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Quantum Lower Bounds for Approximate Counting via Laurent Polynomials*

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Abstract

We study quantum algorithms that are given access to trusted and untrusted quantum witnesses. We establish strong limitations of such algorithms, via new techniques based on *Laurent polynomials* (i.e., polynomials with positive and negative integer exponents). Specifically, we resolve the complexity of *approximate counting*, the problem of multiplicatively estimating the size of a nonempty set $S \subseteq [N]$, in two natural generalizations of quantum query complexity.

Our first result holds in the standard Quantum Merlin–Arthur (QMA) setting, in which a quantum algorithm receives an untrusted quantum witness. We show that, if the algorithm makes T quantum queries to S, and also receives an (untrusted) m-qubit quantum witness, then either $m = \Omega(|S|)$ or $T = \Omega\left(\sqrt{N/|S|}\right)$. This is optimal, matching the straightforward protocols where the witness is either empty, or specifies all the elements of S. As a corollary, this resolves the open problem of giving an oracle separation between SBP, the complexity class that captures approximate counting, and QMA.

In our second result, we ask what if, in addition to a membership oracle for S, a quantum algorithm is also given "QSamples"—i.e., copies of the state $|S\rangle = \frac{1}{\sqrt{|S|}} \sum_{i \in S} |i\rangle$— or even access to a unitary transformation that enables QSampling? We show that, even then, the algorithm needs either $\Theta\left(\sqrt{N/|S|}\right)$ queries or else $\Theta\left(\min\left\{|S|^{1/3}, \sqrt{N/|S|}\right\}\right)$ QSamples or accesses to the unitary.

Our lower bounds in both settings make essential use of Laurent polynomials, but in different ways.

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1 Introduction

Understanding the power of quantum algorithms has been a central research goal over the last few decades. One success story in this regard has been the discovery of powerful methods that establish limitations on quantum algorithms in the standard setting of *query complexity*. This setting roughly asks, for a specified function f, how many bits of the input must be examined by any quantum algorithm that computes f (see [BdW02] for a survey of query complexity).

A fundamental topic of study in complexity theory is algorithms that are "augmented" with additional information, such as an untrusted witness provided by a powerful prover. For example, the classical complexity class NP is defined this way. In the quantum setting, if we go beyond standard query algorithms, and allow algorithms to receive a quantum state, the model becomes much richer, and we have very few techniques to establish lower bounds for these algorithms. In this paper, we develop such techniques. Our methods crucially use *Laurent polynomials*, which are polynomials with positive and negative integer exponents.

We demonstrate the power of these lower bound techniques by proving optimal lower bounds for the *approximate counting* problem, which captures the following task. Given a nonempty finite set $S \subseteq [N] := \{1, \ldots, N\}$, estimate its cardinality, $|S|$, to within some constant (say, 2) multiplicative accuracy. Approximate counting is a fundamental task with a rich history in computer science. This includes the works of Stockmeyer [Sto85], which showed that approximate counting is in the polynomial hierarchy, and Sinclair and Jerrum [SJ89], which showed the equivalence between approximate counting and approximate sampling that enabled the development of a whole new class of algorithms based on Markov chains. Additionally, approximate counting precisely highlights the limitations of current lower bound techniques for the complexity class QMA (as we explain in Section 1.1).

Formally, we study the following decision version of the problem in this paper:

Problem 1 (Approximate Counting)**.** *In the* $\mathsf{ApxCount}_{N,w}$ *problem, our goal is to decide whether a nonempty set* $S \subseteq [N]$ *satisfies* $|S| \geq 2w$ *(YES) or* $|S| \leq w$ *(NO), promised that one of these is the case.*

In the query model, the algorithm is given a membership oracle for S: one that, for any $i \in [N]$, returns whether $i \in S$. How many queries must we make, as a function of both N and $|S|$, to solve approximate counting with high probability?

For classical randomized algorithms, it is easy to see that $\Theta(N/|S|)$ membership queries are necessary and sufficient. For quantum algorithms, which can query the membership oracle on superpositions of inputs, Brassard et al. [BHT98a, BHMT02] gave an algorithm that makes only $O\big(\sqrt{N/|S|}\big)$ queries. It follows from the optimality of Grover's algorithm (i.e., the BBBV Theorem [BBBV97]) that this cannot be improved. Hence, the classical and quantum complexity of approximate counting with membership queries alone is completely understood. In this paper, we study the complexity of approximate counting in models with untrusted and trusted quantum states.

1.1 First result: QMA complexity of approximate counting

Our first result, presented in Section 3, considers the standard Quantum Merlin–Arthur (QMA) setting, in which the quantum algorithm receives an untrusted quantum state (called the witness). This model is the quantum analogue of the classical complexity class NP, and is of great interest in quantum complexity theory. It captures natural problems about ground states of physical systems,

properties of quantum circuits and channels, noncommutative constraint satisfaction problems, consistency of representations of quantum systems, and more [Boo14].

In a QMA protocol, a skeptical verifier (Arthur) receives a quantum witness state $|\psi\rangle$ from an all-powerful but untrustworthy prover (Merlin), in support of the claim that $f(x) = 1$. Arthur then needs to verify $|\psi\rangle$, via some algorithm that satisfies the twin properties of *completeness* and *soundness*. That is, if $f(x) = 1$, then there must exist some $|\psi\rangle$ that causes Arthur to accept with high probability, while if $f(x) = 0$, then every $|\psi\rangle$ must cause Arthur to reject with high probability. We call such a protocol a QMA (Quantum Merlin–Arthur) protocol for computing f.

In the query complexity setting, there are two resources to consider: the length of the quantum witness, m, and the number of queries, T, that Arthur makes to the membership oracle. A QMA protocol for f is efficient if both m and T are polylog(N).

The known lower bound technique for QMA. Prior to our work, all known QMA lower bounds used the same proof technique.[1] The technique establishes (and exploits) the complexity class containment QMA $\subseteq$ SBQP, where SBQP is a complexity class that models quantum algorithms with tiny acceptance and rejection probabilities. Specifically, we say that a function f has SBQP query complexity at most k if there exists a k-query quantum algorithm that

- outputs 1 with probability $\geq \alpha$ when $f(x) = 1$, and
- outputs 1 with probability $\leq \alpha/2$ when $f(x) = 0$,

for some α that does not depend on the input (but may depend on the input size). Note that when $\alpha = 2/3$, we recover standard quantum query complexity. But α could be also be exponentially small, which makes SBQP algorithms very powerful.

Nevertheless, one can establish significant limitations on SBQP algorithms, by using a variation of the polynomial method of Beals et al. [BBC+01]. If a function f can be evaluated by an SBQP algorithm with k queries, then there exists a real polynomial p of degree $2k$ such that $p(x) \in [0, 1]$ whenever $f(x) = 0$ and $p(x) \geq 2$ whenever $f(x) = 1$. The minimum degree of such a polynomial is also called *one-sided approximate degree* [BT15].

The relationship between SBQP and QMA protocols is very simple: if f has a QMA protocol that receives an m-qubit witness and makes T queries, then it also has an SBQP algorithm that makes $O(mT)$ queries. This was essentially observed by Marriott and Watrous [MW05, Remark 3.9] and used by Aaronson [Aar12] to show an oracle relative to which SZK $\not\subset$ QMA.

Beyond the known lower bound technique for QMA. Our goal is to find a new method of lower bounding QMA, that does not go through SBQP complexity. The natural way to formalize this quest is to find a problem that has an efficient SBQP algorithm, and show that it does not have an efficient QMA protocol. A natural candidate for this is the $\mathsf{ApxCount}_{N,w}$ problem. We know that $\mathsf{ApxCount}_{N,w}$ *does* have a very simple SBQP algorithm of cost 1: the algorithm picks an $i \in [N]$ uniformly at random, and accepts if and only if $i \in S$. Clearly the algorithm accepts with probability greater than $2w/N$ on yes inputs and with probability at most w/N on no inputs.

[1] There is one special case in which it is trivial to lower-bound QMA complexity. Consider the AND_N function on N bits that outputs 1 if and only if all N bits equal 1. For this function, since Merlin wants to convince Arthur that $f(x) = 1$, intuitively there is nothing interesting that Merlin can say to Arthur other than "x is all ones" since that is the only input with $f(x) = 1$. Formally, Arthur can simply create the witness state that an honest Merlin would have sent on the all ones input, and hence Arthur does not need Merlin [RS04]. For such functions, QMA complexity is the same as standard quantum query complexity.

Our first result establishes that $\mathsf{ApxCount}_{N,w}$ does *not* have an efficient QMA protocol.

Theorem 2. *Consider a* QMA *protocol that solves* $\mathsf{ApxCount}_{N,w}$*. If the protocol receives a quantum witness of length* m*, and makes* T *queries to the membership oracle for* S*, then either* $m = \Omega(w)$ *or* $T = \Omega\left(\sqrt{N/w}\right)$*.*

This lower bound proved in Section 3.2 resolves the QMA complexity of $\mathsf{ApxCount}_{N,w}$, as (up to a $\log N$ factor) it matches the cost of two trivial QMA protocols. In the first, Merlin sends $2w$ items claimed to be in S, and Arthur picks a constant number of the items at random and confirms they are all in S with one membership query each. This protocol has witness length $m = O(w \log N)$ (the number of bits needed to specify $2w$ elements out of N) and $T = O(1)$. In the second protocol, Merlin does nothing, and Arthur solves the problem with $T = O\left(\sqrt{N/w}\right)$ quantum queries.

Oracle separation. Our result also yields new oracle separations. The approximate counting problem is complete for the complexity class SBP [BGM06], which is sandwiched between MA (Merlin–Arthur) and AM (Arthur–Merlin). The class SBQP (discussed above), first defined by Kuperberg [Kup15], is a quantum analogue of SBP that contains both SBP and QMA.

By the usual connection between oracle separations and query complexity lower bounds, Theorem 2 implies an oracle separation between SBP and QMA—i.e., there exists an oracle A such that $\mathsf{SBP}^A \not\subset \mathsf{QMA}^A$ (see Corollary 20). Prior to our work, it was known that there exist oracles A, B such that $\mathsf{SBP}^A \not\subset \mathsf{MA}^A$ [BGM06] and $\mathsf{AM}^B \not\subset \mathsf{QMA}^B$, which follows from $\mathsf{AM}^B \not\subset \mathsf{PP}^B$ [Ver92], but the relation between SBP and QMA remained elusive.[2] Figure 1 shows the known inclusion relations among these classes (all of which hold relative to all oracles).

Previous techniques were inherently unable to establish this oracle separation for the reason stated above: all existing QMA lower bounds intrinsically apply to SBQP as well. Since SBP is contained in SBQP, prior techniques cannot establish $\mathsf{SBP}^A \not\subset \mathsf{QMA}^A$, or even $\mathsf{SBQP}^A \not\subset \mathsf{QMA}^A$, for any oracle A. Our analysis also yields the first oracle with respect to which SBQP is not closed under intersection.

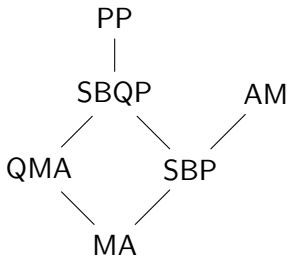

Figure 1: Relationships between complexity classes. An upward line indicates that a complexity class is contained in the one above it relative to all oracles.

Proof overview. To get around the issue of $\mathsf{ApxCount}_{N,w}$ being in SBQP, we use a clever strategy that was previously used by Göös et al. [GLM+16], and that was suggested to us by Thomas Watson (personal communication). Our strategy exploits a structural property of QMA: the fact that QMA is closed under intersection, but (at least relative to oracles, and as we'll show) SBQP is not.

Given a function f, let $\mathsf{AND}_2 \circ f$ be the AND of two copies of f on separate inputs.[3] Then if f has small QMA query complexity, it's not hard to see that $\mathsf{AND}_2 \circ f$ does as well: Merlin simply sends witnesses corresponding to both inputs; then Arthur checks both of them independently. While it's not completely obvious, one can verify that a dishonest Merlin would gain nothing by

[2] It is interesting to note that in the non-relativized world, under plausible derandomization assumptions [MV99], we have $\mathsf{NP} = \mathsf{MA} = \mathsf{SBP} = \mathsf{AM}$. In this scenario, all these classes are equal, and all are contained in QMA.

[3] Because we focus on lower bounds, for a promise problem f (such as $\mathsf{ApxCount}_{N,w}$), we take the promise for $\mathsf{AND}_2 \circ f$ to be that both instances of f must satisfy f's promise. Then, any lower bound also applies to more relaxed definitions, such as only requiring one of the two instances to be in the promise.

entangling the two witness states. Hence if $\mathsf{ApxCount}_{N,w}$ had an efficient QMA protocol, then so would $\mathsf{AND}_2 \circ \mathsf{ApxCount}_{N,w}$, with the witness size and query complexity increasing by only a constant factor.

By contrast, even though $\mathsf{ApxCount}_{N,w}$ does have an efficient SBQP algorithm, we will show that $\mathsf{AND}_2 \circ \mathsf{ApxCount}_{N,w}$ does not. This is the technical core of our proof and proved in Section 3.1.

Theorem 3. *Consider an* SBQP *algorithm for* $\mathsf{AND}_2 \circ \mathsf{ApxCount}_{N,w}$ *that makes* T *queries to membership oracles for the two instances of* $\mathsf{ApxCount}_{N,w}$. *Then* $T = \Omega\left(\min\{w, \sqrt{N/w}\}\right)$.

Theorem 3 is quantitatively optimal, as we'll exhibit a matching SBQP upper bound. Combined with the connection between QMA and SBQP, Theorem 3 immediately implies a QMA lower bound for $\mathsf{AND}_2 \circ \mathsf{ApxCount}_{N,w}$, and by extension $\mathsf{ApxCount}_{N,w}$ itself. However, this QMA lower bound is not quantitatively optimal. To obtain the optimal bound of Theorem 2, we exploit additional analytic properties of the SBQP protocols that are derived from QMA protocols.

At a high level, the proof of Theorem 3 assumes that there's an efficient SBQP algorithm for $\mathsf{AND}_2 \circ \mathsf{ApxCount}_{N,w}$. This assumption yields a low-degree one-sided approximating polynomial for the problem in $2N$ Boolean variables, where N variables come from each $\mathsf{ApxCount}_{N,w}$ instance. We then symmetrize the polynomial (using the standard Minsky–Papert symmetrization argument [MP88]) to obtain a bivariate polynomial in two variables x and y that represent the Hamming weight of the original instances.[4] This yields a polynomial $p(x,y)$ that for *integer pairs* x, y (also called lattice points) satisfies $p(x,y) \in [0,1]$ when either $x \in \{0,\ldots,w\}$ and $y \in \{0,\ldots,w\} \cup \{2w,\ldots,N\}$, or (symmetrically) $y \in \{0,\ldots,w\}$ and $x \in \{0,\ldots,w\} \cup \{2w,\ldots,N\}$. If both $x \in \{2w,\ldots,N\}$ and $y \in \{2w,\ldots,N\}$, then $p(x,y) \geq 2$. This polynomial p is depicted in Figure 2.

One difficulty is that we have a guarantee on the behavior of p at lattice points only, whereas the rest of our proof requires precise control over the polynomial even at non-integer points. We ignore this issue for now and assume that $p(x,y) \geq 2$ for all real values $x, y \in [2w, N]$, and $p(x,y) \in [0,1]$ whenever $x \in [0,w]$ and $y \in [2w,N]$ or vice versa. We outline how we address integrality issues one paragraph hence.

The key remaining difficulty is that we want to lower-bound the degree of a bivariate polynomial, but almost all known lower bound techniques apply only to univariate polynomials. To address this, we introduce a new technique to reduce the number of variables (from 2 to 1) in a degree-preserving way: we pass a *hyperbola* through the xy plane (see Figure 2) and consider the polynomial p restricted to the hyperbola. Doing so gives us a new univariate *Laurent* polynomial $\ell(t) = p(2wt, 2w/t)$, whose positive and negative degree is at most $\deg(p)$. This Laurent polynomial has an additional symmetry, which stems from the fact that $\mathsf{AND}_2 \circ \mathsf{ApxCount}_{N,w}$ is the AND of two identical problems (namely, $\mathsf{ApxCount}_{N,w}$). We leverage this symmetry to view $\ell(t)$, a Laurent polynomial in t, as an ordinary univariate polynomial r in $t + 1/t$ of degree $\deg(p)$. We know that $r(2) = \ell(1) = p(2w, 2w) \geq 2$, while for all $k \in [2.5, N/w + w/N]$, we know that $r(k) \in [0,1]$. It then follows from classical results in approximation theory that this univariate polynomial must have degree $\Omega\left(\sqrt{N/w}\right)$.

Returning to integrality issues, to obtain a polynomial whose behavior we can control at non-integer points, we use a different symmetrization argument (dating back at least to work of Shi

[4] The term "symmetrization" originally referred to the process of averaging a multivariate polynomial over permutations of its inputs to obtain a symmetric polynomial. More recently, authors have used "symmetrization" more generally to refer to any method for turning a multivariate polynomial into a univariate one in a degree non-increasing manner (see, e.g., [She09, She10]). In this paper, we use the term "symmetrization" in this more general sense.

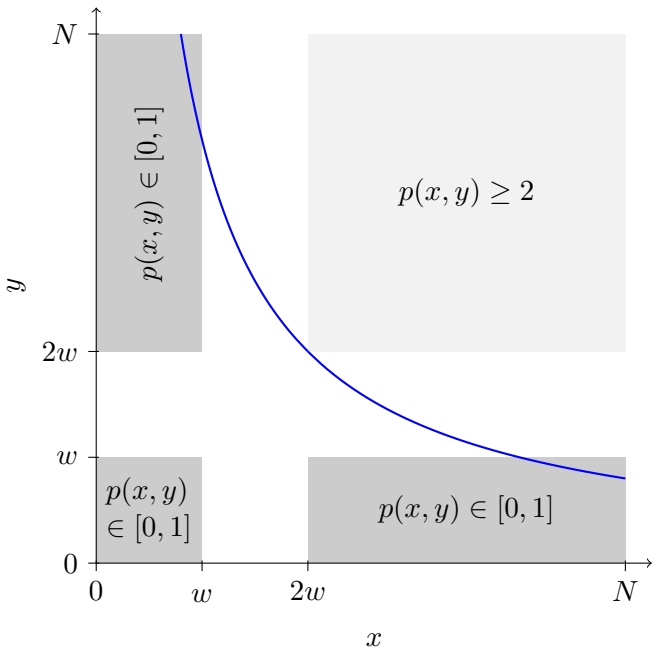

Figure 2: The behavior of the (Minsky–Papert symmetrized) bivariate polynomial $p(x, y)$ at integer points (x, y) in the proof of Theorem 3. The polynomial q obtained by erase-all-subscripts symmetrization is not depicted. We later restrict q to a hyperbola similar to the one drawn in blue.

[Shi02]) that we call "erase-all-subscripts" symmetrization (see Lemma 12). This symmetrization yields a bivariate polynomial q of the same degree as p that is bounded in $[0, 1]$ at all *real-valued* inputs in $[0, N] \times [0, N]$. However, while we have more control over q's values at non-integer inputs relative to p, we have *less* control over q's values at integer inputs relative to p, and this introduces additional challenges. (These challenges are not merely annoyances; they are why the SBQP complexity of $\mathsf{AND}_2 \circ \mathsf{ApxCount}_{N,w}$ is $T = \Theta\big(\min\{w, \sqrt{N/w}\}\big)$, and not $\Theta\big(\sqrt{N/w}\big)$). Ultimately, both types of symmetrization play an important role in our analysis, as we use p to bound q when the polynomials have degree $o(w)$, using tools from approximation theory and Chernoff bounds.

1.2 Second result: Approximate counting with quantum samples

Our second result resolves the complexity of $\mathsf{ApxCount}_{N,w}$ in a different generalization of the quantum query model, in which the algorithm is given access to certain (trusted) quantum states.

Quantum samples. In practice, when trying to estimate the size of a set $S \subseteq [N]$, often we can do more than make membership queries to S. At the least, often we can efficiently generate nearly uniform *samples* from S, for instance by using Markov Chain Monte Carlo techniques. To give two examples, if S is the set of perfect matchings in a bipartite graph, or the set of grid points in a high-dimensional convex body, then we can efficiently sample S using the seminal algorithms of Jerrum, Sinclair, and Vigoda [JSV04] or of Dyer, Frieze, and Kannan [DFK91], respectively.

The natural quantum generalization of uniform sampling from a set S is *QSampling* S—a term coined in 2003 by Aharonov and Ta-Shma [ATS03], and which means that we can approximately

prepare the uniform superposition

$$|S\rangle := \frac{1}{\sqrt{|S|}} \sum_{i \in S} |i\rangle \tag{1}$$

via a polynomial-time quantum algorithm (where "polynomial" here means polylog(N)). Because we need to uncompute garbage, the ability to prepare $|S\rangle$ as a coherent superposition is a more stringent requirement than the ability to classically sample from S. Indeed, Aharonov and Ta-Shma [ATS03] showed that the ability to QSample lends considerable power: all problems in the complexity class SZK (which contains problems that are widely believed be hard on average [GK93, GMR89, MV03, GG00, PV08]) can be efficiently reduced to the task of *QSampling* some set that can be *classically* sampled in polynomial time. To be clear, QSampling supposes that the algorithm is given trusted copies of $|S\rangle$; unlike in the QMA setting, the state need not be "verified" by the algorithm.

On the other hand, Aharonov and Ta-Shma [ATS03], and Grover and Rudolph [GR02], observed that many interesting sets S can be efficiently QSampled as well.[5]

QSampling via unitaries. In many applications (such as when S is the set of perfect matchings in a bipartite graph or grid points in a convex body), the reason an algorithm can QSample S is because it is possible to efficiently construct a quantum circuit implementing a unitary operator U that prepares the state $|S\rangle$. Access to this unitary U potentially conveys substantially more power than QSampling alone. For example, access to U conveys (in a black box manner) the ability not only to QSample, but also to perform reflections about $|S\rangle$: that is, to apply the unitary transformation

$$\mathcal{R}_S := \mathbb{1} - 2|S\rangle\langle S|, \tag{2}$$

which has eigenvalue -1 for $|S\rangle$ and eigenvalue $+1$ for all states orthogonal to $|S\rangle$. More concretely, let U be the unitary that performs the map $U|0\rangle = |S\rangle$, for some canonical starting state $|0\rangle$. Since we know the circuit U, we can also implement $U^\dagger$, by reversing the order of all the gates and replacing all the gates with their adjoints. Then $\mathcal{R}_S$ is simply

$$\mathcal{R}_S = \mathbb{1} - 2|S\rangle\langle S| = U\left(\mathbb{1} - 2|0\rangle\langle 0|\right)U^\dagger. \tag{3}$$

Note that *a priori*, QSamples and reflections about $|S\rangle$ could be incomparable resources; it is not obvious how to simulate either one using the other. On the other hand, it is known how to apply a quantum channel that is ε-close to $\mathcal{R}_S$ (in the diamond norm) using $\Theta(1/\varepsilon)$ copies of $|S\rangle$ [LMR14, KLL+17].

Access to a quantum circuit computing U also permits an algorithm to efficiently apply U on inputs that do not produce the state $|S\rangle$, to construct a controlled version of U, etc.

Results. As previously mentioned, Aharonov and Ta-Shma [ATS03] showed that the ability to QSample lends considerable power, including the ability to efficiently solve SZK-complete problems. It is natural to ask just how much power the ability to QSample conveys. In particular, can one extend the result of Aharonov and Ta-Shma [ATS03] from any problem in SZK to any problem in

[5] In particular, this holds for all sets S such that we can approximately count not only S itself, but also the restrictions of S obtained by fixing bits of its elements. So in particular, the set of perfect matchings in a bipartite graph, and the set of grid points in a convex body, can both be efficiently QSampled. There are other sets that can be QSampled but not because of this reduction. A simple example would be a set S such that $|S| \geq \frac{N}{\text{polylog}\, N}$: in that case we can efficiently prepare $|S\rangle$ using postselection, but approximately counting S's restrictions might be hard.

SBP? Equivalently stated, can one solve approximate counting efficiently, using *any* combination of polylog(N) queries and applications of a unitary U that permits QSampling?[6] In this work, we show that the answer is no. We begin by focusing on the slightly simplified setting where the algorithm is only permitted to perform membership queries, QSamples, and reflections about the state $|S\rangle$.

Theorem 4. *Let Q be a quantum algorithm that makes T queries to the membership oracle for S, and uses a total of R copies of $|S\rangle$ and reflections about $|S\rangle$. If Q decides whether $|S| = w$ or $|S| = 2w$ with high probability, promised that one of those is the case, then either*

$$T = \Omega\left(\sqrt{\frac{N}{w}}\right) \qquad or \qquad R = \Omega\left(\min\left\{w^{1/3}, \sqrt{\frac{N}{w}}\right\}\right). \tag{4}$$

This is proved in Section 4.4. So if (for example) we set $w := N^{3/5}$, then any quantum algorithm must either query S, or use the state $|S\rangle$ or reflections about $|S\rangle$, at least $\Omega(N^{1/5})$ times. Put another way, Theorem 4 means that unless w is very small ($w \leq \text{polylog}(N)$) or extremely large ($w \geq N/\text{polylog}(N)$), the ability to QSample S, reflect about $|S\rangle$, and determine membership in S is not sufficient to approximately count S efficiently. Efficient quantum algorithms for approximate counting will have to leverage additional structure of S, beyond the ability to QSample, reflect about $|S\rangle$, and determine membership in S.

In Theorem 31 of Section 4.6, we then strengthen Theorem 4 to hold not only against algorithms that can QSample and reflect about $|S\rangle$ (in addition to performing membership queries to S), but also against all algorithms that are given access to a specific unitary U that conveys the power to QSample and reflect about $|S\rangle$.[7]

Finally, we prove that the lower bounds in Theorem 4 and Theorem 31 are optimal. As mentioned before, Brassard et al. [BHT98a] gave a quantum algorithm to solve the problem using $T = O(\sqrt{N/w})$ queries alone, which proves the optimality of the lower bound on the number of queries. On the other hand, it's easy to solve the problem using $O\left(\sqrt{w}\right)$ copies of $|S\rangle$ alone, by simply measuring each copy of $|S\rangle$ in the computational basis and then searching for birthday collisions. Alternately, we can solve the problem using $O\left(\frac{N}{w}\right)$ copies of $|S\rangle$ alone, by projecting onto the state $|\psi\rangle = \frac{1}{\sqrt{N}}\left(|1\rangle + \cdots + |N\rangle\right)$ or its orthogonal complement. This measurement succeeds with probability $|\langle S|\psi\rangle|^2 = \frac{|S|}{N}$, so we can approximate $|S|$ by simply counting how many measurements succeed.

In Section 4.2 we improve on these algorithms by using samples *and* reflections, and thereby establish that Theorem 4 and Theorem 31 are tight.

Theorem 5. *There is a quantum algorithm that solves $\mathsf{ApxCount}_{N,w}$ with high probability using R copies of $|S\rangle$ and reflections about $|S\rangle$, where $R = O\left(\min\left\{w^{1/3}, \sqrt{\frac{N}{w}}\right\}\right)$.*

The Laurent polynomial method. In our view, at least as interesting as Theorem 4 is the technique used to achieve it. In 1998, Beals et al. [BBC+01] famously observed that, if a quantum algorithm Q makes T queries to an input X, then Q's acceptance probability can be written as

[6] We thank Paul Burchard (personal communication) for bringing this question to our attention.

[7] To be precise, the unitary U to which the lower bound of Theorem 31 applies maps a canonical starting state to $|S\rangle|S\rangle$. As we explain in Section 4.6, such a unitary suffices to implement QSampling, reflections about $|S\rangle$, etc., since the register containing the second copy of $|S\rangle$ can simply be ignored.

a real multilinear polynomial in the bits of X, of degree at most $2T$. And thus, crucially, if we want to *rule out* a fast quantum algorithm to compute some function $f(X)$, then it suffices to show that any real polynomial p that approximates f pointwise must have high degree. This general transformation, from questions about quantum algorithms to questions about polynomials, has been used to prove many results that were not known otherwise at the time, including the quantum lower bound for the collision problem [Aar02, AS04] and the first direct product theorems for quantum search [Aar05a, KŠdW07].

In our case, even in the simpler model with only queries and samples (and no reflections), the difficulty is that the quantum algorithm starts with many copies of the state $|S\rangle$. As a consequence of this—and specifically, of the $1/\sqrt{|S|}$ normalizing factor in $|S\rangle$—when we write the average acceptance probability of our algorithm as a function of $|S|$, we find that we get a *Laurent polynomial*: a polynomial that can contain both positive and negative integer powers of $|S|$. The degree of this polynomial (the highest power of $|S|$) encodes the sum of the number of queries, the number of copies of $|S\rangle$, and the number of uses of $\mathcal{R}_S$, while the "anti-degree" (the highest power of $|S|^{-1}$) encodes the sum of the number of copies of $|S\rangle$ and number of uses of $\mathcal{R}_S$. This is described more precisely in Section 4.1. We're thus faced with the task of lower-bounding the degree and the anti-degree of a Laurent polynomial that's bounded in $[0,1]$ at integer points and that encodes the approximate counting problem.

We then lower bound the degree of Laurent polynomials that approximate $\mathsf{ApxCount}_{N,w}$, showing that degree $\Omega\big(\min\big\{w^{1/3}, \sqrt{N/w}\big\}\big)$ is necessary. We give two very different lower bound arguments. The first approach, which we call the "explosion argument," is shorter but yields suboptimal lower bounds, whereas the second approach using "dual polynomials" yields the optimal lower bound.

There are two aspects of this that we find surprising: first, that Laurent polynomials appear at all, and second, that they seem to appear in a completely different way than they appear in our other result about QMA (Theorem 3), despite the close connection between the two statements. For Theorem 4, Laurent polynomials are needed just to describe the quantum algorithm's acceptance probability, whereas for Theorem 3, ordinary (bivariate) polynomials sufficed to describe this probability; Laurent polynomials appeared only when we restricted a bivariate polynomial to a hyperbola in the plane. In any case, the coincidence suggests that the "Laurent polynomial method" might be useful for other problems as well.[8]

Before describing our techniques at a high level, observe that there are *rational* functions[9] of degree $O(\log(N/w))$ that approximate $\mathsf{ApxCount}_{N,w}$. This follows, for example, from Aaronson's $\mathsf{PostBQP} = \mathsf{PP}$ theorem [Aar05b], or alternately from the classical result of Newman [New64] that for any $k > 0$, there is a rational polynomial of degree $O(k)$ that pointwise approximates the sign function on domain $[-n,-1] \cup [1,n]$ to error $1 - n^{-1/k}$. Thus, our proof relies on the fact that Laurent polynomials are an extremely special kind of rational function.

We also remark that in the randomized classical setting, the complexity of $\mathsf{ApxCount}_{N,w}$ with queries and uniform (classical) samples is easily characterized without such powerful techniques. Either $O(N/w)$ queries or $O(\sqrt{w})$ samples are sufficient, and furthermore either $\Omega(N/w)$ queries or $\Omega(\sqrt{w})$ samples are necessary. For completeness, we provide a sketch of these bounds in Section 4.5.

[8] Since writing this, a third application of the Laurent polynomial method was discovered by the third author [Kre19]: a simple proof that the AND-OR tree $\mathsf{AND}_m \circ \mathsf{OR}_n$ has approximate degree $\widetilde{\Omega}(\sqrt{mn})$.

[9] A rational function of degree d is of the form $\frac{p(x)}{q(x)}$, where p and q are both real polynomials of degree at most d.

Overview of the explosion argument. Our first proof (in Section 4.3) uses an "explosion argument" that, as far as we know, is new in quantum query complexity. We separate out the purely positive degree[10] and purely negative degree parts of our Laurent polynomial as $q(|S|) = u(|S|) + v(1/|S|)$, where u and v are ordinary polynomials. We then show that, if u and v both have low enough degree, namely $\deg(u) = o\left(\sqrt{N/w}\right)$ and $\deg(v) = o\left(w^{1/4}\right)$, then we get "unbounded growth" in their values. That is: for approximation theory reasons, either u or v must attain large values, far outside of $[0,1]$, at some integer values of $|S|$. But that means that, for q itself to be bounded in $[0,1]$ (and thus represent a probability), the other polynomial must *also* attain large values. And that, in turn, will force the first polynomial to attain even larger values, and so on forever—thereby proving that these polynomials could not have existed.

Overview of the method of dual polynomials. Our second argument (in Section 4.4) obtains the (optimal) lower bound stated in Theorem 4, via a novel adaptation of the so-called *method of dual polynomials*.

With this method, to lower-bound the approximate degree of a Boolean function f, one exhibits an explicit *dual polynomial* ψ for f, which is a dual solution to a certain linear program. Roughly speaking, a dual polynomial ψ is a function mapping the domain of f to $\mathbb{R}$ that is (a) uncorrelated with any polynomial of degree at most d, and (b) well-correlated with f.

Approximating a univariate function g via low-degree Laurent polynomials is also captured by a linear program, but the linear program is more complicated because Laurent polynomials can have negative-degree terms. We analyze the value of this linear program in two steps.

In Step 1, we transform the linear program so that it refers only to ordinary polynomials rather than Laurent polynomials. Although simple, this transformation is crucial, as it lets us bring techniques developed for ordinary polynomials to bear on our goal of proving Laurent polynomial degree lower bounds.

In Step 2, we explicitly construct an optimal dual witness to the transformed linear program from Step 1. We do so by first identifying two weaker dual witnesses: ψ_1, which witnesses that *ordinary* (i.e., purely positive degree) polynomials encoding approximate counting require degree at least $\Omega\left(\sqrt{N/w}\right)$, and ψ_2, which witnesses that purely negative degree polynomials encoding approximate counting require degree $\Omega(w^{1/3})$. The first witness is derived from prior work of Bun and Thaler [BT13] (who refined earlier work of Špalek [Špa08]), while the second builds on a non-constructive argument of Zhandry [Zha12].

Finally, we show how to "glue together" ψ_1 and ψ_2, to get a dual witness ψ showing that any general Laurent polynomial that encodes approximate counting must have either positive degree $\Omega\left(\sqrt{N/w}\right)$ or negative degree $\Omega(w^{1/3})$.

Overview of the upper bound. To recap, Theorem 4 shows that any quantum algorithm for $\mathsf{ApxCount}_{N,w}$ needs either $\Theta(\sqrt{N/w})$ queries or $\Theta\left(\min\left\{w^{1/3}, \sqrt{N/w}\right\}\right)$ samples and reflections. Since we know from the work of Brassard, Høyer, Tapp [BHT98a] that the problem can be solved with $O(\sqrt{N/w})$ queries alone, it remains only to show the matching upper bound using samples and reflections, which we describe in Section 4.2.

First we describe a simple algorithm that uses $O(\sqrt{N/w})$ samples and reflections. If we take one copy of $|S\rangle$, and perform a projective measurement onto $|\psi\rangle = \frac{1}{\sqrt{N}}(|1\rangle + \cdots + |N\rangle)$ or its

[10] Throughout this paper we allow any "purely positive degree" Laurent polynomial and any "purely negative degree" Laurent polynomial to include a constant (degree zero) term.

orthogonal complement, the measurement will succeed with probability $|\langle S|\psi\rangle|^2 = |S|/N$. Thus $O(N/w)$ repetitions of this will allow us to distinguish the probabilities w/N and $2w/N$. We can improve this by using amplitude amplification [BHMT02] and only make $O(\sqrt{N/w})$ repetitions.

Our second algorithm solves the problem with $O(w^{1/3})$ reflections and samples and is based on the quantum collision-finding algorithm [BHT98b]. We first use $O(w^{1/3})$ copies of $|S\rangle$ to learn $w^{1/3}$ distinct elements in S. We now know a fraction of elements in S, and this fraction is either $w^{-2/3}$ or $\frac{1}{2}w^{-2/3}$. We then use amplitude amplification (or quantum counting) to distinguish these two cases, which costs $O(w^{1/3})$ repetitions, where each repetition uses a reflection about $|S\rangle$.

2 Preliminaries

In this section we introduce some definitions and known facts about polynomials and complexity classes.

2.1 Approximation theory

We will use several results from approximation theory, each of which has previously been used (in some form) in other applications of the polynomial method to quantum lower bounds. We start with the basic inequality of A.A. Markov [Mar90].

Lemma 6 (Markov). *Let p be a real polynomial, and suppose that*

$$\max_{x,y\in[a,b]} |p(x) - p(y)| \leq H. \tag{5}$$

Then for all $x \in [a,b]$, we have

$$\left|p'(x)\right| \leq \frac{H}{b-a}\deg(p)^2, \tag{6}$$

where $p'(x)$ is the derivative of p at x.

We'll also need a bound that was explicitly stated by Paturi [Pat92], and which amounts to the fact that, among all degree-d polynomials that are bounded within a given range, the Chebyshev polynomials have the fastest growth outside that range.

Lemma 7 (Paturi). *Let p be a real polynomial, and suppose that $|p(x)| \leq 1$ for all $|x| \leq 1$. Then for all $x \leq 1+\mu$, we have*

$$|p(x)| \leq \exp\left(2\deg(p)\sqrt{2\mu+\mu^2}\right). \tag{7}$$

We now state a useful corollary of Lemma 7, which says (in effect) that slightly shrinking the domain of a low-degree real polynomial can only modestly shrink its range.

Corollary 8. *Let p be a real polynomial of degree d, and suppose that*

$$\max_{x,y\in[a,b]} |p(x) - p(y)| \geq H. \tag{8}$$

Let $\varepsilon \leq \frac{1}{100d^2}$ and $a' := a + \varepsilon(b-a)$. Then

$$\max_{x,y\in[a',b]} |p(x) - p(y)| \geq \frac{H}{2}. \tag{9}$$

Proof. Suppose by contradiction that

$$|p(x) - p(y)| < \frac{H}{2} \tag{10}$$

for all $x, y \in [a', b]$. By affine shifts, we can assume without loss of generality that $|p(x)| < \frac{H}{4}$ for all $x \in [a', b]$. Then by Lemma 7, for all $x \in [a, b]$ we have

$$|p(x)| < \frac{H}{4} \cdot \exp\left(2d\sqrt{2\left(\frac{1}{1-\varepsilon} - 1\right) + \left(\frac{1}{1-\varepsilon} - 1\right)^2}\right) \leq \frac{H}{2}. \tag{11}$$

But this violates the hypothesis. ■

We will also need a bound that relates the range of a low-degree polynomial on a discrete set of points to its range on a continuous interval. The following lemma generalizes a result due to Ehlich and Zeller [EZ64] and Rivlin and Cheney [RC66], who were interested only in the case where the discrete points are evenly spaced.

Lemma 9. *Let p be a real polynomial of degree at most $\sqrt{k}$, and let $0 = z_1 < \cdots < z_M = k$ be a list of points such that $z_{i+1} - z_i \leq 1$ for all i (the simplest example being the integers $0, \ldots, k$). Suppose that*

$$\max_{x,y \in [0,k]} |p(x) - p(y)| \geq H. \tag{12}$$

Then

$$\max_{i,j} |p(z_i) - p(z_j)| \geq \frac{H}{2}. \tag{13}$$

Proof. Suppose by contradiction that

$$|p(z_i) - p(z_j)| < \frac{H}{2} \tag{14}$$

for all i, j. By affine shifts, we can assume without loss of generality that $|p(z_i)| < \frac{H}{4}$ for all i. Let

$$c := \max_{x \in [0,k]} \frac{|p(x)|}{H/4}. \tag{15}$$

If $c \leq 1$, then the hypothesis clearly fails, so assume $c > 1$. Suppose that the maximum, $|p(x)| = \frac{cH}{4}$, is achieved between z_i and z_{i+1}. Then by basic calculus, there exists an $x^* \in [z_i, z_{i+1}]$ such that

$$|p'(x^*)| > \frac{2(c-1)}{z_{i+1} - z_i} \cdot \frac{H}{4} \geq \frac{(c-1)H}{2}. \tag{16}$$

So by Lemma 6,

$$\frac{(c-1)H}{2} < \frac{cH/4}{k} \deg(p)^2. \tag{17}$$

Solving for c, we find

$$c < \frac{2k}{2k - \deg(p)^2} \leq 2. \tag{18}$$

But if $c < 2$, then $\max_{x \in [0,k]} |p(x)| < \frac{H}{2}$, which violates the hypothesis. ■

We also use a related inequality due to Coppersmith and Rivlin [CR92] that bounds a polynomial on a continuous interval in terms of a bound on a discrete set of points, but now with the weaker assumption that the degree is at most k, rather than $\sqrt{k}$. This gives a substantially weaker bound.

Lemma 10 (Coppersmith and Rivlin)**.** *Let p be a real polynomial of degree at most k, and suppose that $|p(x)| \leq 1$ for all integers $x \in \{0, 1, \ldots, k\}$. Then there exist universal constants a, b such that for all $x \in [0, k]$, we have*

$$|p(x)| \leq a \cdot \exp\left(b \deg(p)^2/k\right). \tag{19}$$

2.2 Symmetric polynomials

Univariate symmetrizations. Our starting point is the well-known *symmetrization lemma* of Minsky and Papert [MP88] (see also Beals et al. [BBC+01] for its application to quantum query complexity), by which we can often reduce questions about multivariate polynomials to questions about univariate ones.

Lemma 11 (Minsky–Papert symmetrization)**.** *Let $p : \{0,1\}^N \to \mathbb{R}$ be a real multilinear polynomial of degree d, and let $q : \{0, 1, \ldots, N\} \to \mathbb{R}$ be defined as*

$$q(k) := \mathbb{E}_{|X|=k}[p(X)]. \tag{20}$$

Then q can be written as a real polynomial in k of degree at most d.

We now introduce a different, lesser known notion of symmetrization, which we call the *erase-all-subscripts* symmetrization for reasons to be explained shortly. This symmetrization previously appeared in [Shi02] under the name "linearization," and it is also equivalent to the noise operator used in analysis of Boolean functions [O14, Definition 2.46].

Lemma 12 (Erase-all-subscripts symmetrization)**.** *Let $p : \{0,1\}^N \to \mathbb{R}$ be a real multilinear polynomial of degree d, and for any real number $k \in [0, 1]$, let M_k denote the distribution over $\{0,1\}^N$, wherein each coordinate is selected independently to be 1 with probability k. Let $q : [0, 1] \to \mathbb{R}$ be defined as*

$$q(k) := \mathbb{E}_{X \sim M_k}[p(X)]. \tag{21}$$

Then q can be written as a real polynomial in k of degree at most d.

Proof. (see, for example, [STT12, Proof of Theorem 3]). Given the multivariate polynomial expansion of p, we can obtain q easily just by "erasing all the subscripts in each variable". For example, if $p(x_1, x_2, x_3) = 2x_1x_2 + x_2x_3 + x_2$, we replace every x_i with k to obtain $q(k) = 2k \cdot k + k \cdot k + k = 3k^2 + k$. This follows from linearity of expectation along with the fact that M_k is defined to be the product distribution wherein each coordinate has expected value k. ■

We highlight the following key difference between Minsky–Papert symmetrization and the erase-all-subscripts symmetrization. Let $p : \{0,1\}^N \to [0, 1]$ be a real multivariate polynomial whose evaluations at Boolean inputs are in $[0, 1]$, i.e., for all $x \in \{0,1\}^n$, we have $p(x) \in [0, 1]$. If q is the erase-all-subscripts symmetrization of p, then q takes values in $[0, 1]$ at all *real-valued* inputs in $[0, 1]$: $q(k) \in [0, 1]$ for all $k \in [0, 1]$. If q is the Minsky–Papert symmetrization of p, then it is only guaranteed to take values in $[0, 1]$ at *integer-valued* inputs in $[0, N]$, i.e., $q(k) \in [0, 1]$ is only guaranteed to hold at $k \in \{0, 1, \ldots, N\}$. This is the main reason we use erase-all-subscripts symmetrization in this work.

Bivariate symmetrizations. In this paper, it will be convenient to consider bivariate versions of both Minsky–Papert and erase-all-subscripts symmetrization, and their applications to oracle separations. To this end, define $X \in \{0,1\}^N$, the "characteristic string" of the set $S \subseteq [N]$, by $x_i = 1$ if $i \in S$ and $x_i = 0$ otherwise. Let $\mathcal{O}_S$ denote the unitary that performs a membership query to S, defined as

$$\mathcal{O}_S |i\rangle |b\rangle = (1 - 2bx_i) |i\rangle |b\rangle \tag{22}$$

for any index $i \in [N]$ and bit $b \in \{0,1\}$.

Because we study oracle intersection problems, it is often convenient to think of an algorithm as having access to *two* oracles, wherein the first bit in the oracle register selects the choice of oracle. As a consequence, we need a slight generalization of a now well-established fact in quantum complexity: that the acceptance probability of a quantum algorithm with an oracle can be expressed as a polynomial in the bits of the oracle string.

Lemma 13 (Symmetrization with two oracles)**.** *Let $Q^{\mathcal{O}_{S_0},\mathcal{O}_{S_1}}$ be a quantum algorithm that makes T queries to a pair of membership oracles for sets $S_0, S_1 \subseteq [N]$. Let D_μ denote the distribution over subsets of $[N]$ wherein each element is selected independently with probability $\frac{\mu}{N}$. Then there exist bivariate real polynomials $q(s,t)$ and $p(x,y)$ of degree at most $2T$ satisfying:*

$$\textit{for all real numbers } s,t \in [0,N], \quad q(s,t) = \mathbb{E}_{\substack{S_0 \sim D_s, \\ S_1 \sim D_t}} \left[\Pr[Q^{\mathcal{O}_{S_0},\mathcal{O}_{S_1}} \textit{ accepts}]\right], \textit{ and}$$

$$\textit{for all integers } x,y \in \{0,1,\ldots,N\}, \quad p(x,y) = \mathbb{E}_{\substack{|S_0|=x, \\ |S_1|=y}} \left[\Pr[Q^{\mathcal{O}_{S_0},\mathcal{O}_{S_1}} \textit{ accepts}]\right].$$

Proof. Take $X = X_0|X_1$ to be the concatenation of the characteristic strings of the two oracles, and let $S \subseteq [2N]$ be such that X is the characteristic string of S. Then, Lemma 4.2 of Beals et al. [BBC+01] tells us that there is a real multilinear polynomial $r(X)$ of degree at most $2T$ in the bits of X such that $r(X) = \Pr[Q^{\mathcal{O}_S} \text{ accepts}]$.

Observe that r has a meaningful probabilistic interpretation over arbitrary inputs in $[0,1]$. A vector $X \in [0,1]^{2N}$ of probabilities corresponds to a distribution over $\{0,1\}^{2N}$ wherein each bit is chosen from a Bernoulli distribution with the corresponding probability. Because r is multilinear, r in fact computes the expectation of the acceptance probability over this distribution. In particular, the polynomial

$$q(s,t) = r\Bigg(\underbrace{\frac{s}{N},\ldots,\frac{s}{N}}_{N \text{ times}},\underbrace{\frac{t}{N},\ldots,\frac{t}{N}}_{N \text{ times}}\Bigg) = \mathbb{E}_{\substack{S_0 \sim D_s, \\ S_1 \sim D_t}} \left[\Pr[Q^{\mathcal{O}_{S_0},\mathcal{O}_{S_1}} \text{ accepts}]\right] \tag{23}$$

corresponds to selecting $S_0 \sim D_s$ and $S_1 \sim D_t$. The total degree of q is obviously at most the degree of r, by the same reasoning as in the proof of Lemma 12.

To construct p, we apply the symmetrization lemma of Minsky and Papert [MP88] to symmetrize r, first with respect to X_0, then with respect to X_1:

$$p_0(x, X_1) = \mathbb{E}_{|S_0|=x}\, r(X_0, X_1) = \mathbb{E}_{|S_0|=x} \left[\Pr[Q^{\mathcal{O}_{S_0},\mathcal{O}_{S_1}} \text{ accepts}]\right] \tag{24}$$

$$p(x,y) = \mathbb{E}_{|S_1|=y}\, p_0(x, X_1) = \mathbb{E}_{\substack{|S_0|=x, \\ |S_1|=y}} \left[\Pr[Q^{\mathcal{O}_{S_0},\mathcal{O}_{S_1}} \text{ accepts}]\right] \tag{25}$$

The degree of p is at most the degree of r, due to Lemma 11. ∎

We remark that, as a consequence of their definitions in Lemma 13, p and q satisfy:

$$q(s,t) = \mathbb{E}\left[p(X,Y)\right], \tag{26}$$

where X and Y are drawn from N-trial binomial distributions with means s and t, respectively.

Symmetric Laurent polynomials. Finally, we state a useful fact about Laurent polynomials:

Lemma 14 (Symmetric Laurent polynomials)**.** *Let $\ell(x)$ be a real Laurent polynomial of positive and negative degree d that satisfies $\ell(x) = \ell(1/x)$. Then there exists a (ordinary) real polynomial q of degree d such that $\ell(x) = q(x + 1/x)$.*

Proof. $\ell(x) = \ell(1/x)$ implies that the coefficients of the x^i and x^{-i} terms are equal for all i, as otherwise $\ell(x) - \ell(1/x)$ would not equal the zero polynomial. Thus, we may write $\ell(x) = \sum_{i=0}^{d} a_i \cdot (x^i + x^{-i})$ for some coefficients a_i. So, it suffices to show that $x^i + x^{-i}$ can be expressed as a polynomial in $x + 1/x$ for all $0 \leq i \leq d$.

We prove by induction on i. The case $i = 0$ corresponds to constant polynomials. For $i > 0$, by the binomial theorem, observe that $(x + 1/x)^i = x^i + x^{-i} + r(x)$ where r is a degree $i - 1$ real Laurent polynomial satisfying $r(x) = r(1/x)$. By the induction assumption, r can be expressed as a polynomial in $x + 1/x$, so we have $x^i + x^{-i} = (x + 1/x)^i - r(x)$ is expressed as a polynomial in $x + 1/x$. ■

2.3 Complexity classes

Definition 15. *The complexity class* QMA *consists of the languages L for which there exists a quantum polynomial time verifier V with the following properties:*

1. *Completeness: if $x \in L$, then there exists a quantum witness state $|\psi\rangle$ on $\mathrm{poly}(|x|)$ qubits such that $\Pr\left[V(x, |\psi\rangle) \text{ accepts}\right] \geq \frac{2}{3}$.*
2. *Soundness: if $x \notin L$, then for any quantum witness state $|\psi\rangle$ on $\mathrm{poly}(|x|)$ qubits, $\Pr\left[V(x, |\psi\rangle) \text{ accepts}\right] \leq \frac{1}{3}$.*

A quantum verifier that satisfies the above promise for a particular language will be referred to as a QMA verifier or QMA protocol throughout.

Though SBP and SBQP can be defined in terms of counting complexity functions, for our purposes it is easier to work with the following equivalent definitions (see Böhler et al. [BGM06]):

Definition 16. *The complexity class* SBP *consists of the languages L for which there exists a probabilistic polynomial time algorithm M and a polynomial σ with the following properties:*

1. *If $x \in L$, then $\Pr\left[M(x) \text{ accepts}\right] \geq 2^{-\sigma(|x|)}$.*
2. *If $x \notin L$, then $\Pr\left[M(x) \text{ accepts}\right] \leq 2^{-\sigma(|x|)}/2$.*

The complexity class SBQP *is defined analogously, wherein the classical algorithm is replaced with a quantum algorithm.*

A classical (respectively, quantum) algorithm that satisfies the above promise for a particular language will be referred to as an SBP (respectively, SBQP) algorithm throughout. Using these definitions, a query complexity relation between QMA protocols and SBQP algorithms follows from the procedure of Marriott and Watrous [MW05], which shows that one can exponentially improve the soundness and completeness errors of a QMA protocol without increasing the witness size. This relationship is now standard; see for example [MW05, Remark 6] or [ST19, Proposition 4.2] for a proof of the following lemma:

Lemma 17. *Suppose there is a* QMA *protocol for some problem that makes T queries and receives an m-qubit witness. Then there is a quantum query algorithm Q for the same problem that makes $O(mT)$ queries, and satisfies the following:*

1. *If $x \in L$, then $\Pr[Q(x) \text{ accepts}] \geq 2^{-m}$.*
2. *If $x \notin L$, then $\Pr[Q(x) \text{ accepts}] \leq 2^{-10m}$.*

3 QMA complexity of approximate counting

This section establishes an optimal lower bound on the QMA complexity of approximate counting. We first lower bound the SBQP complexity of the $\mathsf{AND}_2 \circ \mathsf{ApxCount}_{N,w}$ problem (Theorem 3). This implies a QMA lower bound for $\mathsf{ApxCount}_{N,w}$ via Lemma 17, but it is not quantitatively optimal. We prove the optimal QMA lower bound (Theorem 2) via Lemma 19, which leverages additional properties of the SBQP protocol derived via Lemma 17 from any QMA protocol with small witness length. Finally, Corollary 20 describes new oracle separations that are immediate consequences of Theorem 2 and Theorem 3.

3.1 Lower bound for SBQP algorithms

Our lower bound on the SBQP complexity of $\mathsf{AND}_2 \circ \mathsf{ApxCount}_{N,w}$ hinges on the following theorem. The theorem uses Laurent polynomials to prove a degree lower bound for bivariate polynomials that satisfy an upper bound on an "L"-shaped pair of rectangles and a lower bound at a nearby point:

Theorem 18. *Let $0 < w < 32w < N$ and $M \geq 1$. Let $R_1 = [4w, N] \times [0, w/2]$ and $R_2 = [0, w/2] \times [4w, N]$ be disjoint rectangles in the plane, and let $L = R_1 \cup R_2$. Let $p(x, y)$ be a real polynomial of degree d with the following properties:*

1. *$p(4w, 4w) \geq 1.5 \cdot M$.*
2. *$0 \leq p(x, y) \leq 1$ for all $(x, y) \in L$.*

Then $d = \Omega(\sqrt{N/w} \cdot \log M)$.

Proof. Observe that if $p(x, y)$ satisfies the statement of the theorem, then so does $p(y, x)$. This is because the constraints in the statement of the theorem are symmetric in x and y (in particular, because R_1 and R_2 are mirror images of one another along the line $x = y$; see Figure 3). As a result, we may assume without loss of generality that p is symmetric, i.e., $p(x, y) = p(y, x)$. Else, we may replace p by $\frac{p(x,y)+p(y,x)}{2}$ because the set of polynomials that satisfy the inequalities in the statement of the theorem are closed under convex combinations.

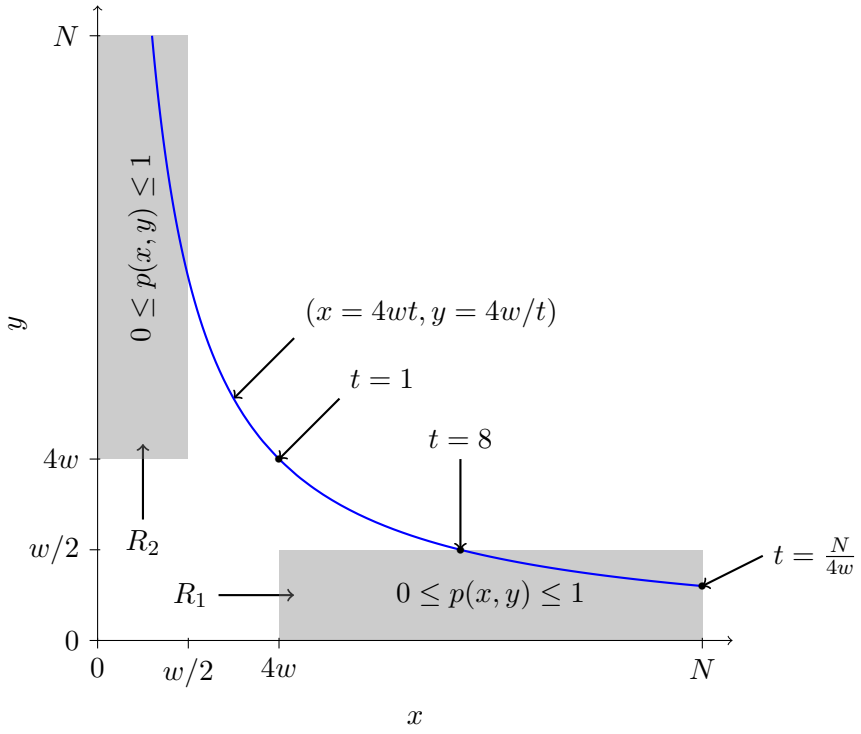

Figure 3: Diagram of Theorem 18 (not drawn to scale).

Consider the hyperbolic parametric curve $(x = 4wt, y = 4w/t)$ as it passes through R_1 (see Figure 3). We can view the restriction of $p(x, y)$ to this curve as a Laurent polynomial $\ell(t) = p(4wt, 4w/t)$ of positive and negative degree d. The bound of $p(x, y)$ on all of R_1 implies that $|\ell(t)| \le 1$ when $t \in [8, \frac{N}{4w}]$ and that $\ell(1) \ge 1.5$ (see Figure 3). Moreover, the condition that $p(x, y)$ is symmetric implies that $\ell(t) = \ell(1/t)$.

By Lemma 14 for symmetric Laurent polynomials, $\ell(t)$ can be viewed as a degree d polynomial $q(t + 1/t)$. Under the transformation $s = t + 1/t$, q satisfies $|q(s)| \le 1$ for $s \in [8 + 1/8, \frac{N}{4w} + \frac{4w}{N}]$ and $q(2) \ge 1.5M$. Note that the length of the interval $[8 + 1/8, \frac{N}{4w} + \frac{4w}{N}]$ is $\Theta(N/w)$ because $w < N$. By an appropriate affine transformation of q, we can conclude from Lemma 7 with $\mu = \Theta(w/N)$ that $d = \Omega(\sqrt{N/w} \cdot \log M)$. ■

Why is Theorem 18 useful? One may be tempted to apply this theorem directly to the polynomial $p(x, y)$ obtained in Lemma 13 to conclude a degree lower bound (and thus a query complexity lower bound), as the "L"-shaped pair of rectangles $L = R_1 \cup R_2$ correspond to "no" instances of $\mathsf{AND}_2 \circ \mathsf{ApxCount}_{N,w}$, while $(4w, 4w)$ corresponds to a "yes" instance. However, even though $p(x, y)$ is bounded at lattice points in L, it need not be bounded along the entirety of L.[11]

To obtain a lower bound, we instead use the connection between the polynomials $p(x, y)$ and

[11] One can nevertheless use this intuition to obtain a nontrivial (though suboptimal) lower bound by inspecting p alone. Using the Markov brothers' inequality (Lemma 6), if $\deg(p) = o(\sqrt{w})$, then the bounds on $p(x, y)$ at lattice points in L imply that $|p(x, y)| \le 1 + o_w(1)$ for all $(x, y) \in L$. Thus, Theorem 18 applies if $\deg(p) = o(\sqrt{w})$, so overall we get a lower bound of $\Omega\left(\min\left\{\sqrt{w}, \sqrt{N/w}\right\}\right)$ for the SBQP query complexity of $\mathsf{AND}_2 \circ \mathsf{ApxCount}_{N,w}$. See arXiv:1902.02398 for details.

$q(s,t)$ from Lemma 13, and establish Theorem 3 from the introduction, restated for convenience:

Theorem 3. *Consider an* SBQP *algorithm for* $\mathsf{AND}_2 \circ \mathsf{ApxCount}_{N,w}$ *that makes* T *queries to membership oracles for the two instances of* $\mathsf{ApxCount}_{N,w}$. *Then* $T = \Omega\left(\min\{w, \sqrt{N/w}\}\right)$.

Proof. Let $N > 32w$ (otherwise the theorem holds trivially). Since Q is an SBQP algorithm, we may suppose that Q accepts with probability at least 2α on a "yes" instance and with probability at most α on a "no" instance (note that α may be exponentially small in N). Take $p(x,y)$ and $q(s,t)$ to be the symmetrized bivariate polynomials of degree at most $2T$ defined in Lemma 13. Define $L' = ([0,w] \times [0,w]) \cup ([0,w] \times [2w,N]) \cup ([2w,N] \times [0,w])$. The conditions on the acceptance probability of Q for all S_0, S_1 that satisfy the $\mathsf{ApxCount}_{N,w}$ promise imply that $p(x,y)$ satisfies these corresponding conditions:

1. $1 \geq p(x,y) \geq 2\alpha$ for all $(x,y) \in ([2w,N] \times [2w,N]) \cap \mathbb{Z}^2$.
2. $0 \leq p(x,y) \leq \alpha$ for all $(x,y) \in L' \cap \mathbb{Z}^2$.

Our strategy is to show that if $T = o(w)$, then these conditions on p imply that the polynomial $q(s,t) \cdot \frac{0.9}{\alpha}$ satisfies the statement of Theorem 18 for all sufficiently large w. This in turn implies $T = \Omega(\sqrt{N/w})$. This allows us conclude that either $T = \Omega(w)$ or $T = \Omega(\sqrt{N/w})$, which proves the theorem.

Suppose $T = o(w)$, so that $p(x,y)$ and $q(s,t)$ both have degree $d = o(w)$. We begin by upper bounding $p(x,y)$ at the lattice points (x,y) outside of L'. We claim the following:

(a) $|p(x,y)| \leq \alpha \cdot a \cdot \exp(bd^2/w) \leq \alpha \cdot a \cdot \exp(bd)$ whenever $(x,y) \in L'$ and either x or y is an integer, where a and b are the constants from Lemma 10. This follows from Lemma 10 by fixing either x or y to be an integer and viewing the resulting restriction of $p(x,y)$ as a univariate polynomial in the other variable.

(b) $|p(x,y)| \leq \alpha \cdot a \cdot \exp(bd) \cdot \exp(2\sqrt{3}d) = \alpha \cdot a \cdot \exp((b+2\sqrt{3})d)$ whenever $x \in [w,2w]$, $y \in [0,w]$, and y is an integer. This follows Lemma 7: consider the univariate polynomial $p(\cdot,y)$ on the intervals $[0,w]$ and $[2w,3w]$, where it is bounded by (a).

(c) $|p(x,y)| \leq \alpha \cdot a \cdot \exp((b+2\sqrt{3})d) \cdot a \cdot \exp(bd^2/w) \leq \alpha \cdot a^2 \cdot \exp((2b+2\sqrt{3})d)$ whenever $x \in [w,2w]$ and $y \in [0,w]$. This follows from Lemma 10: consider the univariate polynomial $p(x,\cdot)$ on the interval $[0,w]$, where it is bounded at integer points by (b).

(d) $|p(x,y)| \leq \alpha \cdot a^2 \cdot \exp((2b+2\sqrt{3})d) \cdot \exp(4dy/w) = \alpha \cdot a^2 \cdot \exp((2b+2\sqrt{3}+4y/w)d)$ whenever $x \in [0,N]$, $y \in [w+1,N]$, and x is an integer. This follows from Lemma 7: consider the univariate polynomial $p(x,\cdot)$ on the interval $[0,w]$, where it is bounded by (a) when $x \in [0,w]$ or $x \in [2w,N]$, or bounded by (c) when $x \in [w,2w]$. By an affine shift, this corresponds to applying Lemma 7 with $\mu = 2y/w - 2$, with the observation that $\sqrt{2\mu+\mu^2} < \mu + 2$.

We now use this to upper bound $q(s,t)$ when $s \in [4w,N]$ and $t \in [0,w/2]$. Let X and Y be drawn from N-trial binomial distributions with means s and t, respectively, so that $q(s,t) = \mathbb{E}[p(X,Y)]$. Using the above bounds and basic probability, we have

$$0 \leq q(s,t) = \mathbb{E}[p(X,Y)] \leq \alpha \cdot \left(\Pr[X \geq 2w, Y \leq w] + \Pr[X \leq 2w, Y \leq w] \cdot a \cdot \exp\left(\left(b + 2\sqrt{3}\right)d\right)\right.$$

$$+ \sum_{y=w+1}^{N} \Pr[Y = y] \cdot a^2 \cdot \exp\left(\left(2b + 2\sqrt{3} + 4y/w\right) d\right)\Bigg) \tag{27}$$

$$\leq \alpha \cdot \Bigg(1 + \Pr[X \leq 2w] \cdot a \cdot \exp\left(\left(b + 2\sqrt{3}\right) d\right)$$

$$+ \sum_{y=w+1}^{N} \Pr[Y \geq y] \cdot a^2 \cdot \exp\left(\left(2b + 2\sqrt{3} + 4y/w\right) d\right)\Bigg). \tag{28}$$

The probabilities above are easily bounded with a Chernoff bound:

$$q(s,t) = \mathbb{E}[p(X,Y)] \leq \alpha \cdot \Bigg(1 + a \cdot \exp\left(\left(b + 2\sqrt{3}\right) d - w/2\right)$$

$$+ \sum_{y=w+1}^{N} a^2 \cdot \exp\left(\left(2b + 2\sqrt{3} + 4y/w\right) d - y/6\right)\Bigg). \tag{29}$$

Because a and b are universal constants from Lemma 10, when $d = o(w)$, the first exponential term becomes arbitrarily small for all sufficiently large w. Moreover, for all sufficiently large w, the remaining sum becomes bounded by a geometric sum. For some constant c, we have

$$\begin{aligned}\sum_{y=w+1}^{N} a^2 \cdot \exp\left(\left(2b + 2\sqrt{3} + 4y/w\right) d - y/6\right) &\leq \sum_{y=w+1}^{\infty} c \cdot \exp\left(-y/12\right) \\ &\leq \frac{c}{1 - \exp(-1/12)} \cdot \exp(-w/12) \\ &= o_w(1).\end{aligned}$$

Thus we conclude that $0 \leq q(s,t) \leq \alpha \cdot (1 + o_w(1))$ when $s \in [4w, N]$ and $t \in [0, w/2]$ (i.e., $(s,t) \in R_1$ in the statement of Theorem 18). By symmetry, we can conclude the same bound when $s \in [0, w/2]$ and $t \in [4w, N]$ (i.e., $(s,t) \in R_2$ in the statement of Theorem 18).

Now, we lower bound $q(4w, 4w)$. Let X and Y be drawn from independent N-trial binomial distributions with mean $4w$, so that $q(4w, 4w) = \mathbb{E}\left[p(X,Y)\right]$. Then we have

$$\begin{aligned}\mathbb{E}\left[p(X,Y)\right] &\geq 2\alpha \cdot \Pr[X \geq 2w, Y \geq 2w] \\ &\geq 2\alpha \cdot (1 - \Pr[X \leq 2w] - \Pr[Y \leq 2w]) \\ &\geq 2\alpha \cdot (1 - 2\exp(-w/2)) \\ &\geq 2\alpha \cdot (1 - o_w(1))\end{aligned}$$

We conclude that $q(s,t) \cdot \frac{0.9}{\alpha}$ satisfies the statement of Theorem 18 (with $M = 1$) for all sufficiently large w. ■

We remark that this lower bound is tight, i.e., there exists an SBQP algorithm that makes $O\left(\min\left\{w, \sqrt{N/w}\right\}\right)$ queries. The $O(\sqrt{N/w})$ upper bound follows from the BQP algorithm of Brassard, Høyer, and Tapp [BHT98a]. The $O(w)$ upper bound is in fact an SBP upper bound with the following algorithmic interpretation: first, guess $w+1$ items randomly from each of S_0 and S_1. Then, verify using the membership oracle that the first $w+1$ items all belong to S_0 and that the latter $w+1$ items all belong to S_1, accepting if and only if this is the case. Clearly, this accepts with nonzero probability if and only if $|S_0| \geq w+1$ and $|S_1| \geq w+1$.

3.2 Lower bound for QMA

In this section, we establish the optimal QMA lower bound (Theorem 2). We begin by quantitatively improving the SBQP lower bound for $\mathsf{AND}_2 \circ \mathsf{ApxCount}_{N,w}$ of Theorem 3, under the stronger assumption that the parameter α in the SBQP protocol is not smaller than 2^{-w}. (In addition to a stronger conclusion, this assumption also permits a considerably simpler analysis than was required to prove Theorem 3).

Lemma 19. *Consider any quantum query algorithm $Q^{\mathcal{O}_{S_0},\mathcal{O}_{S_1}}$ for $\mathsf{AND}_2 \circ \mathsf{ApxCount}_{N,w}$ that makes T queries to the membership oracles $\mathcal{O}_{S_0}$ and $\mathcal{O}_{S_1}$ for the two instances of $\mathsf{ApxCount}_{N,w}$ and satisfies the following. For some $m = o(w)$, $\alpha = 2^{-m}$, and $M \in [1, \alpha^{-1}]$:*

1. *If $x \in L$, then $\Pr[Q(x) \text{ accepts}] \geq \alpha$.*
2. *If $x \notin L$, then $\Pr[Q(x) \text{ accepts}] \leq \alpha/(2M)$.*

Then $T = \Omega\left(\sqrt{N/w} \cdot \log M\right)$

Proof. As in the proof of Theorem 3, define $L' = ([0,w]\times[0,w]) \cup ([0,w]\times[2w,N]) \cup ([2w,N]\times[0,w])$, and take $p(x,y)$ and $q(s,t)$ to be the symmetrized bivariate polynomials of degree at most $2T$ defined in Lemma 13. $p(x,y)$ satisfies the following properties.

(a) $1 \geq p(x,y) \geq \alpha$ for all $(x,y) \in ([2w,N]\times[2w,N]) \cap \mathbb{Z}^2$.

(b) $0 \leq p(x,y) \leq \alpha/(1.5M)$ for all $(x,y) \in L' \cap \mathbb{Z}^2$.

(c) $0 \leq p(x,y) \leq 1$ for all $(x,y) \in ([0,N]\times[0,N]) \cap \mathbb{Z}^2$.

We use these properties to upper bound $q(s,t)$ when $s \in [4w,N]$ and $t \in [0,w/2]$. Let X and Y be drawn from N-trial binomial distributions with means s and t, respectively, so that $q(s,t) = \mathbb{E}[p(X,Y)]$. Using the above bounds and basic probability, we have

$$\begin{aligned} 0 \leq q(s,t) = \mathbb{E}[p(X,Y)] &\leq \alpha/(2M)\Pr[X \geq 2w, Y \leq w] + (1 - \Pr[X \geq 2w, Y \leq w]) \\ &\leq \alpha/(2M) + 2^{-\Omega(w)} \leq (1+o(1))\alpha/(2M) \end{aligned}$$

Here, the first inequality holds by Properties (a)-(c) above, while the second follows from a Chernoff Bound, and the third holds because $\alpha/(2M) \geq 2^{-o(w)}$.

Thus we conclude that $0 \leq q(s,t) \leq \alpha/(2M) \cdot (1 + o_w(1))$ when $s \in [4w,N]$ and $t \in [0,w/2]$ (i.e., $(s,t) \in R_1$ in the statement of Theorem 18). By symmetry, we can conclude the same bound when $s \in [0,w/2]$ and $t \in [4w,N]$ (i.e., $(s,t) \in R_2$ in the statement of Theorem 18).

Now, we lower bound $q(4w,4w)$. Let X and Y be drawn from independent N-trial binomial distributions with mean $4w$, so that $q(4w,4w) = \mathbb{E}[p(X,Y)]$. Then we have

$$\begin{aligned} \mathbb{E}[p(X,Y)] &\geq \alpha \cdot \Pr[X \geq 2w, Y \geq 2w] \\ &\geq \alpha \cdot (1 - \Pr[X \leq 2w] - \Pr[Y \leq 2w]) \\ &\geq \alpha \cdot (1 - 2\exp(-w/2)) \\ &\geq \alpha \cdot (1 - o_w(1)) \end{aligned}$$

We conclude that $q(s,t) \cdot \frac{1.8M}{\alpha}$ satisfies the statement of Theorem 18 for all sufficiently large w. Hence, $T = \Omega\left(\sqrt{N/w} \cdot \log M\right)$ as claimed. ■

We now establish Theorem 2 from the introduction, which quantitatively lower bounds the QMA complexity of $\mathsf{ApxCount}_{N,w}$. The analysis exploits two key properties of the SBQP protocols that result from applying Lemma 17 to a QMA protocol with witness length m: (1) the parameter α of the SBQP protocol is not too small (at least 2^{-m}) and (2) the multiplicative gap between acceptance probabilities when $f(x) = 0$ vs. $f(x) = 1$ is at least 2^m, which may be much greater than 2.

Theorem 2. *Consider a* QMA *protocol that solves* $\mathsf{ApxCount}_{N,w}$. *If the protocol receives a quantum witness of length m, and makes T queries to the membership oracle for S, then either $m = \Omega(w)$ or $T = \Omega\left(\sqrt{N/w}\right)$.*

Proof. Consider a QMA protocol for $\mathsf{ApxCount}_{N,w}$ with witness size m and query cost T. If $m = \Omega(w)$, the theorem is vacuous, so suppose that $m = o(w)$. Running the verifier, Arthur, a constant number of times with fresh witnesses to reduce the soundness and completeness errors, one obtains a verifier with soundness and completeness errors $1/6$ that receives an $O(m)$-length witness and makes $O(T)$ queries. Repeating twice with two oracles and computing the AND, one obtains a QMA verifier $V'^{\mathcal{O}_{S_0},\mathcal{O}_{S_1}}$ for $\mathsf{AND}_2 \circ \mathsf{ApxCount}_{N,w}$ with soundness and completeness errors $1/3$ that receives an $O(m)$-length witness and makes $O(T)$ queries. Applying Lemma 17 to V', there exists a quantum query algorithm $Q^{\mathcal{O}_{S_0},\mathcal{O}_{S_1}}$ for $\mathsf{AND}_2 \circ \mathsf{ApxCount}_{N,w}$ that makes $O(m \cdot T)$ queries and satisfies the hypothesis of Lemma 19 with $M = 2^{-\Theta(m)}$. Theorem 3 tells us that $m \cdot T = \Omega\left(\sqrt{N/w} \cdot m\right)$. Equivalently, $T = \Omega\left(\sqrt{N/w}\right)$. ■

Theorem 3 also implies several oracle separations:

Corollary 20. *There exists an oracle A and a pair of languages L_0, L_1 such that:*

1. $L_0, L_1 \in \mathsf{SBP}^A$
2. $L_0 \cap L_1 \notin \mathsf{SBQP}^A$.
3. $\mathsf{SBP}^A \not\subset \mathsf{QMA}^A$.

Proof. For an arbitrary function $A : \{0,1\}^* \to \{0,1\}$ and $i \in \{0,1\}$, define $A_i^n = \{x \in \{0,1\}^n : A(i,x) = 1\}$. Define the unary language $L_i^A = \{1^n : |A_i^n| \geq 2^{n/2}\}$. Observe that as long as A satisfies the promise $|A_i^n| \geq 2^{n/2}$ or $|A_i^n| \leq 2^{n/2-1}$ for all $n \in \mathbb{N}$, then $L_i^A \in \mathsf{SBP}^A$. Intuitively, the oracles A that satisfy this promise encode a pair of $\mathsf{ApxCount}_{N,w}$ instances $|A_0^n|$ and $|A_1^n|$ for every $n \in \mathbb{N}$ where $N = 2^n$ and $w = 2^{n/2-1}$.

Theorem 3 tells us that an SBQP algorithm Q that makes $o(2^{n/4})$ queries fails to solve $\mathsf{AND}_2 \circ \mathsf{ApxCount}_{N,w}$ on *some* pair (S_0, S_1) that satisfies the promise. Thus, one can construct an A such that $L_0, L_1 \in \mathsf{SBP}^A$ and $L_0 \cap L_1 \notin \mathsf{SBQP}^A$, by choosing (A_0^n, A_1^n) so as to diagonalize against all SBQP algorithms.

Because QMA^A is closed under intersection for any oracle A, and because $\mathsf{QMA}^A \subseteq \mathsf{SBQP}^A$ for any oracle A, it must be the case that either $L_0 \notin \mathsf{QMA}^A$ or $L_1 \notin \mathsf{QMA}^A$. ■

4 Approximate counting with quantum samples and reflections

4.1 The Laurent polynomial method

By using Minsky–Papert symmetrization (Lemma 11), we now prove the key fact that relates quantum algorithms, of the type we're considering, to real Laurent polynomials in one variable.

The following lemma generalizes the connection between quantum algorithms and real polynomials established by Beals et al. [BBC+01].

Lemma 21. *Let Q be a quantum algorithm that makes T queries to $\mathcal{O}_S$, uses R_1 copies of $|S\rangle$, and makes R_2 uses of the unitary $\mathcal{R}_S$. Let $R := R_1 + 2R_2$. For $k \in \{1, \ldots, N\}$, let*

$$q(k) := \mathbb{E}_{|S|=k}\left[\Pr\left[Q^{\mathcal{O}_S,\mathcal{R}_S}\left(|S\rangle^{\otimes R_1}\right) \text{ accepts}\right]\right]. \tag{30}$$

Then q can be written a univariate Laurent polynomial, with maximum exponent at most $2T + R$ and minimum exponent at least $-R$.

Proof. Let $|\psi_{\text{initial}}\rangle$ denote the initial state of the algorithm, which we can write as

$$|\psi_{\text{initial}}\rangle = |S\rangle^{\otimes R_1} = \left(\frac{1}{\sqrt{|S|}}\sum_{i\in S}|i\rangle\right)^{\otimes R_1} = \left(\frac{1}{\sqrt{|S|}}\sum_{i\in[N]}x_i|i\rangle\right)^{\otimes R_1}$$
$$= \frac{1}{|S|^{R_1/2}}\sum_{i_1,\ldots,i_{R_1}\in[N]} x_{i_1}\cdots x_{i_{R_1}}|i_1,\ldots,i_{R_1}\rangle.$$

Thus, each amplitude is a complex multilinear polynomial in $X = (x_1, \ldots, x_N)$ of degree R_1, divided by $|S|^{R_1/2}$.

Throughout the algorithm, each amplitude will remain a complex multilinear polynomial in X divided by some power of $|S|$. Since $x_i^2 = x_i$ for all i, we can always maintain multilinearity without loss of generality.

Like Beals et al. [BBC+01], we now consider how the polynomial degree of each amplitude and the power of $|S|$ in the denominator change as the algorithm progresses. We have to handle 3 different kinds of unitaries that the quantum circuit may use: the membership query oracle $\mathcal{O}_S$, unitaries independent of the input, and the reflection unitary $\mathcal{R}_S$.

The first two cases are handled as in Beals et al. Since $\mathcal{O}_S$ is a unitary whose entries are degree-1 polynomials in X, each use of this unitary increases a particular amplitude's degree as a polynomial by 1 and does not change the power of $|S|$ in the denominator. Second, input-independent unitary transformations only take linear combinations of existing polynomials and hence do not increase the degree of the amplitudes or the power of $|S|$ in the denominator. Finally, we consider the reflection unitary $\mathcal{R}_S = \mathbb{1} - 2|S\rangle\langle S|$. The $(i,j)^{\text{th}}$ entry of this operator is $\delta_{ij} - \frac{2x_ix_j}{|S|} = \frac{\delta_{ij}|S| - 2x_ix_j}{|S|}$, where δ_{ij} is the Kronecker delta function. Since $|S| = \sum_i x_i$, this is a degree-2 polynomial divided by $|S|$. Hence applying this unitary will increase the degree of the amplitudes by 2 and increase the power of $|S|$ in the denominator by 1.

In conclusion, we start with each amplitude being a polynomial of degree R_1 divided by $|S|^{R_1/2}$. T queries to the membership oracle will increase the degree of each amplitude by at most T and leave the power of $|S|$ in the denominator unchanged. R_2 uses of the reflection unitary will increase the degree by at most $2R_2$ and the power of $|S|$ in the denominator by R_2. It follows that Q's final state has the form

$$|\psi_{\text{final}}\rangle = \sum_z \alpha_z(X)|z\rangle, \tag{31}$$

where each $\alpha_z(X)$ is a complex multilinear polynomial in X of degree at most $R_1 + 2R_2 + T = R + T$, divided by $|S|^{R_1/2+R_2} = |S|^{R/2}$. Since X itself is real-valued, it follows that the real and imaginary

parts of $\alpha_z(X)$, considered individually, are real multilinear polynomials in X of degree at most $R+T$ divided by $|S|^{R/2}$.

Hence, if we let

$$p(X) := \Pr\left[Q^{\mathcal{O}_S,\mathcal{R}_S}\left(|S\rangle^{\otimes R_1}\right) \text{ accepts}\right], \tag{32}$$

then

$$p(X) = \sum_{\text{accepting } z} |\alpha_z(X)|^2 = \sum_{\text{accepting } z} \left(\mathrm{Re}^2\,\alpha_z(X) + \mathrm{Im}^2\,\alpha_z(X)\right) \tag{33}$$

is a real multilinear polynomial in X of degree at most $2(R+T)$, divided through (in every monomial) by $|S|^R = |X|^R$.

Now consider

$$q(k) := \mathbb{E}_{|X|=k}[p(X)]. \tag{34}$$

By Lemma 11, this is a real univariate polynomial in $|X|$ of degree at most $2(R+T)$, divided through (in every monomial) by $|S|^R = |X|^R$. Or said another way, it's a real Laurent polynomial in $|X|$, with maximum exponent at most $R+2T$ and minimum exponent at least $-R$. ■

4.2 Upper bounds

Before proving our lower bounds on the degree of Laurent polynomials approximating $\mathsf{ApxCount}_{N,w}$, we establish some simpler *upper bounds*. We show upper bounds on Laurent polynomial degree and in the queries, samples, and reflections model.

Laurent polynomial degree of approximate counting. We now describe a *purely negative* degree Laurent polynomial of degree $O(w^{1/3})$ for approximate counting. This upper bound will serve as an important source of intuition when we prove the (matching) lower bound of Theorem 4 (see Section 4.4.3). We are thankful to user "fedja" on MathOverflow for describing this construction.[12]

Lemma 22 (fedja)**.** *For all w, there is a real polynomial p of degree $O\left(w^{1/3}\right)$ such that:*

1. $0 \leq p(1/k) \leq \frac{1}{3}$ *for all* $k \in [w]$.
2. $\frac{2}{3} \leq p(1/k) \leq 1$ *for all integers* $k \geq 2w$.
3. $0 \leq p(1/k) \leq 1$ *for all* $k \in \{w+1, w+2, \ldots, 2w-1\}$.

Proof. Assuming for simplicity that w is a perfect cube, consider

$$u(x) := (1-x)(1-2x)\cdots\left(1-w^{1/3}x\right). \tag{35}$$

Notice that $\deg(u) = w^{1/3}$ and $u\left(\frac{1}{k}\right) = 0$ for all $k \in \left[w^{1/3}\right]$. Furthermore, we have $u(x) \in [0,1]$ for all $x \in \left[0, \frac{1}{w^{1/3}}\right]$, and also $u(x) \in \left[1 - O\left(\frac{1}{w^{1/3}}\right), 1\right]$ for all $x \in \left[0, \frac{1}{w}\right]$. Now, let v be the Chebyshev polynomial of degree $w^{1/3}$, affinely adjusted so that $v(x) \in [0,1]$ for all $x \in \left[0, \frac{1}{w^{1/3}}\right]$ (rather than in $[-1,1]$ for all all $|x| \leq 1$), and with a large jump between $\frac{1}{2w}$ and $\frac{1}{w}$. Then the product, $p(x) := u(x)\,v(x)$, has degree $2w^{1/3}$ and satisfies all the requirements, except possibly that the constants $\frac{1}{3}$ and $\frac{2}{3}$ in the first two requirements may be off. Composing with a constant degree polynomial corrects this, and gives a polynomial of degree $O(w^{1/3})$ that satisfies all three requirements. ■

[12] See https://mathoverflow.net/questions/302113/real-polynomial-bounded-at-inverse-integer-points

Interestingly, if we restrict our attention to purely negative degree Laurent polynomials, then a matching lower bound is not too hard to show. In the same MathOverflow post, user fedja also proves the following, which can also be shown using earlier work of Zhandry [Zha12, Proof of Theorem 7.3]):

Lemma 23. *Let p be a real polynomial, and suppose that $|p(1/k)| \leq 1$ for all $k \in [2w]$, and that $p\left(\frac{1}{w}\right) \leq \frac{1}{3}$ while $p\left(\frac{1}{2w}\right) \geq \frac{2}{3}$. Then $\deg(p) = \Omega\left(w^{1/3}\right)$.*

Section 4.3 and Section 4.4 below take the considerable step of extending Lemma 23 from purely negative degree Laurent polynomials to general Laurent polynomials.

Upper bounds in the queries, samples, and reflections model. Although we showed that there is a purely negative degree Laurent polynomial of degree $O(w^{1/3})$ for $\mathsf{ApxCount}_{N,w}$, this does not imply the existence of a quantum algorithm in the queries, samples, and reflections model with similar complexity.

We now show that our lower bounds in the queries, samples, and reflections model (in Theorem 4) are tight (up to constants). This is Theorem 5 in the introduction, restated here for convenience:

Theorem 5. *There is a quantum algorithm that solves $\mathsf{ApxCount}_{N,w}$ with high probability using R copies of $|S\rangle$ and reflections about $|S\rangle$, where $R = O\left(\min\left\{w^{1/3}, \sqrt{\frac{N}{w}}\right\}\right)$.*

Proof. We describe two quantum algorithms for this problem with the two stated complexities.

The first algorithm uses $O(w^{1/3})$ samples and reflections. This algorithm is reminiscent of the original collision finding algorithm of Brassard, Høyer, and Tapp [BHT98b]. We first use $O(w^{1/3})$ copies of $|S\rangle$ to learn a set $M \subset S$ of size $w^{1/3}$ by simply measuring copies of $|S\rangle$ in the computational basis. Now we know that the ratio $|S|/|M|$ is either $w^{2/3}$ or $2w^{2/3}$. Now consider running Grover's algorithm on the set S where the elements in M are considered the "marked" elements. Grover's algorithm alternates reflections about the uniform superposition over the set being searched, S, with an operator that reflects about the marked elements in M. The first reflection is simply $\mathcal{R}_S$, which we have access to. The second unitary can be constructed since we have an explicit description of the set M. Now Grover's algorithm can be used to distinguish whether the fraction of marked elements is $1/w^{2/3}$ or half of that, and the cost will be $O(w^{1/3})$.

The second algorithm uses $O(\sqrt{N/w})$ reflections only and no copies of $|S\rangle$. Consider running the standard approximate counting algorithm [BHMT02] that uses membership queries to S and distinguishes $|S| \leq w$ from $|S| \geq 2w$ using $O(\sqrt{N/w})$ membership queries. Observe that this algorithm starts with the state $|\psi\rangle = \frac{1}{\sqrt{N}}(|1\rangle + \cdots + |N\rangle)$, which is in $\mathrm{span}\{|S\rangle, |\bar{S}\rangle\}$, and only uses reflections about $|\psi\rangle$ and membership queries to $|S\rangle$ in the form of a unitary that maps $|i\rangle$ to $-|i\rangle$ when $i \in S$. This means the state of the algorithm remains in $\mathrm{span}\{|S\rangle, |\bar{S}\rangle\}$ at all times. Within this subspace, a membership query to S is the same as a reflection about $|S\rangle$. Hence we can replace membership queries with the reflection operator to get an approximate counting algorithm that only uses $O(\sqrt{N/w})$ reflections and no copies of $|S\rangle$. ∎

Note that both the algorithms presented above generalize to the situation where we want to distinguish $|S| = w$ from $|S| = (1+\varepsilon)w$. For the first algorithm, we now pick a subset M of size $w^{1/3}/\varepsilon^{2/3}$. Now we want to $(1+\varepsilon)$-approximate the fraction of marked elements, which is either $1/(w\varepsilon)^{2/3}$ or $(1+\varepsilon)^{-1}$ times that. This can be done with approximate counting [BHMT02, Theorem

15], and the cost will be $O\left(\frac{1}{\varepsilon}(w\varepsilon)^{1/3}\right) = O\left(\frac{w^{1/3}}{\varepsilon^{2/3}}\right)$. The second algorithm is simpler to generalize, since we simply plug in the query complexity of ε-approximate counting, which is $O\left(\frac{1}{\varepsilon}\sqrt{\frac{N}{w}}\right)$.

4.3 Lower bound using the explosion argument

We now show a weaker version of Theorem 4 using the explosion argument described in the introduction. The difference between the following theorem and Theorem 4 is the exponent of w in the lower bound.

Theorem 24. *Let Q be a quantum algorithm that makes T queries to the membership oracle for S, and uses a total of R copies of $|S\rangle$ and reflections about $|S\rangle$. If Q decides whether $|S| = w$ or $|S| = 2w$ with success probability at least $2/3$, promised that one of those is the case, then either*

$$T = \Omega\left(\sqrt{\frac{N}{w}}\right) \qquad or \qquad R = \Omega\left(\min\left\{w^{1/4}, \sqrt{\frac{N}{w}}\right\}\right). \tag{36}$$

Proof. Since we neglect multiplicative constants in our lower bounds, let us allow the algorithm to use up to R copies of $|S\rangle$ and R uses of $\mathcal{R}_S$. Let

$$q(k) := \mathbb{E}_{|S|=k}\left[\Pr\left[Q^{\mathcal{O}_S,\mathcal{R}_S}\left(|S\rangle^{\otimes R}\right) \text{ accepts}\right]\right]. \tag{37}$$

Then by Lemma 21, we can write q as a Laurent polynomial:

$$q(k) = u(k) + v(1/k), \tag{38}$$

where u is a real polynomial in k with $\deg(u) = O(T+R)$, and v is a real polynomial in $1/k$ with $\deg(v) = O(R)$. So to prove the theorem, it suffices to show that either $\deg(u) = \Omega\left(\sqrt{\frac{N}{w}}\right)$, or else $\deg(v) = \Omega\left(w^{1/4}\right)$. To do so, we'll assume that $\deg(u) = o\left(\sqrt{\frac{N}{w}}\right)$ and $\deg(v) = o\left(w^{1/4}\right)$, and derive a contradiction.

Our high-level strategy is as follows: we'll observe that, if approximate counting is being successfully solved, then either u or v must attain a large first derivative somewhere in its domain. By the approximation theory lemmas that we proved in Section 2.1, this will force that polynomial to have a large range—even on a subset of integer (or inverse-integer) points. But the sum, $u(k) + v(1/k)$, is bounded in $[0,1]$ for all $k \in [N]$. So if one polynomial has a large range, then the other does too. But this forces the *other* polynomial to have a large derivative somewhere in its domain, and therefore (by approximation theory) to have an even larger range, forcing the first polynomial to have an even larger range to compensate, and so on. As long as $\deg(u)$ and $\deg(v)$ are both small enough, this endless switching will force both u and v to attain *unboundedly* large values—with the fact that one polynomial is in k, and the other is in $1/k$, crucial to achieving the desired "explosion." Since u and v are polynomials on compact sets, such unbounded growth is an obvious absurdity, and this will give us the desired contradiction.

In more detail, we will study the following quantities.

$$\begin{aligned}
G_u &:= \max_{x,y\in\left[\sqrt{w},2w\right]} |u(x)-u(y)| & G_v &:= \max_{x,y\in\left[\frac{1}{N},\frac{1}{w}\right]} |v(x)-v(y)| \\
\Delta_u &:= \max_{x\in\left[\sqrt{w},2w\right]} |u'(x)| & \Delta_v &:= \max_{x\in\left[\frac{1}{N},\frac{1}{w}\right]} |v'(x)| \\
H_u &:= \max_{x,y\in\left[\sqrt{w},N\right]} |u(x)-u(y)| & H_v &:= \max_{x,y\in\left[\frac{1}{N},\frac{1}{\sqrt{w}}\right]} |v(x)-v(y)| \\
I_u &:= \max_{x,y\in[w,N]} |u(x)-u(y)| & I_v &:= \max_{x,y\in\left[\frac{1}{2w},\frac{1}{\sqrt{w}}\right]} |v(x)-v(y)| \\
L_u &:= \max_{x,y\in\{w,\ldots,N\}} |u(x)-u(y)| & L_v &:= \max_{x,y\in\left\{\sqrt{w},\ldots,2w\right\}} \left|v\left(\tfrac{1}{x}\right)-v\left(\tfrac{1}{y}\right)\right|
\end{aligned} \tag{39}$$

We have $0 \le q(k) \le 1$ for all $k \in [N]$, since in those cases $q(k)$ represents a probability. Since Q solves approximate counting, we also have $q(w) \le \frac{1}{3}$ and $q(2w) \ge \frac{2}{3}$. This means in particular that either

(i) $u(2w) - u(w) \ge \frac{1}{6}$, and hence $G_u \ge \frac{1}{6}$, or else

(ii) $v\left(\frac{1}{2w}\right) - v\left(\frac{1}{w}\right) \ge \frac{1}{6}$, and hence $G_v \ge \frac{1}{6}$.

We will show that either case leads to a contradiction.

We have the following inequalities regarding u:

$$\begin{aligned}
G_u &\ge L_v - 1 && \text{by the boundedness of } q \\
\Delta_u &\ge \tfrac{G_u}{2w} && \text{by basic calculus} \\
H_u &\ge \tfrac{\Delta_u\left(N-\sqrt{w}\right)}{\deg(u)^2} && \text{by Lemma 6} \\
I_u &\ge \tfrac{H_u}{2} && \text{by Corollary 8} \\
L_u &\ge \tfrac{I_u}{2} && \text{by Lemma 9}
\end{aligned} \tag{40}$$

Here the fourth inequality uses the fact that, setting $\varepsilon := \frac{\sqrt{w}}{N}$, we have $\deg(u) = o\left(\frac{1}{\sqrt{\varepsilon}}\right)$ (thereby satisfying the hypothesis of Corollary 8), while the fifth inequality uses the fact that $\deg(u) = o\left(\sqrt{N}\right)$.

Meanwhile, we have the following inequalities regarding v:

$$\begin{aligned}
G_v &\ge L_u - 1 && \text{by the boundedness of } q \\
\Delta_v &\ge G_v w && \text{by basic calculus} \\
H_v &\ge \tfrac{\Delta_v\left(\frac{1}{\sqrt{w}}-\frac{1}{N}\right)}{\deg(v)^2} && \text{by Lemma 6} \\
I_v &\ge \tfrac{H_v}{2} && \text{by Corollary 8} \\
L_v &\ge \tfrac{I_v}{2} && \text{by Lemma 9}
\end{aligned} \tag{41}$$

Here the fourth inequality uses the fact that, setting $\varepsilon := \frac{1/2w}{1/\sqrt{w}} = \frac{1}{2\sqrt{w}}$, we have $\deg(v) = o\left(\frac{1}{\sqrt{\varepsilon}}\right)$ (thereby satisfying the hypothesis of Corollary 8). The fifth inequality uses the fact that, if we set $V(x) := v(x/w)$, then the situation satisfies the hypothesis of Lemma 9: we are interested in the range of V on the interval $\left[\frac{1}{2}, \sqrt{w}\right]$, compared to its range on discrete points $\frac{w}{\sqrt{w}}, \frac{w}{\sqrt{w}+1}, \ldots, \frac{w}{2w}$ that are spaced at most 1 apart from each other; and we also have $\deg(V) = \deg(v) = o\left(w^{1/4}\right)$.

All that remains is to show that, if we insert either $G_u \geq \frac{1}{6}$ or $G_v \geq \frac{1}{6}$ into the coupled system of inequalities above, then we get unbounded growth and the inequalities have no solution. Let us collapse the two sets of inequalities to

$$L_u \geq \frac{1}{4}\frac{N-\sqrt{w}}{\deg(u)^2}\frac{G_u}{2w} = \Omega\left(\frac{N}{w\deg(u)^2}G_u\right),$$
$$L_v \geq \frac{1}{4}\frac{\frac{1}{\sqrt{w}}-\frac{1}{N}}{\deg(v)^2}G_v w = \Omega\left(\frac{\sqrt{w}}{\deg(v)^2}G_v\right).$$

Hence

$$G_u \geq L_v - 1 = \Omega\left(\frac{\sqrt{w}}{\deg(v)^2}G_v\right) - 1,$$
$$G_v \geq L_u - 1 = \Omega\left(\frac{N}{w\deg(u)^2}G_u\right) - 1.$$

By the assumption that $\deg(v) = o\left(w^{1/4}\right)$ and $\deg(u) = o\left(\sqrt{\frac{N}{w}}\right)$, we have $\frac{\sqrt{w}}{\deg(v)^2} \gg 1$ and $\frac{N}{w\deg(u)^2} \gg 1$. Plugging in $G_u \geq \frac{1}{6}$ or $G_v \geq \frac{1}{6}$, this is enough to give us unbounded growth. ■

4.4 Lower bound using dual polynomials

In this section we use the method of dual polynomials to establish our main result, Theorem 4, restated for convenience:

Theorem 4. *Let Q be a quantum algorithm that makes T queries to the membership oracle for S, and uses a total of R copies of $|S\rangle$ and reflections about $|S\rangle$. If Q decides whether $|S| = w$ or $|S| = 2w$ with high probability, promised that one of those is the case, then either*

$$T = \Omega\left(\sqrt{\frac{N}{w}}\right) \qquad \text{or} \qquad R = \Omega\left(\min\left\{w^{1/3}, \sqrt{\frac{N}{w}}\right\}\right). \tag{4}$$

Let $p(r)$ be a univariate Laurent polynomial of negative degree D_1 and positive degree D_2. That is, let $p(r)$ be of the form

$$p(r) = a_0/r^{D_1} + a_1/r^{D_1-1} + \cdots + a_{D_1-1}/r + a_{D_1} + a_{D_1+1}\cdot r + \cdots + a_{D_2+D_1}\cdot r^{D_2}. \tag{42}$$

Theorem 4 follows by combining the Laurent polynomial method (Lemma 21) and the following theorem.

Theorem 25. *Let $\varepsilon < 1$. Suppose that p has negative degree D_1 and positive degree D_2 and satisfies the following properties.*

- $|p(w) - 1| \leq \varepsilon$
- $|p(2w) + 1| \leq \varepsilon$
- $|p(\ell)| \leq 1 + \varepsilon$ *for all* $\ell \in \{1, 2, \ldots, n\}$

Then either $D_1 \geq \Omega\left(w^{1/3}\right)$ *or* $D_2 \geq \Omega\left(\sqrt{N/w}\right)$.

In fact, our proof of Theorem 25 will show that the lower bound holds even if $|p(\ell)| \leq 1+\varepsilon$ only for $\ell \in \{w^{1/3}, w^{1/3}+1, \ldots, w\} \cup \{2w, 2w+1, \ldots, N\}$. We refer to a Laurent polynomial p satisfying the three properties of Theorem 25 as an *approximation for approximate counting*.

Proof of Theorem 25. Let p be any Laurent polynomial satisfying the hypothesis of Theorem 25. We begin by transforming p into a (standard) polynomial q in a straightforward manner. This transformation is captured in the following lemma, whose proof is so simple that we omit it.

Lemma 26. *If* p *satisfies the properties of Theorem 25, then the polynomial* $q(r) = p(r) \cdot r^{D_1} = a_0 + a_1 r + \cdots + a_{D_1+D_2} r^{D_1+D_2}$ *is a (standard) polynomial of degree at most* $D_1 + D_2$*, and* q *satisfies the following three properties.*

- $\left|q(w) - w^{D_1}\right| \leq \varepsilon \cdot w^{D_1}$
- $\left|q(2w) + (2w)^{D_1}\right| \leq \varepsilon \cdot (2w)^{D_1}$
- $|q(\ell)| \leq (1+\varepsilon)\,\ell^{D_1}$ *for all* $\ell \in \{1, 2, \ldots, N\}$

We now turn to showing that, for any constant $\varepsilon < 1$, no polynomial q can satisfy the conditions of Lemma 26 unless $D_1 \geq \Omega(w^{1/3})$ or $D_2 \geq \Omega\left(\sqrt{N/w}\right)$.

Consider the following linear program. The variables of the linear program are ε, and the $D_2 + D_1 + 1$ coefficients of q.

$$\begin{array}{ll} \text{minimize} & \varepsilon \\ \text{such that} & \\ & |q(w) - w^{D_1}| \leq \varepsilon \cdot w^{D_1} \\ & |q(2w) + (2w)^{D_1}| \leq \varepsilon \cdot (2w)^{D_1} \\ & |q(\ell)| \leq (1+\varepsilon) \cdot \ell^{D_1} \text{ for all } \ell \in \{1, 2, \ldots, N\} \\ & \varepsilon \geq 0 \end{array} \tag{43}$$

Standard manipulations reveal the dual.

$$\begin{array}{l} \text{maximize } \phi(w) \cdot w^{D_1} - \phi(2w) \cdot (2w)^{D_1} - \sum_{\ell \in \{1,\ldots,N\}, \ell \notin \{w, 2w\}} |\phi(\ell)| \cdot \ell^{D_1} \\ \text{such that} \\ \sum_{\ell=1}^{N} \phi(\ell) \cdot \ell^j = 0 \text{ for } j = 0, 1, 2, \ldots, D_1 + D_2 \\ \sum_{\ell=1}^{N} |\phi(\ell)| \cdot \ell^{D_1} = 1 \\ \phi \colon \mathbb{R} \to \mathbb{R} \end{array} \tag{44}$$

Theorem 25 will follow if we can exhibit a solution ϕ to the dual linear program achieving value $\varepsilon > 0$, for some setting of $D_1 \geq \Omega(w^{1/3})$ and $D_2 \geq \Omega\left(\sqrt{N/w}\right)$.[13] We now turn to this task.

[13] We will alternatively refer to such dual solutions ϕ as *dual witnesses*, since they act as a witness to the fact that any low-degree Laurent polynomial p approximating the approximate counting problem must have large error.

4.4.1 Constructing the dual solution

For a set $T \subseteq \{0, 1, \ldots, N\}$, define

$$Q_T(t) = \prod_{i=0,1,\ldots,N, i \notin T} (t - i). \tag{45}$$

Let $c > 2$ be an integer constant that we will choose later (the bigger we choose c to be, the better the objective value achieved by our final dual witness. But choosing a bigger c will also lower the degrees D_1, D_2 of Laurent polynomials against which our lower bound will hold).

We now define two sets T_1 and T_2. The size of T_1 will be

$$d_1 := \lfloor (w/c)^{1/3} \rfloor = \Theta\left(w^{1/3}\right) \tag{46}$$

and the size of T_2 will be d_2 for

$$d_2 := \lfloor \sqrt{N/(cw)} \rfloor = \Theta\left(\sqrt{N/w}\right). \tag{47}$$

Let

$$T_1 = \left\{ \lfloor w/(ci^2) \rfloor \colon i = 1, 2, \ldots, d_1 \right\} \tag{48}$$

and

$$T_2 = \left\{ c \cdot i^2 \cdot w \colon i = 1, 2, \ldots, d_2 := \sqrt{N/(cw)} \right\}. \tag{49}$$

Finally, define

$$T = \{w, 2w\} \cup T_1 \cup T_2. \tag{50}$$

At last, define $\Phi \colon \{0, 1, \ldots, N\} \to \mathbb{R}$ via

$$\Phi(t) = (-1)^t \cdot \binom{N}{t} \cdot Q_T(t). \tag{51}$$

Our final dual solution ϕ will be a scaled version of Φ. Specifically, Φ itself does not satisfy the second constraint of the dual linear program, that $\sum_{\ell=1}^{N} |\Phi(\ell)| \cdot \ell^{D_1} = 1$. So letting

$$C = \sum_{\ell=1}^{N} |\Phi(\ell)| \cdot \ell^{D_1}, \tag{52}$$

our final dual witness ϕ will be Φ/C.

The sizes of T_1 and T_2. Clearly, under the above definition of T_2, $|T_2| = d_2$ as claimed above. It is not as immediately evident that $|T_1| = d_1$: to establish this, we must show that for distinct $i, j \in \{1, 2, \ldots, d_1\}$, $\lfloor w/(ci^2) \rfloor \neq \lfloor w/(cj^2) \rfloor$. This is handled in the following easy lemma.

Lemma 27. *Let $i \neq j$ be distinct numbers in $\{1, \ldots, d_1\}$ and $c > 2$ be a constant. Then as long as $d_1 < (w/c)^{1/3}$, it holds that $\lfloor w/(ci^2) \rfloor \neq \lfloor w/(cj^2) \rfloor$.*

Proof. Assume without loss of generality that $i > j$. Then $w/(cj^2) - w/(ci^2)$ is clearly minimized when $i = d_1$ and $j = i - 1$. For the remainder of the proof, fix $i = d_1$. In this case,

$$w/(cj^2) - w/(ci^2) \geq w/\left(c(i-1)^2\right) - w/\left(ci^2\right) = \frac{wi^2 - w\left(i-1\right)^2}{c \cdot i^2 \cdot \left(i-1\right)^2}$$
$$= \frac{w}{c} \cdot \frac{2i-1}{i^2\left(i-1\right)^2} \geq \frac{w}{c} \cdot \frac{2i-1}{i^4} \geq \frac{w}{ci^3} \geq 1. \tag{53}$$

Here, the final inequality holds because $i^3 = d_1^3 \leq w/c$.

Equation (53) implies the lemma, as two numbers whose difference is at least 1 cannot have the same integer floor. ■

Lemma 27 is false for $d_1 = \omega(w^{1/3})$, highlighting on a technical level why one cannot choose d_1 larger than $\Theta(w^{1/3})$ without the entire construction and analysis of Φ breaking down.

4.4.2 Intuition: "gluing together" two simpler dual solutions

Before analyzing the dual witnesses Φ and ϕ constructed in Equation (51) and Equation (52), in this subsection and the next, we provide detailed intuition for why the definitions of Φ and ϕ are natural, and briefly overview their analysis.

A dual witness for purely positive degree (i.e., approximate degree). Suppose we were merely interested in showing an approximate degree lower bound of $\Omega(\sqrt{N/w})$ for approximate counting (i.e., a lower bound on the degree of traditional polynomials that distinguish input w from $2w$, and are bounded at all other integer inputs in $1, \ldots, N$). This is equivalent to exhibiting a solution to the dual linear program with $D_1 = 0$. A valid dual witness ϕ_1 for this simpler case is to also use Equation (51), but to set

$$T = \{w, 2w\} \cup T_2, \tag{54}$$

rather than $T = \{w, 2w\} \cup T_1 \cup T_2$.

We will explain intuition for why Equation (54) is a valid dual solution for the approximate degree of approximate counting in the next subsection. For now, we wish to explain how this construction relates to prior work. In [BT13], for any constant $\delta > 0$, a dual witness is given for the fact that the $(1-\delta)$-approximate degree of OR is $\Omega(\sqrt{N})$. This dual witness *nearly* corresponds to the above, with $w = 1$. Specifically, Bun and Thaler [BT13] use the set $T = \{0, 1\} \cup \{ci^2 : i = 1, 2, \ldots, \sqrt{N/c}\}$, and they show that almost all of the "mass" of this dual witness is located on the inputs 0 and 1, i.e.,

$$|\Phi(0)| + |\Phi(1)| \geq (1-\delta) \cdot \sum_{i=2}^{N} |\Phi(i)|. \tag{55}$$

Here, the bigger c is chosen to be, the smaller the value of δ for which Equation (55) holds.

In the case of $w = 1$, our dual witness for approximate counting differs from this only in that $\{0, 1\}$ is replaced with $\{1, 2\}$. This is because, in order to show a lower bound for distinguishing input $w = 1$ from input $2w = 2$, we want almost all of the mass to be on inputs $\{1, 2\}$ rather than $\{0, 1\}$ (this is what will ensure that the objective function of the dual linear program is large).

For general w, we want most of the mass of ψ to be concentrated on inputs w and $2w$. Accordingly, relative to the $w = 1$ case, we effectively multiply *all* points in T by w, and one can show that this does not affect the calculation regarding concentration of mass.

A dual witness for purely negative degree. Now, suppose we were merely interested in showing that Laurent polynomials of *purely negative* degree require degree $\Omega(w^{1/3})$ to approximate the approximate counting problem. This is equivalent to exhibiting a solution to the dual linear program with $D_2 = 0$. Then a valid dual witness ϕ_2 for this simpler case is to also use Equation (51), but to set

$$T = \{w, 2w\} \cup T_1. \tag{56}$$

Again, we will give intuition for why this is a valid dual solution in the next subsection (Section 4.4.3). For now, we wish to explain how this construction relates to prior work. Essentially, the $\Omega\left(w^{1/3}\right)$-degree lower bound for Laurent polynomials with *only negative* powers was proved by Zhandry [Zha12, Theorem 7.3]. Translating Zhandry's theorem into our setting is not entirely trivial, and he did not explicitly construct a solution to our dual linear program. However (albeit with significant effort), one can translate his argument to our setting to show that Equation (56) gives a valid dual solution to prove a lower bound against Laurent polynomials with only negative powers.

Gluing them together. The above discussion explains that the key ideas for constructing dual solutions ϕ_1, ϕ_2 witnessing degree lower bounds for Laurent polynomials of *only negative* or *only positive* powers were essentially already known, or at least can be extracted from prior work with enough effort. In this work, we are interested in proving lower bounds for Laurent polynomials with both positive and negative powers. Our dual solution Φ essentially just "glues together" the dual solutions that can be derived from prior work. By this, we mean that the set T of integer points on which our Φ is nonzero is the *union* of the corresponding sets for ϕ_1 and ϕ_2 individually. Moreover, this union is nearly disjoint, as the only points in the intersection of the two sets being unioned are w and $2w$.

Overview of the analysis. To show that we have constructed a valid solution to the dual linear program (Equation (44)), we must establish that (a) Φ is uncorrelated with every polynomial of degree at most $D_1 + D_2$ and (b) Φ is well-correlated with any function g that evaluates to $+1$ on input w, to -1 on input $2w$, and is bounded in $[-1, 1]$ elsewhere. In (b), the correlation is taken with respect to an appropriate weighting of the inputs, that on input $\ell \in [N]$ places mass proportional to ℓ^{D_1}.

The definition of Φ as a "gluing together" of ϕ_1 and ϕ_2 turns out, in a straightforward manner, to ensure that Φ is uncorrelated with polynomials of degree at $D_1 + D_2$. All that remains is to show that Φ is well-correlated with g under the appropriate weighting of inputs. This turns out to be technically demanding, but ultimately can be understood as stemming from the fact that ϕ_1 and ϕ_2 are individually well-correlated with g (albeit, in the case of ϕ_2, under a *different* weighting of the inputs than the weighting that is relevant for Φ).

4.4.3 Intuition via complementary slackness

We now attempt to lend some insight into why the dual witnesses ϕ_1 and ϕ_2 for the purely positive degree and purely negative degree take the form that they do. This section is deliberately slightly imprecise in places, and builds on intuition that has been put forth in prior works proving approximate degree lower bounds via dual witnesses [BT13, Tha16, BKT18].

Notice that ϕ_1 is precisely defined so that $\phi_1(i) = 0$ for any $i \notin \{w, 2w\} \cup T_2$, and similarly $\phi_2(i) = 0$ for any $i \notin \{w, 2w\} \cup T_1$. The intuition for why this is reasonable comes from complementary slackness, which states that an optimal dual witness should equal 0 except on inputs that correspond to primal constraints that are *made tight by an optimal primal solution*. By "constraints made tight by an optimal primal solution", we mean constraints that, for the optimal primal solution, hold with equality rather than (strict) inequality.

Unpacking that statement, this means the following. Suppose that q is an optimal solution to the primal linear program of Section 4.4, meaning it minimizes the error ε amongst all polynomials of the same same degree. The constraints made tight by q are precisely those inputs ℓ at which q hits its "maximum error" (e.g., an input ℓ such that $|q(\ell)| = (1+\varepsilon) \cdot \ell^{D_1}$). We call these inputs *maximum-error* inputs for q. Complementary slackness says that there is an optimal solution to the dual linear program (Equation (44)) that equals 0 at all inputs that are not maximum-error inputs for q.

In both the purely positive degree case, and the purely negative degree case, we know roughly what primal optimal solutions q look like, and moreover we know what roughly their maximum-error points look like. In the first case, the maximum-error points are well-approximated by the points in T_2, and in the purely negative degree case, the maximum error points are well-approximated by the points in T_1. Let us explain.

Purely positive degree case. Let T_d be the degree d Chebyshev polynomial of the first kind. It can be seen that $P(\ell) = T_{\sqrt{N}}(1 + 2/N - \ell/N)$ satisfies $P(1) \geq 2$, while $|P(\ell)| \leq 1$ for $\ell = 2, 3, \ldots, N$. That is, up to scaling, P approximates the approximate counting problem for $w = 1$, and its known that its degree is within a constant factor of optimal.

It is known that the extreme points of T_d are of the following form, for $k = 1, \ldots, d$:

$$\cos\left(\frac{(2k-1)}{2d}\pi\right) \approx 1 - k^2/(2d^2), \tag{57}$$

where the approximation uses the Taylor expansion of the cosine function around 0. Equation (57) means that the extreme points of P are roughly those inputs ℓ such that $1+2/N-\ell/N \approx 1-k^2/(2d^2)$, where $d = \sqrt{N}$. Such ℓ are roughly of the form $\ell \approx c \cdot i^2$ for some constant c, as i ranges from 1 up to $\Theta(N^{1/2})$.

More generally, when $w \geq 1$, an asymptotically optimal approximation for distinguishing input w from $2w$ is $P(\ell) = T_{\sqrt{N/w}}(1 + 2w/N - \ell/(wN))$. The extreme points of P are roughly of the form $\ell \approx c \cdot i^2 \cdot w$ for some constant c, as i ranges from 1 up to $\Theta(\sqrt{N/w})$, which is exactly the form of the points in our set T_2.

Purely negative degree case. In Lemma 22, we exhibited a simple, purely negative degree Laurent polynomial p (i.e., $p(\ell)$ is a standard polynomial in $1/\ell$) with degree $D_1 = w^{1/3}$ that solves the approximate counting problem (the construction is due to MathOverflow user "fedja"). Roughly speaking, p can be written as a product $p(\ell) = u(\ell) \cdot v(\ell)$, where $u(\ell)$ has the roots $\ell = 1, 2, \ldots, w^{1/3}$,

and $v(\ell)$ is (an affine transformation) of a Chebyshev polynomial of degree $w^{1/3}$, applied to $1/\ell$. One can easily look at this construction and see that $p(\ell)$ outputs *exactly* the correct value on inputs $\{1, 2, \ldots, w^{1/3}\}$, so these are not maximum error points for p. Moreover, the analysis of the maximum error points for Chebyshev polynomials above can be applied to show that the maximum error points of p are roughly of the form ℓ such that $1/\ell = c \cdot i^2/w$ for some constant c, with i ranging from 1 up to $\Theta(w^{1/3})$. This means that the extreme points are roughly of the form $\ell \approx \frac{w}{ci^2}$, which is why our set T_1 consists of points of the form $\lfloor \frac{w}{ci^2} \rfloor$ (the floors are required because we are proving lower bounds against polynomials whose behavior is only constrained at integer inputs).

4.4.4 Analysis of the dual solution Φ

Lemma 28. *Let $d_1 = |T_1|$ and $d_2 = |T_2|$. Then for any $j = 0, 1, \ldots, d_1 + d_2$, it holds that*

$$\sum_{\ell=1}^{N} \Phi(\ell) \cdot \ell^j = 0.$$

Proof. A basic combinatorial fact is that for any polynomial Q of degree at most $N-1$, the following identity holds:

$$\sum_{\ell=0}^{N} \binom{N}{\ell} (-1)^\ell Q(\ell) = 0. \tag{58}$$

Observe that for any $j \leq d_1 + d_2 + 1$,

$$Q_T(\ell) \cdot \ell^j \text{ is a polynomial in } \ell \text{ of degree at most } N-1. \tag{59}$$

Furthermore, $\Phi(0) = 0$, because $0 \notin T$. Hence

$$\sum_{\ell=0}^{N} \binom{N}{\ell} (-1)^\ell Q_T(\ell) \cdot \ell^j = \sum_{\ell=1}^{N} \binom{N}{\ell} (-1)^\ell Q_T(\ell) \cdot \ell^j. \tag{60}$$

Thus, we can calculate:

$$\begin{aligned}\sum_{\ell=1}^{N} \Phi(\ell) \cdot \ell^j &= \sum_{\ell=1}^{N} (-1)^\ell \cdot \binom{N}{\ell} \cdot Q_T(\ell) \cdot \ell^j \\ &= \sum_{\ell=0}^{N} (-1)^\ell \cdot \binom{N}{\ell} \cdot Q_T(\ell) \cdot \ell^j = 0.\end{aligned}$$

Here, the second equality follows from Equation (60), while the third follows from Equations (58) and (59). ■

Let us turn to analyzing Φ's value on various inputs. Clearly the following condition holds:

$$\Phi(\ell) = 0 \text{ for all } \ell \notin T. \tag{61}$$

Next, observe that for any $r \in T$,

$$|\Phi(r)| = N! \cdot \frac{1}{\prod_{j \in T, j \neq r} |r - j|}.$$

Consider any quantity $c \cdot i^2 \cdot w \in T_2$. Then

$$\begin{aligned}
\left|\Phi(c \cdot w \cdot i^2)\right| / \left|\Phi(w)\right| &= \frac{\prod_{j \in T, j \neq w} |w - j|}{\prod_{j \in T, j \neq c \cdot i^2 \cdot w} |w \cdot c \cdot i^2 - j|} \\
&= \frac{|w - 2w| \cdot \left(\prod_{j=1}^{d_2} \left|w - c \cdot j^2 \cdot w\right|\right) \cdot \left(\prod_{j=1}^{d_1} \left(w - \left\lfloor \frac{w}{cj^2} \right\rfloor\right)\right)}{|c \cdot i^2 \cdot w - w| \cdot |c \cdot i^2 \cdot w - 2w| \cdot \left(\prod_{j=1, j \neq i}^{d_2} |w \cdot c \cdot i^2 - w \cdot c \cdot j^2|\right) \cdot \left(\prod_{j=1}^{d_1} \left(w \cdot c \cdot i^2 - \left\lfloor \frac{w}{c \cdot j^2} \right\rfloor\right)\right)} \\
&= \frac{c^{d_2} \cdot \left(\prod_{j=1}^{d_2} \left(j^2 - \frac{1}{c}\right)\right) \cdot \prod_{j=1}^{d_1} \left(w - \left\lfloor \frac{w}{c \cdot j^2} \right\rfloor\right)}{(ci^2 - 1) \cdot (ci^2 - 2) \cdot c^{d_2 - 1} \cdot \left(\prod_{j=1, j \neq i}^{d_2} |i^2 - j^2|\right) \cdot \left(\prod_{j=1}^{d_1} \left(w \cdot c \cdot i^2 - \left\lfloor \frac{w}{c \cdot j^2} \right\rfloor\right)\right)} \\
&\leq \frac{c \cdot \left(\prod_{j=1}^{d_2} \left(j^2 - \frac{1}{c}\right)\right) \cdot \prod_{j=1}^{d_1} \left(w - \left\lfloor \frac{w}{c \cdot j^2} \right\rfloor\right)}{(ci^2 - 1) \cdot (ci^2 - 2) \cdot \left(\prod_{j=1, j \neq i}^{d_2} |i^2 - j^2|\right) \cdot \left(\prod_{j=1}^{d_1} \left(w \cdot c \cdot i^2 - \frac{w}{c \cdot j^2}\right)\right)} \qquad (62)
\end{aligned}$$

Now, observe that

$$\begin{aligned}
\prod_{j=1}^{d_1} \left(w - \left\lfloor \frac{w}{c \cdot j^2} \right\rfloor\right) &\leq \prod_{j=1}^{d_1} \left(w - \frac{w}{cj^2} + 1\right) = \prod_{j=1}^{d_1} w \cdot \left(1 - \frac{1}{cj^2}\right) \cdot \left(1 + \frac{1}{w \cdot \left(1 - \frac{1}{cj^2}\right)}\right) \\
&\leq \prod_{j=1}^{d_1} w \cdot \left(1 - \frac{1}{cj^2}\right) \left(1 + \frac{1}{(1 - 1/c) \cdot w}\right) \leq \left(\prod_{j=1}^{d_1} w \cdot \left(1 - \frac{1}{cj^2}\right)\right) \cdot (1 + o(1)) \,. \qquad (63)
\end{aligned}$$

Hence, we see that Expression (62) is bounded by

$$\begin{aligned}
&\frac{c \cdot \left(\prod_{j=1}^{d_2} \left(j^2 - \frac{1}{c}\right)\right) \cdot \left(\prod_{j=1}^{d_1} \left(1 - \frac{1}{c \cdot j^2}\right)\right) \cdot (1 + o(1))}{(ci^2 - 1) \cdot (ci^2 - 2) \cdot \left(\prod_{j=1, j \neq i}^{d_2} |i^2 - j^2|\right) \cdot \left(\prod_{j=1}^{d_1} \left(c \cdot i^2 - \frac{1}{c \cdot j^2}\right)\right)} \\
&\leq \frac{c \cdot (d_2!)^2 \cdot \left(\prod_{j=1}^{d_1} \left(1 - \frac{1}{c \cdot j^2}\right)\right) \cdot (1 + o(1))}{(ci^2 - 1) \cdot (ci^2 - 2) \cdot \left(\prod_{j=1, j \neq i}^{d_2} |i - j| \, |i + j|\right) \cdot (c \cdot i^2)^{d_1} \cdot \left(\prod_{j=1}^{d_1} \left(1 - \frac{1}{c^2 \cdot i^2 \cdot j^2}\right)\right)} \\
&= \frac{c \cdot (d_2!)^2 \cdot 2i^2 \cdot \left(\prod_{j=1}^{d_1} \left(1 - \frac{1}{c \cdot j^2}\right)\right) \cdot (1 + o(1))}{(ci^2 - 1) \cdot (ci^2 - 2) \cdot (d_2 + i)! \, (d_2 - i)! \cdot (c \cdot i^2)^{d_1} \cdot \left(\prod_{j=1}^{d_1} \left(1 - \frac{1}{c^2 \cdot i^2 \cdot j^2}\right)\right)} \\
&\leq \frac{c \cdot 2i^2 \cdot (d_2!)^2 \cdot (1 + o(1))}{(ci^2 - 1) \, (ci^2 - 2) \cdot (d_2 + i)! \, (d_2 - i)! \cdot (c \cdot i^2)^{d_1}} \leq \frac{2 \, (1 + o(1))}{\left(1 - \frac{1}{c \cdot i^2}\right) \cdot (c \cdot i^2 - 2) \cdot (c \cdot i^2)^{d_1}} . \qquad (64)
\end{aligned}$$

In the penultimate inequality, we used the fact that $\frac{(d_2!)^2}{(d_2+i)!(d_2-i)!} = \frac{\binom{2d_2}{d_2+i}}{\binom{2d_2}{d_2}} \leq 1$.

Next, consider any quantity $\left\lfloor \frac{w}{c \cdot i^2} \right\rfloor \in T_1$. Then

$$\left|\Phi\left(\left\lfloor \frac{w}{c \cdot i^2} \right\rfloor\right)\right| / \left|\Phi(w)\right|$$

$$
\begin{aligned}
&= \frac{|w-2w|\left(\prod_{j=1}^{d_2}|w-cj^2w|\right)\left(\prod_{j=1}^{d_1}\left(w-\left\lfloor\frac{w}{cj^2}\right\rfloor\right)\right)}{\left(w-\left\lfloor\frac{w}{c\cdot i^2}\right\rfloor\right)\cdot\left(2w-\left\lfloor\frac{w}{c\cdot i^2}\right\rfloor\right)\left(\prod_{j=1}^{d_2}\left(w\cdot c\cdot j^2-\left\lfloor\frac{w}{c\cdot i^2}\right\rfloor\right)\right)\prod_{j=1,j\neq i}^{d_1}\left|\left\lfloor\frac{w}{c\cdot i^2}\right\rfloor-\left\lfloor\frac{w}{c\cdot j^2}\right\rfloor\right|} \\
&\leq \frac{|w-2w|\left(\prod_{j=1}^{d_2}|w-cj^2w|\right)\left(\prod_{j=1}^{d_1}\left(w-\frac{w}{cj^2}\right)\right)}{\left(w-\frac{w}{c\cdot i^2}\right)\cdot\left(2w-\frac{w}{c\cdot i^2}\right)\left(\prod_{j=1}^{d_2}\left(w\cdot c\cdot j^2-\frac{w}{c\cdot i^2}\right)\right)\prod_{j=1,j\neq i}^{d_1}\left|\left\lfloor\frac{w}{c\cdot i^2}\right\rfloor-\left\lfloor\frac{w}{c\cdot j^2}\right\rfloor\right|} \\
&\leq \frac{|w-2w|\left(\prod_{j=1}^{d_2}|w-cj^2w|\right)\left(\prod_{j=1}^{d_1}\left(w-\frac{w}{cj^2}\right)\right)\cdot(1+o(1))}{\left(w-\frac{w}{c\cdot i^2}\right)\cdot\left(2w-\frac{w}{c\cdot i^2}\right)\left(\prod_{j=1}^{d_2}\left(w\cdot c\cdot j^2-\frac{w}{c\cdot i^2}\right)\right)\prod_{j=1,j\neq i}^{d_1}\left|\left\lfloor\frac{w}{c\cdot i^2}\right\rfloor-\left\lfloor\frac{w}{c\cdot j^2}\right\rfloor\right|}
\end{aligned}
\tag{65}
$$

Here, the final inequality used Equation (63). Let us consider the expression $\prod_{j=1,j\neq i}^{d_1}\left|\left\lfloor\frac{w}{c\cdot i^2}\right\rfloor-\left\lfloor\frac{w}{c\cdot j^2}\right\rfloor\right|$. This quantity is *at least*

$$
\begin{aligned}
\prod_{j=1,j\neq i}^{d_1}\left(\left|\frac{w}{c\cdot i^2}-\frac{w}{c\cdot j^2}\right|-1\right) &= w^{d_1-1}\cdot\prod_{j=1,j\neq i}^{d_1}\frac{\left|j^2-i^2\right|-\frac{ci^2j^2}{w}}{ci^2j^2} \\
&= w^{d_1-1}\cdot\prod_{j=1,j\neq i}^{d_1}\frac{|j-i|\cdot|j+i|-\frac{ci^2j^2}{w}}{ci^2j^2} \\
&= \left(\frac{w}{ci^2}\right)^{d_1-1}\cdot\prod_{j=1,j\neq i}^{d_1}\frac{|j-i|\cdot|j+i|-\frac{ci^2j^2}{w}}{j^2}
\end{aligned}
\tag{66}
$$

We claim that Expression (66) is at least

$$
\left(\frac{w}{ci^2}\right)^{d_1-1}\cdot\frac{1}{2}. \tag{67}
$$

In the case that $c=2$ and d_1 is (at most) $w^{1/3}$, this is precisely [Zha12, Claim 4]. We will ultimately take c to be a constant strictly greater than 2 and hence $d_1=\lfloor(w/c)^{1/3}\rfloor$ is a constant factor smaller than $w^{1/3}$. The proof of [Zha12, Claim 4] works with cosmetic changes in this case. For completeness, we present a derivation of the claim in Appendix A.

Equation (67) implies that Expression (65) is at most:

$$
\begin{aligned}
&\frac{|w-2w|\left(\prod_{j=1}^{d_2}|w-cj^2w|\right)\left(\prod_{j=1}^{d_1}\left(w-\frac{w}{cj^2}\right)\right)\cdot(1+o(1))}{\left(w-\frac{w}{c\cdot i^2}\right)\cdot\left(2w-\frac{w}{c\cdot i^2}\right)\left(\prod_{j=1}^{d_2}\left(w\cdot c\cdot j^2-\frac{w}{c\cdot i^2}\right)\right)\left(\frac{w}{ci^2}\right)^{d_1-1}\cdot\frac{1}{2}} \\
&= \frac{2\left(\prod_{j=1}^{d_2}|1-cj^2|\right)\left(\prod_{j=1}^{d_1}\left(1-\frac{1}{cj^2}\right)\right)\cdot(1+o(1))}{\left(1-\frac{1}{c\cdot i^2}\right)\cdot\left(2-\frac{1}{c\cdot i^2}\right)\left(\prod_{j=1}^{d_2}\left(c\cdot j^2-\frac{1}{c\cdot i^2}\right)\right)\left(\frac{1}{ci^2}\right)^{d_1-1}} \\
&= \frac{2\left(\prod_{j=1}^{d_2}(j^2-1/c)\right)\left(\prod_{j=1}^{d_1}\left(1-\frac{1}{cj^2}\right)\right)\cdot(1+o(1))}{\left(1-\frac{1}{c\cdot i^2}\right)\cdot\left(2-\frac{1}{c\cdot i^2}\right)\left(\prod_{j=1}^{d_2}\left(j^2-\frac{1}{c^2\cdot i^2}\right)\right)\left(\frac{1}{ci^2}\right)^{d_1-1}} \\
&\leq \frac{2\,(1+o(1))}{\left(1-\frac{1}{c\cdot i^2}\right)\cdot\left(2-\frac{1}{c\cdot i^2}\right)\left(\frac{1}{ci^2}\right)^{d_1-1}} \leq 4\cdot\left(ci^2\right)^{d_1-1}.
\end{aligned}
\tag{68}
$$

Summarizing Equations (64) and (68), we have shown that: for any quantity $c \cdot i^2 \cdot w \in T_2$,

$$\left|\Phi(c \cdot w \cdot i^2)\right| / \left|\Phi(w)\right| \leq \frac{2\left(1+o(1)\right)}{\left(1-\frac{1}{c \cdot i^2}\right) \cdot (c \cdot i^2 - 2) \cdot (c \cdot i^2)^{d_1}} \tag{69}$$

and for any quantity $\left\lfloor \frac{w}{c \cdot i^2} \right\rfloor \in T_1$,

$$\left|\Phi\left(\left\lfloor \frac{w}{c \cdot i^2} \right\rfloor\right)\right| / \left|\Phi(w)\right| \leq 4 \cdot \left(ci^2\right)^{d_1 - 1}. \tag{70}$$

Let $\phi = \Phi / C$, where C is as in Equation (52). Let $D_1 = d_1$ and $D_2 = d_2$. Lemma 28 implies that ϕ is a feasible solution for the dual linear program of Section 4.4.1. We now show that, for any constant $\delta > 0$, by choosing c to be a sufficiently large constant (that depends on δ), we can ensure that ϕ achieves objective value $1 - 2\delta$.

Let

$$\begin{aligned} A &= |\Phi(w)| \cdot w^{D_1}, \\ B &= |\Phi(2w)| \cdot (2w)^{D_1}, \end{aligned}$$

and

$$E = \sum_{i=1}^{d_1} |\Phi(\lfloor w/ci^2 \rfloor)| \cdot \left(\lfloor w/ci^2 \rfloor\right)^{D_1} + \sum_{i=1}^{d_2} |\Phi(\lfloor w \cdot ci^2 \rfloor)| \cdot \left(w \cdot c \cdot i^2\right)^{D_1}.$$

By Equation (61), $C = A + B + E$.

Moreover, observe that $\mathrm{sgn}(\Phi(w)) = -\mathrm{sgn}(\Phi(2w))$, so without loss of generality we may assume $\Phi(w) \geq 0$ and $\Phi(2w) \leq 0$ (if not, then replace Φ with $-\Phi$ throughout).

We now claim that by choosing c to be a sufficiently large constant, we can ensure that $E \leq \delta \cdot A$. To see this, observe that Equations (69) and (70), along with the fact that $D_1 = d_1$ and $D_2 = d_2$ implies that

$$\begin{aligned} E/A &\leq \frac{1}{w^{D_1}} \left[\left(\sum_{i=1}^{d_1} \left(\lfloor w/ci^2 \rfloor\right)^{D_1} \cdot 4 \cdot \left(ci^2\right)^{d_1-1} \right) + \left(\sum_{i=1}^{d_2} \left(w \cdot c \cdot i^2\right)^{D_1} \frac{2\left(1-\frac{1}{c \cdot i^2}\right)\left(1+o(1)\right)}{(c \cdot i^2 - 2) \cdot (c \cdot i^2)^{d_1}} \right) \right] \\ &\leq \frac{1}{w^{D_1}} \left[\left(\sum_{i=1}^{d_1} \left(w/ci^2\right)^{D_1} \cdot 4 \cdot \left(ci^2\right)^{d_1-1} \right) + \left(\sum_{i=1}^{d_2} \left(w \cdot c \cdot i^2\right)^{D_1} \frac{2\left(1-\frac{1}{c \cdot i^2}\right)\left(1+o(1)\right)}{(c \cdot i^2 - 2) \cdot (c \cdot i^2)^{d_1}} \right) \right] \\ &\leq 4 \left(\sum_{i=1}^{d_1} \frac{1}{c \cdot i^2} \right) + \left(\sum_{i=1}^{d_2} \frac{2\left(1+o(1)\right)}{\left(1-\frac{1}{c \cdot i^2}\right)(c \cdot i^2 - 2)} \right) \end{aligned}$$

Since $\sum_{i=1}^{\infty} 1/(ci^2) \leq \frac{\pi^2}{6c}$, we see that choosing c to be a sufficiently large constant depending on δ ensures that $E/A \leq \delta$ as desired.

Hence, ϕ achieves objective value at least

$$\begin{aligned} \phi(w) \cdot w^{D_1} - \phi(2w) \cdot (2w)^{D_1} - \sum_{\ell \in \{1,\ldots,N\}, \ell \notin \{w, 2w\}} |\phi(\ell)| \cdot \ell^{D_1} \\ \geq \frac{A + B - E}{A + B + E} \geq \frac{(1-\delta)A + B}{(1+\delta)A + B} \geq 1 - 2\delta. \end{aligned}$$

4.5 Approximate counting with classical samples

For completeness, in this section, we sketch classical counterparts of Theorem 4 and Theorem 5. That is, we show tight bounds on classical randomized algorithms for $\mathsf{ApxCount}_{N,w}$ that make membership queries and have access to uniform random samples from the set being counted.

Proposition 29. *There is a classical randomized algorithm that solves* $\mathsf{ApxCount}_{N,w}$ *with high probability using either* $O(N/w)$ *queries to the membership oracle for* S*, or else using* $O(\sqrt{w})$ *uniform samples from* S*.*

Proof sketch. By reducing approximate counting to the problem of estimating the mean of a biased coin, $O(N/w)$ queries are sufficient.

Alternatively, if we take R samples, then the expected number of birthday collisions is $\binom{R}{2} \cdot \frac{1}{|S|}$ and the variance is $\binom{R}{2} \cdot \frac{1}{|S|}\left(1 - \frac{1}{|S|}\right)$. So, taking $O(\sqrt{w})$ samples and computing the number of birthday collisions is sufficient to distinguish $|S| \leq w$ from $|S| \geq 2w$ with $\frac{2}{3}$ success probability. ■

Proposition 30. *Let* M *be a classical randomized algorithm that makes* T *queries to the membership oracle for* S*, and takes a total of* R *uniform samples from* S*. If* M *decides whether* $|S| = w$ *or* $|S| = 2w$ *with high probability, promised that one of those is the case, then either* $T = \Omega(N/w)$ *or* $R = \Omega(\sqrt{w})$*.*

Proof sketch. Note that without loss of generality, we may assume that the algorithm first takes all of the samples it needs, and then queries random elements of $[N]$ that did not appear in the samples. Suppose the algorithm takes $R = o(\sqrt{w})$ samples and then makes $T = o(N/w)$ queries. Consider what happens when the algorithm tries to distinguish a random subset of size w from a random subset of size $2w$ of $[N]$. By a union bound, the probability that the algorithm sees any collisions in the samples is $o(1)$, and the probability that the algorithm finds any additional elements of S via queries is also $o(1)$. So, if the set has size either w or $2w$, with $1 - o(1)$ probability, the algorithm's view of the samples is just a random subset of size R of $[N]$ drawn without replacement, and the algorithm's view of the queries is just T "no" answers to membership queries. Hence, the algorithm fails to distinguish random sets of size w and size $2w$ with any constant probability of success. ■

4.6 Extending the lower bound to QSampling unitarily

So far in this section we have proved upper and lower bounds on the power of quantum algorithms for approximate counting that have access to two resources (in addition to membership queries): copies of $|S\rangle$, and the unitary transformation that reflects about $|S\rangle$. The assumption of access to the reflection unitary is justified by the argument that, if we had access to a unitary that prepared $|S\rangle$, then it could be used to reflect about $|S\rangle$ as well.

Giving the algorithm access to just the two resources above is an appealing model to use for upper bounds, since it does not assume anything about the method by which copies of $|S\rangle$ are prepared. This means algorithms derived in this model work in many different settings. For example, the algorithm may be able to QSample because someone else simply handed the algorithm copies of $|S\rangle$, or perhaps several copies of $|S\rangle$ just happen to be stored in the algorithm's quantum memory as a side effect of the execution of some earlier quantum algorithm. The upper bound given in Theorem 5 applies in any of these settings.

On the other hand, since only permitting access to QSamples and reflections about $|S\rangle$ ties the algorithm's hands, lower bounds for this model (e.g., Theorem 4) could be viewed as weaker than is desirable. In particular, our original justification for allowing access to reflections about $|S\rangle$ was that access to a unitary that prepared the state $|S\rangle$ would in particular allow such reflections to be done. Given this justification, it is very natural to wonder whether our lower bounds extend beyond just QSamples and reflections, to algorithms that are given access to *some* unitary process that permits both QSampling and reflections about $|S\rangle$.

Note that an algorithm with access to such a unitary could potentially exploit the unitary in ways other than QSamples and reflections to learn information about $|S\rangle$. For example, the algorithm could choose to run the unitary on inputs that do not produce $|S\rangle$. More generally, given a quantum circuit that implements a unitary, it is possible to construct, in a completely black-box manner, the inverse of this unitary, and also a controlled version of the unitary. The algorithm may choose to run the inverse on a state other than $|S\rangle$ to learn some additional information that is not captured by access to QSamples and reflections alone.

In summary, in this section we ask whether we can we extend the lower bound of Theorem 4 to work in a model where the algorithm is given access to some unitary operator that conveys the power to both QSample and reflect about $|S\rangle$.[14] Via Theorem 31 below, we explain that the answer is yes.

It may seem convenient to assume that the unitary transformation preparing $|S\rangle$ maps the all-zeros state to $|S\rangle$. But this is not the most general method of preparing $|S\rangle$ by a unitary. A unitary U that maps the all-zeros state to $|S\rangle|\psi\rangle$ would also suffice to create copies of $|S\rangle$, since the register containing $|\psi\rangle$ can simply be ignored for the remainder of the computation. More formally, assume U behaves as

$$U|0^m\rangle = |S\rangle|\psi\rangle, \tag{71}$$

where $|S\rangle|\psi\rangle$ is some m-qubit state. Clearly we can use U to create as many copies of $|S\rangle$ as we like, which as a by-product also creates copies of $|\psi\rangle$. This unitary also lets us reflect about $|S\rangle$. To see how, first use this unitary to create a copy of $|\psi\rangle$, and then consider the action of the unitary $U(\mathbb{1} - 2|0^m\rangle\langle 0^m|)U^\dagger$ on the state $|\phi\rangle|\psi\rangle$ for any state $|\phi\rangle$. We claim that this unitary acts as a reflection about $|S\rangle$ when restricted to the first register. This establishes that any U of this form subsumes the power of both QSamples and reflections about $|S\rangle$.

Let us also assume without loss of generality that $|S\rangle|\psi\rangle$ is orthogonal to $|0^m\rangle$ from now on. This can be achieved by adding an additional qubit to the input that is always negated by the unitary. That is, we could instead consider the map $(U \otimes X)|0^m\rangle|0\rangle = |S\rangle|\psi\rangle|1\rangle$, which is orthogonal to the starting state by construction, and only increases the value of m by 1.

Of course, the requirement that $U|0^m\rangle = |S\rangle|\psi\rangle$ does not fully specify U, as it does not prescribe how U behaves on other input states. A reasonable prescription is that U should behave "trivially" on other input states, so that it does not leak information about S by its behavior on other states. In tension with this prescription is the fact the rest of the unitary must depend on S, since the first column of the unitary contains $|S\rangle$, and the rest of the columns have to be orthogonal to this.

Alexander Belov (personal communication) brought to our attention a very simple construction of such a unitary that leaks minimal additional information about S. Consider the unitary U that satisfies $U|0^m\rangle = |S\rangle|\psi\rangle$ and $U|S\rangle|\psi\rangle = |0^m\rangle$, with U acting as identity outside $\mathrm{span}\{|0^m\rangle, |S\rangle|\psi\rangle\}$. U is simply a reflection about the state $\frac{1}{\sqrt{2}}\big(|0^m\rangle - |S\rangle|\psi\rangle\big)$. This state is correctly normalized

[14] We thank Alexander Belov (personal communication) for raising this question.

because we assumed that $|S\rangle|\psi\rangle$ is orthogonal to $|0^m\rangle$. Clearly U is now fully specified on the entire domain (once we have fixed $|\psi\rangle$) and it does not seem to leak any additional information about S.

In order to prove concrete lower bounds on the cost of algorithms for approximate counting given access to U, we need to fix $|\psi\rangle$. To answer the question posed in this section, we only need to establish that there exists *some* choice of $|\psi\rangle$ for which our algorithms cannot be improved. (Note that we cannot hope to establish lower bounds for arbitrary $|\psi\rangle$, since $|\psi\rangle$ could just contain the answer to the problem we are solving.)

To this end we make the specific choice of $|\psi\rangle = |S\rangle$ and consider the unitary V that acts as the unitary U above with $|\psi\rangle = |S\rangle$. In other words, V maps $|0^m\rangle$ to $|S\rangle|S\rangle$, $|S\rangle|S\rangle$ to $|0^m\rangle$, and acts as identity on the rest of the space. We also assume that $|0^m\rangle$ is orthogonal to $|S\rangle|S\rangle$. In other words, V simply reflects about the state $\frac{1}{\sqrt{2}}\big(|0^m\rangle - |S\rangle|S\rangle\big)$.

As previously discussed, granting an algorithm access to this unitary V lends the algorithm at least as much power the ability to QSample and perform reflections about $|S\rangle$. How efficiently can we solve approximate counting with membership queries and uses of the unitary V?

We can use our Laurent polynomial method to establish optimal lower bounds in this model as well and we obtain lower bounds identical to Theorem 4.

Theorem 31. *Let Q be a quantum algorithm that makes T queries to the membership oracle for S, and makes R uses of the unitary V defined above (and its inverse and controlled-V). If Q decides whether $|S| = w$ or $|S| = 2w$ with high probability, promised that one of those is the case, then either*

$$T = \Omega\left(\sqrt{\frac{N}{w}}\right) \qquad or \qquad R = \Omega\left(\min\left\{w^{1/3}, \sqrt{\frac{N}{w}}\right\}\right). \tag{72}$$

Proof. We follow the same strategy as in the proof of Theorem 4. Recall that $x \in \{0,1\}^N$ denotes the indicator vector of the set S. We only need to show that such a quantum algorithm gives rise to a Laurent polynomial in $|S| := \sum_{i=1}^n x_i$, with maximum exponent $O(T+R)$ and minimum exponent at least $-O(R)$ (as shown in Lemma 21 for the QSamples and reflections model).

We can prove this exactly the same way as Lemma 21 is established. Our quantum algorithm starts out from a canonical starting state that does not depend on the input and hence each entry of the starting state is a degree-0 polynomial. Membership queries involve multiplication with an oracle whose entries are ordinary polynomials of degree at most 1. The only thing that remains is understanding what the entries of the unitary V look like. We claim that the entries of V are given by a polynomial of degree at most 2 in the entries of the input x, with all coefficients of this degree-2 polynomial equal to either a constant, or a constant multiple of $|S|^{-1}$.

To see this, note that V is simply a reflection about the state

$$\frac{1}{\sqrt{2}}\big(|0^m\rangle - |S\rangle|S\rangle\big) = \frac{1}{\sqrt{2}}\left(|0^m\rangle - \frac{1}{|S|}\Big(\sum_i x_i|i\rangle\Big)\Big(\sum_j x_j|j\rangle\Big)\right). \tag{73}$$

The coefficient in front of $|0^m\rangle$ is a degree-0 polynomial and the other nonzero coefficients are a polynomial of degree at most 2 in the entries of the input x, with each coefficient of this polynomial equal to a constant multiple of $|S|^{-1}$.

Hence, each entry of the unitary V is also a polynomial of degree at most 2 in the entries of the input x, with each coefficient of this degree-2 polynomial equal to either a constant, or a constant

multiple of $|S|^{-1}$. The same also holds for controlled-V, since that unitary is just the direct sum of identity with V. V is also self-inverse, so we do not need to account for that separately.

After the algorithm has made all the membership queries and uses of V, each amplitude of the final quantum state can be expressed as a polynomial of degree $O(T+R)$ in the input x, in which all coefficients are constant multiples of $|S|^{-R}$. The acceptance probability $p(x)$ of this algorithm will be a sum of squares of such polynomials. Exactly as in the proof of Theorem 4, Lemma 11 implies that there is a univariate polynomial q of degree at most $O(T+R)$, with coefficients that are multiples of the coefficients of p, such that for all integers $k \in \{0,\ldots,N\}$,

$$q(k) := \mathbb{E}_{|X|=k}\left[p(X)\right]. \tag{74}$$

Since the coefficients of $p(X)$ are constant multiples of $|X|^{-2R}$, q is in fact a real Laurent polynomial in k, with maximum exponent at most $O(R+T)$ and minimum exponent at least $-2R$. The theorem follows by a direct application Theorem 25 to q. ■

5 Discussion and open problems

5.1 Approximate counting with QSamples and queries only

If we consider the model where we only have membership queries and samples (but no reflections), then the best upper bound we can show is $O\left(\min\left\{\sqrt{w}, \sqrt{N/w}\right\}\right)$, using the sampling algorithm that looks for birthday collisions, and the quantum counting algorithm. It would be interesting to improve the lower bound further in this case, but it is clear that the Laurent polynomial approach cannot do so, since it hits a limit at $w^{1/3}$. Hence a new approach is needed to tackle the model without reflections.

We now give what we think is a viable path to solve this problem. Specifically, we observe that our problem—of lower-bounding the number of copies of $|S\rangle$ *and* the number of queries to $\mathcal{O}_S$ needed for approximate counting of S—can be reduced to a pure problem of lower-bounding the number of copies of $|S\rangle$. To do so, we use a hybrid argument, closely analogous to an argument recently given by Zhandry [Zha19] in the context of quantum money.

Given a subset $S \subseteq [L]$, let $|S\rangle$ be a uniform superposition over S elements. Then let

$$\rho_{L,w,k} := \mathbb{E}_{S\subseteq[L]\,:\,|S|=w}\left[\left(|S\rangle\langle S|\right)^{\otimes k}\right] \tag{75}$$

be the mixed state obtained by first choosing S uniformly at random subject to $|S|=w$, then taking k copies of $|S\rangle$. Given two mixed states ρ and σ, recall also that the *trace distance*, $\|\rho-\sigma\|_{\mathrm{tr}}$, is the maximum bias with which ρ can be distinguished from σ by a single-shot measurement.

Theorem 32. *Let $2w \leq L \leq N$. Suppose $\|\rho_{L,w,k} - \rho_{L,2w,k}\|_{\mathrm{tr}} \leq \frac{1}{10}$. Then any quantum algorithm Q requires either $\Omega\left(\sqrt{\frac{N}{L}}\right)$ queries to $\mathcal{O}_S$ or else $\Omega(k)$ copies of $|S\rangle$ to decide whether $|S|=w$ or $|S|=2w$ with success probability at least $2/3$, promised that one of those is the case.*

Proof. Choose a subset $S \subseteq [N]$ uniformly at random, subject to $|S|=w$ or $|S|=2w$, and consider S to be fixed. Then suppose we choose $U \subseteq [N]$ uniformly at random, subject to both $|U|=L$ and $S \subseteq U$. Consider the hybrid in which Q is still given R copies of the state $|S\rangle$, but now gets

oracle access to $\mathcal{O}_U$ rather than $\mathcal{O}_S$. Then so long as Q makes $o\left(\sqrt{\frac{N}{L}}\right)$ queries to its oracle, we claim that Q cannot distinguish this hybrid from the "true" situation (i.e., the one where Q queries $\mathcal{O}_S$) with $\Omega(1)$ bias. This claim follows almost immediately from the BBBV Theorem [BBBV97]. In effect, Q is searching the set $[N] \setminus S$ for any elements of $U \setminus S$ (the "marked items," in this context), of which there are $L - |S|$ scattered uniformly at random. In such a case, we know that $\Omega\left(\sqrt{\frac{N-|S|}{L-|S|}}\right) = \Omega\left(\sqrt{\frac{N}{L}}\right)$ quantum queries are needed to detect the marked items with constant bias.

Next suppose we first choose $U \subseteq [N]$ uniformly at random, subject to $|U| = L$, and consider U to be fixed. We then choose $S \subseteq U$ uniformly at random, subject to $|S| = w$ or $|S| = 2w$. Note that this produces a distribution over (S, U) pairs identical to the distribution that we had above. In this case, however, since U is fixed, queries to $\mathcal{O}_U$ are no longer relevant. The only way to decide whether $|S| = w$ or $|S| = 2w$ is by using our copies of $|S\rangle$—of which, by assumption, we need $\Omega(k)$ to succeed with constant bias, even after having fixed U. ■

One might think that Theorem 32 would lead to immediate improvements to our lower bound for the queries and samples model. In practice, however, the best lower bounds that we currently have, even purely on the number of copies of $|S\rangle$, come from the Laurent polynomial method (Theorem 4)! Having said that, we are optimistic that one could obtain a lower bound that beats Theorem 4 at least when w is small, by combining Theorem 32 with a brute-force computation of trace distance.

5.2 Approximate counting to multiplicative factor $1 + \varepsilon$

Throughout, we considered the task of approximating $|S|$ to within a multiplicative factor of 2. But suppose our task was to distinguish the case $|S| \leq w$ from the case $|S| \geq (1+\varepsilon)\, w$; then what is the optimal dependence on ε?

In the model with quantum membership queries only, the algorithm of Brassard et al. [BHMT02, Theorem 15] makes $O\left(\frac{1}{\varepsilon}\sqrt{\frac{N}{w}}\right)$ queries, which is optimal [NW99]. The algorithm uses amplitude amplification, the basic primitive of Grover's search algorithm [Gro96]. The original algorithm of Brassard et al. also used quantum phase estimation, in effect *combining* Grover's algorithm with Shor's period-finding algorithm. However, one can remove the phase estimation, and adapt Grover search with an unknown number of marked items to get an approximate count of the number of marked items [AR19].

One can also show without too much difficulty that in the queries+QSamples model, the problem can be solved with

$$O\left(\min\left\{\frac{\sqrt{w}}{\varepsilon^2}, \frac{1}{\varepsilon}\sqrt{\frac{N}{w}}\right\}\right) \tag{76}$$

queries and copies of $|S\rangle$. As observed after Theorem 5, the problem can also be solved with

$$O\left(\min\left\{\frac{w^{1/3}}{\varepsilon^{2/3}}, \frac{1}{\varepsilon}\sqrt{\frac{N}{w}}\right\}\right) \tag{77}$$

samples and reflections. On the lower bound side, what generalizations of Theorem 4 can we prove that incorporate ε? We note that the explosion argument doesn't automatically generalize; one would need to modify something to continue getting growth in the polynomials u and v after the

first iteration. The lower bound using dual polynomials should generalize, but back-of-the-envelope calculations show that the lower bound does not match the upper bound.

5.3 Other questions

Non-oracular example of our result. Is there any interesting real-world example of a class of sets for which QSampling and membership testing are both efficient, but approximate counting is not? (I.e., is there an interesting non-black-box setting that appears to exhibit the behavior that this paper showed can occur in the black-box setting?)

The Laurent polynomial connection. At a deeper level, is there is any meaningful connection between our two uses of Laurent polynomials? And what other applications can be found for the Laurent polynomial method?

6 Followup work

Since this work was completed, Belovs and Rosmanis [BR20] obtained essentially tight lower bounds on the complexity of approximate counting with access to membership queries, QSamples, reflections, and a unitary transformation that prepares the QSampling state, for all possible tradeoffs between these different resources. Additionally, they resolve the ε-dependence of approximate counting to multiplicative factor $1+\varepsilon$. The techniques involved are quite different from ours: Belovs and Rosmanis use a generalized version of the quantum adversary bound that allows for multiple oracles, combined with tools from the representation theory of the symmetric group.

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A Establishing Equation 67

A.1 A clean calculation establishing a loose version of equation 67

For clarity of exposition, we begin by presenting a relatively clean calculation that establishes a slightly loose version of Equation (67). Using just this looser bound, we would be able to establish that Equation (67) holds (with the constant $1/2$ replaced by a slightly smaller constant) so long as we set d_1 to be $\Theta\left(w^{1/3}/\log w\right)$. A slightly more involved calculation (cf. Appendix A.2) is required to establish Equation (67) for our desired value of $d_1 = \lfloor (w/c)^{1/3} \rfloor$.

Expression (66) equals

$$\left(\frac{w}{ci^2}\right)^{d_1-1} \cdot \frac{i^2}{((d_1)!)^2} \cdot \prod_{j=1,j\neq i}^{d_1} \left(|j-i| \cdot |j+i| - \frac{ci^2j^2}{w}\right)$$

$$= \left(\frac{w}{ci^2}\right)^{d_1-1} \cdot \frac{i^2}{((d_1)!)^2} \cdot \prod_{j=1,j\neq i}^{d_1} (|j-i| \cdot |j+i|) \cdot \left(1 - \frac{ci^2j^2}{w \cdot |j-i||j+i|}\right)$$

$$= \left(\frac{w}{ci^2}\right)^{d_1-1} \cdot \frac{(d_1+i)!(d_1-i)!}{2\left((d_1)!\right)^2} \cdot \prod_{j=1,j\neq i}^{d_1} \left(1 - \frac{ci^2j^2}{w \cdot |j-i||j+i|}\right) \tag{78}$$

$$\geq \left(\frac{w}{ci^2}\right)^{d_1-1} \cdot \frac{1}{2} \cdot \prod_{j=1,j\neq i}^{d_1} \left(1 - \frac{ci^2j^2}{w \cdot |j-i||j+i|}\right)$$

$$\geq \left(\frac{w}{ci^2}\right)^{d_1-1} \cdot \frac{1}{2} \cdot \left(1 - \sum_{j=1,j\neq i}^{d_1} \frac{ci^2j^2}{w \cdot |j-i||j+i|}\right)$$

$$\geq \left(\frac{w}{ci^2}\right)^{d_1-1} \cdot \frac{1}{2} \cdot \left(1 - \frac{ci^2}{w} \sum_{j=1,j\neq i}^{d_1} \frac{j^2}{|j-i||j+i|}\right). \tag{79}$$

Let us consider the expression $\sum_{j=1,j\neq i}^{d_1} \frac{j^2}{|j-i||j+i|}$. If $i^2 \notin [j^2/2, 3j^2/2]$, then the j'th term in this sum is at most 2. Hence, letting H_i denote the ith Harmonic number and using the fact that $H_i \leq \ln(i+1)$,

$$\sum_{j=1,j\neq i}^{d_1} \frac{j^2}{|j-i||j+i|}$$

$$\leq 2 \cdot d_1 + \sum_{j=\lfloor\sqrt{2/3}i\rfloor}^{\lfloor\sqrt{2}i\rfloor} \frac{j^2}{|j-i||j+i|}$$

$$\leq 2 \cdot d_1 + \sum_{j=\lfloor\sqrt{2/3}\cdot i\rfloor}^{\lceil\sqrt{2}\cdot i\rceil} \frac{j}{|j-i|}$$

$$\leq 2d_1 + \sqrt{2} \cdot i \cdot \sum_{j=1}^{(\sqrt{2}-1)\cdot i} 2/j$$

$$\leq 2d_1 + 2\sqrt{2} \cdot i \cdot H_i \leq 2d_1 + 2\sqrt{2}i\ln(i+1). \tag{80}$$

We conclude that if d_1 were set to a value less than $w^{1/3}/(100 \cdot c^2 \cdot \ln(w))$ (rather than to $\lfloor(w/c)^{1/3}\rfloor$), then Expression (79) is at least

$$\left(\frac{w}{ci^2}\right)^{d_1-1} \cdot \frac{1-1/c}{2}. \tag{81}$$

A.2 The tight bound

To obtain the tight bound, we need a tighter sequence of inequalities following Expression (78). Specifically, Expression (78) is bounded below by:

$$\begin{aligned}
&\geq \left(\frac{w}{ci^2}\right)^{d_1-1} \cdot \frac{1}{2}\left(1+\frac{i}{2d_1}\right)^{i} \cdot \prod_{j=1,j\neq i}^{d_1}\left(1-\frac{ci^2j^2}{w\cdot|j-i||j+i|}\right) \\
&\geq \left(\frac{w}{ci^2}\right)^{d_1-1} \cdot \frac{1}{2}\cdot e^{i^2/(2d_1)} \cdot \prod_{j=1,j\neq i}^{d_1}\left(1-\frac{ci^2j^2}{w\cdot|j-i||j+i|}\right) \\
&\geq \left(\frac{w}{ci^2}\right)^{d_1-1} \cdot \frac{1}{2}\cdot e^{i^2/(2d_1)} \cdot \prod_{j=1,j\neq i}^{d_1}\left(1-\frac{ci^2j^2}{w\cdot|j-i||j+i|}\right) \qquad (82)
\end{aligned}$$

The rough idea of how to proceed is as follows. Equation (80) implies that for $i \ll w^{1/3}/\ln w$, the factor

$$F_1 := \prod_{j=1,j\neq i}^{d_1}\left(1-\frac{ci^2j^2}{w\cdot|j-i||j+i|}\right)$$

is at some a positive constant, and hence Expression (82) is bounded below by the desired quantity. If $i \gtrsim w^{1/3}/\ln w$, then Equation (80) does not yield a good bound on this factor, leaving open the possibility that this factor is subconstant. But in this case, the factor $F_2 := e^{i^2/(2d_1)} \geq e^{\tilde{\Omega}(d_1)}$, and the largeness of F_2 dominates the smallness of F_1.

In more detail, let $x_{i,j} = \frac{ci^2j^2}{w\cdot|i-j|j+i||}$. Then for all $i \neq j$ such that $i, j \leq d_1$,

$$x_{i,j} \leq \frac{c\cdot d_1^2(d_1-1)^2}{(2d_1-1)\cdot w} \leq \frac{c\cdot d_1^3}{2w} \leq 1/2, \qquad (83)$$

where in the final inequality we used the fact that $d_1 \leq (w/c)^{1/3}$.

Using the fact that $1-x \geq e^{-x-x^2}$ for all $x \in [0, 1/2]$, we can write

$$F_1 \geq \prod_{j=1,j\neq i}^{d_1} e^{-x_{i,j}-x_{i,j}^2}.$$

Hence,

$$F_1\cdot F_2 \geq \exp\left(i^2/(2d_1) - \sum_{j=1,j\neq i}^{d_1} -x_{i,j}-x_{i,j}^2\right).$$

From Equations (80) and (83), we know that

$$\sum_{j=1,j\neq i}^{d_1} x_{i,j}+x_{i,j}^2 \leq \frac{ci^2}{w}\cdot\left(3d_1+3\sqrt{2}i\ln(i+1)\right) \leq \frac{ci^2}{w}\cdot(4d_1\ln(d_1)).$$

Hence,

$$\begin{aligned} F_1 \cdot F_2 &\geq \exp\left(i^2/(2d_1) - \frac{ci^2}{w} \cdot 4c\ln(d_1)\right) \\ &= \exp\left(i^2\left(\frac{1}{2d_1} - \frac{4c^2\ln(d_1)}{w}\right)\right) \\ &\geq \exp\left(i^2 \cdot \frac{1}{2d_1} \cdot (1 - o(1))\right) \\ &\geq 1. \end{aligned}$$

Equation (67) follows.

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