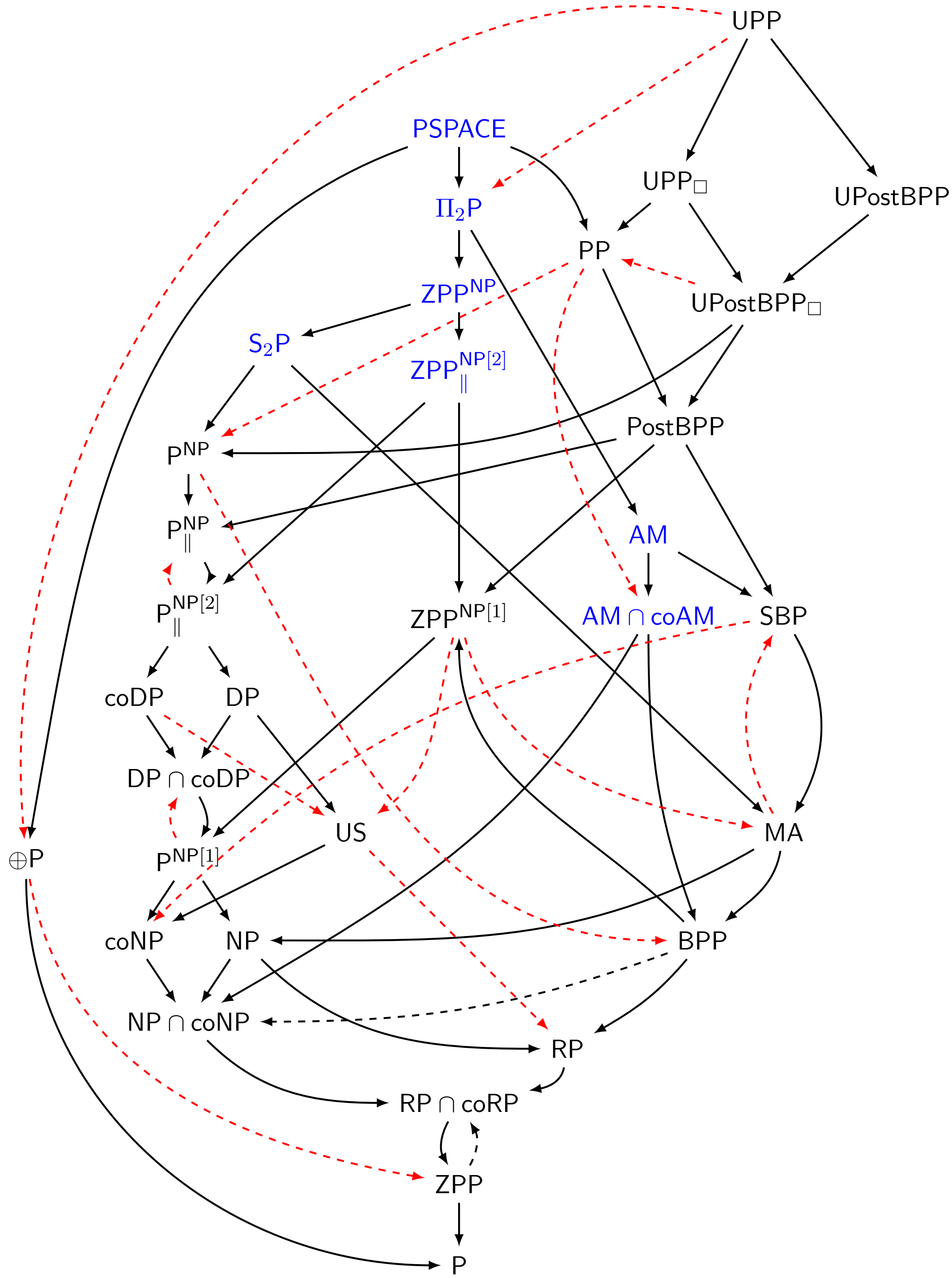

Figure 1: $A \longrightarrow B$ denotes $B_{\mathrm{op}} \subseteq A_{\mathrm{op}}$, and $A \dashrightarrow B$ denotes $B_{\mathrm{op}} \not\subseteq A_{\mathrm{op}}$. Red indicates new results and Blue indicates classes for which no explicit lower bounds are known.

and $B \longrightarrow C$, then $A \longrightarrow C$ is an unnecessary arrow.

Inclusions Establishing inclusions is much easier in the case of best-communication, since it is easy to see that (Lemma 2.2) that if A and B are communication complexity classes and $A \subseteq B$, then $A_{\text{op}} \subseteq B_{\text{op}}$. Note that this is not true for an intersection of communication complexity classes e.g. $\mathsf{NP} \cap \mathsf{coNP} = \mathsf{P}$ for total functions, but $\mathsf{NP}_{\text{op}} \cap \mathsf{coNP}_{\text{op}} \neq \mathsf{P}_{\text{op}}$ [9] since NP and coNP protocols may use different partitions.

Noninclusions

$\mathsf{NP}_{\text{op}} \cap \mathsf{coNP}_{\text{op}} \not\subseteq \mathsf{BPP}_{\text{op}}$: This result was proven by Jukna [9].

$\mathsf{RP}_{\text{op}} \cap \mathsf{coRP}_{\text{op}} \not\subseteq \mathsf{ZPP}_{\text{op}}$: This result was also proven by Jukna [9].

$\mathsf{DP}_{\text{op}} \cap \mathsf{coDP}_{\text{op}} \neq \mathsf{P}_{\text{op}}^{\mathsf{NP}[1]}$: Using ideas similar to the ideas of Jukna we prove that there is a total function separating these classes (Theorem 4.29). Note that in the fixed-partition model such a separation is unknown.

The following results are proven using the transformations that lift communication complexity from the fixed-partition model to the best-partition model.

Separation	Fixed-partition Model	Best-partition Model
$\mathsf{BPP}_{\text{op}} \not\subseteq \mathsf{P}_{\text{op}}^{\mathsf{NP}}$	Papakonstantinou et al. [15, Lemma 14]	Corollary 4.2
$\mathsf{MA}_{\text{op}} \not\subseteq \mathsf{ZPP}_{\text{op}}^{\mathsf{NP}[1]}$	Göös et al. [6, Theorem 1]	Corollary 4.4
$\mathsf{US}_{\text{op}} \not\subseteq \mathsf{ZPP}_{\text{op}}^{\mathsf{NP}[1]}$	Göös et al. [6, Theorem 2]	Corollary 4.7
$\mathsf{US}_{\text{op}} \not\subseteq \mathsf{coDP}_{\text{op}}$	Göös et al. [6, Theorem 3]	Corollary 4.9
$\mathsf{RP}_{\text{op}} \not\subseteq \mathsf{US}_{\text{op}}$	Göös et al. [6, Observation 26]	Corollary 4.11
$\mathsf{ZPP}_{\text{op}} \not\subseteq \oplus\mathsf{P}_{\text{op}}$	Göös et al. [6, Observation 27]	Corollary 4.13
$\mathsf{P}_{\text{op}}^{\mathsf{NP}} \not\subseteq \mathsf{PP}_{\text{op}}$	Buhrman et al. [3, Section 3.2]	Corollary 4.15
$\oplus\mathsf{P}_{\text{op}} \not\subseteq \mathsf{UPP}_{\text{op}}$	Forster [4, Corollary 2.2]	Corollary 4.17
$\Pi_2\mathsf{P}_{\text{op}} \not\subseteq \mathsf{UPP}_{\text{op}}$	Razborov and Sherstov [16, Corollary 1.2]	Corollary 4.19
$\mathsf{SBP}_{\text{op}} \not\subseteq \mathsf{MA}_{\text{op}}$	Göös et al. [5, Theorem 3]	Corollary 4.21
$\mathsf{PP}_{\text{op}} \not\subseteq \mathsf{UPostBPP}_{\square\text{op}}$	Göös et al. [6, Theorem 6]	Corollary 4.23
$\mathsf{coNP}_{\text{op}} \not\subseteq \mathsf{SBP}_{\text{op}}$	Göös and Watson [7, Corollary 2]	Corollary 4.26
$\mathsf{AM}_{\text{op}} \cap \mathsf{coAM}_{\text{op}} \not\subseteq \mathsf{PP}_{\text{op}}$	Klauck [12, Theorem 5]	Corollary 4.28

2 Preliminaries

We denote the set of all partitions Π of the numbers $[n]$ into two sets Π_0 and Π_1 by $\mathcal{P}_n$ and the set of all functions from $\{0,1\}^n$ to $\{0,1\}$ by $\mathcal{B}_n$. We say that a partition $\Pi = (\Pi_0, \Pi_1)$ is balanced iff $||\Pi_0| - |\Pi_1|| \leq 1$.

A formal communication complexity measure μ is a function assigning to each pair of a Boolean function and a partition of its inputs a natural number such that the following constraints are satisfied.

- The measure cannot increase if we replace some variables by constants and flip some other variables i.e. for every function $f : \{0,1\}^n \to \{0,1\}$, partition $\Pi \in \mathcal{P}_n$, and sequence $\{\rho_i : \{0,1\} \to \{0,1\}\}_{i=1}^n$ of Boolean functions, $\mu(f,\Pi) \geq \mu(h,\Pi)$, where $h(x_1,\ldots,x_n) = f(\rho_1(x_1),\ldots,\rho_n(x_n))$.

- The measure cannot change if we add "dummy" variables i.e. for every function $f : \{0,1\}^n \to \{0,1\}$, partition $\Pi \in \mathcal{P}_n$, and set $I = \{i_1 < i_2 < \cdots < i_n\} \subseteq [k]$ $\mu(f,\Pi) = \mu(g,\Pi')$, where $g(x_1,\ldots,x_k) = f(x_{i_1},\ldots,x_{i_n})$ and Π' is a partition such that $j \in \Pi'_b$ iff $i_j \in \Pi_b$ for all $j \in [n]$ and $b \in \{0,1\}$.

- The measure preserves under permutations of input variables i.e. for every function $f : \{0,1\}^n \to \{0,1\}$, partition $\Pi \in \mathcal{P}_n$, and permutation $\pi \in S_n$[1], $\mu(f,\Pi) = \mu(g,\Pi')$, where $g(x_1,\ldots,x_n) = f(x_{\pi(1)},\ldots,x_{\pi(n)})$ and $\Pi' = (\{i : \pi(i) \in \Pi_0\}, \{i : \pi(i) \in \Pi_1\})$.

- The measure decreases at most linearly if a part of the input is revealed i.e. for every function $f : \{0,1\}^{n+\ell} \to \{0,1\}$ and partition $\Pi \in \mathcal{P}_{n+\ell}$, $\mu(f,\Pi) \leq \max_{c_1,\ldots,c_\ell \in \{0,1\}} \mu(g_{c_1,\ldots,c_\ell},\Pi) + l$, where $g_{c_1,\ldots,c_\ell}(x_1,\ldots,x_{n+\ell}) = f(x_1,\ldots,x_n,c_1,\ldots,c_\ell)$.

It is easy to see that all the standard communication complexity measures are formal communication complexity measures indeed (one may found all the definitions of the measures and explanations that these measures are formal communication measures in the appendix).

Remark 2.1. *Let μ be a formal communication complexity measure, $c_1,\ldots,c_n \in \{0,1\}$ be Boolean numbers, $i_1 < i_2 < \cdots < i_k$ be integers, $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function, and $\Pi \in \mathcal{P}_n$ be a partition. Then, $\mu(f,\Pi) \geq \mu(g,\Pi')$, where $g(x_1,\ldots,x_k) = f(c_1,\ldots,c_{i_1-1},x_1,c_{i_1+1},\ldots,c_n)$ and $\Pi' = (\{j : i_j \in \Pi_0\}, \{j : i_j \in \Pi_1\})$.*

For a formal communication complexity measure μ, a function $f \in \mathcal{B}_n$, and a partition $\Pi \in \mathcal{P}_n$ we say that $\mu(f,\Pi)$ is a μ communication complexity of f with respect to Π. Additionally, for a function $f : \{0,1\}^n \times \{0,1\}^m \to \{0,1\}$, we denote μ communication complexity of f with respect to the partition $([1,n],[n+1,n+m])$ by $\mu(f)$.

If μ is a formal communication complexity measure, we call μ best-communication complexity of a function $f : \{0,1\}^n \to \{0,1\}$ the minimal $\mu(f,\Pi)$ over all balanced partitions $\Pi \in \mathcal{P}_n$ and denote it as $\mu_{\mathrm{op}}(f)$.

As an abuse of notation we denote by μ not only the formal communication measure itself but also a class of all families $\{(f_n,\Pi_n)\}_{n\in\mathbb{N}}$ such that $\mu(f_n,\Pi_n) = \mathrm{poly}(\log n)$. Additionally, we denote by μ_{op} not only the formal communication complexity measure itself but also a class of all families $\{f_n\}_{n\in\mathbb{N}}$ such that $\mu_{\mathrm{op}}(f_n) = \mathrm{poly}(\log n)$.

Lemma 2.2. *Let μ and ν be some formal communication complexity measures. If $\mu \subseteq \nu$, then $\mu_{\mathrm{op}} \subseteq \nu_{\mathrm{op}}$.*

Proof. Let us consider some family $\{f_n\}_{n\in\mathbb{N}} \in \mu_{\mathrm{op}}$. Note that $\mu(f_n,\Pi_n) = \mathrm{poly}(\log n)$ for every family of balanced partitions Π_n since $\{f_n\}_{n\in\mathbb{N}} \in \mu_{\mathrm{op}}$. Hence, $\nu(f_n,\Pi_n) = \mathrm{poly}(\log n)$. As a result, $\nu_{\mathrm{op}}(f_n) = \mathrm{poly}(\log n)$, i.e., $\{f_n\}_{n\in\mathbb{N}} \in \nu_{\mathrm{op}}$. $\square$

[1] S_n denotes the set of all permutations of the numbers $[n]$.

3 Transformations

In this section we explain the intuition behind the Shift and Segerlinds transformations. Consider some set of permutations $P \subseteq S_N$ and a surjective function $\alpha : \{0,1\}^\ell \to P$ for $N \geq n$ and $\ell = \lceil \log |P| \rceil$.

Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function. Then $\mathrm{perm}_{P,f} : \{0,1\}^\ell \to \{0,1\}$ is a Boolean function such that $\mathrm{perm}_{P,f}(z_1, \dots, z_\ell, x_1, \dots, x_N)$ is equal to $f(x_{\pi(1)}, \dots, x_{\pi(n)})$, where $\alpha(z_1, \dots, z_\ell) = \pi$.

Let μ be a formal communication complexity measure, Π be a partition of the variables of $\mathrm{perm}_{S_n,f}$. It is easy to see that

$$\mu(\mathrm{perm}_{S_n,f}, \Pi) \geq \mu(f, \Gamma),$$

for Γ such that $|\Gamma_i| = |\{j \in [n] \ : \ (j + \ell) \in \Pi_i\}|$ (an observation similar to this was used in [10] to prove separations between classical OBDDs and their quantum counterparts). Indeed, application of a partial substitution $z_k = a_k$ for each $k \in [\ell]$ to $\mathrm{perm}_{P,f}(z_1, \dots, z_\ell, x_1, \dots, x_n)$ yields the function $g(x_1, \dots, x_n) = f(x_{\pi(1)}, \dots, x_{\pi(n)})$ where $\pi = \alpha(a_1, \dots, a_\ell)$. Hence, if π is a permutation such that $\{j \in [n] \ : \ \pi(j) \in \Gamma_i\} = \{j \in [n] \ : \ (j + \ell) \in \Pi_i\}$, then

$$\mu(\mathrm{perm}_{P,f}, \Pi) \geq \mu(g, \Pi') \geq \mu(f, \Gamma),$$

where $\Pi' = (\{j \in [n] \ : \ (j + \ell) \in \Pi_0\}, \{j \in [n] \ : \ (j + \ell) \in \Pi_1\})$.

Nonetheless, $\{j \in [n] \ : \ (j + \ell) \in \Pi_i\}$ may be an empty set. In order to solve this issue we need P to be a small set of permutations (note that $|\{j \in [n] \ : \ (j + \ell) \in \Pi_i\}| \geq |\Pi_i| - \ell$). In this case we encounter another problem: the permutation π from the previous argument may not belong to the set P. Shift transformation and Segerlinds transformation propose two different approaches to the former problem. However, both approaches follow the same plan: we choose some set of permutations P and show that if there is a "good" permutation α for partitions $\Pi \in \mathcal{P}_{N+\ell}$ and $\Gamma \in \mathcal{P}_n$ (in each case we formalize later what it means to be "good"), then $\mu(\mathrm{perm}_{P,f}, \Pi) \geq \mu(f, \Gamma)$. Afterwards, we show that for every pair of balanced partitions $\Pi \in \mathcal{P}_{N+\ell}$ and $\Gamma \in \mathcal{P}_n$ such a "good" permutation exists.

3.1 Shift Transformation

Let $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ be a Boolean function. Then $\mathrm{shift}_f : \{0,1\}^{2n + \lceil \log n \rceil}$ is a Boolean function, such that

$$\mathrm{shift}_f(z_1, \dots, z_t, x_1 \dots, x_n, y_1, \dots, y_n) = f(x_1, \dots, x_n, y_{1 + \mathrm{bin}(z_1, \dots, z_t) \bmod n}, \dots, y_{n + \mathrm{bin}(z_1, \dots, z_t) \bmod n}),$$

where $t = \lceil \log n \rceil$ and $\mathrm{bin}(z_1, \dots, z_t)$ is an integer with a binary representation $z_1 \dots z_t$. In other words, $\mathrm{shift}_f = \mathrm{perm}_{P,f}$, where $P = \{\pi_j \in S_n \ : \ j \in [1,n]\}$ and for all $i, j \in [1,n]$, $\pi_j(i) = i$ and $\pi_j(n+i) = n + ((i+j) \mod n)$.

In this section we generalize the result of Lam and Ruzzo [14] and show that if a paddable function f is hard for some formal communication complexity measure in the fixed-partition model, then shift_f is hard for the same formal communication complexity measure in the best-partition model. The initial definition of paddable functions was complicated, however, for all known examples we may use a much simpler definition. We say that a family of functions $\{f_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}\}_{n \in \mathbb{N}}$ is easily paddable iff for every $n \in \mathbb{N}$ and $I = \{i_1 < \dots < i_k\} \subseteq [n]$, there are $c_1, d_1, \dots, c_{i_1 - 1}, d_{i_1 - 1}, c_{i_1 + 1}, d_{i_1 + 1}, \dots, c_n, d_n \in \{0,1\}$, such that

$$f_n(c_1, \dots, c_{i_1 - 1}, x_1, c_{i_1 + 1}, \dots, c_n, d_1, \dots, d_{i_1 - 1}, y_1, d_{i_1 + 1}, \dots, d_n) = f_k(x_1, \dots, x_k, y_1, \dots, y_k).$$

Remark 3.1. *Let μ be a formal communication complexity measure and $\{f_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}\}_{n\in\mathbb{N}}$ be an easily paddable family of Boolean functions. Then $\mu(f_m) \leq \mu(f_n)$ for all $m \leq n$.*

Theorem 3.2. *Let μ be a formal communication complexity measure. If a family of Boolean functions $\{f_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}\}_{n\in\mathbb{N}}$ is easily paddable, then $\mu_{\mathrm{op}}(\mathrm{shift}_{f_n}) \geq \mu(f_{n/8})$.*

Before we start to prove this theorem, let us prove the following technical lemmas.

Lemma 3.3. *Let μ be a formal communication complexity measure. If a family of Boolean functions $\{f_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}\}_{n\in\mathbb{N}}$ is easily paddable, then for every partition $\Pi \in \mathcal{P}_{2n}$ such that $|\{i \ : \ i \in \Pi_0, (i+n) \in \Pi_1\}| \geq k$,*

$$\mu(f_n, \Pi) \geq \mu(f_k).$$

Proof. Let us consider $I = \{i_1 < \cdots < i_k\} \subseteq \{i \ : \ i \in \Pi_0, (i+n) \in \Pi_1\}$. Since f_n is easily paddable there are $c_1, d_1, \ldots, c_{i_1-1}, d_{i_1-1}, c_{i_1+1}, d_{i_1+1}, \ldots, c_n, d_n \in \{0,1\}$, such that

$$f_n(c_1, \ldots, c_{i_1-1}, x_1, c_{i_1+1}, \ldots, c_n, d_1, \ldots, d_{i_1-1}, y_1, d_{i_1+1}, \ldots, d_n) = f_k(x_1, \ldots, x_k, y_1, \ldots, y_k).$$

Hence, by Remark 2.1, $\mu(f_n, \Pi) \geq \mu(f_k, \Pi')$, where $\Pi' = ([1,k],[k+1,2k])$. As a result, $\mu(f_n, \Pi) \geq \mu(f_k)$. ☐

Lemma 3.4. *Let $A, B \subseteq [n]$ be sets of size k. Then there is s, such that*

$$|A \cap \{i \ : \ (i+s \mod n) \in B\}| \geq \frac{k^2}{n}.$$

Proof. Let us note that

$$\sum_{s\in[n]} |A\cap\{i \ : \ (i+s \mod n) \in B\}| = \sum_{i\in A, s\in[n]} 1 = \sum_{i\in A} |\{s \ : \ (i+s \mod n) \in B\}| = |A|\cdot|B| = k^2.$$

Hence, by pigeonhole principal there is s, such that

$$|A \cap \{i \ : \ (i+s \mod n) \in B\}| \geq \frac{k^2}{n}.$$

☐

Proof of Theorem 3.2. Let us consider a balanced partition $\Pi \in \mathcal{P}_{2n+\lceil \log n\rceil}$ of the input of the function shift_{f_n}. Note that both parts of Π contains at least $n - \frac{1}{2}\log n$ indices from $[2n]$. Without loss of generality Π_0 contains at least $\frac{n}{2}$ indices from $[1,n]$ and Π_1 contains at least $\frac{n}{2} - \frac{1}{2}\log n$ indices from $[n+1, 2n]$.

Let $A \subseteq [n]$ be a set of $\lfloor \frac{n}{2} - \frac{1}{2}\log n\rfloor$ indices i from $[n]$, such that $i \in \Pi_0$, $B \subseteq [n]$ be a set of $\lfloor \frac{n}{2} - \frac{1}{2}\log n\rfloor$ indices i from $[n]$, such that $(i+n) \in \Pi_1$.

Note that by Lemma 3.4, there is s, such that $\ell = |A\cap\{i \ : \ (i+s \mod n) \in B\}| \geq \frac{(n-\log n)^2}{4n} = \frac{n}{4} - \frac{\log n}{2} + \frac{\log^2 n}{4n} \geq \frac{n}{8}$. Let us consider $c_1, \ldots, c_t \in \{0,1\}$, such that $s = \mathrm{bin}(c_1, \ldots, c_t)$. In this case,

$$\mathrm{shift}_{f_n}(x_1, \ldots, x_n, y_1, \ldots, y_n, c_1, \ldots, c_t) = f_n(x_1, \ldots, x_n, y_{(1+s) \mod n}, \ldots, y_{(n+s) \mod n}).$$

Let us consider a Boolean function $g : \{0,1\}^{2n} \to \{0,1\}$, such that

$$g(x_1, \dots, x_n, y_1, \dots, y_n) = \mathrm{shift}_{f_n}(x_1, \dots, x_n, y_1, \dots, y_n, c_1, \dots, c_t)$$

and a partition $\Pi' \in \mathcal{P}_{2n}$, such that $\Pi'_0 = \Pi_0 \cap [2n]$ and $\Pi'_1 = \Pi_1 \cap [2n]$. By Remark 2.1, $\mu(\mathrm{shift}_{f_n}, \Pi) \geq \mu(g, \Pi')$. Additionally, since μ is a communication complexity measure, $\mu(g, \Pi') = \mu(f, \Pi'')$, where $\Pi'' \in \mathcal{P}_{2n}$ is a partition such that $\Pi''_0 = (\Pi'_0 \cap [n]) \cup \{n+i \ : \ n + (i+s \mod n) \in \Pi'_0\}$ and $\Pi''_1 = (\Pi'_1 \cap [n]) \cup \{n+i \ : \ n+(i+s \mod n) \in \Pi'_1\}$.

Note that $A \subseteq \Pi''_0$ and $\{n+i \ : \ (i+s \mod n) \in B\} \subseteq \Pi''_1$. Hence, by Lemma 3.3, $\mu(f_n, \Pi'') \geq \mu(f_\ell)$. Therefore, $\mu(\mathrm{shift}_{f_n}, \Pi) \geq \mu(f_\ell) \geq \mu(f_{n/8})$. □

Theorem 3.5. *Let μ be a formal communication complexity measure. For every $f_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$, $\mu_{\mathrm{op}}(\mathrm{shift}_{f_n}) \leq \mu(f_n) + \lceil \log n \rceil$.*

Proof. Let us consider an arbitrary partition $\Pi \in \mathcal{P}_{2n+t}$, such that $[1,n] \subseteq \Pi_0$ and $[n+1, 2n] \subseteq \Pi_1$. For each $s \in [n]$, let us define

$$f_{n,s}(x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_t) = f_n(x_1, \dots, x_n, y_{1+s \mod n}, \dots, y_{n+s \mod n}).$$

Note that $\Pi = (\{i \ : \ \pi_s(i) \in \Pi_0\}, \{i \ : \ \pi_s(i) \in \Pi_1\})$ for every $s \in [n]$. Hence, $\mu(f_n, \Pi) = \mu(f_{n,s}, \Pi)$ and

$$\mu(\mathrm{shift}_{f_n}, \Pi) \leq \max_{s \in [n]} \mu(f_{n,s}, \Pi) + t = \mu(f_n) + \lceil \log n \rceil$$

since μ is a formal communication complexity measure. □

3.2 Segerlinds Transformation

In this section we propose an alternative approach originated in ideas of Segerlind [17]. The idea of this transformation is to add "clones" of the variables and as a result increase the number of reachable permutations. To define this transformation we need to define a composition of functions. Let $f : \{0,1\}^n \to \{0,1\}$ and $g : \{0,1\}^k \to \{0,1\}$ be Boolean functions. We denote by $f \circ g : \{0,1\}^{nk} \to \{0,1\}$ a Boolean function, such that

$$(f \circ g)(y_{1,1}, \dots, y_{1,k}, \dots, y_{n,1}, \dots, y_{n,k}) = f(g(y_{1,1}, \dots, y_{1,k}), \dots, g(y_{n,1}, \dots, y_{n,k})).$$

Let $t \in \mathbb{N}$ be given, $\mathbb{F}$ be a field of size 2^t. Define the set $P_t \subseteq S_{2^t}$ to be a set of all mappings given by $x \mapsto ax+b$ with $a, b \in \mathbb{F}$ and $a \neq 0$. We also fix some surjective function $\alpha_t : \{0,1\}^{2t} \to P_t$.

Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function and $t = \lceil \log(nk) \rceil$. Then Segerlinds transform of $f : \{0,1\}^n \to \{0,1\}$ parameterized by $g : \{0,1\}^k \to \{0,1\}^n$ is $\mathcal{S}_g(f) = \mathrm{perm}_{P_t, f \circ g}$.

In the paper [17], Segerlind used the transformation $\mathcal{S}_{\wedge^m}$ to prove lower bounds on an OBDD based proof system, later in the paper [13] it was noticed that this transformation allows to lift lower bounds for deterministic communication complexity from the fixed-partition model to the best-partition model. However, it is easy to see that the transformation $\mathcal{S}_{\wedge^m}$ does not allow us to prove the upper bound on complexity of $\mathcal{S}_{\wedge_m}(f)$; e.g. deterministic communication complexity of $\oplus_n$ is a constant for every partition of inputs, but $\mathcal{S}_{\wedge_2}(\oplus_n)$ is equal to $\mathrm{perm}_{P_{\log 2n}, \mathtt{IP}_n}$ ($\mathtt{IP}_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ is the inner product function and $\wedge_2 : \{0,1\}^2 \to \{0,1\}$ is a conjunction of two bits) and communication complexity of this function is equal to $\Omega(n)$ for at least one partition. In the application of the following theorem we use $g = \oplus_k$, i.e, the parity function. However, we prove the general theorem for the sake of completeness.

Theorem 3.6. *Let μ be a formal communication complexity measure. Then for every n large enough, Boolean function $f : \{0,1\}^n \to \{0,1\}$, $k \geq 100n$, Boolean function $g : \{0,1\}^k \to \{0,1\}$, and partition $\Pi \in \mathcal{P}_n$, $\mu_{\mathrm{op}}(\mathcal{S}_g(f)) \geq \mu(f, \Pi)$ providing that g depends on all its inputs.*

Proof of this theorem is based on Lemmas 3.7 and 3.10. We prove these lemmas first and after that return to the proof of this theorem.

Lemma 3.7. *Let n and k be some integers such that $k \geq 100n$, $t = \log nk$, and $\Gamma = (\Gamma_0, \Gamma_1)$ be a partition of $[2^t]$, such that $2^t \geq |\Gamma_0|, |\Gamma_1| \geq 2^{t-1} - 2t$. There is $\pi \in P_t$, such that for every $i \in [n]$ and $b \in \{0,1\}$, $\pi((i-1)\cdot k + j) \in \Gamma_b$ for some $j \in [k]$.*

We prove this lemma using the probabilistic principle, hence, we need two following lemmas.

Lemma 3.8 ([18])**.** *For every t, $|P_t| = 2^t \cdot (2^t - 1)$, every mapping from P_t is a permutation, and for any $x_1, x_2, y_1, y_2 \in [2^t]$ if $x_1 \neq x_2$ and $y_1 \neq y_2$, then $\Pr\limits_{\pi \in P_t}[\pi(x_1) = y_1, \pi(x_2) = y_2] = \frac{1}{2^t(2^t-1)}$.*

Lemma 3.9 (Chebyshev's inequality)**.** *If $X_1, \ldots, X_t$ are random Boolean variables and $Y = \sum\limits_{i=1}^{t} X_i$, then*

$$\Pr[Y = 0] \leq \frac{\mathbb{E}[Y] + \sum\limits_{i \neq j \in [t]} \mathrm{Cov}(X_i, X_j)}{(\mathbb{E}[Y])^2}.$$

Proof of Lemma 3.7. Choose uniformly random $\pi \in P_t$ and for $b \in \{0,1\}$ and consider random variables $\chi^b_{i,j}$ and Y^b_i, such that $\chi^b_{i,j} = 1$ iff $\pi((i-1)\cdot k + j) \in \Gamma_b$ and $Y^b_i = \sum\limits_{j=1}^{k} \chi^b_{i,j}$.

By Lemma 3.8, expectation of $\chi^b_{i,j}$ equals $\frac{|\Gamma_b|}{2^t}$ and by additivity of expectation, expectation of Y^b_i is equal to $\frac{k|\Gamma_b|}{2^t}$. Note that

$$\begin{aligned}
\mathrm{Cov}(\chi^b_{i,j_0}, \chi^b_{i,j_1}) &= \mathbb{E}[\chi^b_{i,j_0} \cdot \chi^b_{i,j_1}] - \mathbb{E}[\chi^b_{i,j_0}]\mathbb{E}[\chi^b_{i,j_1}] \\
&= \sum_{u \neq v \in \Gamma_b} \Pr[\pi((i-1)\cdot k + j_0) = u, \pi((i-1)\cdot k + j_1) = v] - \frac{|\Gamma_k|^2}{2^{2t}} \\
&= \frac{|\Gamma_b|(|\Gamma_b| - 1)}{2^t(2^t-1)} - \frac{|\Gamma_b|^2}{2^{2t}} < \frac{|\Gamma_b|^2}{2^t}\left(\frac{1}{2^t - 1} - \frac{1}{2^t}\right) \\
&= \frac{|\Gamma_b|^2}{2^{2t}(2^t-1)} = \frac{\left(\mathbb{E}[Y^b_i]\right)^2}{k^2(2^t-1)}.
\end{aligned}$$

Hence, by Lemma 3.9,

$$\begin{aligned}
\Pr[Y^b_i = 0] &\leq \frac{\mathbb{E}[Y^b_i] + \sum\limits_{i \neq j \in [n]} \mathrm{Cov}(\chi^b_{i,j_0}, \chi^b_{i,j_1})}{\left(\mathbb{E}[Y^b_i]\right)^2} \\
&\leq \frac{2^t}{k|\Gamma_b|} + \frac{k(k-1)}{k^2(2^t-1)} \leq \frac{2^t}{k\left(2^{t-1} - 2t\right)} + \frac{1}{2^t - 1} \\
&\leq \frac{2^t}{k2^{t-2}} + \frac{1}{2^t - 1} = \frac{4}{k} + \frac{1}{2^t - 1}.
\end{aligned}$$

Therefore, by union bound, $\Pr\left[\exists i, b\ Y_i^b = 0\right] \leq \frac{8n}{k} + \frac{2n}{2^t-1} \leq 1$. As a result, there is a permutation $\pi \in P_t$, such that for any $i \in [n]$ and $b \in \{0,1\}$ there is $j \in [k]$, such that $\pi((i-1)\cdot k + j) \in \Gamma_b$. $\square$

Lemma 3.10. *Let $f : \{0,1\}^n \to \{0,1\}$ and $g : \{0,1\}^k \to \{0,1\}$ be Boolean functions, such that g depends on all its inputs, and $\Pi' \in \mathcal{P}_{nk}$ and $\Pi \in \mathcal{P}_n$ be partitions. If for every $i \in [n]$ and $b \in \{0,1\}$ there is $j \in [k]$ such that $(i-1)\cdot k + j \in \Pi'_b$, then $\mu(f \circ g, \Pi') \geq \mu(f, \Pi)$.*

Proof. Let $b_1, \dots, b_n \in \{0,1\}$ be Boolean numbers such that for every $i \in [n]$, $i \in \Pi_{b_i}$. Let us denote j such that $(i-1)\cdot k + j \in \Pi'_{b_i}$ by j_i. Since g depends on all its inputs, including j_i, there are $c_{i,1}, \dots, c_{i,n}, s_i \in \{0,1\}$, such that

$$g(c_{i,1}, \dots, c_{j_i-1}, s_i \oplus x, c_{j_i+1}, \dots, c_{i,n}) = x.$$

Note that

$$(f \circ g)(c_{1,1}, \dots, c_{j_1-1}, s_1 \oplus x_1, c_{j_1+1}, \dots, c_{1,n}, c_{2,1}, \dots) = f(x_1, \dots, x_n).$$

Hence, $\mu(f \circ g, \Pi') \geq \mu(f, \Pi)$. $\square$

Proof of Theorem 3.6. Let N be an integer such that $f \circ g : \{0,1\}^N \to \{0,1\}$. Fix two arbitrary balanced partitions Γ and Π of the variables of $\mathcal{S}_g(f)$ and f respectively. We prove that $\mu(\mathcal{S}_g(f), \Gamma) \geq \mu(f, \Pi)$.

Let $t = \lceil \log N \rceil$. By Lemma 3.7, there is a permutation $\pi \in P_t$, such that for any $i \in [n]$ and $b \in \{0,1\}$ there is $j \in [k]$, such that $\pi((i-1)\cdot k + j) \in \Gamma_b$.

Let Γ' be a partition induced by Γ on $[1, 2^t]$. Let us consider a $c_1, \dots, c_{2t} \in \{0,1\}$ such that $\alpha(c_1, \dots, c_{2t}) = \pi$. In this case,

$$\mathcal{S}_g(f)(x_1, \dots, x_{2^t}, c_1, \dots, c_{2t}) = (f \circ g)(x_{\pi(1)}, \dots, x_{\pi(n)})$$

for all $x_1, \dots, x_{2^t} \in \{0,1\}$ and a partition Π', such that $\Pi'_0 = \left\{i \in [2^t] : \pi(i) \in \Gamma'_0\right\}$ and $\Pi'_1 = \left\{i \in [2^t] : \pi(i) \in \Gamma'_1\right\}$. By properties of the formal communication complexity measure, $\mu(\mathcal{S}_g(f), \Gamma) \geq \mu(f \circ g, \Pi')$. Note that for every $i \in [n]$ and $b \in \{0,1\}$ there is $j \in [k]$ such that $(i-1)\cdot k + j \in \Pi'_b$. Hence, by Lemma 3.10, $\mu(f \circ g, \Pi') \geq \mu(f, \Pi)$. $\square$

Theorem 3.11. *Let μ be a formal communication complexity measure, $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function, and $g : \{0,1\}^k \to \{0,1\}$ be a Boolean function. If for every partition $\Pi \in \mathcal{P}_{nk}$, $\mu(f \circ g, \Pi) \leq S$, then $\mu_{\mathrm{op}}(\mathcal{S}_g(f)) \leq S + 2\lceil \log n \rceil$.*

Proof. Let us fix some balanced partition $\Gamma \in \mathcal{P}_{2^t+t}$ and consider $h_\pi(x_1, \dots, x_{2^t}) = (f \circ g)(x_{\pi(1)}, \dots, x_{\pi(nk)})$. Note that $\mu(\mathcal{S}_g(f), \Gamma) \leq \max\limits_{\pi \in P_t} \mu(h_\pi, \Gamma) + 2t \leq S + 2t$. $\square$

4 Separations

In this section we use the separations in the fixed-partition model to prove separations in the best-partition model.

$\mathsf{BPP_{op}} \not\subseteq \mathsf{P^{NP}_{op}}$

Let $\mathtt{GHD}_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ be a partial Boolean function, such that $\mathtt{GHD}_n(x,y) = 1$ if $d_H(x,y) \geq \frac{2n}{3}$[2], $\mathtt{GHD}_n(x,y) = 0$ if $d_H(x,y) \leq \frac{n}{3}$, and for all other x and y the function is not defined.

Theorem 4.1 (Papakonstantinou et al. [15, Lemma 14]). $\mathsf{P^{NP}}(\mathtt{GHD}_n) = \Omega(n)$.

Corollary 4.2. *There is a family of functions $h_n : \{0,1\}^{m_n} \to \{0,1\}$ such that*

- $m_n = O(n^2)$,
- $\mathsf{BPP_{op}}(h_n) = O(\log n)$, *and*
- $\mathsf{P^{NP}_{op}}(h_n) = \Omega(n)$.

Proof. Let $k_n = 200n$ and $h_n = \mathcal{S}_{\oplus_{k_n}}(\mathtt{GHD}_n)$. By Theorem 3.6, $\mathsf{P^{NP}_{op}}(h_n) = \Omega(n)$. Note that $\mathsf{BPP}(\mathtt{GHD}_n \circ \oplus_{k_n}, \Pi) = O(1)$, for every $\Pi \in \mathcal{P}_{2nk_n}$ since the following BPP protocol with cost $O(\log n)$ computes $\mathtt{GHD}_n \circ \oplus_{k_n}$: Alice and Bob pick random $i \in [n]$ and check if $\oplus_{k_n}(y_{i,1}, \ldots, y_{i,k_n}) = 1$. Hence, by Theorem 3.11, $\mathsf{BPP_{op}}(h_n) = O(\log n)$. □

$\mathsf{MA_{op}} \not\subseteq \mathsf{ZPP^{NP[1]}_{op}}$

Let us consider the function $\mathtt{Block\text{-}EQ}_n : \{0,1\}^{2n^2} \to n$, such that

$$\mathtt{Block\text{-}EQ}_n(x_{1,1}, \ldots, x_{1,n}, \ldots, x_{n,1}, \ldots, x_{n,n}, y_{1,1}, \ldots, y_{1,n}, \ldots, y_{n,1}, \ldots, y_{n,n}) = 1$$

iff for some $i \in [n]$, for every $j \in [n]$, $x_{i,j} = y_{i,j}$.

Theorem 4.3 (Göös et al. [6, Theorem 1]). $\mathsf{ZPP^{NP[1]}}(\mathtt{Block\text{-}EQ}_n) = \Omega(n)$.

Corollary 4.4. *There is a family of functions $h_n : \{0,1\}^{m_n} \to \{0,1\}$ such that*

- $m = O(n^4)$,
- $\mathsf{MA_{op}}(h_n) = O(\log n)$, *and*
- $\mathsf{ZPP^{NP[1]}_{op}}(h_n) = \Omega(n)$.

Proof. Let $k_n = 200n^2$ and $h_n = \mathcal{S}_{\oplus_{k_n}}(\mathtt{Block\text{-}EQ}_n)$. By Theorem 3.6, $\mathsf{ZPP^{NP[1]}_{op}}(h_n) = \Omega(n)$. Note that $\mathsf{MA}(\mathtt{Block\text{-}EQ}_n \circ \oplus_{k_n}, \Pi) = O(\log n)$, for every $\Pi \in \mathcal{P}_{2n^2k_n}$. Indeed, the following MA protocol computes $\mathtt{Block\text{-}EQ}_n \circ \oplus_{k_n}$: Merlin sends $i \in [n]$, after that Alice and Bob evaluate $\mathtt{EQ}_n \circ \oplus_{k_n}$ on the ith block of input (it can be done with $O(\log n)$ communication). Hence, by Theorem 3.11, $\mathsf{MA_{op}}(h_n) = O(\log n)$. □

[2] d_H is a Hamming distance.

$\mathsf{US}_{\mathrm{op}} \not\subseteq \mathsf{ZPP}^{\mathsf{NP}[1]}_{\mathrm{op}}$

Let $\texttt{Unique-INTER}_n : \{0,1\}^{2n} \to \{0,1\}$ be a function such that

$$\texttt{Unique-INTER}_n(x_1, \ldots, x_n, y_1, \ldots, y_n) = 1$$

iff $|\,\{i \in [n] \ : \ x_1 = y_i = 1\}\,| = 1$.

Remark 4.5. $\texttt{Unique-INTER}_n$ *is easily paddable.*

Theorem 4.6 (Göös et al. [6, Theorem 2])**.**

- $\mathsf{ZPP}^{\mathsf{NP}[1]}(\texttt{Unique-INTER}_n) = \Omega(n)$ *and*
- $\mathsf{US}(\texttt{Unique-INTER}_n) = O(\log n)$.

Corollary 4.7. *There is a family of functions $h_n : \{0,1\}^{m_n} \to \{0,1\}$ such that*

- $m = O(n)$,
- $\mathsf{US}_{\mathrm{op}}(h_n) = O(\log n)$, *and*
- $\mathsf{ZPP}^{\mathsf{NP}[1]}_{\mathrm{op}}(h_n) = \Omega(n)$.

Proof. Let $h = \mathrm{shift}_{\texttt{Unique-INTER}_n}$. By Theorem 3.5, $\mathsf{US}_{\mathrm{op}}(h_n) = O(\log n)$, and by Theorem 3.2, $\mathsf{ZPP}^{\mathsf{NP}[1]}_{\mathrm{op}}(h_n) = \Omega(n)$. ☐

$\mathsf{US}_{\mathrm{op}} \not\subseteq \mathsf{coDP}_{\mathrm{op}}$

The function , from the previous section can be also used to separate $\mathsf{US}_{\mathrm{op}}$ and $\mathsf{coDP}_{\mathrm{op}}$.

Theorem 4.8 (Göös et al. [6, Theorem 3])**.** $\mathsf{coDP}(\texttt{Unique-INTER}_n) = \Omega(n)$.

Corollary 4.9. *There is a family of functions $h_n : \{0,1\}^{m_n} \to \{0,1\}$ such that*

- $m = O(n)$,
- $\mathsf{US}_{\mathrm{op}}(h_n) = O(\log n)$, *and*
- $\mathsf{coDP}_{\mathrm{op}}((h_n) = \Omega(n)$.

Proof. Let $h = \mathrm{shift}_{\texttt{Unique-INTER}_n}$. By Theorem 3.5, $\mathsf{US}_{\mathrm{op}}(h_n) = O(\log n)$, additionally by Theorem 3.2, $\mathsf{coDP}_{\mathrm{op}}(h_n) = \Omega(n)$. ☐

$\mathsf{RP}_{\mathrm{op}} \not\subseteq \mathsf{US}_{\mathrm{op}}$

Let us also consider a function $\texttt{NEQ}_n : \{0,1\}^{2n} \to \{0,1\}$, such that

$$\texttt{NEQ}_n(x_1, \ldots, x_n, y_1, \ldots, y_n) = 1$$

iff for some $i \in [n]$, $x_i \neq y_i$.

Theorem 4.10 (Göös et al. [6, Observation 26])**.** $\mathsf{US}(\texttt{NEQ}_n) = \Omega(n)$.

Corollary 4.11. *There is a family of functions $h_n : \{0,1\}^{m_n} \to \{0,1\}$ such that*

- $m = O(n)$,
- $\mathsf{RP}_{\text{op}}(h_n) = O(\log n)$, *and*
- $\mathsf{US}_{\text{op}}((h_n) = \Omega(n)$.

Proof. Let $h_n = \text{shift}_{\texttt{NEQ}_n}$. It is well-known that $\mathsf{RP}(\texttt{NEQ}_n) = O(\log n)$. It is also easy to see that $\texttt{NEQ}_n$ is easily paddable. Hence, by Theorem 3.5, $\mathsf{RP}_{\text{op}}(h) = O(\log n)$, and by Theorem 3.2, $\mathsf{US}_{\text{op}}(h) = \Omega(n)$. □

$\mathsf{ZPP}_{\text{op}} \not\subseteq \oplus\mathsf{P}_{\text{op}}$

Let $\texttt{Which-EQ}_n : \{0,1\}^{2n} \to \{0,1\}$ be a function, such that $\texttt{Which-EQ}_n(x_0x_1, y_0y_1) = 1$ iff $x_0 = y_0$ and $\texttt{Which-EQ}_n(x_0x_1, y_0y_1) = 0$ iff $x_1 \neq y_1$, where $|x_0| = |y_0| = \lceil \frac{n}{2} \rceil$.

Theorem 4.12 (Göös et al. [6, Observation 27])**.** $\oplus\mathsf{P}(\texttt{Which-EQ}_n) = \Omega(n)$.

Corollary 4.13. *There is a family of functions* $h_n : \{0,1\}^{m_n} \to \{0,1\}$ *such that*

- $m = O(n^2)$,
- $\mathsf{ZPP}_{\text{op}}(h_n) = O(\log n)$, *and*
- $\oplus\mathsf{P}_{\text{op}}(h_n) = \Omega(n)$.

Proof. Let $k_n = 200n$ and $h_n = \mathcal{S}_{\oplus_{k_n}}(\texttt{Which-EQ}_n)$. By Theorem 3.6, $\oplus\mathsf{P}_{\text{op}}(h_n) = \Omega(\sqrt{n})$. However, $\mathsf{ZPP}(\texttt{Which-EQ}_n \circ \oplus_{k_n}, \Pi) = O(\log n)$ for every $\Pi \in \mathcal{P}_{2nk_n}$ since $\texttt{NEQ}_n \in \mathsf{RP}$ and $\texttt{EQ}_n \in \mathsf{coRP}$. Indeed, we compute $\texttt{NEQ}_{\lceil \frac{n}{2} \rceil}(x_1, y_1)$ using an RP protocol and $\texttt{EQ}_{\lfloor \frac{n}{2} \rfloor}(x_0, y_0)$ using an coRP protocol; if the first one is equal to 1 we return 0, if the second one is equal to 1 we return 1, otherwise we return $\perp$. Note that if $\texttt{Which-EQ}_n(x_0x_1, y_0y_1) = 1$, then the probability that we return 0 is equal to 0 and the probability of $\perp$ is at most 1/2; if $\texttt{Which-EQ}_n(x_0x_1, y_0y_1) = 0$, then the probability that we return 1 is equal to 0 and the probability of $\perp$ is at most 1/2.

Hence, by Theorem 3.11, $\mathsf{ZPP}_{\text{op}}(h_n) = O(\log n)$. □

$\mathsf{P}^{\mathsf{NP}}_{\text{op}} \not\subseteq \mathsf{PP}_{\text{op}}$

Let $\texttt{ODD-MAX-BIT}_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ be a Boolean function, such that $\texttt{ODD-MAX-BIT}_n(x_1, \ldots, x_n, y_1, \ldots, y_n) = 1$ iff $\max\{i : x_i = y_i = 1\} \equiv 1 \pmod 2$.

Theorem 4.14 (Buhrman et al. [3, Section 3.2])**.** $\mathsf{PP}(\texttt{ODD-MAX-BIT}_n) = \Omega(n^{1/3})$.

Corollary 4.15. *There is a family of functions* $h_n : \{0,1\}^{m_n} \to \{0,1\}$ *such that*

- $m = O(n^2)$,
- $\mathsf{P}^{\mathsf{NP}}_{\text{op}}(h_n) = O(\log^2 n)$, *and*
- $\mathsf{PP}_{\text{op}}(h_n) = \Omega(n^{1/3})$.

Proof. Let $k_n = 200n$ and $h_n = \mathcal{S}_{\oplus_{k_n}}(\texttt{ODD-MAX-BIT}_n)$. By Theorem 3.6, $\mathsf{PP}_{\text{op}}(h_n) = \Omega(n^{1/3})$.

Nevertheless, using binary search one may show that $\mathsf{P}^{\mathsf{NP}}(\texttt{ODD-MAX-BIT}_n \circ \oplus_{k_n}, \Pi) = O(\log n)$, for every $\Pi \in \mathcal{P}_{2nk_n}$. To show this, let us first note that if Alice and Bob are given $x_{1,1}, \ldots, x_{n,k_n}$ and $y_{1,1}, \ldots, y_{n,k_n}$, then for every $1 \le a < b \le n$ they may check if there is $i \in [a, b]$ such that $\bigoplus_{j=1}^{k_n} x_{i,j} = \bigoplus_{j=1}^{k_n} y_{i,j} = 1$ using NP communication protocol of cost $O(\log n)$ with respect to Π. Using this observation we may construct a P^{NP} protocol for $\texttt{ODD-MAX-BIT}_n \circ \oplus_{k_n}$. This protocol has at most $\lceil \log n \rceil$ stages, on each stage Alice and Bob consider some segment $[a, b]$: initially the segment is equal to $[1, n]$ and on each iteration Alice and Bob guess (using the NP oracle) $i \in \left[\frac{a+b}{2}, b\right]$ such that $\bigoplus_{j=1}^{k_n} x_{i,j} = \bigoplus_{j=1}^{k_n} y_{i,j} = 1$ and if such i does not exist they go to the segment $\left[a, \frac{a+b}{2}\right]$, otherwise they go to the segment $\left[\frac{a+b}{2}, b\right]$. After at most $\lceil \log n \rceil$ iterations the segment consists of one point i, if i is odd the value of the function is 1, otherwise it is 0. Note that cost of this protocol is $O(\log^2 n)$

As a result, by Theorem 3.11, $\mathsf{P}^{\mathsf{NP}}_{\text{op}}(h_n) = O(\log^2 n)$. ∎

$\oplus\mathsf{P}_{\text{op}} \not\subseteq \mathsf{UPP}_{\text{op}}$

Let H_n be a sequence of matrices $2^n \times 2^n$, such that $H_0 = 1$ and $H_{n+1} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}$. Let $\texttt{HAD}_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ be a Boolean function, such that $\texttt{HAD}_n(x, y) = 1$ iff the (r, c) entry of H_n is 1 where $r = \text{bin}(x)$ and $c = \text{bin}(y)$.

Theorem 4.16 (Forster [4, Corollary 2.2])**.** $\mathsf{UPP}(\texttt{HAD}_n) = \Omega(n^{1/3})$.

Corollary 4.17. *There is a family of functions $h_n : \{0,1\}^{m_n} \to \{0,1\}$ such that*

- $m = O(n^2)$,
- $\oplus\mathsf{P}_{\text{op}}(h_n) = O(\log^2 n)$*, and*
- $\mathsf{UPP}_{\text{op}}(h_n) = \Omega(n^{1/3})$.

Proof. Let $k_n = 200n$ and $h_n = \mathcal{S}_{\oplus_{k_n}}(\texttt{HAD}_n)$. By Theorem 3.6, $\mathsf{UPP}_{\text{op}}(h_n) = \Omega(n^{1/3})$.

Let us now prove that $\oplus\mathsf{P}(\texttt{HAD}_n \circ \oplus_{k_n}, \Pi) = O(\log^2 n)$ for any partition Π. Fix some partition Π of the input $x_{1,1}, \ldots, x_{n,k_n}, y_{1,1}, \ldots, y_{n,k_n}$. Note that if $\bigoplus_{j=1}^{k_n} y_{1,j} = 1$, and $\bigoplus_{j=1}^{k_n} x_{1,j} = 1$,

$$\texttt{HAD}_n \circ \oplus_{k_n}(x_{1,1}, \ldots, x_{n,k_n}, y_{1,1}, \ldots, y_{n,k_n}) = \neg\texttt{HAD}_{n-1} \circ \oplus_{k_n}(x_{2,1}, \ldots, x_{n,k_n}, y_{2,1}, \ldots, y_{n,k_n});$$

otherwise,

$$\texttt{HAD}_n \circ \oplus_{k_n}(x_{1,1}, \ldots, x_{n,k_n}, y_{1,1}, \ldots, y_{n,k_n}) = \texttt{HAD}_{n-1} \circ \oplus_{k_n}(x_{2,1}, \ldots, x_{n,k_n}, y_{2,1}, \ldots, y_{n,k_n}).$$

Hence, $\texttt{HAD}_n \circ \oplus_{k_n} = \bigoplus_{i=1}^{n} \left(\left(\bigoplus_{j=1}^{k_n} x_{i,j} = 1 \right) \wedge \left(\bigoplus_{j=1}^{k_n} y_{i,j} = 1 \right) \right)$. Thus, $\texttt{HAD}_n \circ \oplus_{k_n}$ has $\oplus\mathsf{P}$ communication complexity $O(\log n)$ in any partition Π.

As a result, by Theorem 3.11, $\oplus\mathsf{P}_{\text{op}}(h_n]) = O(\log n)$. ∎

$\Pi_2\mathsf{P}_{\text{op}} \not\subseteq \mathsf{UPP}_{\text{op}}$

Let $\mathtt{RS}_n : \{0,1\}^{n^3} \times \{0,1\}^{n^3} \to \{0,1\}$ be a Boolean function such that

$$\mathtt{RS}_n(x_{1,1},\ldots,x_{n,n^2},y_{1,1},\ldots,y_{n,n^2}) = \bigwedge_{i=1}^{n}\bigvee_{j=1}^{n^2}(x_{i,j} \wedge y_{i,j}).$$

Theorem 4.18 (Razborov and Sherstov [16, Corollary 1.2])**.** $\mathsf{UPP}(\mathtt{RS}_n) = \Omega(n)$.

Corollary 4.19. *There is a family of functions $h_n : \{0,1\}^{m_n} \to \{0,1\}$ such that*

- $m = O(n^6)$,
- $\Pi_2\mathsf{P}_{\text{op}}(h_n) = O(\log n)$, *and*
- $\mathsf{UPP}_{\text{op}}(h_n) = \Omega(n)$.

Proof. Let $k_n = 200n^3$ and $h_n = \mathcal{S}_{\oplus_{k_n}}(\mathtt{RS}_n)$. It is easy to see that $\Pi_2\mathsf{P}(\mathtt{RS}_n \circ \oplus_{k_n}, \Pi) = O(\log n)$, for every $\Pi \in \mathcal{P}_{2n^3k_n}$. Indeed, $(x_{i,j} \wedge y_{i,j}) \circ \oplus_{k_n}$ has deterministic communication complexity $O(1)$ with respect to any partition Π.

As a result, by Theorem 3.11, $\Pi_2\mathsf{P}_{\text{op}}(h_n) = O(\log n)$, and by Theorem 3.6, $\mathsf{UPP}_{\text{op}}(h_n) = \Omega(n)$. $\square$

$\mathsf{SBP}_{\text{op}} \not\subseteq \mathsf{MA}_{\text{op}}$

$\mathtt{WT}_n^\ell : \{0,1\}^n \to \{0,1\}$ is a partial Boolean function such that $\mathtt{WT}_n^\ell(z) = 1$ if $w_H(z) \geq \ell$, $\mathtt{WT}_n^\ell(z) = 0$ if $w_H(z) \leq \ell/2$, and for all other z the function is not defined[3].

Theorem 4.20 (Göös et al. [5, Theorem 3])**.** $\mathsf{MA}(\mathtt{WT}_n^\ell \circ \mathtt{IP}_{100\log n}) = \Omega(n^{1/4}\log n)$.

Corollary 4.21. *There is a family of functions $h_n : \{0,1\}^{m_n} \to \{0,1\}$ such that*

- $m = O((n\log n)^2)$,
- $\mathsf{SBP}_{\text{op}}(h_n) = O(\log n)$, *and*
- $\mathsf{MA}_{\text{op}}(h_n) = \Omega(n^{1/4}\log n)$.

Proof. Let $k_n = 200n\log n$, $b_n = 100\log n$, and $h_n = \mathcal{S}_{\oplus_{k_n}}(\mathtt{WT}_n^\ell \circ \mathtt{IP}_{b_n})$. So, $\mathsf{MA}_{\text{op}}(h_n) = \Omega(n^{1/4}\log n)$. It is easy to see that $\mathtt{IP}_{b_n} \circ \oplus_{k_n}$ has communication complexity $O(b_n)$ with respect to any partition; hence, $\mathsf{SBP}(\mathtt{WT}_n^\ell \circ \mathtt{IP}_{b_n} \circ \oplus_{k_n}, \Pi) = O(\log n)$, for every $\Pi \in \mathcal{P}_{200nk_n\log n}$. Thus, $\mathsf{SBP}_{\text{op}}(h_n) = O(\log n)$. $\square$

[3] $w_h(z)$ is a hamming weight of z

$\mathsf{PP}_{\mathrm{op}} \not\subseteq \mathsf{UPostBPP}_{\Box\mathrm{op}}$

Let $b_n = 100 \log n$ and $\mathtt{MAJ}_n : \{0,1\}^{2n+1} \to \{0,1\}$ be a function, such that $\mathtt{MAJ}_(z) = 1$ iff $w_H(z) \geq n+1$.

Theorem 4.22 (Göös et al. [6, Theorem 6])**.** $\mathsf{UPostBPP}_{\Box}(\mathtt{MAJ}_n \circ \mathtt{IP}_{b_n}) = \Omega(n \log n)$.

Corollary 4.23. *There is a family of functions $h_n : \{0,1\}^{m_n} \to \{0,1\}$ such that*

- $m = O((n \log n)^2)$,
- $\mathsf{PP}_{\mathrm{op}}(h_n) = O(\log n)$, *and*
- $\mathsf{MA}_{\mathrm{op}}(h_n) = \Omega(n^{1/4} \log n)$.

Proof. Let $k_n = 200(2n+1)\log n$ and $h_n = \mathcal{S}_{\oplus_{k_n}}(\mathtt{MAJ}_n \circ \mathtt{IP}_{b_n})$. By Theorem 3.6, $\mathsf{UPostBPP}_{\Box\mathrm{op}}(h_n) = \Omega(n \log n)$. However, $\mathsf{PP}(\mathtt{MAJ}_n \circ \mathtt{IP}_{b_n} \circ \oplus_{k_n}, \Pi) = O(\log n)$, for every $\Pi \in \mathcal{P}_{200nk_n \log n}$. Hence, by Theorem 3.11, $\mathsf{PP}_{\mathrm{op}}(h_n) = O(\log n)$. $\Box$

$\mathsf{coNP}_{\mathrm{op}} \not\subseteq \mathsf{SBP}_{\mathrm{op}}$

$\mathtt{DISJ}_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ is a Boolean function such that $\mathtt{DISJ}_n(x,y) = 1$ iff for some $j \in [n]$, $x_i = y_i = 1$.

Remark 4.24. $\mathtt{DISJ}_n$ *is easily paddable.*

Theorem 4.25 (Göös and Watson [7, Corollary 2])**.** $\mathsf{SBP}(\mathtt{DISJ}_n) = \Omega(n)$.

Corollary 4.26. *There is a family of functions $h_n : \{0,1\}^{m_n} \to \{0,1\}$ such that*

- $m = O(n)$,
- $\mathsf{coNP}_{\mathrm{op}}(h_n) = O(\log n)$, *and*
- $\mathsf{SBP}_{\mathrm{op}}(h_n) = \Omega(n)$.

Proof. Let $h_n = \mathrm{shift}_{\mathtt{DISJ}_n}$. It is well-known that $\mathsf{coNP}(\mathtt{DISJ}_n) = O(\log n)$. Hence, by Theorem 3.2, $\mathsf{SBP}_{\mathrm{op}}(h_n) = \Omega(n)$ and by Theorem 3.5, $\mathsf{coNP}_{\mathrm{op}}(h_n) = O(\log n)$. $\Box$

$\mathsf{AM}_{\mathrm{op}} \cap \mathsf{coAM}_{\mathrm{op}} \not\subseteq \mathsf{PP}_{\mathrm{op}}$

Let M be a matrix in $\{0,1\}^{n \times 4n^2}$, we say that M is good if every row of M contains 1, and we say that M is bad if at least $2n/3$ rows of M contain only 0.

Let $\mathtt{AppMPC}_n : \{0,1\}^{n \times 4n^2} \times \{0,1\}^{n \times 4n^2} \to \{0,1\}$ be a partial function, such that $\mathtt{AppMPC}_n(M_0, M_1) = 1$ iff M_0 is good and M_1 is bad and $\mathtt{AppMPC}_n(M_0, M_1) = 0$ iff M_0 is bad and M_1 is good.

Let $f : \{0,1\}^\ell \to \{0,1\}$. Then pattern matrix P_f of f is a communication problem $P_f : \{0,1\}^{2\ell} \times (\{0,1\}^\ell \times \{0,1\}^\ell) \to \{0,1\}$ such that $P_f(x,y,z) = f(x(y) \oplus z)$, where $\oplus$ is the bitwise xor and denotes ℓ bit string that contains x_{2i-y_i} in position $i \in [\ell]$.

We denote by $\mathtt{PAppMPC}_n$ the pattern matrix of $\mathtt{AppMPC}_n$.

Theorem 4.27 (Klauck [12, Theorem 5]). $\mathsf{PP}(\mathtt{PAppMPC}_n) = \Omega(\sqrt{n})$.

Corollary 4.28. *There is a family of functions $h_n : \{0,1\}^{m_n} \to \{0,1\}$ such that*

- $m = O(n^3)$,
- $\mathsf{AM}_{\mathrm{op}}(h_n) = O(\log n)$, $\mathsf{coAM}_{\mathrm{op}}(h_n) = O(\log n)$, *and*
- $\mathsf{PPop}(h_n) = \Omega(\sqrt{n})$.

Proof. Let $k_n = 3200n^3$ and $h_n = \mathcal{S}_{\oplus_{k_n}}(\mathtt{PAppMPC}_n)$. It is easy to see that $\mathsf{AM}(\mathtt{PAppMPC}_n \circ \oplus_{k_n}, \Pi) = O(\log n)$ and $\mathsf{coAM}(\mathtt{PAppMPC}_n \circ \oplus_{k_n}, \Pi) = O(\log n)$, for every $\Pi \in \mathcal{P}_{3200n^3}$. Let us prove it for the AM case, for coAM the proof is the same. Let the public coin random number i represents a random row of the matrix M and Merlin replies with a position j such that $M_{i,j}$. Hence, Alice and Bob just check that $(x(y) \oplus z)_{i,j} \circ \oplus_{k_n} = 1$ which can be done using $O(\log n)$ bits of communication.

As a result, by Theorem 3.11, $\mathsf{AM}_{\mathrm{op}}(h_n) = O(\log n)$ and $\mathsf{coAM}_{\mathrm{op}}(h_n) = O(\log n)$, and by Theorem 3.6, $\mathsf{PP}(h_n) = \Omega(\sqrt{n})$. □

4.1 $\mathsf{DP}_{\mathrm{op}} \cap \mathsf{coDP}_{\mathrm{op}} \neq \mathsf{P}_{\mathrm{op}}^{\mathsf{NP}[1]}$

In the fixed-partition model it is not known if it is possible to separate these classes on total functions; however, in the best-partition model we can construct a function with small DP and coDP best-communication complexity measures, but with a big $\mathsf{P}^{\mathsf{NP}[1]}$ best-communication complexity measure.

Theorem 4.29. *There is a function f such that $f \in \mathsf{DP}_{\mathrm{op}} \cap \mathsf{coDP}_{\mathrm{op}}$ but $f \notin \mathsf{P}_{\mathrm{op}}^{\mathsf{NP}[1]}$.*

Proof. Let $A \in \{0,1\}^{n^2}$ be a Boolean matrix $n \times n$. We say that a row (a column) of A is good iff there are exactly two 1's in this row (column). We define a function $f_n : \{0,1\}^{n^2} \to \{0,1\}$ such that $f_n(A) = 1$ iff there is exactly one good row in A and the number of bad columns in A is not equal to 1.

Claim 4.29.1. $\mathsf{DP}_{\mathrm{op}}(f_n) = O(\log n)$ *and* $\mathsf{DP}_{\mathrm{op}}(\neg f_n) = O(\log n)$.

Let Π be the partition such that Alice knows all the values of the first $n/2$ columns and Bob knows the rest.

We are going to prove that $\mathsf{DP}(f_n, \Pi) = O(\log n)$. Let $g_{\ell,n} : \{0,1\}^{n^2} \to \{0,1\}$ be the function such that $g_{\ell,n}(A) = 1$ iff A has ℓ good rows. Note that $\mathsf{NP}(g_{\ell,n}, \Pi) = O(\ell \log n)$; indeed, the protocol first guesses these ℓ rows (candidates for good rows). Then, using 3ℓ bits, Alice tells Bob whether the first half of these rows contains none, one, two or more than two 1's. After that Bob has the whole information about the value $g_{\ell,n}(A)$, and can send the answer. Let $h_n : \{0,1\}^{n^2} \to \{0,1\}$ be the function such that $h_n(A) = 1$ iff the number of bad columns in A is not equal to 1. Note that $\mathsf{P}(h_n, \Pi) = O(\log n)$; indeed, Alice may send the number of her bad columns. After that Bob has the whole information about the value $h_n(A)$, and can send the answer. As a result, $f_n(A) = 1$ iff $g_{1,n}(A) = 1 \wedge g_{2,n}(A) = 0 \wedge h_n(A)$ and $\mathsf{DP}(f_n, \Pi) = O(\log n)$.

In order to prove that $\mathsf{DP}(\neg f_n, K) = O(\log n)$ we just need to replace columns by rows and vice versa.

Claim 4.29.2. $\mathsf{P}_{\mathrm{op}}^{\mathsf{NP}[1]}(f_n) = \Omega(n)$.

Without loss of generality we may assume that $n \geq 10$. Let us assume that Alice and Bob choose some balanced partition Γ. We say that a column is mixed iff there are neither Alice nor Bob see all the entries of the column. Let m be the number of mixed columns and consider two cases.

- ($m \leq \frac{n}{2} - 1$) Since each player can see at most $n/2$ columns each player will see at least $n - (n/2 + m) \geq 1$ columns. Choose one column seen by Alice and one column seen by Bob. Let B be the $(n-3) \times 2$ submatrix of the input matrix A formed by these columns and all but three last rows. Let $x, y \in \{0,1\}^{n-3}$. We set all the entries of the last three rows to one (note that after this substitution all the columns are bad and last three rows are bad as well), we set all the remaining entries of A outside B to 0, and set the first column to be equal to x and the second column to be equal to y. Note that $f_n(A) = 1$ iff there is only one good row i.e. $\texttt{Unique-INTER}_{n-3}(x, y) = 1$. Hence, $\mathsf{P}^{\mathsf{NP}[1]}(f_n, \Gamma) = \Omega(n)$.

- ($m \geq \frac{n}{2}$) Let B be the $n \times m$ submatrix of A formed by the mixed columns. For each column $i \in [m]$ of B select an entry a_i known by Alice and and an entry b_i known by Bob. Since we selected $2m \leq 2n$ entries there is a row r such that we selected at most 2 entries in it. Let C be a $n \times (m-3)$ submatrix of B such that nothing is selected in the row r. We set two entries of r outside of C to be equal to 1 and all others to be equal to 0 (note that after this substitution r is good). Additionally, we set all the not substituted entries of B outside of C to be equal to 1 (note that after this all the rows except r are bad and all the columns of B outside of C are bad) and we set all the entries outside of B to be equal to 0 (note that after this all the columns outside of B are bad). Let $x, y \in \{0,1\}^{m-3}$. Finally, for each column $i \in [m-3]$ of C we substitute x_i to a_i and y_i to b_i. As a result, $f_n(A) = 1$ iff the number of bad columns of C is not equal to 1 i.e. $\texttt{Unique-INTER}_{m-3}(x, y) = 0$. Hence, $\mathsf{P}^{\mathsf{NP}[1]}(f_n, \Gamma) = \Omega(n)$.

$\square$

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A List of Communication Complexity Measures

In this section we consider all the communication complexity models from Section 4 and explain why all of them are formal communication complexity measures.

A.1 Deterministic Computations

Let us start from the definition of the rectangles and deterministic communication with an oracle access.

Definition A.1. *Let Π be a partition of $[n]$. We say that a set $R \subseteq \{0,1\}^n$ is a rectangle with respect to Π iff for every $x^{(0)}, x^{(1)} \in R$ and $y \in \{0,1\}^n$,*

$$\left.\begin{array}{l}\forall i \in \Pi_0 \ y_i = x_i^{(0)} \\ \forall i \in \Pi_1 \ y_i = x_i^{(1)}\end{array}\right\} \implies y \in R.$$

Definition A.2. *($\mathsf{P}_{\|}^{\mu[q]}$) Let μ be a formal communication complexity measure.*

Protocol: *A $\mathsf{P}_{\|}^{\mu[q]}$ communication protocol with respect to a partition $\Pi \in \mathcal{P}_n$ is a deterministic communication protocol with respect to Π such that for each leaf v with associated rectangle R_v there are q associated functions $f_{v,i}$ ($i \in [q]$) and an output function $o_v : \{0,1\}^q \to \{0,1\}$.*

Value: *Value of this protocol on $x \in \{0,1\}^n$ is 1 iff $o_v(r) = 1$, where v is a node such that $x \in R_v$ and r is Boolean string such that $r_i = f_{v,i}(x)$ for all $i \in [q]$.*

Cost: *Cost of this protocol is $\max_v \mu(f_1, \Pi) + \cdots + \mu(f_q, \Pi)$ plus the communication complexity of the deterministic protocol.*

Definition A.3. *(P^{μ})*

Protocol: *A P^{μ} communication protocol with respect to a partition $\Pi \in \mathcal{P}_n$ is a protocol tree where each inner node v is labeled by a "query" function $q_v : \{0,1\}^n \to \{0,1\}$.*

Value: *Value of this protocol on $x \in \{0,1\}^n$ is equal to the label of the leaf reached from the root if in each inner node v we go via the edge labeled by 1 iff $q_v(x) = 1$.*

Cost: *Cost of this protocol is the maximum over all possible paths p from the root to a leaf of the sum of $\mu(q_v)$'s for v in p.*

Definition A.4. *($\mathsf{P}^{\mu[q]}$)*

Protocol: *A $\mathsf{P}^{\mu[q]}$ communication protocol is a P^{μ} protocol with at most q nontrivial queries on each path from the source to a leaf (a query in a node v is trivial iff q_v depends only on bits with indices from either Π_0 or Π_1.*

Value: *Value of these protocols is the same as in the case of P^{μ} protocols.*

Cost: *Cost of these protocols is the same as in the case of P^{μ} protocols.*

Definition A.5. *($\mathsf{P}_{\|}^{\mu}$)*

Protocol: *A* $\mathsf{P}^{\mu}_{\|}$ *communication protocol is a* P^{μ} *protocol where the result of each query is not revealed until we reach a leaf of the tree, i.e., for each node* v *with a nontrivial query both subtrees are the same except the labels of the leaves.*

Value: *Value of these protocols is the same as in the case of* P^{μ} *protocols.*

Cost: *Cost of these protocols is the same as in the case of* P^{μ} *protocols.*

Let us prove that $\mathsf{P}(\mathsf{P}^{\mu}$, $\mathsf{P}^{\mu[q]}$, $\mathsf{P}^{\mu[q]}_{\|}$, and $\mathsf{P}^{\mu}_{\|})$ is a formal communication complexity measure. We need to check four properties.

- To check that $\mathsf{P}(\mathsf{P}^{\mu}$, $\mathsf{P}^{\mu[q]}$, $\mathsf{P}^{\mu[q]}_{\|}$, and $\mathsf{P}^{\mu}_{\|})$ complexity cannot increase if we replace some variables by constants and flip some of them note that Alice and Bob may compute $\rho(x_1)$, $\ldots$, $\rho(x_n)$ for their parts of the input and after that run the protocol for f (note that μ is a formal communication complexity measure, hence, the query complexity will not change).

- It is easy to see that the $\mathsf{P}(\mathsf{P}^{\mu}$, $\mathsf{P}^{\mu[q]}$, $\mathsf{P}^{\mu[q]}_{\|}$, and $\mathsf{P}^{\mu}_{\|})$ communication complexity cannot change if we add "dummy" variables since Alice and Bob can simply ignore them (the oracle can do the same, since μ is a formal communication measure).

- Let us now show that the $\mathsf{P}(\mathsf{P}^{\mu}$, $\mathsf{P}^{\mu[q]}$, $\mathsf{P}^{\mu[q]}_{\|}$, and $\mathsf{P}^{\mu}_{\|})$ communication complexity preserves under permutations of input variables. Let $\Pi \in \mathcal{P}_n$ be a partition and π be a permutation of $[n]$. Note that $x_i \in \Pi_0$ iff $x_{\pi(i)} \in \Pi'_0$. Hence, Alice and Bob may run the protocol for the function f using $x_{\pi(i)}$'s instead of x_i's (it will not affect queries to the oracle since μ is a formal communication measure).

- Finally, to check that the $\mathsf{P}(\mathsf{P}^{\mu}$, $\mathsf{P}^{\mu[q]}$, $\mathsf{P}^{\mu[q]}_{\|}$, and $\mathsf{P}^{\mu}_{\|})$ communication complexity decreases at most linearly if a part of the input is revealed we may notice that Alice and Bob may send the revealed bits to each other (the subsection will not increase the complexity of the queries since μ is a formal communication complexity measure).

A.2 Types of Protocols

In this section we give two definitions of general types of protocols and explaining that these protocols describe formal communication measures.

We may note that many of the communication protocols we are going to discuss in the next sections are sets of rectangles $\{P_i : i \in I\}$ with respect to a partition Π such that the value of the protocol on an input x depends only on the function $r_x : I \to \{0,1\}$, where $r(i) = 1$ iff $x \in P_i$ and the complexity of the protocol depends only on I (we call these protocol type I protocols).

Let us fix a function $f : \{0,1\}^n \to \{0,1\}$ and some communication protocol of this type $\{P_i : i \in I\}$ for f with respect to a partition $\Pi \in \mathcal{P}_n$.

We will show that three first properties of the formal communication complexity measure hold for such protocols.

- To check that these communication complexities cannot increase if we replace some variables of the function by constants and flip some of them, note that for every sequence of functions $\{\rho_i : \{0,1\} \to \{0,1\}\}_{i=1}^{n}$, the protocol $\{P_{w_0,w_1} : i \in I\}$ is a protocol for h with respect to Π, where $h(x_1,\ldots,x_n) = f(\rho_1(x_1),\ldots,\rho_n(x_n))$ and P'_i is a rectangle with respect to Π such that

$$(x_1,\ldots,x_n) \in P'_i \iff (\rho_1(x_1),\ldots,\rho_n(x_n)) \in P_i.$$

- Let $I = \{i_1 < i_2 < \cdots < i_n\} \subseteq [k]$. Fix some partition Π' such that $i_j \in \Pi'_b$ iff $j \in \Pi_b$ for all $j \in [n]$ and $b \in \{0,1\}$. It is easy to see that $\{P'_i \ : \ i \in I\}$ is a communication protocol for g with respect to Π, where $g(x_1, \ldots, x_k) = f(x_{i_1}, \ldots, x_{i_n})$ and P'_i is a rectangle with respect to Π' such that
$$(x_1, \ldots, x_k) \in P'_i \iff (x_{i_1}, \ldots, x_{i_n}) \in P_i.$$
- Let us now show that the P communication complexity preserves under permutations of input variables. Let π be a permutation of $[n]$. Note that $\{P'_i \ : \ i \in I\}$ is a communication protocol for g with respect to Π', where $g(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$, $\Pi' = (\{i \ : \ \pi(i) \in \Pi_0\}, \{i \ : \ \pi(i) \in \Pi_1\})$, and
$$(x_1, \ldots, x_k) \in P'_i \iff (x_{\pi(1)}, \ldots, x_{\pi(n)}) \in P_i.$$

Another type of protocols is a family of functions $\{f_i : \{0,1\}^n \to \{0,1\} \ : \ i \in I\}$ such that the value of the protocol depends only on the function $r_x : I \to \{0,1\}$, where $r(i) = f_i(x)$ and the complexity of the protocol depends only on I and the maximal μ communication complexity of f_i's (μ is a formal communication measure); we call these protocol type II μ protocols. Since μ is a formal communication complexity measure the communication complexity measures with respect to protocols of type II μ are formal communication complexity measures as well.

A.3 Models with Nondeterminism and Alternation

Definition A.6. *(*NP*)*

Protocol: *An* NP *communication protocol with respect to a partition* $\Pi \in \mathcal{P}_n$ *is a set* $\left\{R_w \subseteq \{0,1\}^n \ : \ w \in \{0,1\}^k\right\}$ *of rectangles with respect to* Π.

Value: *We say that the protocol is a protocol for a function* $f : \{0,1\}^n \to \{0,1\}$ *iff*
$$f(x) = 1 \iff \exists w \ x \in R_w.$$

Cost: *Cost of this protocol is* k.

Definition A.7. *(*US*)*

Protocol: *A* US *communication protocol with respect to a partition* $\Pi \in \mathcal{P}_n$ *is a set* $\left\{R_w \subseteq \{0,1\}^n \ : \ w \in \{0,1\}^k\right\}$ *of rectangles with respect to* Π.

Value: *We say that the protocol is a protocol for a function* $f : \{0,1\}^n \to \{0,1\}$ *iff*
$$f(x) = 1 \iff \left|\left\{w \in \{0,1\}^k \ : \ x \in R_w\right\}\right| = 1.$$

Cost: *Cost of this protocol is* k.

Definition A.8. *(*$\oplus\mathsf{P}$*)*

Protocol: *A* $\oplus\mathsf{P}$ *communication protocol with respect to a partition* $\Pi \in \mathcal{P}_n$ *is a set* $\left\{R_w \subseteq \{0,1\}^n \ : \ w \in \{0,1\}^k\right\}$ *of rectangles with respect to* Π.

Value: *We say that the protocol is a protocol for a function* $f : \{0,1\}^n \to \{0,1\}$ *iff*

$$f(x) = 1 \iff \left|\left\{w \in \{0,1\}^k \ : \ x \in R_w\right\}\right| \equiv 1 \pmod 2.$$

Cost: *Cost of this protocol is* k.

Definition A.9. *(*$\mathsf{S_2P}$*)*

Protocol: *An* $\mathsf{S_2P}$ *communication protocol with respect to a partition* $\Pi \in \mathcal{P}_n$ *is a set* $\left\{P_{w_0,w_1} \ : \ w_0, w_1 \in \{0,1\}^k\right\}$ *of deterministic protocols with respect to* Π *outputting values from* $\{0,1\}$.

Value: *We say that the protocol is a protocol for a function* $f : \{0,1\}^n \to \{0,1\}$ *iff*

$$f(x) = 1 \implies \exists w_1 \forall w_0 \ P_{w_0,w_1}(x) = 1$$
$$f(x) = 0 \implies \exists w_0 \forall w_1 \ P_{w_0,w_1}(x) = 0$$

Cost: *The maximum communication cost of any constituent deterministic protocol plus* k.

Definition A.10. *(*$\Pi_2\mathsf{P}$*)*

Protocol: *A* $\Pi_2\mathsf{P}$ *communication protocol with respect to a partition* $\Pi \in \mathcal{P}_n$ *is a set* $\left\{R_{w_0,w_1} \ : \ w_0, w_1 \in \{0,1\}^k\right\}$ *of rectangles with respect to* Π.

Value: *We say that the protocol is a protocol for a function* $f : \{0,1\}^n \to \{0,1\}$ *iff*

$$f(x) = 1 \iff \forall w_0 \exists w_1 \ x \in R_{w_0,w_1}$$

Cost: *Cost of this protocol is* k.

Definition A.11. *(*DP*)*

Protocol: *A* DP *communication protocol with respect to a partition* $\Pi \in \mathcal{P}_n$ *is a pair of sets* $\left\{R_w \subseteq \{0,1\}^n \times \{0,1\}^m \ : \ w \in \{0,1\}^k\right\}$ *and* $\left\{T_w \subseteq \{0,1\}^n \times \{0,1\}^m \ : \ w \in \{0,1\}^k\right\}$ *of rectangles with respect to* Π.

Value: *We say that the protocol is a protocol for a function* $f : \{0,1\}^n \to \{0,1\}$ *iff*

$$f(x) = 1 \iff x \in \bigcup_{w \in \{0,1\}^k} R_w \setminus \bigcup_{w \in \{0,1\}^k} T_w.$$

Cost: *Cost of this protocol is* k.

Note that all the protocols described before (except $\mathsf{S_2P}$ protocols) are type I protocols; hence, to prove that these measures are formal communication measure, we need to prove only the last property. Let $f : \{0,1\}^{n+\ell} \to \{0,1\}$ be a Boolean function and $\left\{R_w^{c_1,\dots,c_\ell} \subseteq \{0,1\}^{n+\ell} \ : \ w \in \{0,1\}^k\right\}$ be an NP (US, $\oplus\mathsf{P}$, $\Pi_2\mathsf{P}$, DP) communication protocol for $g_{c_1,\dots,c_\ell}$ with respect to a partition $\Pi \in \mathcal{P}_{n+\ell}$, where $g_{c_1,\dots,c_\ell}(x_1,\dots,x_{n+\ell}) = f(x_1,\dots,x_n,c_1,\dots,c_\ell)$. As a result, $\left\{R_w^{c_1,\dots,c_\ell} \subseteq \{0,1\}^{n+\ell} \ : \ w \in \{0,1\}^k, c_1,\dots,c_\ell \in \{0,1\}\right\}$ is an NP (US, $\oplus\mathsf{P}$, $\Pi_2\mathsf{P}$, DP) communication protocol for f with respect to Π and the cost of this protocol is $k + \ell$

As about $\mathsf{S_2P}$ protocols, they are type II P communication protocols, hence $\mathsf{S_2P}$ is a formal communication complexity measure.

A.4 Probabilistic Models

Definition A.12. *(*ZPP*)*

Protocol: *A* ZPP *communication protocol with respect to a partition* $\Pi \in \mathcal{P}_n$ *is a distribution* $\mathcal{D}$ *over the deterministic protocols with respect to* Π *outputting values from* $\{0, 1, \perp\}$*.*

Value: *We say that the protocol is a protocol for a function* $f : \{0,1\}^n \to \{0,1\}$ *iff*

$$\Pr[\mathcal{D}(x) \in \{f(x), \perp\}] = 1 \textit{ and}$$
$$\Pr[\mathcal{D}(x) = f(x)] \geq \tfrac{3}{4}$$

Cost: *The maximum communication cost of any constituent deterministic protocol.*

Definition A.13. *(*RP*)*

Protocol: *An* RP *communication protocol with respect to a partition* $\Pi \in \mathcal{P}_n$ *is a distribution* $\mathcal{D}$ *over the deterministic protocols with respect to* Π *outputting values from* $\{0, 1\}$*.*

Value: *We say that the protocol is a protocol for a function* $f : \{0,1\}^n \to \{0,1\}$ *iff*

$$f(x) = 1 \implies \Pr[\mathcal{D}(x) = 1] \geq \tfrac{1}{2}$$
$$f(x) = 0 \implies \Pr[\mathcal{D}(x) = 0] = 1.$$

Cost: *The maximum communication cost of any constituent deterministic protocol.*

Definition A.14. *(*BPP*)*

Protocol: *A* BPP *communication protocol with respect to a partition* $\Pi \in \mathcal{P}_n$ *is a distribution* $\mathcal{D}$ *over the deterministic protocols with respect to* Π *outputting values from* $\{0, 1\}$*.*

Value: *We say that the protocol is a protocol for a function* $f : \{0,1\}^n \to \{0,1\}$ *iff*

$$\Pr[\mathcal{D}(x) = f(x)] \geq \frac{3}{4}.$$

Cost: *The maximum communication cost of any constituent deterministic protocol.*

Definition A.15. *(*MA*)*

Protocol: *An* MA *communication protocol with respect to a partition* $\Pi \in \mathcal{P}_n$ *is a set* $\left\{\mathcal{D}_w \ : \ w \in \{0,1\}^k\right\}$ *of distributions over the deterministic protocols with respect to* Π *outputting values from* $\{0, 1\}$*.*

Value: *We say that the protocol is a protocol for a function* $f : \{0,1\}^n \to \{0,1\}$ *iff*

$$f(x) = 1 \implies \exists w \ \Pr[\mathcal{D}_w = 1] \geq \tfrac{3}{4}$$
$$f(x) = 0 \implies \forall w \ \Pr[\mathcal{D}_w = 0] \geq \tfrac{3}{4}$$

Cost: *The maximum communication cost of any constituent deterministic protocol plus k.*

Definition A.16. *(AM)*

Protocol: *An AM communication protocol with respect to a partition $\Pi \in \mathcal{P}_n$ is a distribution $\mathcal{D}$ over the NP protocols with respect to Π.*

Value: *We say that the protocol is a protocol for a function $f : \{0,1\}^n \to \{0,1\}$ iff*

$$\Pr[\mathcal{D}(x) = f(x)] \geq \frac{3}{4}.$$

Cost: *The maximum communication cost of any constituent NP protocol.*

Definition A.17. *($\mathsf{ZPP}_{\|}^{\mathsf{NP}[q]}$)*

Protocol: *An $\mathsf{ZPP}_{\|}^{\mathsf{NP}[q]}$ communication protocol with respect to a partition $\Pi \in \mathcal{P}_n$ is a distribution $\mathcal{D}$ over the $\mathsf{P}_{\|}^{\mathsf{NP}[q]}$ protocols with respect to Π outputting values from $\{0,1,\perp\}$.*

Value: *We say that the protocol is a protocol for a function $f : \{0,1\}^n \to \{0,1\}$ iff*

$$\Pr[\mathcal{D}(x) = f(x)] \geq \frac{3}{4}.$$

Cost: *The maximum communication cost of any constituent $\mathsf{P}_{\|}^{\mathsf{NP}[q]}$ protocol.*

Definition A.18. *($\mathsf{ZPP}_{\|}^{\mathsf{NP}}$)*

Protocol: *An $\mathsf{ZPP}_{\|}^{\mathsf{NP}}$ communication protocol with respect to a partition $\Pi \in \mathcal{P}_n$ is a distribution $\mathcal{D}$ over the $\mathsf{P}_{\|}^{\mathsf{NP}}$ protocols with respect to Π outputting values from $\{0,1,\perp\}$.*

Value: *We say that the protocol is a protocol for a function $f : \{0,1\}^n \to \{0,1\}$ iff*

$$\Pr[\mathcal{D}(x) = f(x)] \geq \frac{3}{4}.$$

Cost: *The maximum communication cost of any constituent $\mathsf{P}_{\|}^{\mathsf{NP}}$ protocol.*

Definition A.19. *($\mathsf{ZPP}^{\mathsf{NP}[q]}$)*

Protocol: *An $\mathsf{ZPP}^{\mathsf{NP}[q]}$ communication protocol with respect to a partition $\Pi \in \mathcal{P}_n$ is a distribution $\mathcal{D}$ over the $\mathsf{P}^{\mathsf{NP}[q]}$ protocols with respect to Π outputting values from $\{0,1,\perp\}$.*

Value: *We say that the protocol is a protocol for a function $f : \{0,1\}^n \to \{0,1\}$ iff*

$$\Pr[\mathcal{D}(x) = f(x)] \geq \frac{3}{4}.$$

Cost: *The maximum communication cost of any constituent $\mathsf{P}^{\mathsf{NP}[q]}$ protocol.*

Definition A.20. *($\mathsf{ZPP}^{\mathsf{NP}}$)*

Protocol: *An* $\mathsf{ZPP}^{\mathsf{NP}}$ *communication protocol with respect to a partition* $\Pi \in \mathcal{P}_n$ *is a distribution* $\mathcal{D}$ *over the* P^{NP} *protocols with respect to* Π *outputting values from* $\{0, 1, \perp\}$.

Value: *We say that the protocol is a protocol for a function* $f : \{0,1\}^n \to \{0,1\}$ *iff*

$$\Pr[\mathcal{D}(x) = f(x)] \geq \frac{3}{4}.$$

Cost: *The maximum communication cost of any constituent* P^{NP} *protocol.*

Definition A.21. *(*SBP*)*

Protocol: *An* SBP *communication protocol with respect to a partition* $\Pi \in \mathcal{P}_n$ *is a set* $\left\{P_w \ : \ w \in \{0,1\}^k\right\}$ *of deterministic protocols with respect to* Π *outputting values from* $\{0,1\}$.

Value: *We say that the protocol is a protocol for a function* $f : \{0,1\}^n \to \{0,1\}$ *iff*

$$\min_{x \in f^{-1}(1)} \Pr_{w \leftarrow U(\{0,1\}^k)} [P_w(x) = 1] > 2 \cdot \max_{x \in f^{-1}(0)} \Pr_{w \leftarrow U(\{0,1\}^k)} [P_w(x) = 1],$$

where $U(\{0,1\}^k)$ *is a uniform distribution over* $\{0,1\}^k$.

Cost: *The maximum communication cost of any constituent deterministic protocol plus* k.

Definition A.22. *(*$\mathsf{PostBPP}$*)*

Protocol: *A* $\mathsf{PostBPP}$ *communication protocol with respect to a partition* $\Pi \in \mathcal{P}_n$ *is a set* $\left\{P_w \ : \ w \in \{0,1\}^k\right\}$ *of deterministic protocols with respect to* Π *outputting values from* $\{0, 1, \perp\}$.

Value: *We say that the protocol is a protocol for a function* $f : \{0,1\}^n \to \{0,1\}$ *iff*

$$\Pr_{w \leftarrow U(\{0,1\}^k)} [P_w(x) = f(x)] > 2 \cdot \Pr_{w \leftarrow U(\{0,1\}^k)} [P_w(x) = 1 - f(x)].$$

Cost: *The maximum communication cost of any constituent deterministic protocol plus* k.

Definition A.23. *(*$\mathsf{UPostBPP}_{\square}$*) Same as Definition A.22 but instead of* $U(\{0,1\}^k)$ *we may consider an arbitrary distribution over* $\{0,1\}^k$.

Definition A.24. *(*$\mathsf{UPostBPP}$*) Same as Definition A.22 but with private-randomness.*

Definition A.25. *(*PP*)*

Protocol: *A* PP *communication protocol with respect to a partition* $\Pi \in \mathcal{P}_n$ *is a set* $\left\{P_w \ : \ w \in \{0,1\}^k\right\}$ *of deterministic protocols with respect to* Π *outputting values from* $\{0,1\}$.

Value: *We say that the protocol is a protocol for a function* $f : \{0,1\}^n \to \{0,1\}$ *iff*

$$\Pr_{w \leftarrow U(\{0,1\}^k)} [P_w(x) = f(x)] > \frac{1}{2}.$$

Cost: *The maximum communication cost of any constituent deterministic protocol plus k.*

Definition A.26. *($\mathsf{UPP}_{\square}$) Same as Definition A.25 but instead of $U(\{0,1\}^k)$ we may consider an arbitrary distribution over $\{0,1\}^k$.*

Definition A.27. *(UPP) Same as Definition A.25 but with private-randomness.*

All the protocols described in this section are type II μ protocols where $\mu \in \left\{\mathsf{P}, \mathsf{NP}, \mathsf{P}^{\mathsf{NP}}, \mathsf{P}_{\parallel}^{\mathsf{NP}}, \mathsf{P}^{\mathsf{NP}[q]}, \mathsf{P}_{\parallel}^{\mathsf{NP}[q]}\right\}$; hence, all these measures are formal communication complexity measures.

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