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Testing Odd Direct Sums Using High Dimensional Expanders

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Abstract

In this work, using methods from high dimensional expansion, we show that the property of k*-direct-sum* is testable for *odd* values of k . Previous work of [9] could inherently deal only with the case that k is even, using a reduction to linearity testing. Interestingly, our work is the first to combine the topological notion of high dimensional expansion (called co-systolic expansion) with the combinatorial/spectral notion of high dimensional expansion (called colorful expansion) to obtain the result.

The classical k-direct-sum problem applies to the complete complex; Namely it considers a function defined over *all* k-subsets of some n sized universe. Our result here applies to any collection of k-subsets of an n-universe, assuming this collection of subsets forms a high dimensional expander.

1 Introduction

Given a collection X of k-subsets of $[n]$, a function $F : X \to \{0,1\}$ is a k-direct-sum if there exists a function $f : [n] \to \{0,1\}$ such that for every A in X: $F(A) = \sum_{a\in A} f(a)$ (where the sum is performed modulo 2). A (Q,E)-tester for k-direct-sums is an algorithm that queries F on Q inputs from X, accepts k-direct-sums and rejects with probability of at least ξ, every function whose distance from the k-direct-sums is at least $E\xi$ (see Definition 2.15 for distance and [8] for a survey on property testing). In this work we present a new novel method for testing k-direct-sums using high dimensional expanders. Our method is the first to deal with k-direct-sums for odd constant values of k.

The question of testing whether a function is a k-direct-sum, as well as the entire area of testability, has strong relations to PCP constructions. For example, one can consider the gap amplification proof of the PCP theorem [5]. This proof uses two steps: First powering the graph which results in every node having an "opinion" about its neighbors' color (which increases the alphabet size) and then reducing the alphabet. A better understanding of the direct sum problem could potentially help in replacing the direct product done in the graph powering phase, and might even allow omitting the alphabet reduction stage which would yield a simpler proof to the PCP theorem and, possibly, better parameters.

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Previous Work There were several works on k-direct-sums, but none of them could deal with the odd constant case due to inherent limitations of their methods: The first work to link direct-sums and high dimensional expanders was done by Kaufman and Lubotzky [9], who showed a test for the 2-direct-sum problem on any simplicial complex that is a high dimensional expander. Their proof is tailored to the case where $k = 2$. Following the work of Kaufman and Lubotzky was a work by David, Dinur, Goldenberg, Kindler, and Shinkar [4] that proposed a tester for k-direct-sums on the for the case where the input set is $\binom{[n]}{k}$. Their tester is based on linearity testing: It picks $x, y \in X$ such that $x\Delta y \in X$, and tests whether $f(x) + f(y) = f(x\Delta y)$ (for more papers on linearity testing see [1, 2, 3]). But in order to get $x, y, x\Delta y \in X$, $\boldsymbol{k}$ **must be even**. In a recent work by Dinur and Kaufman [6], it is shown that the result of David et al. [4] can be applied to testing functions whose inputs are taken from a subset of $\binom{[n]}{k}$ that forms a high dimensional expander. However the limitation above still stands.

In this paper we introduce a new method for testing k-direct-sums that can tackle the odd case for the first time. Specifically we show:

Theorem 1.1 (Main Theorem Informal, for formal see Theorem 5.2)**.** *If X is a collection of subsets that forms a high dimensional expander then there is an $\left(O\left(k^2\right), O\left(k^2\right)\right)$-tester for the k-direct-sums where k is an odd constant.*

Interestingly we combine two notions of high dimensional expanders, a topological notion and a combinatorial notion, to obtain this result. This is the first time that both notions were used together.
In order to describe our strategy we will first have to introduce a generalization of graphs to higher dimensions (called simplicial complexes) as well as both notions of high dimensional expanders:

Simplicial Complexes A simplicial complex can be thought of as a hypergraph with a closure property, meaning that if F is a hyperedge in the hypergraph then so is every subset of F. We also define the dimension of a hyperedge F to be $|F| - 1$, and denote the set of i-dimensional edges of a complex X as $X(i)$. For example: In a graph, the vertices are considered the 0-dimensional hyperedges, and the edges are considered the 1-dimensional hyperedges. Now that we have defined the dimension of a hyperedge, we can define the dimension of the complex as the dimension of the maximal hyperedge. For example: A 2-dimensional simplicial complex is a simplicial complex that contains 2-dimensional hyperedges, often called the "triangles" (note that these hyperedges contain 3 vertices). Throughout this paper, we will use a standard weighted counting norm denoted as $\|\cdot\|$ (which will be defined in 2.7). In this work we will be interested in simplicial complexes whose maximal hyperedges are all of the same dimension (which are called "pure simplicial complexes").

As previously discussed, we will use two generalizations of expansion that apply to simplicial complexes: The first will be co-systolic expanders and the second will be colorful expanders. In order to discuss these notions of expansion, it will be useful to reexamine the Cheeger constant in the 1-dimensional case (aka graphs):

$$\min_{S\neq\emptyset,V}\left\{\frac{\left\|E(S,\bar{S})\right\|}{\min\left\{\|S\|,\left\|\bar{S}\right\|\right\}}\right\}$$

In any higher dimensional analogue, we would still like the essence of this constant to hold, every set of hyperedges of some dimension i (in graphs - vertices) has a number of out-going hyperedges of dimension $i+1$ (in graphs - edges) relative to its size. Because we are dealing with multi-dimensional objects we would also like this bound to apply in every dimension. The only question remaining is how to generalize the notion of an out-going edge to higher dimensions.

Co-systolic Expanders The first notion of expansion we will introduce is the co-systolic expansion, which is the more topological of the two. In this form of expansion, a hyperedge of dimension $i+1$, is said to be going out of a set E of hyperedges of dimension i, if it has an odd number of i-dimensional sub-edges in the set E[1]. We denote the set of hyperedges that are going out of a set E, according to this notion, as δE. Note that the Cheeger constant is normalized over the distance of E from a set that has no neighbors (in the 1-dimensional case the only sets with no neighbors are the empty set and the entire graph). Therefore we normalize our new high-dimensional analogue accordingly and receive the following definition:

$$\epsilon^i(X) = \min_{\substack{S \in \{0,1\}^{X(i)} \\ \delta S \neq \emptyset}} \left\{ \frac{\|\delta S\|}{dist(S, \{Z | \delta Z = \mathbb{0}\})} \right\}$$

A simplicial complex is a co-systolic expander if there exists some ϵ such that in every dimension i: $\epsilon^i(X) \geq \epsilon$. Note that there is another property that a simplicial complex must fulfill in order to be a co-systolic expander. However, it is not required in the proof of this paper and can be found in definition 2.19.

Colorful Expanders The other notion of expansion we will introduce is the colorful expansion, which is the more combinatorial of the two. In this form of expansion, a hyperedge of dimension $i+1$, is said to be going out of a set E of hyperedges of dimension i, if it has at least one i-dimensional sub-edge in E and at least one i-dimensional sub-edge outside of E. We denote the set of hyperedges that are going out of a set E, according to this notion, as $c(E)$. Using this definition of out-going edges, we get the following generalization of the Cheeger constant (for the i-th dimension):

$$\sigma^i(X) = \min_{S \neq \mathbb{0}, X(i)} \left\{ \frac{\|c(S)\|}{\min\{\|S\|, \|X(i) \setminus S\|\}} \right\}$$

A simplicial complex is a σ-colorful-expander if in every dimension i: $\sigma^i(X) \geq \sigma$.

1.1 Proof Layout

We will start by defining the property of being a k-direct-sum again, this time using the language of simplicial complexes: Given a simplicial complex X, a function $F : X(k-1) \to \{0,1\}$ is called a k-direct-sum if there exists a function $f : X(0) \to \{0,1\}$ such that for every A in $X(k-1)$: $F(A) = \sum_{a \in A} f(a)$. Note that we define a k-direct-sum to be a function from the $(k-1)$-dimensional hyperedges of the complex, and not the k-dimensional hyperedges of the complex, because we want the k to represent the size of the set and not the dimension of the face.
We will show that the following algorithm tests whether a given function is a k-direct-sum for odd

[1] In the 1-dimensional case we say that an edge crosses a cut if it has exactly one vertex in the cut. Note that if there is an odd number of vertices in an edge the odd number must be one since edges are of cardinality 2.

constant values of k:

Algorithm 1: $T_{assembled-k-direct-sum}$

1 **pick** *one of the following options uniformly:*
2   Test whether δF is a $(k+1)$-direct-sum using a known test for even sized sets[a].
3   **pick** $m \in X(k+1)$ *randomly:*
4   Check whether $F|_m$ is a k-direct-sum.

[a] Note that $k+1$ is even and, whenever the known test asks to query $\delta F(a)$, the algorithm queries every set in $\binom{a}{k}$ and returns the sum of the results.

In order to analyze this test, it would first be useful to deconstruct F into three functions $F = D + Z + G$ where D is a k-direct-sum, $\delta Z = \mathbb{0}$, and the remainder G.

Bounding the Norm of G Using Co-Systolic expansion First we will show that the rejection probability of step (2) bounds $\|G\|$ from above. In order to do so we must first consider the following two properties of δ:

- δ is linear.
- If D is a k-direct-sum (and k is odd) then δD is a $(k+1)$-direct-sum (See Lemma 3.6).

Combining these properties with the fact that the complex is a co-systolic expander, yields an upper bound for G. Specifically: Because $\delta F = \delta D + \delta G$, the test performed in step (2) gives an upper bound to $\|\delta G\|$ (since δD is a direct sum) and co-systolic expansion implies that $\|G\| \le \epsilon \|\delta G\|$.

Bounding the Norm of Z Using Colorful Expansion Secondly, we will show how to bound $\|Z\|$ from above. Alas, step (3) does not bound $\|Z\|$ from above unconditionally, but if we assume that $G = \mathbb{0}$, we can bound $\|Z\|$ from above using the rejection probability of step (3). We do that in two steps: The first is noting that $\|Z\|$ is bounded from above by all the $(k+1)$-faces that Z "touches", namely $\{m \in X(k+1) | Z|_m \ne \mathbb{0}\}$ due to a property of the norm (Lemma 2.9). We then show the following property: Step (3) rejects every $(k+1)$-dimensional face m on which $Z|_m \notin \{\mathbb{0}, \mathbb{1}\}$. In expander graphs, given a set of vertices S, the set of edges that are going out of S bounds from above the edges that connect two vertices within S. Similarly in higher dimensional colorful expanders, the set of edges that stay within a set S is bounded from above by the set of edges that are going out of S. We think of an edge m on which $Z|_m = \mathbb{1}$ as an edge that connects vertices within S, and an edge m on which $Z|_m \notin \{\mathbb{0}, \mathbb{1}\}$ as an edge that is going out of S. Thus by colorful expansion we can bound $\|\{m \in X(k+1) | Z|_m = \mathbb{1}\}\|$ using $\|\{m \in X(k+1) | Z|_m \notin \{\mathbb{0}, \mathbb{1}\}\}\|$. We conclude that $\|Z\|$ can be bounded from above as follows:

$$\begin{aligned}\|Z\| \le \|\{m \in X(k+1) | \quad & Z|_m \ne \mathbb{0}\}\| = \\ & \|\{m \in X(k+1) | \quad Z|_m \notin \{\mathbb{0}, \mathbb{1}\}\}\| + \|\{m \in X(k+1) | \quad Z|_m = \mathbb{1}\}\| \le \\ & \|\{m \in X(k+1) | \quad Z|_m \notin \{\mathbb{0}, \mathbb{1}\}\}\| + c\,\|\{m \in X(k+1) | \quad Z|_m \notin \{\mathbb{0}, \mathbb{1}\}\}\| = \\ & (1+c)\,\|\{m \in X(k+1) | \quad Z|_m \notin \{\mathbb{0}, \mathbb{1}\}\}\|\end{aligned}$$

which is bounded from above by the probability that step (3) rejects.
We end the proof by showing a way to combine both bounds. Note this is not trivial since the bound on $\|Z\|$ is dependent on the fact that $G = \mathbb{0}$. However, we can mitigate for this dependency, since G can be bounded independently of Z.

2 Preliminaries

Notation 2.1. *Given a set S and an integer k denote by* $\binom{S}{k} = \{s \subseteq S| \quad |s| = k\}$.

2.1 Simplicial Complexes

We are now going to provide formal definitions of simplicial complexes and a norm on them:

Definition 2.2 (Simplicial complex)**.** *A simplicial complex is a pair $X = (V, E)$ such that: $E \subseteq P(V)$, and if $F \in E$ then every $F' \subseteq F$ is in E as well. Elements in the set E are called the faces of X.*

Definition 2.3 (Dimension of a face)**.** *Let m be a face in X. Define the dimension of m to be:*

$$dim(m) := |m| - 1$$

Also, define the set $X(i)$ to be the set of all faces of dimension i (note that $X(-1) = \{\emptyset\}$).

Notation 2.4. *Let X be a d-dimensional simplicial complex, given $-1 \leq i < j \leq d$, a function $F : X(i) \to \{0, 1\}$, and $m \in X(j)$. Denote by $F|_m$ the function $F|_m : \binom{X(j)}{i+1} \to \{0, 1\}$ such that $\forall q \in X(i) : F|_m(q) = F(q)$.*

Definition 2.5 (Dimension of a simplicial complex)**.** *Let $X = (V, F)$ be a simplicial complex. Define the dimension of X to be:*

$$dim(X) := \max_{f \in F} dim(f)$$

Definition 2.6 (Pure simplicial complex)**.** *A d-dimensional simplicial complex X is called pure if all of its maximal faces are of dimension d.*

Definition 2.7 (Norm over the faces)**.** *Let X be a pure simplicial complex of dimension d. Define the weight of the face a to be:*

$$w(a) = \frac{|\{F \in X(d)| a \subseteq F\}|}{\binom{d+1}{|a|} \cdot |X(d)|}$$

and the norm $\|.\| = \|.\|^k : P(X(i)) \to [0, 1]$ to be: $\|A\| := \sum_{a \in A} w(A)$.

We will show in Appendix A that w defines a distribution on every dimension where the probability of a face to be chosen is equal to its norm. For the rest of the paper, when an algorithm chooses a face (unless a distribution is explicitly specified), it chooses a face with the distribution implied by w.

Definition 2.8 (Container)**.** *Let X be a d-dimensional simplicial complex, let $-1 \leq i \leq r \leq d$ and let $A \subseteq X(i)$. Define $\Gamma^r(A) := \{a \in X(r) | \exists b \in A : b \subseteq a\}$.*

Lemma 2.9. *Let X be a d-dimensional simplicial complex, and let $-1 \leq i \leq j \leq d$. Then for any $A \subseteq X(i)$:*

$$\|A\| \leq \left\|\Gamma^j(A)\right\| \leq \binom{j+1}{i+1} \|A\|$$

Lemma 2.10. *Let $A \subseteq X(i)$: $\forall j : \left\| \left\{ A' \in X(i+j) \middle| \binom{A'}{i} \subseteq A \right\} \right\| \leq \|A\|$*

The proofs of Lemma 2.9 and Lemma 2.10 can be found in Appendix B.

Notation 2.11. *Given a complex X, and a test T whose random choice is some $m \in X$, denote the result of the test T when testing the function F and the random face chosen is m from the complex X by $T_X^F(m)$.*

2.2 Co-systolic Expansion

We will now present the first notion of expansion used in this paper, namely - co-systolic expanders. Co-systolic expansion was introduced by Evra and Kaufman in [7] and is the more topological notion of expansion we will use in this paper. In order to define this notion of expansion we must first define some spaces and operators over simplicial complexes:

Definition 2.12 (Co-chains)**.** *Let X be a simplicial complex, define the i-co-chains of X to be $C^i(X) = \{0,1\}^{X(i)}$.*

Note that the norm defined in Definition 2.7 implies a norm on the co-chains by setting the norm of a co-chain to be the norm of set of faces on which it returns 1. Formally:

Definition 2.13 (Extension of the norm to co-chains)**.** *For every $C \in C^i\left(X\right)$ define:*

$$\|C\| := \|\{a \in X(i) | C(a) = 1\}\|$$

Now that we have defined the co-chains and a norm on them, we can also define the distance between co-chains as well as the distance of a co-chain from the k-direct-sums.

Definition 2.14 (Distance between co-chains)**.** *Given $C_1, C_2 \in C^k\left(X\right)$, the distance between C_1 and C_2 is:*

$$dist(C_1, C_2) = \|C_1 + C_2\|$$

Definition 2.15. *We define the distance of a co-chain $C \in C^k\left(X\right)$ to the k-direct-sum to be:*

$$\min_{D \in \{k\text{-}direct\text{-}sum\}} \{dist(C, D)\}$$

Definition 2.16 (Co-boundary operator)**.** *Let $\delta_i : C^i(X) \to C^{i+1}(X)$ be the following function:*

$$\delta_i(F)(m) = \sum_{q \in \binom{m}{i-1}} F(q)$$

Note that $F : X(i) \to \{0,1\}$ and $m \in X(i+1)$.

Lastly we will define two more spaces over the faces of the simplicial complex:

Definition 2.17 (Co-cycles and co-boundaries)**.** *Let X be a simplicial complex, define the following spaces:*

- *The i-co-cycles: $Z^i(X) = Ker(\delta_i) = \left\{ Z \in C^i(X) \middle| \delta_i Z = \mathbb{0} \right\}$.*

- *The i-co-boundaries:* $B^i(X) = Im(\delta_{i-1}) = \{B \in C^i(X) \big| \exists B' \in C^{i-1}(X) : B = \delta_{i-1}B'\}$.

Fact 2.18. *For every dimension i:* $B^i(X) \subseteq Z^i(X) \subseteq C^i(X)$.

A complex X is an (ϵ, μ)-co-systolic expander if any i-co-chain that is far from being a co-cycle "touches" an odd number of times many $(i+1)$-co-chains. In addition to that, any co-cycle that is not a co-boundary must be large. Formally:

Definition 2.19 (Co-systolic expander)**.** *Let X be a d-dimensional simplicial complex and let $\epsilon, \mu > 0$. X is an (ϵ, μ)-co-systolic-expander if for every $i = 0, 1, ..., d-1$:*

$$exp^i(X) = \min \left\{ \frac{\|\delta_i(f)\|}{\min_{z \in Z^i(X)} \{\|f + z\|\}} \middle| f \in C^i(X) \setminus Z^i(X) \right\} \geq \epsilon$$

and

$$syst^i(X) = \min \{\|z\| \big| z \in Z^i(X) \setminus B^i(X)\} \geq \mu$$

Note that $\min_{z \in Z^i(X)} \{\|f + z\|\}$ *is the distance of f from being a co-cycle.*

This notion of expansion implies that the simplicial complex has the topological overlapping property (which is explained in detail in [7]). In this paper, we will use this definition of expansion in order to estimate the non-co-cyclic part of the difference between the function given to us and its closest k-direct-sum. We will do that by first applying the co-boundary operator to the function given to us, and then test whether the result is a $(k+1)$-direct-sum (we will see why this suffices in section 3).

2.3 Colorful Expansion

The other form of high dimensional expansion we use is a combinatorial one. It was first introduced by Kaufman and Mass in [10]. This notion of expansion considers every face on which the i-co-chain is equal to 1 as if it is colored in one color, and every face on which the i-co-chain is equal to 0 as if is it colored in a different color. Then we look at all the $(i+1)$-faces that are not monochromatic. More formally:

Definition 2.20 (Colorful Operator)**.** *Let $c_i : C^i(X) \to C^{i+1}(X)$ be the following function:*

$$c_i(F)(m) = \begin{cases} 1 & \exists a, b \in \binom{m}{k-1} : F(a) = 1 \text{ and } F(b) = 0 \\ 0 & \text{otherwise} \end{cases}$$

Note that $F : X(i) \to \{0, 1\}$ and $m \in X(i+1)$.

A simplicial complex is a colorful expander if every sufficiently small i-co-chain implies a lot of non-monochromatic $(i+1)$-faces. Formally:

Definition 2.21 (Colorful Expander)**.** *Let X be a d-dimensional simplicial complex. We say that X is a σ-colorful-expander if for any $W \in C^i(X)$ $(0 \leq i < d)$ such that $\|W\| \leq 0.5$:*

$$\frac{\|c_i(W)\|}{\|W\|} \geq \sigma$$

This notion of expansion deals with random walks - consider the random walk that moves between two i-faces through a common $(i+1)$-face that contains them both. In [10] it was shown that such random walks converge rapidly to the stationary distribution. In this paper we will use this notion of expansion in order to estimate the co-cyclic part of the difference between the function given to us and its closest k-direct-sum (which would be impossible to do using the other notion of expansion).

3 Properties of Direct Sums

We will now present what the k-direct-sums are and show some useful properties of k-direct-suns.

Definition 3.1 (k-direct-sum). *A co-chain $D : X(k-1) \to \{0,1\}$ is called a k-direct-sum if there is some function $d : X(0) \to \{0,1\}$ such that $D(a) = \sum_{v \in a} d(v)$ (The sum is performed modulo 2).*

Definition 3.2 (Origin function). *Let $D : X(k) \to \{0,1\}$ be a k-direct-sum. An origin function of D is any function $d : X(0) \to \{0,1\}$ such that $D(a) = \sum_{v \in a} d(v)$.*

In the rest of this chapter we will explore properties of the k-direct-sums. We will start by finding a set of functions that spans the k-direct-sums. Then we will use these functions in order to show how direct-sums behave when applying the co-boundary operator to them. We will start by showing that the set of k-direct-sums is linear:

Lemma 3.3 (Direct sums are closed under addition). *Let F and G be two k-direct-sums whose origin functions are f and g respectively then $F + G$ is a k-direct-sum and its origin function is $f + g$.*

Proof. We know that $F(a) = \sum_{b \in a} f(b)$ and $G(a) = \sum_{b \in a} g(b)$. It is easy to see that $F + G = \sum_{b \in a} f(b) + \sum_{b \in a} g(b) = \sum_{b \in a} f(b) + g(b) = \sum_{b \in a} (f+g)(b)$. Therefore $F + G$ is a k-direct-sum and $f + g$ is its origin function. $\square$

We will now wish to find a set of functions that spans the k-direct-sum so:

Definition 3.4 (Spanning set of the k-direct-sums). *Let $u \in X(0)$. Define $H_u^k : X(k-1) \to \{0,1\}$ to be:*

$$H_u^k(a) = \begin{cases} 1 & \textit{if } u \in a \\ 0 & \textit{otherwise} \end{cases}$$

One can easily check that $\forall k : H_u^k$ is a k-direct-sum whose origin function is:

$$h_u^k(v) = \begin{cases} 1 & \textit{if } v = u \\ 0 & \textit{otherwise} \end{cases}$$

We can now prove that $\left\{H_u^k\right\}$ spans the set of k-direct-sums:

Lemma 3.5. *The set of k-direct-sums is spanned by $\left\{H_u^k | u \in X(0)\right\}$*

Proof. Let F be a k-direct-sum. By definition there exists $f : X(0) \to \{0,1\}$ such that $F(a) = \sum_{b \in a} f(b)$. Consider the support of f: $\sup(f) = \{u \in X(0) | f(u) = 1\}$, and define $G = \sum_{u \in \sup(f)} H_u^k$. It is easy to see that $F(a) = \sum_{b \in a} f(b) = \sum_{b \in a} H_u^k(b)$ and therefore $F \in span\left\{H_u^k | u \in X(0)\right\}$.

Let $F \in span\left\{H_u^k \middle| u \in X(0)\right\}$ therefore there exists some set $I \subseteq X(0)$ such that $F = \sum_{u\in I} H_u^k$. We know that $\left\{H_u^k\right\}_{u,k}$ are k-direct-sums, therefore F is a sum of k-direct-sums and, due to Lemma 3.3, F is a k-direct-sum as well. □

We will now show a connection between the k-direct-sums in the odd dimensions and the k-direct-sums in the even dimensions:

Lemma 3.6. *For odd values of k: $\delta_k H_u^k = H_u^{k+1}$*

Proof.

$$\delta_k H_u^k(a) = \sum_{\substack{b\subset a\\ |b|=|a|-1}} H_u^k(B) = |\left\{b | b \subset a, |b| = |a|-1, u \in b\right\}| = \begin{cases} \binom{k}{k-1} & \text{if } v_i \in a \\ 0 & \text{otherwise} \end{cases} =$$
$$\begin{cases} k & \text{if } u \in a \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } u \in A \\ 0 & \text{otherwise} \end{cases} = H_u^{k+1}$$

□

Lemma 3.7. *For odd values of k, if F is k-direct-sum then δF is a $(k+1)$-direct-sum.*

Proof. F is a k-direct-sum therefore there exists some $I \subseteq X(0)$ such that $F = \sum_{u\in I} H_u^k$. And thus $\delta F = \delta(\sum_{u\in I} H_u^k) = \sum_{u\in I} \delta H_u^k = \sum_{u\in I} H_u^{k+1} \in span\left\{H_u^{k+1} | u \in X(0)\right\}$ and δF is a $(k+1)$-direct-sum. □

Lemma 3.8. *For even values of k, if F is a k-direct-sum then $F \in B^{k+1}(X) \subseteq Z^{k-1}(X)$.*

Proof. F is a k-direct-sum therefore there exists $I \subseteq X(0)$ such that $F = \sum_{u\in I} H_u^k = \sum_{u\in I} \delta H_u^{k-1}$. Finally we get that $F \in B^{k-1}(X) \subseteq Z^{k-1}(X)$. □

Note that the previous two Lemmas imply that Lemma 3.7 is true for any value of k.

4 Definition of Components Appearing in the Tester

In this section, we will provide some definitions that will help us build the test for the k-direct-sum problem.
We would first like to define a relaxed version of the k-direct-sum, namely the k-co-cycle-indifferent-direct-sum:

Definition 4.1 (Co-cycle indifferent direct sum)**.** *Define the property of being a k-co-cycle-indifferent-direct-sum to be:*

$$CI = \left\{F = D + Z \middle| D \text{ is a } k\text{-direct-sum and } Z \in Z^{k-1}(X)\right\}$$

In section 6 will show that this property is testable for odd values of k.
We would also want to define a separator which helps in separating k-direct-sums from non k-direct-sums. Unlike tests, in which the rejection probability is linear in the distance from the property, separators reject with (at least) constant probability when their input is not in the property.

Definition 4.2 (Direct sum separator)**.** *Let X be a simplicial complex. An algorithm T is called an (n,k,Q,η)-direct-sum-separator if, for the complete complex on $n-1$ nodes (denoted by X_{n-1}), when given $f \in C^{k+1}(X_{n-1})$, the following applies:*

- *If f is a k-direct-sum then $Pr[T^f = 1] = 1$.*
- *If f is not a k-direct-sum then $Pr[T^f = 0] \geq \eta$.*
- *T queries f on at most Q faces in $X_{n-1}(k-1)$.*

In appendix C we will show an explicit separator whose error probability is 0 and queries the entire complex. We will also show how to construct a separator from a test. It is important to note that one can reduce the query complexity of the test presented in this paper by providing a different separator with lower query complexity (using, for example, Lemma C.3).

5 Presenting A Test for Being a k-direct-sum

In this section, we will prove the main theorem. But first recall the definition of a (Q,E)-test for being a k-direct-sum:

Definition 5.1 ((Q,E)-test for being a k-direct-sum)**.** *A (Q,E)-test for being a k-direct-sum is an algorithm that:*

- *Queries F on Q inputs from X.*
- *Accepts k-direct-sums.*
- *Rejects with probability of at least ξ every function whose distance from the k-direct-sums is at least $E\xi$.*

Theorem 5.2 (Main Theorem)**.** *Let X be a d-dimensional pure simplicial complex, and $0 < k \leq d-2$ be an odd constant. Also assume there exists a (Q,E)-test for being a $(k+1)$-direct-sum on X and let $F : X(k-1) \to \{0,1\}$ be a function. Then, if X is an (ϵ,μ)-co-systolic expander and a σ-colorful-expander, there exists a test T such that:*

- *T queries F a maximum of $\max\left\{(k+1)\cdot Q, \binom{k+2}{k}\right\}$ times.*
- *F is a k-direct-sum $\Leftrightarrow \Pr[T$ accepts $F] = 1$.*
- *If $\Pr[T$ rejects $F] \leq \xi$ then there exists a k-direct-sum F' such that*

$$dist(F,F') \leq \left(\left(1+\frac{1}{\sigma}\right)\left(\binom{k+2}{k}\frac{E}{\epsilon}+1\right)+\frac{E}{\epsilon}\right)\xi.$$

As a corollary we show that the k-direct-sum problem on the complete complex is testable with $O(k^2)$-queries for odd k.

Corollary 5.3 (k-direct-sum is testable on the complete complex for odd k's)**.** *On the complete complex there exists a $(O(k^2),E)$-test for being a k-direct-sum where E is constant and k is odd.*

The proof of this corollary will be presented in Appendix C. We also show that:

Corollary 5.4. *For any dimension d, there exists a family of* ***bounded*** *degree simplicial complexes X such that the property of k-direct-sum is testable on X.*

Proof. We will show that Ramanujan complexes satisfy the conditions of Theorem 5.2:

- In [7] it was shown that for any dimension d there exists q_0, such that for any prime power $q > q_0$, there are $\mu = \mu(d)$ and $\epsilon = \epsilon(d, q)$ such that if X is the the d-dimensional complex induced by a q-thick Ramanujan complex then X is an (ϵ, μ)-co-systolic expander.

- In addition to that in [10] it was proven that for any dimension d, there exists a constant $q_0' = q_0'(d)$ such that, if X is a d-dimensional q'-thick Ramanujan complex for $q' > q_0'$, then there are $\sigma = \sigma(d, q')$ such that X is a σ-colorful expander.

We end this proof by noting that it was shown in [11] that there is an explicit construction of Ramanujan complexes (and therefore there is an explicit construction for complexes that are both co-systolic expanders and colorful expanders). □

We will prove the main theorem using a (Q_{CI}, ζ)-test for the k-co-cycle-indifferent-direct-sum problem called T_{CI} and a $(k+2, k, Q_{sep}, \eta)$-direct-sum-separator T_{sep}. Specifically, we will prove that the following is a tester for the k-direct-sum problem:

Algorithm 2: $T_{direct-sum}$

```
pick one of the following options uniformly:
    Run T_CI and return its result.
    pick m ∈ X(k+1) randomly:
        Run T_sep on m with F|_m and return its result.
```

Formally we will prove that:

Theorem 5.5 (k-direct-sums are testable)**.** *On any complex that is a σ-colorful-expander and for any constant odd value of k, given:*

- *T_{sep} - A $(k+2, k, Q_{sep}, \eta)$-direct-sum-separator for the complete complex.*

- *T_{CI} - A (Q_{CI}, ζ)-test for the k-co-cycle-indifferent-direct-sum.*

We can construct $T_{direct-sum}$ as shown above such that $T_{direct-sum}$ is a:

$$\left(\max\{Q_{CI}, Q_{sep}\}, \left(1 + \frac{1}{\sigma}\right)\left(\frac{1}{\eta} + \binom{k+2}{k}\zeta\right) + \zeta\right)\text{-test}$$

for the k-direct-sum problem.

In order to understand why the test works, consider a deconstruction of F into three parts: $F = D + Z + G$. In this deconstruction we assume that:

- G is minimal with regards to the k-co-cycle-indifferent-direct-sum.

- Z is the minimal co-cycle with regards to the k-direct-sum problem.

- D is a k-direct-sum.

In sub-section 5.1 we will show that the rejection probability of step (2) bounds from above $\|G\|$. In sub-section 5.2 we will show that, when ignoring G, step (3)'s rejection probability bounds $\|Z\|$ from above. Finally in sub-section 5.3 we will show how the combination of both steps provides a test for being a k-direct-sum. Note that unlike step (2) (in which there is no assumption on Z), the analysis of step (3) assumes that $G = \mathbb{0}$.

5.1 Step (2) of the test estimates the Norm of G

Lemma 5.6. *Let T_{CI} be a (Q_{CI}, ζ)-test for the k-co-cycle-indifferent-direct-sum then:*

$$\|G\| \leq \zeta \cdot \Pr\left[\text{step (2) rejects}\right]$$

Proof. $\|G\| = dist(F, CI) \leq \zeta \cdot \Pr\left[T_{CI} = 0\right] = \zeta \cdot \Pr\left[\text{step (2) rejects}\right]$ The second inequality holds due to the definition of T_{CI}. □

5.2 Step (3) of the Test Estimates Norm of Z Assuming That There is No Remainder

In step (3) we pick a $(k+1)$-dimensional face randomly and then check whether the function is a k-direct sum on that specific face. In this section, we will show that the failure probability of doing so bounds $\|Z\|$ from above. We will do that by first observing that given m, a $(k+1)$-dimensional face, either $F|_m$ is not a k-direct-sum or $Z|_m \in \{\mathbb{0}, \mathbb{1}\}$:

Lemma 5.7. *Let $F = D + Z$ such that D is a k-direct-sum and $Z \in Z^{k-1}(X)$ then for every odd value k and $m \in X(k+1)$: If $F|_m$ is a k-direct-sum on m then: $Z|_m \in \{\mathbb{0}, \mathbb{1}\}$.*

Proof. $F|_m$ is a k-direct-sum and, because $G = \mathbb{0}$, so is $Z|_m$ as $Z|_m = F|_m + D|_m$. Assuming that $Z|_m \notin \{\mathbb{0}, \mathbb{1}\}$, let $z : m \to \{0, 1\}$ be an origin function of Z and let $A_i = \{v \in m | z(v) = i\}$. Pick the largest possible set (of up to $k+1$ elements) of odd size out of A_1 (the set is not empty because otherwise $Z|_m = \mathbb{0}$) and name it A. Add to that set $k + 1 - |A|$ items from A_0 (which cannot be empty since $Z|_m \neq \mathbb{1}$) to form a $(k+1)$-face which we will denote as t. It is easy to see that $\delta Z|_m(t) = \sum_{v \in t} z(v) = \sum_{v \in A} z(v) = 1$ (the last equality holds because $|A|$ is odd) which contradicts the fact that $Z \in Z^{k-1}(X)$ □

We now observe that the set of $(k+1)$-dimensional faces on which $Z|_m \neq \mathbb{0}$ can be split into two sets:

- The set of all $m \in X(k+1)$ on which $F|_m$ is not a k-direct-sum.
- The set of all $m \in X(k+1)$ such that $Z|_m = \mathbb{1}$ (which we will denote as S).

It is easy to see that the rejection probability of step (3) bounds the first set (since step (3) fails on every face in the set). We will spend the majority of this sub-section proving that $\|S\|$ can also be bounded from above using the rejection probability of step (3). We will end this sub-section by combining the aforementioned bounds.
Before discussing how to bound $\|S\|$ from above, it will be useful to present Lemma 5.7 again, this time with the new terminology described above:

Corollary 5.8. *Let $m \in X(k+1)$ then:*

$$F|_m \textit{ is not a } k\textit{-direct-sum} \Leftrightarrow m \in \Gamma^{k+1}(Z) \setminus S$$

Proof. $m \in \Gamma^{k+1}(Z)$ iff $Z|_m \neq \mathbb{0}$ (due to the definition of Γ) and $m \notin S$ iff $Z|_m \neq \mathbb{1}$ (due to the definition of S) therefore:

$$F|_m \text{ is not a } k\text{-direct-sum} \Leftrightarrow Z \notin \{\mathbb{0}, \mathbb{1}\} \Leftrightarrow m \in \Gamma^{k+1}(Z) \setminus S$$

□

In order to bound $\|S\|$ we will look at a different function whose norm bounds $\|S\|$ from above, specifically:

Definition 5.9. *Define $E : X(k) \to \{0, 1\}$ to be the following function:*

$$E(a) = \begin{cases} 1 & \textit{if } Z|_a = \mathbb{1} \\ 0 & \textit{otherwise} \end{cases}$$

This function helps in bounding $\|S\|$ from above because every face in S is comprised solely of k-dimensional faces on which E returns 1. Combining this fact with Lemma 2.10 yields that $\|S\| \leq \|E\|$.
All we have to do now is to bound $\|E\|$. This will be done by first showing that step (3) of the test rejects every non-monochromatic $(k+1)$-face (where E is considered the coloring). We will then show that $\|E\| < 0.5$ which will allow to use the colorful expansion in order to bound $\|E\|$.

Lemma 5.10 (Step (3) Fails on the Non-Monochromatic Faces)**.** *Let $m \in Z(k+1)$. If $c(E)(m) = 1$ then $F|_m$ is not a k-direct-sum.*

Proof. $c(E)(m) = 1 \Rightarrow \exists a, b \in \binom{m}{k+1} : E(a) = 1$ and $E(b) = 0$. Using the definition of E we get that:

- $E(b) = 0 \Rightarrow \exists c \in \binom{b}{k} : Z(c) = 0$
- $E(a) = 1 \Rightarrow \forall t \in \binom{a}{k} : Z(t) = 1$

Therefore $Z|_m \notin \{\mathbb{0}, \mathbb{1}\}$ and $F|_m$ is not a k-direct-sum (Lemma 5.7). □

Lemma 5.11. *For every function of the form $F = D + Z + G$ it holds that $\|Z\| \leq 0.5$ (Note that this lemma is true even if $G \neq 0$).*

Proof. It is easy to see that the function $f(v) = 1$ is the origin function of $\mathbb{1}$ and therefore $\mathbb{1}$ is a k-direct-sum. Now, assuming that $\|Z\| > 0.5$ we conclude that $\|\mathbb{1} + Z\| \leq 0.5$ and $(\mathbb{1}+D)+(\mathbb{1}+Z) = D+Z$. Also $(\mathbb{1}+D)$ is a k-direct-sum. We conclude that $\|\mathbb{1} + Z\| < \|Z\|$ and $F+G+(\mathbb{1}+Z) = \mathbb{1}+D$ which is a k-direct-sum. This contradicts the fact that Z is minimal. □

Corollary 5.12. $\|E\| \leq 0.5$.

Proof. By the definition of E if $E(a) = 1$ then $\forall a' \in \binom{a}{k} : Z(a') = 1$. Using Lemma 2.10 yields that $\|E\| \leq \|Z\|$ which finishes the proof. □

We are now finally ready to bound E using the colorful expansion of X:

Lemma 5.13 (Estimating E)**.** *On every σ-colorful expander X:*

$$\|E\| \leq \frac{1}{\sigma} \|\{m \in X(k+1)| \quad F|_m \textit{ is not a } k\textit{-direct-sum}\}\|$$

Proof. X is a colorful expander and $\|E\| \leq 0.5$ therefore $\sigma \leq \frac{\|c(E)\|}{\|E\|}$ which in turn means that:

$$\sigma \|E\| \leq \|c(E)\| \leq \|\{m \in X(k+1)| \quad F|_m \text{ is not a } k\text{-direct-sum}\}\|$$

(the second inequality is due to Lemma 5.10) and therefore:

$$\|E\| \leq \frac{1}{\sigma} \|\{m \in X(k+1)| \quad F|_m \text{ is not a } k\text{-direct-sum}\}\|$$

□

Lemma 5.14 (Estimating Z)**.** *Let X be a σ-colorful-expander, and let F be a function of the form $F = D + Z$ such that D is a k-direct-sum and Z is a co-cycle then:*

$$\|Z\| \leq \frac{1}{\eta}\left(1 + \frac{1}{\sigma}\right) \Pr\left[\text{step (3) rejects}\right]$$

Proof.

$$\begin{aligned}\|Z\| \leq \left\|\Gamma^{k+1}(Z)\right\| \leq \left\|\left(\Gamma^{k+1}(Z) \setminus S\right) \cup S\right\| = \left\|\Gamma^{k+1}(Z) \setminus S\right\| + \|S\| \leq \\ \left\|\Gamma^{k+1}(Z) \setminus S\right\| + \|E\| \leq \left\|\Gamma^{k+1}(Z) \setminus S\right\| + \frac{1}{\sigma}\left\|\Gamma^{k+1}(Z) \setminus S\right\| = \\ \left(1 + \frac{1}{\sigma}\right)\left\|\Gamma^{k+1}(Z) \setminus S\right\| = \\ \left(1 + \frac{1}{\sigma}\right)\|\{m \in X(k+1)| \quad F|_m \text{ is not a } k\text{-direct-sum}\}\|\end{aligned}$$

Note that the inequality found at the end of the first row is due to Lemma 2.10, the inequality in the second row is due to Lemma 5.13 and the last equality is due to Corollary 5.8.
Note that:

$$\begin{aligned}&\Pr\left[\text{step (3) rejects}\right] = \\ &\quad \Pr\left[F|_m \text{ is not a } k\text{-direct-sum}\right] \cdot \Pr\left[T_{sep} \text{ rejects}| \quad F|_m \text{ is not a } k\text{-direct-sum}\right] = \\ &\quad \|\{m \in X(k+1)| \quad F|_m \text{ is not a } k\text{-direct-sum}\}\| \cdot \eta\end{aligned}$$

All the probabilities are over a choice of $m \in X(k+1)$.
We conclude by noting that this yields that:

$$\|Z\| \leq \frac{1}{\eta}\left(1 + \frac{1}{\sigma}\right) \Pr\left[\text{step (3) rejects}\right]$$

□

5.3 Combining the Estimations

Now that we know how to estimate both $\|G\|$ and $\|Z\|$ (with the assumption that $G = \mathbb{0}$), it is finally time to combine both estimations in order to estimate $\|Z + G\|$. Note that our estimation of $\|Z\|$ is dependent on our estimation of $\|G\|$. We will deal with this dependency by bounding the interference of G using our estimation of it. We will then estimate $\|Z\|$ as if wherever G would have interfered, step (3) rejected.

Lemma 5.15. *Let $F = D + Z + G$ such that D is a k-direct-sum, Z is a $(k-1)$-co-cycle and G is the remainder. Then if* $\Pr\left[T^F_{direct-sum} \text{ rejects}\right] \leq \xi$ *then* $\|Z\| \leq \left(1 + \frac{1}{\sigma}\right)\left(\frac{1}{\eta} + \binom{k+2}{k}\zeta\right)\xi$.

Proof. First note that because $\Pr\left[T^F_{direct-sum} \text{ rejects}\right] \leq \xi$ we know that
$\Pr[\text{step (2) rejects } F] \leq \xi$ and $\Pr[\text{step (3) rejects } F] \leq \xi$. Also, consider what happens when we run the test on $F' = D + Z$. Note that on F' the bound found in Lemma 5.14 holds. Also note the the co-cyclic part of F and F' is Z. Therefore if we could bound the rejection probability of step (3) on F' using the rejection probability of steps (2) and (3) on F we would have a bound for $\|Z\|$. We will start by bounding the set of $(k+1)$-faces on which F' is not a k-direct-sum:

$$\begin{aligned}
&\left\{m \in X(k+1) \middle| \quad F'|_m \text{ is not a } k\text{-direct-sum}\right\} \subseteq \\
&\quad \left\{m \in X(k+1) \middle| \quad (F'+G)|_m \text{ is not a } k\text{-direct-sum and } G|_m = \mathbb{0}\right\} \cup \\
&\quad \{m \in X(k+1) | \quad G|_m \neq \mathbb{0}\} = \\
&\quad \{m \in X(k+1) | \quad F|_m \text{ is not a } k\text{-direct-sum}\} \cup \{m \in X(k+1) | \quad G|_m \neq \mathbb{0}\} = \\
&\quad \{m \in X(k+1) | \quad F|_m \text{ is not a } k\text{-direct-sum}\} \cup \Gamma^{k+1}(G)
\end{aligned}$$

Knowing this, we get that:

$$\begin{aligned}
&\Pr\left[\text{step (3) rejects } m \text{ when testing } F'\right] = \\
&\quad \eta \left\|\left\{m \in X(k+1) \middle| \quad F'|_m \text{ is not a } k\text{-direct-sum}\right\}\right\| \leq \\
&\quad \eta \left\|\{m \in X(k+1) | \quad F|_m \text{ is not a } k\text{-direct-sum}\}\right\| + \eta \left\|\Gamma^{k+1}(G)\right\|
\end{aligned}$$

Using Lemma 5.6, we know that $\|G\| \leq \zeta \cdot \Pr[\text{step (2) rejects}]$ and therefore, using Lemma 2.9 we get that $\left\|\Gamma^{k+1}(G)\right\| \leq \binom{k+2}{k}\|G\| \leq \binom{k+2}{k}\zeta \cdot \Pr[\text{step (2) rejects}]$. Therefore:

$$\begin{aligned}
&\Pr\left[\text{step (3) rejects } m \text{ when testing } F'\right] \leq \\
&\quad \eta \left\|\{m \in X(k+1) | \quad F|_m \text{ is not a } k\text{-direct-sum}\}\right\| + \eta \left\|\Gamma^{k+1}(G)\right\| \leq \\
&\quad \Pr[\text{step (3) rejects } m \text{ when testing } F] + \binom{k+2}{k}\eta\zeta \cdot \Pr[\text{step (2) rejects}] \leq \\
&\quad \left(1 + \binom{k+2}{k}\eta\zeta\right)\xi
\end{aligned}$$

We will now use the bound obtained in Lemma 5.14 on F' which would yield:

$$\eta\frac{1}{1+\frac{1}{\sigma}}\|Z\| \leq \left(1 + \binom{k+2}{k}\eta\zeta\right)\xi \Rightarrow \|Z\| \leq \left(1 + \frac{1}{\sigma}\right)\left(\frac{1}{\eta} + \binom{k+2}{k}\zeta\right)\xi$$

□

We are now finally ready to prove Theorem 5.5:

Proof of Theorem 5.5. First, consider the number of queries performed by $T_{direct-sum}$. If step (2) is chosen then $T_{direct-sum}$ performs Q_{CI} queries and if step (3) is chosen then $T_{direct-sum}$ performs Q_{sep} queries. Therefore $T_{direct-sum}$ performs, at most,

$\max\{Q_{CI}, Q_{sep}\}$ queries.
Suppose that $\Pr\left[T^F_{direct-sum} \text{ rejects}\right] \leq \xi$ then, using Lemma 5.6 and Lemma 5.15 we get that:

$$\|Z+G\| \leq \|Z\| + \|G\| \leq \left(1+\frac{1}{\sigma}\right)\left(\frac{1}{\eta} + \binom{k+2}{k}\zeta\right)\xi + \zeta\xi =$$
$$\left(\left(1+\frac{1}{\sigma}\right)\left(\frac{1}{\eta} + \binom{k+2}{k}\zeta\right) + \zeta\right)\xi$$

Now all that is left to prove is that a k-direct-sum will always pass the test. If step (2) is chosen then, because a k-direct-sum is also a k-co-cycle-indifferent-direct-sum, the test will always accept. Otherwise, if step (3) is chosen then, because the function is a k-direct-sum, it will be a k-direct-sum on any sub-complex of dimension $k+1$ and therefore step (3) will always accept as well. □

We can now prove the main theorem using Theorem 5.5:

Proof of Theorem 5.2. Combining Lemma 6.2 and Lemma C.2 we get that there exists a $\left(\max\left\{(k+1)\cdot Q, \binom{k+2}{k}\right\}, \left(\left(1+\frac{1}{\sigma}\right)\left(\binom{k+2}{k}\frac{E}{\epsilon}+1\right)+\frac{E}{\epsilon}\right)\right)$-test for k-direct-sum for odd values of k. □

6 Providing a Test for Being a k-co-cycle-indifferent-direct-sum

In this section we will show how to obtain a test for being a k-co-cycle-indifferent-direct-sum using a test for being a $(k+1)$-direct-sum. We will do that by considering the expansion of k-direct sums under co-systolic expansion.

Lemma 6.1. *For any function $F = D + Z + G$ (G is minimal) on an ϵ-co-systolic expander: $\|G\| \leq \frac{1}{\epsilon} dist(\delta F, k\text{-direct-sum})$*

Proof. First note that $\delta F = \delta D + \delta Z + \delta G = \delta D + \delta G$. In addition, because the complex is an ϵ-co-systolic expander and G is minimal: $\|G\| \leq \frac{1}{\epsilon}\|\delta G\|$. Also $\|\delta G\| = dist(\delta F, k\text{-direct-sum})$ and therefore $\|G\| \leq \frac{1}{\epsilon} dist(\delta F, k\text{-direct-sum})$. □

We are now ready to provide the actual test:

Lemma 6.2. *Let X be an (ϵ,μ)-co-systolic-expander. If there is a (Q,ξ)-test for being a $(k+1)$-direct-sum (denoted by T) on X, then there is also a $((k+1)\cdot Q, \frac{\xi}{\epsilon})$-test for being a k-co-cycle-indifferent-direct-sum on X.*

Proof. Consider the following test:

Algorithm 3: T_{CI}

1 Return the result of T on δF (whenever T queries δF, calculate it and send the result).

It is easy to see that:

$$dist(F, k\text{-co-cycle-indifferent-direct-sum}) = \|G\| \leq$$
$$\frac{1}{\epsilon} dist(\delta F, k\text{-direct-sum}) \leq \frac{\xi}{\epsilon}\Pr\left[T_{CI} = 0\right]$$

Also, F is a k-co-cycle-indifferent-direct-sum $\Leftrightarrow G = \mathbb{0} \Leftrightarrow \Pr\left[T_{CI} \text{ accepts } F\right] = 1$
For any query T makes, T_{CI} makes $\binom{k+1}{k} = (k+1)$ queries and therefore T_{CI} performs at most $(k+1)\cdot Q$ queries. □

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A Sampling According to the Norm

In this section we will show how to pick a face with probability that equals to its norm. Consider the following sampling algorithm:

Algorithm 4: *Sample*(l, r)

```
1 pick uniformly (using the random bits from r) m ∈ X(d).
2 while |m| > l + 1 do
3     pick uniformly (using the random bits from r) v ∈ m.
4     m ← m \ {v}.
5 end
6 return m.
```

note that steps 3 and 4 are equivalent to choosing a sub-face of m of dimension $\dim(m) - 1$. We are going to requite a way to denote a specific value of m during the run of the sampling algorithm:

Definition A.1. *Given a single run of* $Sample(l, r)$*, define for every* $l + 1 < i < d + 1$*:* M_i^r *to be the value of* m *when* $|m| = i$ *and the random bits chosen by the algorithm are* r*.*

It is easy to see that these sets satisfy the following properties:

- $\forall r \forall i : M_i^r \subset M_{i+1}^r$.
- $\forall i \forall a \in X(i-1) : \Pr[M_i^r = a] = \Pr[Sample(i-1, r) = a]$.

Lemma A.2. *Let* X *be a simplicial complex of dimension* d *and let* $-1 \le l \le d$ *then:*

$$\forall a \in X(l) : \Pr[Sample(l, r) = a] = w(a)$$

And also:

$$\forall A \in P(X(i)) : \|A\| = \Pr[Sample(i, r) \in A]$$

Where Sample is the algorithm 4.

Proof. We will prove this lemma using induction:
Base case: $l = d$: Notice that $\forall a \in X(d) : w(a) = \frac{1}{\binom{d+1}{d+1}|X(d)|}$.
The lemma holds because $Sample(d + 1, r)$ simply chooses a face of dimension d uniformly.
Assuming that $\forall a \in X(l+1) : \Pr[Sample(l + 1, r) = a] = w(a)$ we will now prove that $\forall a \in X(l) : \Pr[Sample(l, r) = a] = w(a)$ (where M_i^r are the values defined in definition A.1):

$$\begin{aligned}
&\forall a \in X(l) : \Pr[Sample(l, r) = a] = \\
&\quad \sum_{b \in \{b \in X(l+1) | a \subseteq b\}} \Pr\left[Sample(l, r) = a \middle| M_{l+1}^r = b\right] \cdot \Pr\left[M_{l+1}^r = b\right] = \\
&\quad \sum_{b \in \{b \in X(l+1) | a \subseteq b\}} \frac{1}{l+2} w(b) = \sum_{b \in \{b \in X(l+1) | a \subseteq b\}} \frac{1}{l+2} \frac{|\{q \in X(d) | b \subseteq q\}|}{\binom{d+1}{|b|} \cdot |X(d)|} = \\
&\quad \sum_{b \in \{b \in X(l+1) | a \subseteq b\}} \frac{1}{d-l} \frac{|\{q \in X(d) | b \subseteq q\}|}{\binom{d+1}{|b|-1} \cdot |X(d)|} = \\
&\quad \frac{1}{(d-l)\binom{d+1}{|a|} \cdot |X(d)|} \sum_{b \in \{b \in X(l+1) | a \subseteq b\}} |\{q \in X(d) | b \subseteq q\}| \\
&\quad = \frac{|\{q \in X(d) | b \subseteq q\}|}{\binom{d+1}{|a|} \cdot |X(d)|} = w(a)
\end{aligned}$$

The fourth equation holds because:

$$(l+2)\binom{d+1}{l+2} = (l+2)\frac{(d+1)!}{(l+2)!(d-l-1)!} = \frac{(d+1)!}{(l+1)!(d-l-1)!}$$
$$= (d-l)\frac{(d+1)!}{(l+1)!(d-l)!} = (d-l)\binom{d+1}{l+1}$$

The sixth equation holds because every maximal face that contains a is counted $\binom{d+1-(l+1)}{1} = d-l$ times. Finally we can see that:

$$\Pr\left[Sample(i,r) \in A\right] = \Pr\left[\bigvee_{a\in A} Sample(i,r) = a\right] = \sum_{a\in A} \Pr\left[Sample(i,r) = a\right] =$$
$$\sum_{a\in A} w(\{a\}) = \|A\|$$

$\square$

B Proofs of Bounds on the Norm

Proof of Lemma 2.9. First consider how a single face behaves under Γ^j:

$$\forall a \in X(i) : \left\|\Gamma^j(\{a\})\right\| = \sum_{\substack{b\in X(j)\\ a\subseteq b}} w(b) = \sum_{\substack{b\in X(j)\\ a\subseteq b}} \frac{|\{q \in X(d) | b \subseteq q\}|}{\binom{d+1}{|b|}\cdot|X(d)|}$$
$$= \sum_{\substack{b\in X(j)\\ a\subseteq b}} \sum_{\substack{q\in X(d)\\ b\subseteq q}} \frac{1}{\binom{d+1}{j+1}\cdot|X(d)|} = \sum_{\substack{q\in X(d)\\ a\subseteq q}} \sum_{\substack{b\in X(j)\\ a\subseteq b\subseteq q}} \frac{1}{\binom{d+1}{j+1}\cdot|X(d)|}$$
$$= \sum_{\substack{q\in X(d)\\ a\subseteq q}} \frac{\binom{d-i}{j-i}}{\binom{d+1}{j+1}\cdot|X(d)|} = \frac{\binom{d-i}{j-i}\cdot|\{q\in X(d)|a\subseteq q\}|}{\binom{d+1}{j+1}\cdot|X(d)|} = \frac{\binom{d-i}{j-i}\cdot\binom{d+1}{i+1}}{\binom{d+1}{j+1}} w(\{a\})$$
$$= \binom{j+1}{i+1} w(\{a\})$$

Note that the last equation holds because:

$$\frac{\binom{d-i}{j-i}\cdot\binom{d+1}{i+1}}{\binom{d+1}{j+1}} = \frac{\frac{(d+1)!(d-i)!}{(j-i)!(d-j)!(i+1)!(d-i)!}}{\frac{(d+1)!}{(d-j)!(j+1)!}} = \frac{(j+1)!}{(j-i)!(i+1)!} = \binom{j+1}{i+1}$$

Now one can easily check that:

$$\forall A \subseteq X(i) : \left\|\Gamma^j(A)\right\| = \left\|\bigcup_{a\in A}\Gamma^j(\{a\})\right\| \leq \sum_{a\in A}\left\|\Gamma^j(\{a\})\right\| = \sum_{a\in A}\binom{j+1}{i+1} w(\{a\})$$
$$= \binom{j+1}{i+1}\sum_{a\in A} w(\{a\}) = \binom{j+1}{i+1}\|A\|$$

The other direction can be achieved by looking at the algorithm presented in Lemma A.2: Consider the set of values M_i^r defined in definition A.1. Note that for every co-chain A: If $M_{i+1}^r \in A$ then $M_{j+1}^r \in \Gamma^j(A)$ (because $M_{i+1}^r \subseteq M_{j+1}^r$). Now we can see that:

$$\begin{aligned}\forall A \subseteq X(i) : \|A\| &= \Pr\left[Sample(i,r) \in A\right] = \Pr\left[M_{i+1}^r \in A\right] \leq \Pr\left[M_{j+1}^r \in \Gamma^j(A)\right] \\ &= \Pr\left[Sample(j,r) \in \Gamma^j(A)\right] = \left\|\Gamma^j(A)\right\|\end{aligned}$$

□

Proof of Lemma 2.10. First denote $U = \left\{A' \in X(i+j) \middle| \binom{A'}{i} \subseteq A\right\}$ and let M_i^r be the values defined in definition A.1. Due to Lemma A.2 we know that:

$$\begin{aligned}&\|A\| = \Pr\left[Sample(i,r) \in A\right] = \Pr\left[M_{i+1}^r \in A\right] = \\ &\quad \Pr\left[M_{i+1}^r \in A \middle| M_{i+j}^r \in U\right] \cdot \Pr\left[M_{i+j}^r \in U\right] + \\ &\quad \Pr\left[M_{i+1}^r \in A \middle| M_{i+j}^r \notin U\right] \cdot \Pr\left[M_{i+j}^r \notin U\right] \geq \\ &\quad \Pr\left[M_{i+1}^r \in A \middle| M_{i+j}^r \in U\right] \cdot \Pr\left[M_{i+j}^r \in U\right] = \\ &\quad \Pr\left[M_{i+j}^r \in U\right] = \Pr\left[Sample(i+j,r) \in U\right] = \|U\|\end{aligned}$$

□

C Direct Sum Separators

In this section we will provide two direct sum separators: One using reconstructing the origin-function of F, and the other using a test for being a k-direct-sum. The first method provided here yields a separator that separates a k-direct-sum from other functions with probability 1. The other method, allows reducing the query complexity while increasing the error margin.

C.1 Direct Sum Separator Using Reconstruction

In this section we will provide a simple direct sum separator that, given F, attempts to reconstruct the origin function of F and accepts whenever it succeeds.

Lemma C.1. *Let X_{k+2} be a the complete simplicial complex on $k+2$ nodes and $F : X_{k+2}(k-1) \to \{0,1\}$. Define f to be a function that, given $v \in X_{k+2}(0)$, picks a $q \in X_{k+2}(k-1)$ such that $v \notin q$ and returns $\sum_{w \in \binom{q}{k-1}} F(w \cup \{v\})$. We will show that: F is a k-direct-sum $\Leftrightarrow$ f is an origin function of F.*

Proof. $\Rightarrow$ F is a k-direct-sum therefore there exists an origin function to F denoted by f'.

$$\begin{aligned}&\forall q : f(v) = \sum_{w \in \binom{q}{k-1}} F(w \cup \{v\}) = \sum_{w \in \binom{q}{k-1}} \left(f'(v) + \sum_{v' \in w} f'(v')\right) = \\ &\binom{k}{k-1} f'(v) + \sum_{v' \in q} \binom{k-1}{k-2} f'(v') = k \cdot f'(v) + \sum_{v' \in q} (k-1) \cdot f'(v') = f'(v)\end{aligned}$$

And therefore f is an origin function of F.
$\Leftarrow$ F has an origin function and therefore it is a k-direct-sum. □

This lemma allows us to create the following $\left(k+2, k, \binom{k+2}{k}, 1\right)$-separator:

Lemma C.2 (Direct Sum Separator for Odd Values of k). *The following is a $\left(k+2, k, \binom{k+2}{k}, 1\right)$-direct-sum-separator (given a function $F \in C^k(X)$ on a simplicial complex X):*

Algorithm 5: T_{sep}

---

foreach *node* $v \in X(0)$ **do**

  Calculate $f(v)$.

end

foreach *face* $q \in X(k)$ **do**

  Check whether $F(q) = \sum_{e \in q} f(e)$*, if it is not return* 0.

end

Return 1

Proof. The algorithm returns $1 \Leftrightarrow F$ is a k-direct-sum on X due to Lemma C.1.
It is easy to see that the separator queries the entire function (Therefore it uses $\binom{k+2}{k}$ queries). □

C.2 Obtaining a Direct Sum Separator From a Test

In this section we will show how to construct a separator out of a test for the k-direct-sums over a $k+1$ dimensional complex. This will help reduce query complexity.

Lemma C.3 (Separator from Test). *If there is a (Q, E)-test (denoted by T) for being a k-direct-sum on a $k+1$ dimensional complex then there is a $(k+2, k, Q, \rho)$-direct-sum-separator such that $\rho = min_{F \in C^{k-1}(X) \setminus \{k\text{-direct-sums}\}} \left\{Pr[T_X^G = 0]\right\}$ where X is the complete $(k{+}1)$-dimensional complex.*

Proof. Consider the following tester:

Algorithm 6: T'_{sep}

Run T on F and return its output.

It is east to see that the algorithm queries F exactly Q times.
All we have to prove is that if F is not a k-direct-sum than the algorithm returns false with probability of at least ρ. $F \in C^{k-1}(X) \setminus \{k\text{-direct-sums}\}$ and therefore:

$$Pr[T_X^G = 0] \geq min_{G \in C^{k-1}(X) \setminus \{k\text{-direct-sums}\}} \left\{Pr[T_X^G = 0]\right\} = \rho$$

Note that $\rho > 0$ because if $\rho = 0$ then there would exist a function F' such that $F' \in C^{k-1}(X) \setminus \{k\text{-direct-sums}\}$ and $Pr[T_X^{F'} = 0] = 0$. Note that T is a test and therefore if $Pr[T_X^{F'} = 0] = 0$ then F' is a k-direct-sum which contradicts the assumption about F'. □

Lastly, we can prove corollary 5.3:

Proof of Corollary 5.3. Combining the second test provided in [4] and Lemma C.3 we get a $(k+2, k, O(k), \rho)$-separator. From the first test provided in [4] and Lemma 6.2 we get a $(3k+3, E')$-test for being a k-co-cycle-indifferent-direct-sum. Combining both of these results with Theorem 5.5 yields the desired result. □

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