

Bounded-depth Frege complexity of Tseitin formulas for all graphs

Nicola Galesi¹, Dmitry Itsykson², Artur Riazanov², and Anastasia Sofronova^{2,3}

¹Dipartimento di Informatica, *Sapienza Università di Roma*

²St. Petersburg Department of V.A. Steklov Institute of Mathematics of the Russian Academy of Sciences

³St. Petersburg State University

nicola.galesi@uniroma1.it, dmitrits@pdmi.ras.ru, ariazanov@gmail.com, ana.a.sofronova@gmail.com

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Abstract

We prove that there is a constant K such that *Tseitin* formulas for an undirected graph G requires proofs of size $2^{\text{tw}(G)^{\Omega(1/d)}}$ in depth- d Frege systems for $d < \frac{K \log n}{\log \log n}$, where $\text{tw}(G)$ is the treewidth of G . This extends Håstad recent lower bound for the grid graph to any graph. Furthermore, we prove tightness of our bound up to a multiplicative constant in the top exponent. Namely, we show that if a Tseitin formula for a graph G has size s , then for all large enough d , it has a depth- d Frege proof of size $2^{\text{tw}(G)^{\mathcal{O}(1/d)}} \text{poly}(s)$. Through this result we settle the question posed by M. Alekhovich and A. Razborov of showing that the class of Tseitin formulas is quasi-automatizable for resolution.

1 Introduction

Propositional proof complexity is motivated by the result of Cook and Reckhow [13] saying that if there is a propositional proof system in which any unsatisfiable formula F has a short proof of unsatisfiability (of size polynomial in the size of F), then $\text{NP} = \text{coNP}$. In the last 30 years the complexity of proofs was investigated for several proof systems with the aim of finding concrete evidence, and eventually a proof, that for all proof systems there is a propositional formula which is not efficiently provable, i.e. requires super-polynomial proof size. The approach followed to prove such lower bounds was essentially borrowed from circuit complexity. Lines in a proof are Boolean formulas and we can define different proof system according to the circuit complexity of such formulas. For example resolution, a well-known refutational system for CNFs, corresponds to a system where formulas are of depth 1. In circuit complexity we keep on trying to strength lower bounds to computationally more powerful class of circuits. In proof complexity we follow the analogous approach: to strength lower bounds to systems working on formulas computationally more powerful. The hope is that techniques used to prove lower bounds for classes of Boolean circuits could be lifted to work with proof systems operating with formulas in the same circuit class. At present however we are far from such ideal situation and in fact, in terms of circuit classes, lower bounds for proof systems are well below those for Boolean circuits.

The complexity of proofs in resolution is largely studied. The first lower bound for (a restriction of) resolution was given by Tseitin in [35]. To obtain his result Tseitin introduced a class of formulas (nowadays known as *Tseitin formulas*) encoding a generalisation of the principle that the sum of the degrees of all vertices in a graph is an even number. A Tseitin formula $\text{T}(G, f)$ is defined for every undirected graph $G(V, E)$ and a charging function $f : V \rightarrow \{0, 1\}$. We introduce a propositional variable for every edge of G so that $\text{T}(G, f)$ is a CNF representation of a linear system over the field $\text{GF}(2)$ that for every vertex $v \in V$ states that the sum of all edges incident to v equals $f(v)$. Tseitin formulas, usually defined on graphs with good expansion properties, are among the main examples we could prove lower bounds for in different

proof systems. For unrestricted resolution it was Urquhart in [36] and later Ben-Sasson and Wigderson [9] who proved exponential lower bounds for Tseitin formulas over constant-degree expander graphs. Another example of an important principle largely studied in proof complexity is the Pigeonhole principle, PHP_n . Haken [19], Beame and Pitassi [5] and Ben-Sasson Wigderson [9] were proving exponential resolution lower bounds for CNF encoding of the negation of PHP_n , which were later generalized and improved in several other works [10, 14, 24, 30, 32, 33].

Bounded-depth Frege extends resolution since the formulas in the line of proofs are computable by AC^0 circuits, i.e. constant-depth circuits with unbounded fan-in gates. The importance of understanding the complexity of proofs in bounded-depth-Frege systems was due at least to two reasons: (1) for general Frege systems, where formulas have no restrictions, i.e. are of depth $\mathcal{O}(\log n)$, Buss in [11] proved that the Pigeonhole principle can be proved in polynomial size, hence obtaining an exponential separation with resolution. (2) Lower bounds for AC^0 -circuits were known [16, 21] and hence we could hope for applying lower bound techniques for AC^0 to lower bounds to bounded-depth Frege. Studying the complexity of proofs in bounded-depth Frege is of the utmost importance since it is a frontier proof system, i.e. is the strongest propositional proof system with known significant lower bounds at the moment. Any advance is then a step towards proving lower bounds for $\text{AC}^0[2]$ -Frege, i.e. a bounded-depth Frege admitting also formulas with parity gates, which are unknown at the moment, though we know since a long time exponential lower bounds for $\text{AC}^0[2]$ circuits [25, 31, 34]. In this work we contribute to the complexity of proofs in bounded-depth Frege proving new lower bounds for Tseitin formulas.

Ajtai in [1] was the first to prove a lower bound in bounded-depth Frege. He showed that a proof of PHP_n must have super-polynomial size. His result was later followed by several results simplifying his technique [6] and improving the lower bound [26, 27] showing that any polynomial-size Frege proof of PHP_n must have depth $\Omega(\log \log n)$. The proof complexity of Tseitin formulas in bounded-depth Frege was first considered by Urquhart and Fu in [37], a work where they simplified and adapted the lower bound for the PHP_n to the case of Tseitin formulas over a complete graph. Ben-Sasson in [8], proved exponential lower bounds for the Tseitin formulas over constant-degree expander graphs using a new reduction from the pigeonhole principle [37]. All these lower bounds are adaptation of the technique of [26, 27], hence vanish when the depth of formulas in the proof is more than $\log \log n$. In a very recent major breakthrough [28] showed that Tseitin formulas over a 3-expander graph of n nodes requires super-polynomial bounded-depth Frege proofs at depth $\mathcal{O}(\sqrt{\log n})$. Their result was later improved to depth up to $\frac{C \log n}{\log \log n}$ by Håstad in [22] but for Tseitin formulas defined only on the 2-dimensional grid, where C is a positive constant.

Proofs of $\text{T}(G, f)$ were studied in terms of the treewidth of G , $\text{tw}(G)$, for resolution [2, 17] and for OBDD proof systems [18]. We use Håstad result to prove tight bounds on the complexity of proofs in bounded-depth Frege of $\text{T}(G, f)$ over any graph G in terms of the treewidth of G . Our main result is the following theorem:

Theorem 1. *There is a constant K such that for any graph G over n nodes and for all $d \leq K \frac{\log n}{\log \log n}$, every depth- d Frege proof of $\neg \text{T}(G, f)$ has size at least $2^{\text{tw}(G)^{\Omega(1/d)}}$. Furthermore, for all large enough d there exist depth- d Frege proofs of $\neg \text{T}(G, f)$ of size $2^{\text{tw}(G)^{\mathcal{O}(1/d)}} \text{poly}(|\text{T}(G, f)|)$.*

A class of unsatisfiable CNF F_n is (quasi-)automatizable in a proof systems S , if there exists a deterministic algorithm that, given F in F_n returns a proof in S in time which is (quasi-)polynomial in $|F| + |\tau_F|$, where $|\tau_F|$ is the size of shortest proof of F in S . Theorem 1, together with the results from [2, 3, 17, 20] implies that for any graph G , the class of Tseitin formulas is *quasi-automatizable* in all systems between treelike resolution and constant-depth Frege. This answers the problem of [2] of extending to all graphs the quasi-automatizability of $\text{T}(G, f)$ in resolution, known only for graphs with bounded cyclicity [2] (see Section 5).

Using a result in [3, 23] we can also prove that the size of proofs of $\text{T}(G, f)$ in proof systems between tree-like resolution and bounded-depth Frege are quasi-polynomially correlated, i.e. *if $\text{T}(G, f)$ has a proof of size S in bounded-depth Frege, then it has a proof of size at most $2^{\text{poly}(\log S)}$ in treelike resolution and vice versa* (see Corollary 34). This result provides evidence to the conjecture of Alasdair Urquhart that the shortest resolution proofs of $\text{T}(G, f)$ are regular. Finally other consequences of Theorem 1 are: (1) It gives polynomial size Frege proofs of $\text{T}(G, f)$ of depth $\log(\text{tw}(G))$. (2) It improves the lower bounds of [8, 28] since expanders have treewidth $\Omega(n)$ and on such graphs our lower bound is $2^{n^{\Omega(1/d)}}$, which works for larger d than

[8, 28]; (3) Induces a strict depth-hierarchy for the proof complexities of Tseitin formulas over an infinite sequence of graphs G_n .

Overview of the proof technique. In Theorem 18 we prove the lower bound from Theorem 1. The proof is based on the improvement of the Excluded Grid Theorem by Robertson and Seymour recently obtained by Chuzhoy [12]: the arbitrary graph G contains as a minor a $r \times r$ grid, where $r = \Omega(\text{tw}(G)^{1/37})$. More precisely we use the corollary of this result (see Corollary 9) stating that any graph G has a *wall* of size r as a *topological minor* (i.e. can be obtained from G by several removing of vertices, edges and *suppressions*, see Fig. 1 and Fig. 3). Our proof consists of two parts: at first, we show that if H is a topological minor of G , then any bounded-depth Frege proof of a Tseitin formula $T(G, f)$ can be transformed to a proof of a $T(H, f')$, with constant increase in depth and polynomial increase in size. And then we prove a lower bound on the size of depth- d Frege proof of Tseitin formulas based on walls. In this proof we use the lower bound for grid graphs proved by Håstad [22].

In Theorem 19 we prove the upper bound from Theorem 1. We consider the *compact representation* of linear functions $\mathbb{F}_2^n \rightarrow \mathbb{F}_2$ from variables x_1, x_2, \dots, x_n by propositional formulas of depth d and of size $2^{n^{\mathcal{O}(1/d)}}$. We show that for linear functions f and g if the equations $f(x) = a$ and $g(x) = b$ are given in our representation, then there is a derivation of $(h + g)(x) = a + b$ of depth d and of size $2^{n^{\mathcal{O}(1/d)}}$. We also show that if a linear equation represented in CNF, then it is possible to infer its compact representation with depth d and size $2^{n^{\mathcal{O}(1/d)}}$. Since a Tseitin formula is an unsatisfiable system of linear equations written in CNF, hence it is possible to prove a Tseitin formula in size $2^{m^{\mathcal{O}(1/d)}}$ and depth d , where m is the number of edges in G . However we wish to have the treewidth of G instead of m . We consider a tree-partition of a graph G , the vertices of G are split into bags and there exists a tree such that bags are nodes of this tree and if two vertices of G are connected, then they are either in one bag or in adjacent bags. It is known that there is a tree partition where the size of bags are at most $\mathcal{O}(\text{tw}(G)\Delta(G))$ [38]. Since the number of edges touching a given bag is $\mathcal{O}(\text{tw}(G)\Delta(G)^2)$ we can use the compact representation to take care of the equations involving the parity of sum of adjacent bags with proofs growing in terms of the treewidth of G .

Organization. The paper is divided into four sections. After the Preliminary section, we have Section 3 for the lower bound (Theorem 18), Section 4 for the upper bound (Theorem 19) and a final Section 5 for the consequences of our result.

2 Preliminaries

Formulas and restrictions. We consider propositional formulas over binary \vee and \wedge , unary \neg and Boolean constants $\mathbf{0}, \mathbf{1}$. We represent formulas as rooted trees such that internal vertices are labeled with connectives and leaves are labeled with propositional variables or Boolean constants. The *depth* of a formula is the maximal number of alternations of types of connectives over all the paths from the root to a leaf plus one.

We assume that disjunctions with unbounded fanin are represented via binary disjunctions. By default we mean that $\bigvee_{i=1}^n x_i$ is right-associative, i.e. it denotes $(\dots(x_1 \vee x_2) \vee \dots) \vee x_{n-1} \vee x_n$; we also assume the same for \bigwedge .

We denote by $\text{vars}(F)$ the set of variables of a formula F . A *partial assignment* α for a formula F is mapping from $\text{vars}(F) \rightarrow \{0, 1, *\}$, where $\alpha(x) = *$ if x is unassigned. We denote by $\text{dom}(\alpha) = \alpha^{-1}(\{0, 1\})$ be the set of variables in F which α assigns a Boolean value.

Pudlák-Buss games. We use the game interpretation of Frege proofs introduced by Pudlák and Buss [29]. Let us define a game with two players Pavel and Sam. The game starts with initial conditions of the form $\varphi_1 = a_1, \dots, \varphi_k = a_k$, where $\varphi_1, \varphi_2, \dots, \varphi_k$ are propositional formulas and $a_1, a_2, \dots, a_k \in \{0, 1\}$ such that $\bigwedge_{i=1}^k (\varphi_i = a_i)$ is identically false. Sam claims that he knows an assignment of variables that satisfies $\bigwedge_{i=1}^k (\varphi_i = a_i)$, the goal of Pavel is to convict Sam. At each his move Pavel asks Sam the value of a propositional formula and Sam gives an answer. The game stops when Pavel convict Sam, namely Pavel finds an immediate contradiction among initial conditions and Sam's answers. An immediate contradiction with a Boolean connective \circ of arity t is a set of $(t + 1)$ formulas $\alpha_1, \dots, \alpha_t$ and $\circ(\alpha_1, \dots, \alpha_t)$ with claimed

values a_1, \dots, a_t and b such that $\circ(a_1, \dots, a_k) \neq b$. In particular, $\mathbf{0}$ with claimed value 1 is an immediate contradiction.

A strategy of Pavel is a function that maps initial conditions and the history of a game to a propositional formula (request). A winning strategy is a strategy that allows Pavel to convict Sam for any behaviour of Sam. A winning strategy of Pavel can be represented as a binary tree whose nodes are labeled with Pavel's requests and edges correspond to Sam's answers. A leaf of the tree corresponds to a immediate contradiction among initial conditions and equalities corresponding to the path from the root to this leaf.

A derivation of a formula ψ from formulas $\varphi_1, \varphi_2, \dots, \varphi_s$ is a tree of a Pavel's winning strategy in a game with initial conditions $\varphi_1 = 1, \varphi_2 = 1, \dots, \varphi_s = 1, \psi = 0$. We are interested in the two complexity parameters of derivations: 1) the size of a derivation S that equals the total size of formula ψ and all formulas that are used as labels of nodes; 4) the depth of a derivation d is the maximum depth of ψ and formulas that are used as labels of nodes. We use the notation $\varphi_1, \dots, \varphi_s \vdash_d \psi$ for a derivation of ψ from $\varphi_1, \varphi_2, \dots, \varphi_s$ of depth at most d . A derivation of φ is a derivation of φ from the empty set of formulas.

Lemma 2. *Assume that there is a derivation $\varphi_1, \dots, \varphi_k \vdash_{d_1} \psi_1$ of size S_1 and also there is a derivation $\varphi_1, \dots, \varphi_k, \psi_1 \vdash_{d_2} \psi_2$ of size S_2 , then there is a derivation $\varphi_1, \dots, \varphi_k \vdash_{\max\{d_1, d_2\}} \psi_2$ with size $S_1 + S_2$.*

Proof. Let us create the new tree with the root labelled with ψ_1 such that edge from the root labelled with 0 goes to the root of the first derivation and edge labelled with 1 goes to the root of the second derivation. \square

Lemma 3. *1. If a formula φ has a Frege derivation of size S and depth d , then φ has a Pudlák-Buss game derivation with number of size $\mathcal{O}(S^2)$ and depth d . 2. If φ has a Pudlák-Buss game derivation of size S and depth d , then φ a Frege derivation of size $\mathcal{O}(S^3)$ and depth $d + \mathcal{O}(1)$.*

Proof. For proof see Appendix A. \square

Lemma 4. *Let ψ_1 and ψ_2 be two formulas of depth at most d such that $|\text{vars}(\psi_1) \cup \text{vars}(\psi_2)| = k$ and ψ_1 semantically implies ψ_2 . Then there exists a derivation $\psi_1 \vdash_d \psi_2$ of size at most $2^k (|\psi_1|^2 + |\psi_2|^2)$.*

Proof. Consider a game with initial conditions $\psi_1 = 1, \psi_2 = 0$.

Let $|\text{vars}(\psi_1) \cup \text{vars}(\psi_2)| = \{x_1, \dots, x_k\}$. In the first k rounds of the game Pavel asks Sam x_1, \dots, x_k . There are 2^k possible combinations of Sam's answers. For each combination Pavel asks Sam all subformulas of ψ_1 (the value of ψ_1 is 1 from initial conditions) starting with the deeper ones and then asks all subformulas of ψ_2 starting with the deeper ones (the value of ψ_2 is 0 from initial conditions). Since all variables are known, one of possible answers to each of these questions yields an immediate contradiction. The total size of all subformulas of ψ_1 and ψ_2 is at most $|\psi_1|^2 + |\psi_2|^2$. Consider the branch of the game tree, where we still no get an immediate contradiction. In that case Sam $\psi_1 = 1$ and $\psi_2 = 0$ are consistent with the values of x_1, \dots, x_k , but it is impossible since ψ_1 semantically implies ψ_2 . Hence, there should be an immediate contradiction. \square

Corollary 5. *let ψ_1 and ψ_2 be two formulas of depth d_1 such that $\text{vars}(\psi_1) \cup \text{vars}(\psi_2) = \{x_1, x_2, \dots, x_k\}$ and ψ_2 is semantically implied by ψ_1 . Let $\varphi_1, \dots, \varphi_k$ be formulas of depth d_2 . Let $\psi_i[x_1 \leftarrow \varphi_1, \dots, x_k \leftarrow \varphi_k]$ for $i \in \{1, 2\}$ be the formula obtained by substitution $\varphi_1, \dots, \varphi_k$ instead of the variables of x_1, x_2, \dots, x_k . Then there exists a derivation*

$$\psi_1[x_1 \leftarrow \varphi_1, \dots, x_k \leftarrow \varphi_k] \vdash_{d_1+d_2} \psi_2[x_1 \leftarrow \varphi_1, \dots, x_k \leftarrow \varphi_k]$$

of size at most $2^k (|\psi_1|^2 + |\psi_2|^2) \left(\sum_{i=1}^k |\varphi_i| \right)$

Proof. Use Lemma 4 for ψ_1 and ψ_2 and then substitute $\varphi_1, \dots, \varphi_k$ instead of the variables x_1, \dots, x_k . \square

A *shortcut contradiction* for the disjunction is a situation where Pavel asks Sam formulas $\bigvee_{i=1}^k \alpha_i$ and α_j for $j \in [k]$ and gets the answers 0 and 1 respectively. Similarly a shortcut contradiction for the conjunction is a situation where Pavel asks Sam formulas $\bigwedge_{i=1}^k \alpha_i$ and α_j for $j \in [k]$ and gets the answers 1 and 0.

Lemma 6. Consider a derivation with of size S and of depth d that uses shortcut contradictions in leaves. Then there is an ordinary derivation of size at most S^3 and of depth d .

Proof. We transform a tree with shortcut contradictions to a tree without them. Let us consider a leaf with a shortcut contradiction for the disjunction (case of conjunction is similar), which means that on the path from root to that leaf Pavel has asked Sam formulas $\bigvee_{i=1}^k \alpha_i$ and α_j for $j \in [k]$ and got the answers 0 and 1 respectively. $\bigvee_{i=1}^k \alpha_i$ is represented as $\bigvee_{i=1}^q \alpha_i \vee \bigvee_{i=q+1}^k \alpha_i$ for some index q , where \vee is binary disjunction. Pavel asks Sam formulas $\bigvee_{i=1}^q \alpha_i$ and $\bigvee_{i=q+1}^k \alpha_i$.

1. If one of Sam's answers is 1, we get an immediate contradiction.
2. If both Sam's answers are 0, then if $j \leq q$, we continue this process with formula $\bigvee_{i=1}^q \alpha_i$ otherwise with formula $\bigvee_{i=q+1}^k \alpha_i$.

In the end of that process we get an immediate contradiction. We repeat this for all leaves with shortcut contradictions, the size of the resulting tree is at most S^3 since there are at most S leaves and the size of added formulas is at most S^2 for each leaf. \square

Tseitin Formulas. Let $G(V, E)$ be an undirected graph and $v \in V$. We denote by $E(v)$ the set of edges in E incident with v and by $N(v)$ the set of neighbours $u \in V$ of v , i.e. the u such that $(u, v) \in E(v)$.

A vertex-charging for $G(V, E)$ is a mapping $f : V \rightarrow \{0, 1\}$. We say that f is an odd-charging of G if $\sum_{v \in V} f(v) \equiv 1 \pmod{2}$. The *Tseitin formulas* defined on G using variables $x_e, e \in E$ are the formulas: $\top(G, f) := \bigwedge_{v \in V} \text{Par}(v)$, where $\text{Par}(v)$ is a CNF formula representing $\bigoplus_{e \in E(v)} x_e = f(v)$.

Lemma 7 ([36]). $\top(G, f)$ is unsatisfiable if and only if there is a connected component U of G such that the restriction of f on U is odd-charging.

In this work we will work with the tautological form of Tseitin formulas in the form of $\neg \top(G, f)$.

Grids, Walls, Minors, Topological Minors and Treewidth. We consider 4 structural operations on undirected graphs $G = (V, E)$ possibly with parallel edges, but without loops. We follow [7, 15].

- *edge removal* of $e \in E$. It produces the graph $[G \setminus e] = (V, E \setminus \{e\})$.
- *vertex removal* of $v \in V$. It produces the graph $[G \setminus v] = (V \setminus \{v\}, E \setminus E(v))$, where $E(v)$ is the set of edges in E incident with $v \in V$.
- *edge contraction* of $e = (uv) \in E$. Is the replacement of u and v with a single vertex such that edges incident to the new vertex are the edges other than e that were incident with u or v . The resulting graph $G \star e$ has one edge less than G .
- *vertex suppression* of a vertex v in G of degree 2. Let u and w be v 's neighbours in G . The suppression of v is obtained by deleting the two edges (uv) and (vw) and adding a new edge (wu) (possibly parallel to an existing one). The resulting graph $[G \setminus_s v]$ has one vertex less than G . See Figure 1.

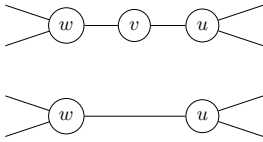


Figure 1: Suppression of v from G

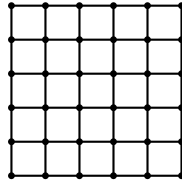


Figure 2: The grid $\mathcal{H}_{5,5}$

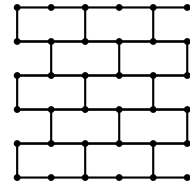


Figure 3: The wall \mathcal{W}_5

A graph H is a *minor* of G if H can be obtained from G by a sequences of edge and vertex removals and edge contractions. A graph H is a *topological minor* of G if H can be obtained from G by a sequence of edge removals, vertex removals and by vertex suppressions [7, 15].

The grid $\mathcal{H}_{m,n}$ is the graph of the cellular rectangle $m \times n$; it has $(m+1)(n+1)$ vertices and $n(m+1) + m(n+1)$ edges, among them $n(m+1)$ horizontal and $m(n+1)$ vertical edges. See fig. 2.

The wall \mathcal{W}_n is a subgraph of $\mathcal{H}_{n,n}$ that is obtained by the removing of several vertical edges. Vertical edges of $\mathcal{H}_{n,n}$ are in n rows and we enumerate them in every row from the left to the right. In the odd rows we remove all vertical edges with even numbers and in even rows we remove all vertical edges with odd numbers. See fig. 3.

A *tree decomposition* of an undirected graph $G(V, E)$ is a tree $T = (V_T, E_T)$ such that every vertex $u \in V_T$ corresponds to a set $X_u \subseteq V$ and it satisfies the following properties: 1. The union of X_u for $u \in V_T$ equals V . 2. For every edge $(a, b) \in E$ there exists $u \in V_T$ such that $a, b \in X_u$. 3. If a vertex $a \in V$ is in the sets X_u and X_v for some $u, v \in V_T$, then it is also in X_w for all w on the path between u and v in T .

The *width of a tree decomposition* is the maximum $|X_u|$ for $u \in V_T$ minus one. A *treewidth* of a graph G is the minimal value of the width among all tree decompositions of the graph G .

Recall the following Theorem proved in [12].

Theorem 8 ([12]). *If G has treewidth t , then it has the grid \mathcal{H}_r as a minor, where $r = \Omega(t^{1/37})$.*

The following Corollary was mentioned in [7].

Corollary 9. *If G has treewidth t , then it has the wall \mathcal{W}_r as a topological minor, where $r = \Omega(t^{1/37})$.*

Proof. Since \mathcal{W}_r is a minor of \mathcal{H}_r , \mathcal{W}_r is a minor of G . Consider a transformation T from G to \mathcal{W}_r made by (1) edge removals, (2) vertex removals and (3) edge contractions that has the *minimal* number of edge contractions. Since operations (1), (2) and (3) commute, we may assume that in T no removal follows an edge contraction (i.e. we remove everything at first, and then contract edges). Now it is easy to understand that all edge contractions should be actually suppressions. Indeed, assume that we contract (u, v) and both u and v has degrees at least 3, then we will get a vertex (uv) with degree at least 4 and this is a contradiction since \mathcal{W}_r does have degrees at most 3 and edge contractions can decrease degree only in one case when we contract (u, v) and v has degree 1, but in this case we may just remove v and it decreases the number of edge contractions. \square

3 Lower bound

3.1 Topological Minor Theorem on Tseitin Formulas

Let φ be a formula and let α be a partial assignment to variables of φ . Define $\varphi[\alpha]$ to be the formula obtained from φ substituting each variable x in the domain of α , with the constant assigned to x by α . Notice that φ and $\varphi[\alpha]$ have the same size and depth.

Lemma 10. *Let Φ_a and Φ'_a for $a \in A$ be propositional formulas of depth at most d such that $|\text{vars}(\Phi_a) \cup \text{vars}(\Phi'_a)| \leq k$. Assume that for all $a \in A$, Φ_a is semantically equivalent to Φ'_a . Then $\neg \bigwedge_{a \in A} \Phi'_a \vdash_{d+\mathcal{O}(1)} \neg \bigwedge_{a \in A'} \Phi_a$ of size at most $2^k \text{poly}(\sum_{a \in A} (|\Phi_a| + |\Phi'_a|))$, where $A' = \{a \in A \mid \Phi_a \text{ is not identically true}\}$.*

Proof. At first Pavel asks $\bigwedge_{a \in A} \Phi'_a$. The answer 1 leads to an immediate contradiction. Then Pavel proceeds to ask Φ'_a for each $a \in A$. If all the answers are 1, Pavel asks all subformulas of the big conjunction $\bigwedge_{a \in A} \Phi'_a$ starting with the deeper ones and at each step one of Sam's answers leads to an immediate contradiction.

If Sam claim that $\Phi'_{a_0} = 0$ for some $a_0 \in A$, we use Lemma 4 to construct a derivation $\neg \Phi'_{a_0} \vdash_d \neg \Phi_{a_0}$ of size at most $2^k (|\Phi'_{a_0}|^2 + |\Phi_{a_0}|^2)$. If $a_0 \notin A'$, then Φ_{a_0} is identically true, thus, by Lemma 4, there is a derivation $\mathbf{1} \vdash_d \Phi_{a_0}$ of size at most $2^k (|\Phi_{a_0}|^2 + 1)$, using it we get an immediate contradiction.

If $a_0 \in A'$, then Pavel asks $\bigwedge_{a \in A'} \Phi_a$. If Sam's answer is 0, we get a shortcut contradiction with Φ_{a_0} and proceed with Lemma 6, otherwise we get an immediate contradiction with the initial condition $\neg \bigwedge_{a \in A'} \Phi_a = 0$. \square

Lemma 11 ([18]). *Let $G(V, E)$ be a connected graph and $H(V', E')$ be a connected subgraph of G with $E' \neq \emptyset$ that is obtained from G by the deletion of some vertices and edges. For every unsatisfiable Tseitin*

formula $\top(G, f)$ there exists a partial assignment α on variables x_e for $e \in E \setminus E'$ such that α does not falsify any clause of $\top(G, f)$.

Lemma 12. *Let $G(V, E)$ be a connected graph and $H(V', E')$ be a connected subgraph of G . Assume that there is a derivation $\vdash_d \neg \top(G, f)$ of size S . Then for some f' there is a derivation $\vdash_{d+\mathcal{O}(1)} \neg \top(H, f')$ of size $S + \text{poly}(|\top(G, f)|)$.*

Proof. Let T be the game tree of $\vdash_d \neg \top(G, f)$. Let α be given by Lemma 11 that is defined on all variables x_e for $e \in E \setminus E'$ and does not falsify any clause of $\top(G, f)$. $T[\alpha]$ be the tree obtained from T applying the substitution α to all the queried formulas. Size and depth do not change, hence $T[\alpha]$ defines a derivation $\vdash_d \neg \top(G, f)[\alpha]$ of size S . $\neg \top(G, f)$ has the form $\neg \bigwedge_{v \in V} \text{Par}(v)$, where $\text{Par}(v)$ is a parity condition of the vertex v . Hence, $\neg \top(G, f)[\alpha]$ is of the form $\neg \bigwedge_i \text{Par}(v)[\alpha]$. If $v \notin V'$, then α assigns values to all variables from $\text{Par}(v)$, since α does not falsify $\text{Par}(v)$, α satisfies $\text{Par}(v)$, hence $\text{Par}(v)[\alpha]$ is identically true. If $v \in V'$, then $\text{Par}(v)[\alpha]$ is a parity statement depending on variables x_e , where $e \in E'$ is incident to v . Hence, for $v \in V'$, $\text{Par}(v)[\alpha]$ is semantically equivalent to a parity condition of a Tseitin formula $\top(H, \varphi')$ for some charging φ' . Let Δ be the maximal degree of G . Then every parity condition of $\top(H, \varphi')$ or $\top(G, \varphi)$ depends on at most Δ variables. Notice that since we represent parities in CNF, $|\top(G, f)| \geq 2^\Delta$. By Lemma 10, there is a derivation $\neg \top(G, f)[\alpha] \vdash_{\mathcal{O}(1)} \neg \top(H, f')$ of size $\text{poly}(|\top(G, f)|)$. The claim follows using the size S , depth d derivation of $\neg \top(G, f)[\alpha]$ together with Lemma 2. \square

A 1-substitution for a formula φ is a partial function mapping variables of φ into its literals. After applying a 1-substitution σ to φ , the depth of the new formula $\varphi[\sigma]$ can increase by one. However 1-substitutions are closed under composition: if σ_1 maps $[y \mapsto \neg z]$ and σ_2 maps $[x \mapsto \neg y]$, then $\sigma = \sigma_1 \circ \sigma_2$ is the 1-substitution $[x \mapsto z, y \mapsto \neg z]$. We use 1-substitutions to handle in $\top(G, f)$ the operation of *vertex suppression* on the graph G . Let $G = (V, E)$ be a graph and $v \in V$ be a node and let $\top(G, f)$ a Tseitin formula on G . Let v be a degree-2 vertex v in G with neighbours u and w . Consider the following 1-substitution σ_v and the charge function f_v for $[G \setminus_s v]$:

$$\sigma_v = \begin{cases} [x_{vw} \mapsto x_{wu}, x_{vu} \mapsto x_{wu}] & f(v) = 0 \\ [x_{vw} \mapsto x_{wu}, x_{vu} \mapsto \neg x_{wu}] & f(v) = 1 \end{cases} \quad f_v(z) = \begin{cases} f(z) & z \in V \setminus \{u, v\} \\ f(u) + f(v) & z = u \end{cases}$$

Let $G(V, E)$ be a graph and $f : V \rightarrow \{0, 1\}$ be a charging. Let A be a finite set. We say that a formula Ψ is a *pseudo Tseitin formula based on G and f with fake vertices in A* , and we write Ψ is $\top_A^*(G, f)$, if Ψ has the form $\bigwedge_{v \in V \cup A} \psi_v$, where

1. for all $v \in V$, ψ_v is a propositional formula depending on variables x_e for all edges e incident to v . And ψ_v is semantically equivalent to parity condition $\text{Par}(v)$ of $\top(G, f)$.
2. for all $v \in A$, ψ_v is a tautology.

Lemma 13. *Let $G(V, E)$ be a connected constant-degree graph over n vertices. Let $[G \setminus_s v]$ be the graph obtained after the suppression of the degree-2 vertex v in G . If Ψ is $\top_A^*(G, f)$, then $\Psi[\sigma_v]$ is $\top_{A \cup \{v\}}^*([G \setminus_s v], f_v)$.*

Proof. Assume that v is linked to the two vertices w and u in G . Let A be the set of fake vertices of Ψ so Ψ has the form $\bigwedge_{x \in V \cup A} \psi_x$, hence $\Psi[\sigma_v]$ is $\bigwedge_{x \in V \cup A} \psi_x[\sigma_v]$. For $x \in A$, $\psi(x)$ is a tautology, hence $\psi_x[\sigma_v]$ is also a tautology. By the definition of σ_v , $\psi_v[\sigma_v]$ is a tautology. It is not hard to verify that for $x \in V \setminus \{v\}$, $\psi_x[\sigma_v]$ is equivalent to parity condition of $\top([G \setminus_s v], f_v)$. Hence, $\Psi[\sigma_v]$ is $\top_{A \cup \{v\}}^*([G \setminus_s v], f_v)$. \square

Lemma 14. *Let $G(V, E)$ be a graph and $f : V \rightarrow \{0, 1\}$ and $W = \{v_1, \dots, v_k\}$ be degree 2 nodes in V suppressed in that order from G and $[G \setminus_s W]$ be the resulting graph. Let σ_i be the corresponding 1-substitutions and let $\sigma = \sigma_k \circ \dots \circ \sigma_1$. There is a charging f_k of G such that if Ψ is $\top_A^*(G, f)$, then $\Psi[\sigma]$ is $\top_{A \cup W}^*([G \setminus_s W], f_k)$.*

Proof. By induction on $k \geq 0$ and using Lemma 13 we prove that $\Psi[\sigma_1] \dots [\sigma_k]$ is $\top_{A \cup W}^*([G \setminus_s W], f_k)$, for f_k given by repeated applications of Lemma 13.

Since the composition of 1-substitutions removes double negations, then the only difference between $\Psi[\sigma_1] \dots [\sigma_k]$ and $\Psi[\sigma]$ can be only in the number of negations in front of the same occurrence of a variable,

though the parity of the number of negations is preserved. Hence for all $v \in V$ the semantical equivalence of the ψ_v 's in $\Psi[\sigma_1] \dots [\sigma_k]$ and $\Psi[\sigma]$ is preserved. Then $\Psi[\sigma]$ too is $\mathsf{T}_{A \cup W}^*([G \setminus_s W], f_k)$. \square

Lemma 15. *Let G be a connected graph on n vertices and maximal degree at most 3. Let H be obtained from G by several suppressions. Assume that there is a derivation of $\neg \mathsf{T}(G, f)$ of size S and depth d . Then for some charging f_k there is a derivation of $\neg \mathsf{T}(H, f_k)$ of size $\mathcal{O}(S) + \text{poly}(n)$ and depth $d + \mathcal{O}(1)$.*

Proof. Assume that, in order, to get H from G we have to apply suppressions for vertices $W = \{v_1, \dots, v_k\}$. Let σ_i be the 1-substitutions corresponding to the suppression of v_i , and let $\sigma = \sigma_k \circ \dots \circ \sigma_1$. $\mathsf{T}(G, f)$ is $\mathsf{T}_{\emptyset}^*(G, f)$. Let f_k be the charging given by Lemma 14 applied to $\mathsf{T}(G, f)$ and $[G \setminus_s W] = H$. Then $\mathsf{T}(G, f)[\sigma]$ is $\mathsf{T}_W^*(H, f_k)$. We apply the 1-substitution σ to the given derivation of $\neg \mathsf{T}(G, f)$ and we get a derivation of $\neg \mathsf{T}(G, f)[\sigma]$ of size $\mathcal{O}(S)$ and depth at most $d + 1$. By Lemma 10, applied on $\mathsf{T}(G, f)[\sigma]$ and $\mathsf{T}(H, f_k)$, there is a derivation $\neg \mathsf{T}(G, f)[\sigma] \vdash_{d+\mathcal{O}(1)} \neg \mathsf{T}(H, f_k)$ of size $\text{poly}(n)$. Combining the two derivations together by Lemma 2 we obtain a derivation $\vdash_{d+\mathcal{O}(1)} \mathsf{T}(H, f_k)$ of size $\mathcal{O}(s) + \text{poly}(n)$. \square

3.2 From Walls To Grids

Lemma 16. *If there exists a derivation $\vdash_d \neg \mathsf{T}(\mathcal{W}_n, f)$ of size S , then there exists a derivation $\vdash_{d+\mathcal{O}(1)} \neg \mathsf{T}(M_n, f')$ of size $\mathcal{O}(S) + \text{poly}(n)$, where M_n is a connected constant-degree graph that contains $\mathcal{H}_{n, \lfloor \frac{n-1}{2} \rfloor}$ as a subgraph.*

Proof. Consider a set I of all the horizontal edges of \mathcal{W}_n that belong to odd columns (on fig. 4 and 5 edges from I are red). I is a matching, i.e. no two edges from I are incident to the same vertex. If we contract all edges from I , we get the graph M_n that for odd n coincides with $\mathcal{H}_{n, \frac{n-1}{2}}$ and for even n coincides with a graph that is obtained from $\mathcal{H}_{n, \lfloor \frac{n}{2} \rfloor}$ by the removal of several edges from the last vertical (see fig. 4 and 5). For every $e \in I$ we denote its left vertex by u_e and the right vertex by v_e . Let E_{u_e} be the set of edges of \mathcal{W}_n incident to u_e except e . Let τ_e denote a CNF formula encoding $\bigoplus_{f \in E_{u_e}} x_f = f(u_e)$.

Consider a game tree T for the derivation of the Tseitin tautology $\neg \mathsf{T}(\mathcal{W}_n, f)$ of size S and depth d . To every formula used in this tree we apply the substitution that replaces every occurrence of x_e with τ_e . We denote the resulting tree by T' .

Notice that T' is a correct game tree of a derivation $\vdash_{d+\mathcal{O}(1)} \neg F$, where F is obtained from $\mathsf{T}(\mathcal{W}_n, f)$ by the same substitution. The depth of this derivation is increased by at most a constant since in several leaves we hang a formula of constant depth; here we also use that I is a matching and thus we do not add new occurrences of variables corresponding edges from I . The size of τ_e is $\mathcal{O}(1)$, hence any formula from the derivation is increased in at most a constant factor, thus the size of the derivation defined by the tree T' is $\mathcal{O}(S)$.

We define a function f' on vertices of M_n as follows. If a vertex w of the graph M_n is obtained by merging the vertices w', w'' of the graph \mathcal{W}_n , then $f'(w) = (f(w') + f(w'')) \bmod 2$. If the vertex w of $\mathcal{H}_{n, \lfloor n/2 \rfloor}$ is obtained from the vertex w of \mathcal{W}_n , then $f'(w) = f(w)$.

Now we show how to derive $\neg \mathsf{T}(M_n, f')$ from $\neg F$. $\mathsf{T}(\mathcal{W}_n, f)$ is a Tseitin formula and it has the following structure: $\bigwedge_{v \in V} \psi_v$, where V is the set of vertices of \mathcal{W}_n and ψ_v is a CNF formula encoding a parity condition for the vertex v . F differs from $\mathsf{T}(\mathcal{W}_n, f)$ only in conditions corresponding to vertices that are incident to an edge from I (if n is even, then there are vertices in \mathcal{W}_n that are not incident to any edge from I). Notice that F has the form $\bigwedge_{v \in V} \psi'_v$ where ψ'_v is obtained by substitution from ψ_v . Let $w = u_e$ for some $e \in I$, then the formula ψ'_w is identically true. If $w = v_e$, then the condition ψ'_w is equivalent to the parity condition of the merged vertex $\{u_e, v_e\}$ in the Tseitin formula $\mathsf{T}(M_n, f')$, but ψ'_w is not written in canonical form.

Since all degrees in M_n are at most 4, then by Lemma 10 there exists a derivation $\neg F \vdash_{d+\mathcal{O}(1)} \neg \mathsf{T}(M_n, f')$ of size $\text{poly}(n)$. The claim follows by Lemma 2. \square

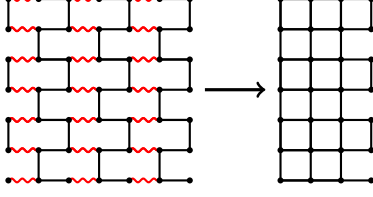


Figure 4: \mathcal{W}_6 is contracted to M_6 .

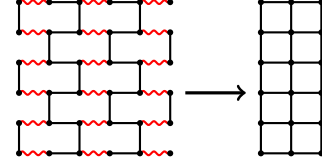


Figure 5: \mathcal{W}_5 is contracted to M_5 .

3.3 Putting it all together

We use Håstad's Theorem from [22].

Theorem 17 ([22]). *There is a constant $K > 0$ such that for $d \leq \frac{K \log n}{\log \log n}$ any depth d derivation of $\neg \mathsf{T}(\mathcal{H}_n, f)$ has size at least $2^{n^{\Omega(1/d)}}$.*

Theorem 18. *There exist constants $K > 0$ and $C > 0$ such that for every connected graph G of treewidth t for every $d \leq \frac{K \log n}{\log \log n} - C$, any depth d derivation of $\neg \mathsf{T}(G, f)$ has size at least $2^{t^{\Omega(1/d)}}$.*

Proof. Suppose that $\neg \mathsf{T}(G, f)$ have a derivation of size S and depth d . By Corollary 9 we know that G contains the wall \mathcal{W}_r as a topological minor, where $r = \Omega(t^{1/37})$. Consider a sequence of operations (edge/vertex removals and suppressions) that transform G to \mathcal{W}_r . Assume that removals do not follow suppressions. And let G' be a subgraph of G that is obtained from G by application of all removals (hence, \mathcal{W}_r can be obtained from G' by application of several suppressions).

By Lemma 12, for some f' there is a derivation of $\neg \mathsf{T}(G', f')$ of size $\text{poly}(|\mathsf{T}(G, f)|) + S$ and depth $d + \mathcal{O}(1)$. Since \mathcal{W}_r can be obtained from G' by application of several suppressions, G' is connected. suppressions can not increase the degrees, hence all degrees in G' are at most 3. By Lemma 15, for some φ'' there is a derivation of $\neg \mathsf{T}(\mathcal{W}_r, f'')$ of size $\text{poly}(|\mathsf{T}(G, f)|) + S$ and depth $d + \mathcal{O}(1)$. By Lemma 16, for some φ''' there is a derivation of $\neg \mathsf{T}(M_r, f''')$ of size $\text{poly}(|\mathsf{T}(G, f)|) + \mathcal{O}(S)$ and depth $d + \mathcal{O}(1)$, where M_r is connected constant-degree graph containing $\mathcal{H}_{\lfloor (r-1)/2 \rfloor}$ as a subgraph. And finally by Lemma 12, for some f'''' there is a Frege derivation of $\mathsf{T}(\mathcal{H}_{\lfloor (r-1)/2 \rfloor}, f''')$ of size $\text{poly}(|\mathsf{T}(G, f)|) + \mathcal{O}(S)$ and depth $d + \mathcal{O}(1)$. Notice that S is the size of a derivation of $\neg \mathsf{T}(G, f)$, hence $S \geq |\mathsf{T}(G, f)|$. Thus, for some constants C and c there is a derivation of $\neg \mathsf{T}(\mathcal{H}_{\lfloor (r-1)/2 \rfloor}, f''')$ of size S^c and depth $d + C$.

By Theorem 17, there is a constant K such that if $d + C \leq \frac{K \log n}{\log \log n}$, then $S^c \geq 2^{\lfloor (r-1)/2 \rfloor^{\Omega(1/(d+C))}}$. Hence $S \geq 2^{r^{\Omega(1/d)}}$ and, thus, $S \geq 2^{t^{\Omega(1/d)}}$. \square

4 Upper bound

In this section we prove the following Theorem:

Theorem 19. *Let $G(V, E)$ be a connected undirected graph and $\mathsf{T}(G, f)$ be an unsatisfiable Tseitin formula. Then for all large enough d the formula $\neg \mathsf{T}(G, f)$ has a derivation of depth d and size $2^{\text{tw}(G)^{\mathcal{O}(1/d)}} \text{poly}(|\mathsf{T}(G, f)|)$.*

In order to prove Theorem 19 we define a *compact representation* of parity by depth- d formulas, then we show that we can efficiently derive the sum of \mathbb{F}_2 -linear equations using the compact representation of parities. And then we prove Theorem 19 using a tree-partition of the graph G .

4.1 A compact representation of parity

Let t_1, t_2, \dots, t_d be natural numbers, where d is a non-negative integer. Let U_0, U_1, \dots, U_d be partitions of a finite set F . We say that a list of partitions $U = (U_0, U_1, \dots, U_d)$ is a (t_1, \dots, t_d) -refinement of F if the following conditions hold:

1. U_0 consists of the only element $U_{0,1} = F$.
2. For every i , U_{i+1} is a subpartition of U_i such that every element of U_i is split into t_{i+1} parts. Hence, U_i split F into m_i parts: $U_{i,1}, U_{i,2}, \dots, U_{i,m_i}$, where $m_i = \prod_{j=1}^i t_j$.
3. All elements of U_d have cardinality at most 1.

Let U be a (t_1, \dots, t_d) -refinement of a set F and let $U_{i,j}$ be one of the blocks of this refinement. Then U induces on each of the blocks $U_{i,j}$ a (t_{i+1}, \dots, t_d) -refinement U' which is obtained by restricting U_{i+1}, \dots, U_d to the set $U_{i,j}$. U' is called a *sub-refinement* of $U_{i,j}$ in U .

Lemma 20. *Let F be a set of size n and $d \geq 0$ be an integer. Let t_1, \dots, t_d be integers such that $t_1 \cdot t_2 \cdot \dots \cdot t_d \geq n$. Then there exists a (t_1, \dots, t_d) -refinement U of F .*

Proof. We define U_d as n singletons each containing one element of F and $t_1 \cdot \dots \cdot t_d - n$ empty sets. For each $i < d$ we construct U_i from U_{i+1} as follows. We split the $t_{i+1} \cdot \dots \cdot t_d$ blocks of the partition U_{i+1} into groups of size t_{i+1} and define $U_{i,1}$ as the union of the elements of the first group, $U_{i,2}$ as the union of the elements of the second group and etc. \square

For $a \in \{0, 1\}$ and natural number n we define a Boolean function $\text{PARITY}_n^a : \{0, 1\}^n \rightarrow \{0, 1\}^n$ such that $\text{PARITY}_n^a(x_1, \dots, x_n) = 1$ iff $\bigoplus_{i=1}^n x_i = a$ for all $x_1, \dots, x_n \in \{0, 1\}$.

Lemma 21. *Let n and d be positive integers and U be a (t_1, t_2, \dots, t_d) -refinement of $[n]$. Then there exists a formula representing PARITY_n^b of depth at most $3d + 1$ and of size $\prod_{i=1}^d 2^{t_i+1} t_i$.*

Proof. Let us prove by backward induction on i from d to 0 that for every $j \in [\prod_{k=1}^i t_k]$, there is a formula representing $\bigoplus_{k \in U_{i,j}} x_k$ of depth $3(d - i)$ and of size $\prod_{q=i+1}^d 2^{t_q+1} t_q$. If $i = d$, then $|U_{d,j}| \leq 1$, hence $\bigoplus_{k \in U_{i,j}} x_k$ is either 0 or a variable x_k and thus has size 1 and depth 0.

Assume that $i < d$. Let $\ell_1, \ell_2, \dots, \ell_{t_{i+1}}$ be such that $U_{i,j} = U_{i+1,\ell_1} \sqcup U_{i+1,\ell_2} \sqcup \dots \sqcup U_{i+1,\ell_{t_{i+1}}}$. Let for $r \in [t_{i+1}]$, β_r be a representation of $\bigoplus_{k \in U_{i+1,\ell_r}} x_k$ of size $\prod_{q=i+2}^d 2^{t_q+1} t_q$ and depth $3(d - i - 1)$ that exists by the induction hypothesis. Consider a CNF-representation of $\beta_1 \oplus \dots \oplus \beta_{t_{i+1}} : \bigoplus_{k \in U_{i,j}} x_k = \bigwedge_{\substack{S \subseteq \{1, \dots, t_{i+1}\} \\ |S| \bmod 2 = 0}} \left(\bigvee_{s \in S} \neg \beta_s \vee \bigvee_{s \notin S} \beta_s \right)$. After the substitution of the representations of $\beta_1, \dots, \beta_{t_{i+1}}$ we obtain a formula of size at most $2^{t_{i+1}} t_{i+1} \cdot \prod_{q=i+2}^d 2^{t_q+1} t_q + 2^{t_{i+1}} t_{i+1} \leq \prod_{q=i+1}^d 2^{t_q+1} t_q$ and of depth $3(d - i - 1) + 3 = 3(d - i)$.

Therefore we have constructed a representation of PARITY_n^1 of the needed size and depth. The representation of PARITY_n^0 could be constructed as $\neg \varphi$ where φ is the obtained representation of PARITY_n^1 . \square

We call the representation of PARITY_n^a obtained by Lemma 21 *the compact representation* of PARITY_n^a with respect to a (t_1, \dots, t_d) -refinement U .

Let us define for $S \subseteq [n]$ and for $a \in \{0, 1\}$, $\text{PARITY}_{n,S}^a(x_1, \dots, x_n) = a \oplus \bigoplus_{i \in S} x_i$. We define a compact representation of $\text{PARITY}_{n,S}^a$ with respect to a (t_1, \dots, t_d) -refinement U as the result of substitutions $x_j := \mathbf{0}$ for all $j \notin S$ to the compact representation of PARITY_n^a with respect to U . We denote the compact representation of $\text{PARITY}_{n,S}^a(x_1, x_2, \dots, x_n)$ w.r.t. U by $\Phi^a(S, U)$.

Lemma 22. *Let U be a (t_1, \dots, t_d) -refinement of $[n]$ and U' be a sub-refinement of $U_{i,j}$ in U . Then for every $S \subseteq U_{i,j}$ there exists a derivation $\Phi^a(S, U') \vdash_{3d+\mathcal{O}(1)} \Phi^a(S, U)$ of size at most $4|\Phi^a(S, U)|^3$.*

Proof. Let $\psi_1 := x$ and ψ_2 be the formula obtained from $\Phi^a(S, U)$ by replacement of all subformulas $\Phi^a(U')$ with x . Clearly ψ_1 and ψ_2 are semantically equivalent. Then let us apply Corollary 5 to $\psi_1, \psi_2, k = 1$ and $\varphi_1 = \Phi^a(S, U')$. Then there exists a derivation of $\psi_1[x \leftarrow \Phi^a(S, U')] \vdash_{3d+\mathcal{O}(1)} \psi_2[x \leftarrow \Phi^a(S, U')]$ of size $2^1(|\psi_1|^2 + |\psi_2|^2)|\varphi_1| = 2(1 + |\Phi^a(S, U)|^2)|\Phi^a(S, U)| \leq 4|\Phi^a(S, U)|^3$. \square

4.2 Summation of linear equations

Let $S\Delta T$ be the symmetric difference of sets S and T i.e. $S\Delta T = (S \cup T) \setminus (S \cap T)$.

Proposition 23. *Let U be a (t_1, \dots, t_d) -refinement of $[n]$. For any $S \subseteq [n]$ there exists a derivation $-\Phi^a(S, U) \vdash_{3d+\mathcal{O}(1)} \Phi^{1\oplus a}(S, U)$ of size at most $2(|\Phi^1(S, U)| + |\Phi^0(S, U)|)$.*

Proof. If $a = 1$ then the premise equals the conclusion, so empty derivation is sufficient. If $a = 0$ we need to remove double negation so Pavel asks $\Phi^0(S, U)$. If the answer is 1 then it is an immediate contradiction with the premise and if the answer is 0 it is an immediate contradiction with the conclusion. \square

Lemma 24. *Let U be a (t_1, \dots, t_d) -refinement of $[n]$. Then there exist a constant c and a derivation*

$$\Phi^a(S, U), \Phi^b(T, U) \vdash_{3d+\mathcal{O}(1)} \Phi^{a\oplus b}(S\Delta T, U)$$

of size at most $c \cdot |\Phi^1(\emptyset, U)|^6$. We write \emptyset here to stress that the size of $\Phi^a(S, U)$ does not depend on S .

Proof. Let us prove this Lemma by induction on d . The base case: $d = 0$. Since U_0 is the last layer of refinement, it contains only singletons on the other hand by the definition $U_0 = \{[n]\}$, thus, $n = 1$ and $\Phi^a(S, U)$, $\Phi^b(T, U)$ and $\Phi^{a\oplus b}(S\Delta T, U)$ all depend on one variable x_1 . Thus by Lemma 4, there exists a derivation $\Phi^a(S, U) \wedge \Phi^b(T, U) \vdash_2 \Phi^{a\oplus b}(S\Delta T, U)$ of constant size. Clearly $\Phi^a(S, U) \wedge \Phi^b(T, U)$ is derivable from $\Phi^a(S, U) \Phi^b(T, U)$ by a derivation of constant depth and size.

If $d > 0$, let V^1, \dots, V^{t_1} be the sub-refinements of $U_{1,1}, \dots, U_{1,t_1}$ in U respectively where $[n] = U_{1,1} \sqcup \dots \sqcup U_{1,t_1}$. Recall that by definition

$$\Phi^1(X, U) = \bigwedge_{\substack{R \subseteq [t_1] \\ |R| \bmod 2 = 0}} \left(\bigvee_{i \in R} -\Phi^1(X \cap U_{1,i}, V^i) \vee \bigvee_{i \notin R} \Phi^1(X \cap U_{1,i}, V^i) \right).$$

We view this representation as CNF in variables $\Phi^1(X \cap U_{1,i}, V^i)$ for $i \in [t_1]$. Recall that by definition $\Phi^0(X, U) = -\Phi^1(S, U)$.

Let us describe a strategy of Pavel in the game with initial conditions $\Phi^a(S, U) = 1$, $\Phi^b(T, U) = 1$ and $\Phi^{a\oplus b}(S\Delta T, U) = 0$.

Pavel asks Sam the values of the following $3t_1$ formulas: $\Phi^1(S \cap U_{1,i}, V^i)$, $\Phi^1(T \cap U_{1,i}, V^i)$, $\Phi^1((S\Delta T) \cap U_{1,i}, V^i)$ for $i \in [t_1]$. There are 2^{3t_1} possible combinations of Sam's answers. The total size of the formulas is at most $2 \cdot 2^{3t_1} \cdot |\Phi^1(\emptyset, V^1)|$. Let us show how Pavel wins in each of them. Let $\alpha_i, \beta_i, \gamma_i$ denote Sam's answers to $\Phi^1(S \cap U_{1,i}, V^i)$, $\Phi^1(T \cap U_{1,i}, V^i)$, $\Phi^1((S\Delta T) \cap U_{1,i}, V^i)$ respectively for $i \in [t_1]$.

If $\bigoplus_{i \in [t_1]} \alpha_i \neq a$, then we can get a contradiction with $\Phi^a(S, U)$. Indeed consider a formula $\psi := \bigwedge_{i=1}^{t_1} \Phi^{\alpha_i}(S \cap U_{1,i}, V^i) \wedge \Phi^a(S, U)$ and let L be the size of ψ if we consider it as a formula in variables $\Phi^1(S \cap U_{1,i}, V^i)$. Clearly $L \leq 2^{t_1+1} \cdot t_1$. By Corollary 5, there is a derivation of $\mathbf{0}$ from ψ of size at most $2^{t_1} L^2 \left(\sum_{i=1}^{t_1} |\Phi^1(\emptyset, V^i)| \right) \leq 2^{3t_1+4} t_1^3 |\Phi^1(\emptyset, V^1)|$. Pavel asks Sam the value of ψ and in both cases gets a contradiction. If the answer is 0 then it is a contradiction with the conjuncts so Pavel proceeds to ask all subformulas of ψ starting with the deeper ones until he gets an immediate contradiction. If the answer is 1 we get a contradiction by Corollary 5. The total size of all such derivation for all possible combinations of Sam's answers is at most $2^{3t_1} \cdot \max\{|\psi|^2, 2^{3t_1+4} t_1^3 |\Phi^1(\emptyset, V^1)|\} \leq 2^{6t_1+4} t_1^3 |\Phi^1(\emptyset, V^1)|$. Adding the size of the $3t_1$ preliminary questions we get $2^{6t_1+4} t_1^3 |\Phi^1(\emptyset, V^1)| + 2 \cdot 2^{3t_1} \cdot |\Phi^1(\emptyset, V^1)| < 20 \cdot 2^{6(t_1+1)} t_1^6 |\Phi^1(\emptyset, V^1)|^6 \leq 20 |\Phi^1(\emptyset, U)|^6$. The cases when $\bigoplus_{i \in [t_1]} \beta_i \neq b$ or $\bigoplus_{i \in [t_1]} \gamma_i \neq a \oplus b$ are similar.

Let us consider the case where $\bigoplus_{i \in [t_1]} \alpha_i = a$, $\bigoplus_{i \in [t_1]} \beta_i = b$ and $\bigoplus_{i \in [t_1]} \gamma_i = 1 \oplus a \oplus b$. In this case there exists $i_0 \in [t_1]$ such that $\alpha_{i_0} + \beta_{i_0} \neq \gamma_{i_0}$. We derive a contradiction using Sam's answers $\Phi^1(S \cap U_{1,i_0}, V^{i_0}) = \alpha_{i_0}$, $\Phi^1(T \cap U_{1,i_0}, V^{i_0}) = \beta_{i_0}$ and $\Phi^1((S\Delta T) \cap U_{1,i_0}, V^{i_0}) = \gamma_{i_0}$. By Proposition 23 we may assume that Sam's answers are $\Phi^{\alpha_{i_0}}(S \cap U_{1,i_0}, V^{i_0}) = 1$, $\Phi^{\beta_{i_0}}(T \cap U_{1,i_0}, V^{i_0}) = 1$ and $\Phi^{\gamma_{i_0}}((S\Delta T) \cap U_{1,i_0}, V^{i_0}) = 1$. By the induction hypothesis there exists a derivation

$$\Phi^{\alpha_{i_0}}(S \cap U_{1,i_0}, V^{i_0}), \Phi^{\beta_{i_0}}(T \cap U_{1,i_0}, V^{i_0}) \vdash_{3d+\mathcal{O}(1)} \Phi^{1\oplus\gamma_{i_0}}((S\Delta T) \cap U_{1,i_0}, V^{i_0})$$

of size at most $c|\Phi^1(\emptyset, V^1)|$ so we get an immediate contradiction with $\Phi^{\gamma_{i_0}}((S\Delta T) \cap U_{1,i_0}, V^{i_0})$. The total size of the derivation is at most

$$\underbrace{2^{3t_1}}_{\text{number of possible values of } \alpha, \beta, \gamma} \cdot \underbrace{c|\Phi^1(\emptyset, V^1)|^6}_{\text{induction hypothesis}} + \underbrace{20|\Phi^1(\emptyset, U)|^6}_{\text{local contradictions}}.$$

Since $|\Phi^1(\emptyset, U)| = 2^{t_1+1}t_1 \cdot |\Phi^1(\emptyset, V^1)|$, for a large enough c the size of derivation does not exceed $c|\Phi^1(\emptyset, U)|^6$. \square

Lemma 25. *Let U be a (t_1, \dots, t_d) -refinement of $[n]$. Let $S_1, S_2, \dots, S_k \subseteq [n]$ and $a_1, \dots, a_k \in \{0, 1\}$. Then there exists a constant c such that:*

1. *There exists a derivation $\Phi^{a_1}(S_1, U), \Phi^{a_2}(S_2, U), \dots, \Phi^{a_k}(S_k, U) \vdash_{3d+\mathcal{O}(1)} \Phi^{a_1 \oplus \dots \oplus a_k}(S_1 \Delta \dots \Delta S_k, U)$ of size at most $c \cdot k \cdot |\Phi^1(\emptyset, U)|^6$.*
2. *If $\bigwedge_{i \in [k]} \left(\bigoplus_{j \in S_i} x_j = a_i \right)$ is unsatisfiable then there exists a derivation $\Phi^{a_1}(S_1, U), \Phi^{a_2}(S_2, U), \dots, \Phi^{a_k}(S_k, U) \vdash_{3d+\mathcal{O}(1)} \mathbf{0}$ of size at most $c \cdot k \cdot |\Phi^1(\emptyset, U)|^6$.*

Proof. The proof of (1) is simply $(k-1)$ consecutive applications of Lemma 24 using Lemma 2. The proof of (2) is based on a well-known fact: a system of linear equations is unsatisfiable iff the contradiction $0 = 1$ can be obtained as a linear combination of the equations. In our case it means that there is a set $R := \{r_1, \dots, r_m\} \subseteq [k]$ such that $S_{r_1} \Delta S_{r_2} \Delta \dots \Delta S_{r_m} = \emptyset$ and $\bigoplus_{i \in R} a_i = 1$.

$\Phi^{a_{r_1} \oplus \dots \oplus a_{r_m}}(S_{r_1} \Delta \dots \Delta S_{r_m}) = \Phi^1(\emptyset, U)$. $\Phi^1(\emptyset, U)$ is identically false, hence by Lemma 4 there is a derivation $\Phi^1(\emptyset, U) \vdash_{3d+1} \mathbf{0}$ of size at most $4|\Phi^1(\emptyset, U)|^2$.

Let us construct Pavel's strategy in the game with initial conditions $\Phi^{a_{r_1}}(S_{r_1}, U) = 1, \dots, \Phi^{a_{r_m}}(S_{r_m}, U) = 1$ and $\Phi^1(\emptyset, U) = 0$. Pavel asks Sam the value of $\Phi^1(\emptyset)$. If Sam answers 0 Pavel uses the strategy corresponding to $\Phi^{a_{r_1}}(S_{r_1}), \Phi^{a_{r_2}}(S_{r_2}), \dots, \Phi^{a_{r_k}}(S_{r_k}) \vdash_{3d+\mathcal{O}(1)} \Phi^{a_{r_1} \oplus a_{r_2} \oplus \dots \oplus a_{r_k}}(S_{r_1} \Delta S_{r_2} \Delta \dots \Delta S_{r_k})$ from (1). If Sam answers 1 Pavel uses the strategy corresponding to the derivation $\Phi^1(\emptyset, U) \vdash_{3d+1} \mathbf{0}$. \square

4.3 Tree-partition-width

Let $G(V, E)$ be an undirected graph and S_1, \dots, S_m be a partition of V . S_1, \dots, S_m is a tree-partition of G if there exists a tree $T([m], E_T)$ such that every edge e of G connects either two vertices from the same part S_i or connects a vertex from S_i and a vertex from S_j , where i and j are adjacent in T , i.e. $(i, j) \in E_T$. A *width* of a tree-partition S_1, S_2, \dots, S_m is the size of the largest set S_i for $i \in [m]$. A *tree-partition width* of a graph G is the smallest width among all tree-partitions of G . We denote tree-partition width of G by $\text{tpw}(G)$.

If we add a new vertex in the middle of every edge (i, j) of the tree T and put the set $S_i \cup S_j$ on it, we will get a tree decomposition of G , hence $\text{tw}(G) \leq 2\text{tpw}(G) - 1$.

The following theorem shows an inequality in the other direction:

Theorem 26 ([38]). *If $\text{tw}(G) \geq 1$, then $\text{tpw}(G) \leq 10\Delta(G)\text{tw}(G)$, where $\Delta(G)$ is the maximum degree of G .*

So, $\text{tw}(G)$ and $\text{tpw}(G)$ coincide up to a multiplicative constants for constant degree graphs.

Theorem 27. *Let $G(V, E)$ be a connected graph and let a Tseitin formula $\text{T}(G, f)$ be unsatisfiable. Then there exists a derivation $\vdash_{3d+\mathcal{O}(1)} \text{T}(G, f)$ of size at most $\text{poly}(|\text{T}(G, f)|) \cdot 2^{(\text{tpw}(G)\Delta(G))^{\mathcal{O}(1/d)}}$, where $\Delta(G)$ is the maximum degree of G .*

Proof. Let S_1, \dots, S_m be a tree-partition of G with width $\text{tpw}(G)$ and let $T([m], E_T)$ be the corresponding tree. W.l.o.g. we assume that T is a rooted tree with root m ; for all $i \in [m-1]$, $p(i)$ denotes its parent and for all $i \in [m]$, $s(i)$ denotes the set of direct successors of i . W.l.o.g. we assume that $p(i) > i$ for all $i \in [m-1]$.

Since $\text{T}(G, f)$ is unsatisfiable and G is connected, $\bigoplus_{v \in V} f(v) = \bigoplus_{i \in [m]} \bigoplus_{v \in S_i} f(v) = 1$. We consider the sum $\bigoplus_{i \in [m]} \bigoplus_{e \in E(S_i, V \setminus S_i)} x_e$. Since each x_e occurs in the sum exactly twice, the sum (modulo 2) is 0 for all

values of x_e . Then for each assignment of $\{x_e\}_{e \in E}$ there exists i_0 such that $\bigoplus_{v \in S_{i_0}} f(v) \neq \bigoplus_{e \in E(S_{i_0}, V \setminus S_{i_0})} x_e$. The first part of Pavel's strategy is to find such i_0 .

Pavel will request parity of the sum of all edges between S_i and S_j for all $(i, j) \in E_T$. In order to represent these formulas in a compact way we now define m different (t_1, \dots, t_d) -refinements W^1, \dots, W^m ; for every i , W^i is a refinement of the set $E\left(S_i, \bigcup_{j \in s(i)} S_j\right)$ of all edges connecting a vertex from S_i with a vertex from $\bigcup_{j \in s(i)} S_j$. We construct appropriate refinements W^i later.

Pavel asks Sam the values of $\bigoplus_{e \in E(S_i, S_{p(i)})} x_e$ represented as $\Phi^1(E(S_i, S_{p(i)}), W^{p(i)})$ for $i \in [m-1]$ in the increasing order until he finds i_0 such that $\bigoplus_{e \in E(S_{i_0}, V \setminus S_{i_0})} x_e \neq \bigoplus_{v \in S_{i_0}} f(v)$.

At the moment when Sam has answered the value of $\Phi^1(E(S_i, S_{p(i)}), W^{p(i)})$ the values of $\bigoplus_{e \in E(S_i, S_j)} x_e$ for each j such that $(i, j) \in E_T$ are all determined, thus, the value of $\bigoplus_{e \in E(S_i, V \setminus S_i)} x_e$ is determined. If $\bigoplus_{e \in E(S_i, V \setminus S_i)} x_e \neq \bigoplus_{v \in S_i} f(v)$ Pavel proceeds to the next part of his strategy. Otherwise he continues to ask Sam similar questions corresponding to the vertices with larger indices.

Now we describe the strategy of Pavel in case if he finds i_0 . We are going to describe this case in terms of derivation using Lemma 2 multiple times. Consider a linear system that consists of the equation $\bigoplus_{e \in E(S_{i_0}, V \setminus S_{i_0})} x_e = 1 \oplus \bigoplus_{v \in S_{i_0}} f(v)$ and all parity conditions of $T(G, f)$ of the vertices from S_{i_0} . This linear system is unsatisfiable. We are going to use Lemma 25. In order to do it we need to derive the representations of these linear equations w.r.t. some refinement Q of a superset of $E(S_{i_0}, V)$.

Let for $i \in [m]$, U^i be a (t_1, t_2, \dots, t_d) -refinement of the set $E(S_i)$ of all edges connecting two vertices from S_i (we construct these refinements in the end of the proof together with the refinements W^i). Let us define a $(3, t_1, \dots, t_d)$ -refinement Q as a union of (t_1, \dots, t_d) -refinements $W^{i_0}, W^{p(i_0)}$ and U^{i_0} such that $Q_1 = \{E(S_{i_0}, \bigcup_{j \in s(i_0)} S_j), E(S_{p(i_0)}, \bigcup_{j \in s(p(i_0))} S_j), E(S_{i_0})\}$ and for every $j \in \{2, 3, \dots, d+1\}$, Q_j is the union of $W_{j-1}^{i_0}, W_{j-1}^{p(i_0)}$ and $U_{j-1}^{i_0}$.

Let a_j be Sam's answer to the question $\bigoplus_{e \in E(S_{i_0}, S_j)} x_e$ for each j that is a neighbour of i_0 in T , hence we may assume that $\Phi^{a_j}(E(S_{i_0}, S_j), W^{i_0})$ for $j \in s(i_0)$ and $\Phi^{a_{p(i_0)}}(E(S_{i_0}, E_{p(i_0)}), W^{p(i_0)})$ are already derived. By Lemma 22, we derive $\Phi^{a_j}(E(S_{i_0}, S_j), Q)$ from $\Phi^{a_j}(E(S_{i_0}, S_j), W^{i_0})$ for $j \in s(i_0)$ and $\Phi^{a_{p(i_0)}}(E(S_{i_0}, S_{p(i_0)}), Q)$ from $\Phi^{a_{p(i_0)}}(E(S_{i_0}, S_{p(i_0)}), W^{p(i_0)})$, where a_j are Sam's answers to the corresponding questions.

By the first part of Lemma 25 we derive $\Phi^{1 \oplus \left(\bigoplus_{v \in S_{i_0}} f(v)\right)}(E(S_{i_0}, V \setminus S_{i_0}), Q)$ from the set of formulas $\{\Phi^{a_j}(E(S_{i_0}, S_j), Q) \mid (i_0, j) \in E_T\}$. We assume that the parity conditions of the vertices of G in $T(G, f)$ represented as CNF are asked at the beginning of the game i.e. for each $v \in V$ we know that the CNF representation of $\bigoplus_{u: (u,v) \in E} x_e$ is true (if any clause of $T(G, f)$ is false Pavel queries all subformulas of $T(G, f)$ except subformulas of the clauses and gets an immediate contradiction, if any of the parity conditions is false it yields an immediate contradiction with the corresponding subset of clauses). Thus, by Lemma 4 we derive the representations of parity conditions of the vertices from S_{i_0} w.r.t. Q . Since the corresponding linear system is unsatisfiable, using the second part of Lemma 25 we get a contradiction.

Claim 28. *The size of the described game tree is at most $m \cdot 2^{3\Delta(G)} \Delta^2(G) \text{tpw}(G) 2^{\mathcal{O}(\sum_{i=1}^d t_i)}$.*

Proof. The game tree consists of m subtrees corresponding to each possible value of i_0 , $m-1$ nodes corresponding to the questions $\Phi^1(E(S_i, S_{p(i)}), W^{p(i)})$ for $i \in [m-1]$ and $\mathcal{O}(|V|)$ nodes corresponding to the initial questions about parity conditions of the vertices of G in CNF. The total size of formulas in these $\mathcal{O}(|V|)$ nodes is $\text{poly}(|T(G, f)|)$. Each of the $m-1$ formulas corresponding to the questions $\Phi^1(E(S_i, S_{p(i)}), W^{p(i)})$ has size $\prod_{i=1}^d 2^{t_i+1} t_i$. Let us estimate the size of each of the m subtrees. It is the sum of the following values:

- The derivation of the formulas $\Phi^1(E(S_{i_0}, S_j), Q)$ for $j \in \{p(i_0)\} \cup s(i_0)$. By Lemma 22 size of each of these derivations is $\mathcal{O}\left(\prod_{i=1}^d 2^{3t_i} t_i^3\right)$.
- The derivation of the representations of parity conditions w.r.t. Q . By Lemma 4 size of each of these derivations is $\mathcal{O}\left(2^{\Delta(G)} (2^{\Delta(G)} \Delta(G) + \prod_{i=1}^d 2^{t_i} t_i^2)\right)$

- The derivation of $\Phi^{1\oplus}\left(\bigoplus_{v \in S_{i_0}} f(v)\right)(E(S_{i_0}, V \setminus S_{i_0}), Q)$. By the first part of Lemma 25 the size of this derivation is $\Delta(G)\text{tpw}(G)2^{\mathcal{O}(\sum_{i=1}^d t_i)}$.
 - The derivation of contradiction by the second part of Lemma 25 has size $(\Delta(G) + 1)\text{tpw}(G)2^{\mathcal{O}(\sum_{i=1}^d t_i)}$.
- Each of these four summands is $2^{3\Delta(G)}\Delta^2(G)\text{tpw}(G)2^{\mathcal{O}(\sum_{i=1}^d t_i)}$. \square

Let us choose $t_i = (\Delta(G)\text{tpw}(G))^{2/d}$ for all $i \in [d]$. Since $|S_i| \leq \text{tpw}(G)$, $|E(S_i)| + \left|E\left(S_i, \bigcup_{j \in \text{es}(i)} S_j\right)\right| \leq \Delta(G)\text{tpw}(G)$. Hence, the condition $\prod_{i=1}^d t_i \geq \Delta(G)\text{tpw}(G) \geq |E(S_i)| + \left|E\left(S_i, \bigcup_{j \in \text{es}(i)} S_j\right)\right|$ holds and, thus, for all $i \in [m]$ the refinements U^i, W^i exist by Lemma 20. If we substitute choosen values in the bound from Clam 28, we get the upper bound $m \cdot 2^{\mathcal{O}(3\Delta(G) + d(\Delta(G)\text{tpw}(G))^{2/d})} = \text{poly}(|T(G, f)|) \cdot 2^{(\Delta(G)\text{tpw}(G))^{\mathcal{O}(1/d)}}$. \square

Now we are ready to prove Theorem 19.

Proof of Theorem 19. Theorem 27 and Theorem 26 imply that there exists a constant c and a derivation $\vdash_{3d+\mathcal{O}(1)} \neg T(G, f)$ of size at most $\text{poly}(|T(G, f)|)2^{(10\Delta^2(G)\text{tw}(G))^{c/d}}$. If $\text{tw}(G) > \Delta(G)$ then we can rewrite our upper bound on the size as $\text{poly}(|T(G, f)|)2^{(10\text{tw}(G))^{3c/d}}$. If $\text{tw}(G) > 1$ then it is $\text{poly}(|T(G, f)|)2^{(\text{tw}(G))^{\mathcal{O}(1/d)}}$. If $\text{tw}(G) = 1$ then it is simply $\text{poly}(|T(G, f)|)$. Otherwise if $\text{tw}(G) \leq \Delta(G)$ we can rewrite the upper bound as $\text{poly}(|T(G, f)|) \cdot 2^{(10\Delta(G))^{3c/d}} = \text{poly}(|T(G, f)|)$ if $3c/d \leq 1$. Thus, for $d \geq 3c$ the upper bound is $\text{poly}(|T(G, f)|) \cdot 2^{\text{tw}(G)^{3c/d}}$. Therefore, for the both cases we have the needed upper bound. \square

5 Consequences of the Main Theorem

The *width* of a clause is the number of literals in it; the width of a resolution refutation is the maximal width of a clause appearing in it. With $w(F \vdash)$ we denote the minimal width to refute F in resolution. By $S(F \vdash)$ we denote the minimal size to refute F in resolution. We abbreviate $w(T(G, f) \vdash)$ with $w(G)$ and $S(T(G, f) \vdash)$ with $S(G)$.

Alekhovich and Razborov in [2] gave the following definition.

Definition 29 ([2]). *A class of unsatisfiable CNF formulas \mathcal{F} is smooth (respectively quasi-smooth) if for all $F \in \mathcal{F}$, $w(F \vdash) \leq \log S(F \vdash)$ (resp. there is a constant C such that $w(F \vdash) \leq \log^C S(F \vdash)$).*

Smoothness and quasi-smoothness are important properties for a class of formulas since, together with width-automatizability imply automatizability (quasi-automotizability) in resolution of that class of formulas (see [2]).

Definition 30. *A class of unsatisfiable CNF formulas \mathcal{F} is (quasi-)automatizable with respect to a proof system Π if there exists a deterministic automatizing algorithm that, given an unsatisfiable formula $F \in \mathcal{F}$, returns its proof in the proof system S in time which is (quasi-)polynomial in $|F| + S_\Pi(F \vdash)$, where $S_\Pi(F \vdash)$ denotes the size of the shortest Π -refutaion of F .*

Beame, Karp, Pitassi and Saks [4] showed that the class of all unsatisfiable CNF formulas is quasi-automatizable with respect to treelike Resolution.

Definition 31. ([2]) *A class \mathcal{F} of unsatisfiable CNFs is width-automatizable if there exists a deterministic algorithm that, given $F \in \mathcal{F}$, returns its resolution refutation of width $\mathcal{O}(w(F \vdash))$ in time which is polynomial in $|F| + 2^{w(F \vdash)}$*

It is known that Tseitin formulas are width-automatizable (see [2]) and that

Theorem 32. (Fact 2.7 [2]) *If a class of unsatisfibale CNFs is both width-automatizable and (quasi-)smooth, then it is (quasi-)automatizable with respect to resolution.*

In [2] they proved that a subclass of Tseitin formulas is quasi-smooth, namely the class defined over graph with *constant cyclicity* ℓ , that is those graphs for which the edges of G can be covered by cycles such a way that every cycle has length at most ℓ and every edge belongs to at most ℓ cycles.

Here, as a consequence of Theorem 1, we show that *for all graphs* the class of Tseitin formulas $\mathsf{T}(G, f)$ is quasi-smooth, hence quasi-automatizable with respect to resolution.

Corollary 33. *For any graph G , $\mathsf{T}(G, f)$ are quasi-smooth, i.e. there is a constant C such that $w(G) \leq \log^C S(G)$. Hence the class of Tseitin formulas is quasi-automatizable.*

Proof. In [2,17] it was established that $w(G) = \max\{\Delta(G), \text{tw}(L(G))\} - 1$, where $L(G)$ is the line graph (graph on edges) of G . It is known that $\max\{\Delta(G), \frac{1}{2}(\text{tw}(G) + 1)\} - 1 \leq \text{tw}(L(G)) \leq (\text{tw}(G) + 1)\Delta(G) - 1$, (see, for example, [20]). By Theorem 1 and the fact that $\mathsf{T}(G, f)$ is a CNF, we have that $S(G) \geq \max\{2^{\Delta(G)}, 2^{\text{tw}(G)^\epsilon}\}$, for some $0 < \epsilon < 1$ given by Theorem 1. Then $S(G) \geq 2^{(\text{tw}(G)^\epsilon + \Delta(G))/2} \geq 2^{(\text{tw}(G)\Delta(G))^{\epsilon/2}} \geq 2^{w(G)^{\epsilon/4}}$. \square

By a result in [3,23] there are tree-like resolution refutations of $\mathsf{T}(G, f)$ of size $n^{\mathcal{O}(w(G))} = 2^{\mathcal{O}(\text{tw}(G)\Delta(G) \log n)}$. Since $\mathsf{T}(G, f)$ is a CNF, then by Theorem 1 $S(G) \geq \max\{2^{\Delta(G)}, 2^{\text{tw}(G)^\epsilon}\}$. Putting the two inequalities together it is not difficult to see that there are Bounded-depth Frege proofs of $\mathsf{T}(G, f)$ of size $2^{\text{poly}(\log S(G))}$.

Corollary 34. *If $\mathsf{T}(G, f)$ have proofs of size S in Bounded-depth Frege, then have proofs of size at most $2^{\text{poly}(\log S)}$ in tree-like Resolution.*

It follows that the size of proofs of $\mathsf{T}(G, f)$ in all the proof systems between tree-like resolution and constant-depth Frege are quasi-polynomially correlated. i.e. if $\mathsf{T}(G, f)$ has a derivation of size S in one of them of size S then it has a proof of size at most $2^{\text{poly}(\log S)}$ in the other.

This observation

1. implies that for all graph G , the regular Resolution proof-search algorithm BWBATP of [2] on input $\mathsf{T}(G, f)$ runs in time $2^{\text{poly}(\log S(G))}$. [2] proved this last result only for $\mathsf{T}(G, f)$ on graphs with a set of edges that can be covered by cycles of constant length;
2. provides positive evidence (i.e. showing it for quasi-polynomial size) to the conjecture of Alasdair Urquhart that the shortest resolution refutations of $\mathsf{T}(G, f)$ are regular;
3. the class of $\mathsf{T}(G, f)$ is quasi-automatizable with respect to all proof systems between tree-like resolution and constant-depth Frege.

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A Frege vs Pudlák-Buss games

A set of formulas Γ *semantically implies* a formula ψ if every assignment that satisfies all formulas from Γ also satisfies ψ .

Every particular Frege system is defined by a finite set of derivation rules. A *derivation rule* has form $\frac{\varphi_1, \dots, \varphi_k}{\varphi}$, where $k \geq 0$, $\varphi_1, \dots, \varphi_k$ and φ are propositional formulas. It is required that every rule is sound, i.e. $\{\varphi_1, \dots, \varphi_k\}$ semantically implies φ .

Let Γ be a list of propositional formulas. We say that φ can be derived from Γ if there exists a sequence of formulas $\psi_1, \psi_2, \dots, \psi_s$ such that $\psi_s = \varphi$ and each ψ_i is either an element of Γ or can be derived from the formulas with smaller numbers by a derivation rule. It is allowed to substitute propositional formulas instead of variables into a derivation rule. A formula τ is derivable if it can be derived from an empty list of formulas.

A system of derivation rules is a Frege system if it is implicationally complete: if some set of formulas Γ semantically implies a formula φ , then φ must be derivable from Γ .

A depth of a derivation is the maximum depth of a formula that occurs in the derivation. A size of a derivation is the sum of sizes of all formulas from the derivation.

Lemma 3. 1. If a formula φ has a Frege derivation of size S and depth d , then φ has a Pudlák-Buss game derivation with number of size $\mathcal{O}(S^2)$ and depth d . 2. If φ has a Pudlák-Buss game derivation of size S and depth d , then φ a Frege derivation of size $\mathcal{O}(S^3)$ and depth $d + \mathcal{O}(1)$.

Proof. 1. Let $\varphi_1, \varphi_2, \dots, \varphi_s$ be a Frege derivation of φ of size S . We construct a strategy for Pavel. Pavel asks Sam values of the formulas $\varphi_1, \varphi_2, \dots$ until he gets the answer 0. $\varphi_s = \varphi$, so at least for φ_s the answer is 0. Assume that Sam has answered 1 for the formulas $\varphi_1, \varphi_2, \dots, \varphi_k$ and 0 for φ_{k+1} where $0 \leq k \leq s - 1$. φ_{k+1} is derived from $\varphi_1, \varphi_2, \dots, \varphi_k$ using one of the rules in Frege system. Let us consider all subformulas of this rule (there is a constant number of such subformulas). Pavel asks Sam values of the formulas corresponding to such subformulas in the application of the rule where φ_{k+1} is derived. Due to correctness of the rules, we get an immediate contradiction. Number of vertices for constructed tree is $\mathcal{O}(S)$. In every vertex Pavel has asked only subformulas of formulas in the Frege derivation, so size of every asked formula does not exceed S and depth does not exceed d . The total size of the game is $\mathcal{O}(S^2)$

2. Let T be a tree for Pudlák-Buss game with the initial condition $\varphi = 0$. Every vertex in the tree corresponds to a sequence of questions and answers on the path from the root to this vertex. For the sequence $\varphi_1 = a_1, \varphi_2 = a_2, \dots, \varphi_j = a_j$ we write a formula $\neg(\neg\varphi \wedge \varphi_1^{a_1} \wedge \dots \wedge \varphi_j^{a_j})$, where ψ^0 denotes $\neg\psi$, and ψ^1 denotes ψ .

We assume that Frege system includes the rule $\frac{\neg(x \wedge y), \neg(x \wedge \neg y)}{\neg x}$ (it can be simulated with derivation of constant size and depth due to implicationally completeness of the system). With that rule, formulas corresponding to the inner vertices of the tree can be derived from the formulas corresponding to their descendants.

We also need to derive formulas in the leaves. Let h be the depth of T . We construct the derivation for a leaf using $\mathcal{O}(h)$ formulas, because in every leaf we have an immediate contradiction. There exist only a constant number of ways to get an immediate contradiction, so we assume that all these axioms are present in our Frege system. In order to use these axioms we need to rearrange the formulas in the big conjunction and place the formulas needed for immediate contradiction first.

For example, if the immediate contradiction is $A = 0, B = 1, A \wedge B = 1$, we need to add an axiom $\overline{\neg(\neg A \wedge B \wedge (A \wedge B) \wedge X)}$.

If the tree T has N vertices then the constructed derivation has $\mathcal{O}(Nh)$ formulas, and every formula has size $\mathcal{O}(S + h)$. Given that $N \leq S$ and $h \leq S$, we get a derivation with total size $\mathcal{O}(S^3)$

□