

Arikan meets Shannon:
Polar codes with near-optimal convergence to channel capacity

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Abstract

Let W be a binary-input memoryless symmetric (BMS) channel with Shannon capacity $I(W)$ and fix any $\alpha > 0$. We construct, for any sufficiently small $\delta > 0$, binary linear codes of block length $O(1/\delta^{2+\alpha})$ and rate $I(W) - \delta$ that enable reliable communication on W with quasi-linear time encoding and decoding. Shannon's noisy coding theorem established the *existence* of such codes (without efficient constructions or decoding) with block length $O(1/\delta^2)$. This quadratic dependence on the gap δ to capacity is known to be best possible. Our result thus yields a constructive version of Shannon's theorem with near-optimal convergence to capacity as a function of the block length. This resolves a central theoretical challenge associated with the attainment of Shannon capacity. Previously such a result was only known for the erasure channel.

Our codes are a variant of Arikan's polar codes based on multiple carefully constructed local kernels, one for each intermediate channel that arises in the decoding. A crucial ingredient in the analysis is a strong converse of the noisy coding theorem when communicating using random linear codes on arbitrary BMS channels. Our converse theorem shows extreme unpredictability of even a single message bit for random coding at rates slightly above capacity.

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1 Introduction

We construct binary linear codes that achieve the Shannon capacity of the binary symmetric channel, and indeed any binary-input memoryless symmetric (BMS) channel, with a near-optimal scaling between the code length and the gap to capacity. Further, our codes have efficient (quasi-linear time) encoding and decoding algorithms. Let us now describe the context of our result and its precise statement in more detail.

The binary symmetric channel (BSC) is one of the most fundamental and well-studied noise models in coding theory. The BSC with crossover probability $p \in (0, 1/2)$ (BSC_p) flips each transmitted bit independently with probability p . By Shannon’s seminal noisy coding theorem [Sha48], we know that the capacity of BSC_p is $1 - h(p)$, where $h(\cdot)$ is the binary entropy function. This means that reliable communication over BSC_p is possible at information rates approaching $1 - h(p)$, and at rates above $1 - h(p)$ this is not possible. More precisely, for any $\delta > 0$, there *exist* codes of rate $1 - h(p) - \delta$ using which one can achieve miscommunication probability at most $2^{-\Omega(\delta^2 n)}$ where n is the block length of the code. In fact, random linear codes under maximum likelihood decoding offer this guarantee with high probability. Thus Shannon’s theorem implies the existence of codes of block length $O(1/\delta^2)$ that can achieve small error probability on BSC_p at rates within δ of capacity. Conversely, by several classical results [Wol57, Str62, Str09, PPV10], we know that the block length has to be at least $\Omega(1/\delta^2)$ in order to approach capacity within δ .

Shannon’s theorem is based on the probabilistic method and does not describe the codes that approach capacity or give efficient algorithms to decode them from errors caused by BSC_p . Thus the codes with rates $1 - h(p) - \delta$ take at least time exponential in $1/\delta^2$ to construct as well as decode. This is also true for concatenated coding schemes [For67] as the inner codes have to be decoded by brute-force, and either have to also be found by a brute-force search or allowed to vary over an exponentially large ensemble (leading to exponentially large block length).

The theoretical challenge of constructing codes of rate $1 - h(p) - \delta$ with construction/decoding complexity scaling polynomially in $1/\delta$ in fact remained wide open for a long time. Finally, around 2013, two independent works [GX15, HAU14] gave an effective finite-length analysis of Arıkan’s remarkable polar codes construction [Arı09]. (Arıkan’s original analysis, as well as follow-ups like [AT09], proved convergence to capacity as the block length grew to infinity but did not quantify the speed of convergence.) Based on this, a construction of polar codes with block length, construction, and decoding complexity all bounded by a polynomial in $1/\delta$ to capacity was obtained in [GX15, HAU14]. The result also applies to any BMS channel, not just the BSC.

If the block length of the code scales as $O(1/\delta^\mu)$ as a function of the gap δ to capacity, we say that μ is the *scaling exponent*. The above results established that the scaling exponent of polar codes is finite. It is worth pointing out that polar codes are the *only* known efficiently decodable capacity-achieving family proven to have a finite scaling exponent. The work [GX15] did not give an explicit upper bound on the scaling exponent of polar codes, whereas [HAU14] showed the bound $\mu \leq 6$. Following some improvements in [GB14, MHU16], the current best known upper bound on μ for the BSC (and any BMS channel) is 4.714.

Note that random linear codes have optimal scaling exponent 2. The above results thus raise the intriguing challenge of constructing codes with scaling exponent close to 2, a goal we could not even dream of till the recent successes of polar codes.

Arıkan’s original polar coding construction is based on a large tensor power of a simple 2×2 matrix, which is called the *kernel* of the construction. For this construction, it was shown in [HAU14] that the scaling exponent μ for Arıkan’s original polar code construction is *lower bounded*

by 3.579, even for the simple binary erasure channel. Given this limitation, one approach to improve μ is to consider polar codes based on $\ell \times \ell$ kernels for larger ℓ . However, better upper bounds on the scaling exponent of polar codes based on larger kernels have not been established except for the simple case of the binary erasure channel (BEC).¹ For the BEC, using large kernels, polar codes with scaling exponent $2+\alpha$ for any desired $\alpha > 0$ were given in the very nice paper [FHMV17] which spurred our work. (We will discuss this and other related works in more detail in Sections 2.2–2.3.)

Our main result in this work is a polynomial time construction of polar codes based on large kernels that approach the optimal scaling exponent of 2 for every BMS channel. Specifically, for any desired $\alpha > 0$, by picking sufficiently large kernels (as a function of α), the block length N can be made as small as $O_\alpha(1/\delta^{2+\alpha})$ for codes of rate $I(W) - \delta$ (the notation $O_\alpha(\cdot)$ hides a constant that depends only on α). The encoding and decoding complexity will be *quasi-linear* in N , and thus can also have a near-quadratic growth with $1/\delta$.

Theorem 1.1 (Main). *Let W be an arbitrary BMS channel with Shannon capacity $I(W)$. For any $c > 0$ and arbitrarily small $\alpha > 0$, if we choose a constant $\ell \geq \ell_0(\alpha)$ to be a power of 2, then there is a code \mathcal{C} generated by the polar coding construction using kernels of size $\ell \times \ell$ such that the following four properties hold when the code length N grows: (1) the code construction has $N^{O_\alpha(1)}$ complexity; (2) both encoding and decoding have $O_\alpha(N \log N)$ complexity; (3) rate of \mathcal{C} is $I(W) - N^{-1/2+(c+6)\alpha}$; (4) decoding error probability is bounded by $O_\alpha(\log N/N^c)$ when \mathcal{C} is used for channel coding over W .*

The above “constructivizes” the quantitative finite-length version of Shannon’s theorem with a small α slack in the speed of convergence to capacity. The lower bound on ℓ can be chosen as $\ell_0(\alpha) = \exp(\alpha^{-1.01})$. Note that a similar lower bound on ℓ also appears in the aforementioned result for the BEC from [FHMV17].

We restrict our attention to binary codes in this paper. We leave the challenge of extending Theorem 1.1 to codes over larger prime alphabets for future work. We suspect a similar proof strategy to ours will work in this case, though the required technical ingredients might get even more complicated.

2 Overview of our construction and analysis

In order to better explain our work and situate it in the rich backdrop of related works on polar codes, we begin with some context and background on the phenomenon of channel polarization that lies at the heart of Arıkan’s polar coding approach.

2.1 Channel transforms, entropy polarization, and polar codes

Consider an arbitrary binary-input memoryless symmetric (BMS)² channel $W : \{0, 1\} \rightarrow \mathcal{Y}$, and an $\ell \times \ell$ invertible binary matrix K (referred to as the *kernel*). Suppose that we are transmitting a binary vector $\mathbf{U} = (U_1, U_2, \dots, U_\ell)$ uniformly chosen from $\{0, 1\}^\ell$ in the following way: first, it is transformed into $\mathbf{X} = \mathbf{U}K$, which is then transmitted through ℓ copies of the channel W to get the output $\mathbf{Y} = W^\ell(\mathbf{X}) \in \mathcal{Y}^\ell$.

¹Polar codes based on $\ell \times \ell$ kernels have much larger block length ℓ^t compared to 2^t for the 2×2 case. So to get an improvement in μ , one has to compensate for the increasing block length via better bounds on the local behavior of the kernel.

²We say that a channel $W : \{0, 1\} \rightarrow \mathcal{Y}$ is a BMS channel if there is a permutation π on the output alphabet \mathcal{Y} satisfying i) $\pi^{-1} = \pi$ and ii) $W(y|1) = W(\pi(y)|0)$ for all $y \in \mathcal{Y}$.

Now imagine decoding the input bits U_i successively in the order of increasing i . This naturally leads to a binary-input channel $W_i : \{0, 1\} \rightarrow \mathcal{Y}^\ell \times \{0, 1\}^{i-1}$, for each $i \in [\ell]$, which is the channel “seen” by the bit U_i when all the previous bits $\mathbf{U}_{<i}$ and all the channel outputs $\mathbf{Y} \in \mathcal{Y}^\ell$ are known. Formally, the transition probabilities of this channel are

$$W_i(\mathbf{Y}, \mathbf{U}_{<i} | U_i) = \frac{1}{2^{\ell-i}} \sum_{\mathbf{V} \in \{0,1\}^{\ell-i}} W^\ell(\mathbf{Y} | (\mathbf{U}_{<i}, U_i, \mathbf{V})K), \quad (1)$$

where $\mathbf{U}_{<i} \in \{0, 1\}^{i-1}$ are the first $(i-1)$ bits of \mathbf{U} , and the sum is over all possible values $\mathbf{V} \in \{0, 1\}^{\ell-i}$ that the last $(\ell-i)$ bits of \mathbf{U} can take. In this paper we will address the channel W_i as “Arıkan’s bit-channel of W with respect to K .”

A *polarization transform* associated with the kernel K is then defined as a transformation that maps ℓ copies of the channel W to the channels W_1, W_2, \dots, W_ℓ . For a BMS channel W , we define $H(W)$ as the conditional entropy of the channel input random variable given the channel output random variable when the channel input has uniform distribution. Since K is invertible, a direct implication of the chain rule for entropy gives *entropy conservation property*, which is

$$\ell \cdot H(W) = H(\mathbf{X} | \mathbf{Y}) = H(\mathbf{U} | \mathbf{Y}) = \sum_{i=1}^{\ell} H(U_i | \mathbf{U}_{<i}, \mathbf{Y}) = \sum_{i=1}^{\ell} H(W_i). \quad (2)$$

If K is invertible and is not upper-triangular under any column permutation (which we refer to as a *mixing matrix*), then the bit-channels W_1, W_2, \dots, W_ℓ start *polarizing*—some of them become better than W (have smaller entropy), and some become worse. The standard approach is then to recursively apply the polarization transform of K to these bit-channels. This naturally leads to an ℓ -ary tree of channels. The t ’th level of the tree corresponds to the linear transformation $K^{\otimes t}$, the t -fold Kronecker product of K .³

In his landmark paper [Ari09], Arıkan proved that when $K = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, at the t ’th level, all but a $o(1)$ fraction of the channels (as $t \rightarrow \infty$) are either almost noiseless (have tiny entropy) or completely useless (have entropy very close to 1). To get capacity-achieving codes from polarization, the idea is to use the almost-noiseless channels, which will constitute $\approx I(W)$ fraction by conservation of entropy, to carry the message bits, and “freeze” the bits in the remaining positions to pre-determined values (eg. all 0s). Thus the generator matrix of the code will consist of those rows of $K^{\otimes t}$ that correspond to the almost-noiseless positions. Arıkan presented a successive cancellation (SC) decoder and proved that it can be implemented using $O(N \log N)$ operations where $N = \ell^t$ is the code length, thanks to the nice recursive structure of $K^{\otimes t}$.

For the parameters of the code, if one shows that at most δ_t fraction of the channels at the t ’th level have entropies in the range $(\zeta_t, 1 - \zeta_t)$, then one (roughly) gets codes of length 2^t , rate $I(W) - \delta_t - \zeta_t$, for which the SC decoder achieves decoding error probability $\zeta_t \ell^t$ for noise caused by W (see, for example [BGN⁺18, Theorem A.3]). Thus, one needs ζ_t sub-exponentially small in t (i.e., at most $\exp(-\omega(t))$) to achieve good decoding error. For Arıkan’s original 2×2 kernel, this was shown in [AT09]. Korada, Sasoglu and Urbanke extended the analysis to arbitrary $\ell \times \ell$ mixing matrices over the binary field [KSU10], and Mori and Tanaka established a similar claim over all finite fields [MT14].

The fraction δ_t of *unpolarized* channels (whose entropies fail to be sub-exponentially close to 0 or 1) governs the gap to capacity of polar codes. The above works established that $\lim_{t \rightarrow \infty} \delta_t = 0$,

³Actually, the analysis is more convenient if one applies a bit-reversal permutation of the U_i ’s, and indeed we do so also in this paper, but this is not important for our current discussion.

and thus polar codes achieve capacity asymptotically as the block length grows to infinity. However, they did not provide any bounds on the speed at which $\delta_t \rightarrow 0$ as a function t , much less quantify a scaling exponent. Note that one would need to show $\delta_t \leq O(\ell^{-t/\mu})$ to establish a scaling exponent of μ , since the code length is ℓ^t .

2.2 Scaling exponents: prior work

For Arıkan’s original kernel $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, two independent works [HAU14, GX15] proved that δ_t drops to 0 exponentially fast in t . This proved that Arıkan’s polar codes have finite scaling exponent (i.e., converge to capacity polynomially fast in the block length), the first codes with this important feature. Blasiok *et al* generalized this result significantly [BGN⁺18], proving that the entire class of polar codes, based on arbitrary mixing matrices over any prime field as kernels, has finite scaling exponent.

For concrete upper bounds on the scaling exponent, the work of Hassani, Alishahi, and Urbanke [HAU14] had proved $\mu \leq 6$ for Arıkan’s original kernel. This was improved to $\mu \leq 5.702$ in [GB14]. Mondelli, Hassani, and Urbanke [MHU16] showed that $\mu \leq 4.714$ for any BMS channel W , and showed a better upper bound $\mu \leq 3.639$ for the case when W is a binary erasure channel (BEC). A *lower bound* $\mu \geq 3.579$ appears in [HAU14] which suggests that polar codes based on Arıkan’s original 2×2 kernel might inherently fall short of the optimal scaling exponent of 2.

For larger kernels, effective upper bounds on the scaling exponent are harder to establish as the local evolution of the channels is more complex. In fact, to the best of our knowledge, there is no such explicit bound in the literature, for any kernel of size bigger than 2.⁴ The analysis of polar codes is a lot more tractable for the case of erasure channels, where symbols get erased (replaced by a ‘?’ but never corrupted). Next we describe some results for erasure channels as well as the difficulty in extending these results to channels such as the BSC.

2.3 Polar codes for erasure channels

For the erasure channel, we have analyses of larger kernels and even codes with scaling exponent approaching 2. Binary $\ell \times \ell$ kernels for powers of two $\ell \leq 64$ optimized for the binary erasure channel appear in [MT12, FV14, YFV19]; a 64×64 kernel achieving $\mu < 3$ is reported in [YFV19].

Pfister and Urbanke proved in [PU16] that the optimal scaling exponent $\mu = 2$ can be approached if one considers transmission over the q -ary erasure channel for large alphabet size q . They used polar codes based on $q \times q$ kernels. Fazeli, Hassani, Mondelli, and Vardy [FHMV17] then established a similar result for the more challenging and also more interesting case of $q = 2$, i.e., for the binary erasure channel, using $\ell \times \ell$ kernels for large ℓ . They pose proving an analogous result for arbitrary BMS channels as an important challenge. Their conjecture that this can be accomplished provided some of the impetus for our work. Our analysis structure follows a similar blueprint to [FHMV17] though the technical ingredients become significantly more complex for channels other than the BEC, as explained next.

The polarization process for erasure channels has a particularly nice structure. If the initial channel W is the binary erasure channel with erasure probability z (denoted $\text{BEC}(z)$), then the Arıkan channels W_i , $i \in [\ell]$, arising from the linear transformation by the kernel are also binary erasure channels (specifically, $\text{BEC}(p_i(z))$ where $p_i(\cdot)$ are some polynomials of degree at most ℓ). Crucially, *all* the channels in the recursive tree remain BEC. Therefore it suffices to prove the

⁴Here we exclude special cases such as a block diagonal matrix with blocks of size at most 2 which can be reduced to the 2×2 case but will only have a worse scaling exponent.

existence of a good polarizing kernel for the class of binary erasure channels, which is parameterized by a single number, the erasure probability, which also equals the entropy of the channel. As shown in [FHMV17], a random kernel works with good probability for all BEC universally. However, fundamentally the calculations for BEC revolve around the rank of various random subspaces, as decoding under the BEC is a linear-algebraic task. Moving beyond the BEC takes us outside the realm of linear algebra into information-theoretic settings where tight quantitative results are much harder to establish.

2.4 The road to BSC: Using multiple kernels

For the case when the initial channel W is a BSC, a fundamental difficulty (among others) is that the channels in the recursion tree will no longer remain BSC (even after the first step). Further, to the best of our knowledge, the various channels that arise do not share a nice common exploitable structure. Therefore, we have to think of the intermediate channels as arbitrary BMS channels, a very large and diverse class of channels. It is not clear (to us) if there exists a single kernel to universally polarize *all* BMS channels at a rapid rate. Even if such a kernel exists, proving so seems out of reach of current techniques. Finally, even for a specific BMS, proving that a random kernel polarizes it fast enough requires some very strong quantitative bounds on the performance and limitations of random linear codes for channel coding. We next describe these issues dealing with which constitutes the core of our contributions.

Since we are not able to establish that a single kernel could work for the whole construction universally, our idea is to pick different kernels, which depend on the channels that we face during the procedure. That way, by picking a suitable kernel for each channel in the tree, we can ensure that polarization is fast enough throughout the whole process.

Though we use different kernels in the code construction, all of them have the same size $\ell \times \ell$. We say that a kernel is *good* if all but a $\tilde{O}(\ell^{-1/2})$ fraction of the bit-channels obtained after polar transform by this kernel have entropy $\ell^{-\Omega(\log \ell)}$ -close to either 0 or 1. Given a BMS channel W , the code construction consists of t steps, from Step 0 to Step $t - 1$. At Step 0, we find a good kernel $K_1^{(0)}$ for the original channel W . After the polar transform of W using kernel $K_1^{(0)}$, we obtain ℓ bit-channels W_1, \dots, W_ℓ . Then in Step 1, we find good kernels for each of these ℓ bit-channels. More precisely, the good kernel for W_i is denoted as $K_i^{(1)}$, and the bit-channels obtained by polar transforms of W_i using kernel $K_i^{(1)}$ are denoted as $\{W_{i,j} : j \in [\ell]\}$; see Figure 1 for an illustration. At Step j , we will have ℓ^j bit-channels $\{W_{i_1, \dots, i_j} : i_1, \dots, i_j \in [\ell]\}$. For each of them, we find a good kernel $K_{i_1, \dots, i_j}^{(j)}$. After polar transform of $\{W_{i_1, \dots, i_j} : i_1, \dots, i_j \in [\ell]\}$ using these kernels, we will obtain ℓ^{j+1} bit-channels. Finally, after the last step (Step $t - 1$), we will obtain $N = \ell^t$ bit-channels $\{W_{i_1, \dots, i_t} : i_1, \dots, i_t \in [\ell]\}$. Using the good kernels we obtained in this code construction procedure, we can build an $N \times N$ matrix (or we can view it as a large kernel) $M^{(t)}$ such that the N bit-channels resulting from the polar transform of the original channel W using this large kernel $M^{(t)}$ are exactly $\{W_{i_1, \dots, i_t} : i_1, \dots, i_t \in [\ell]\}$. We will say a few more words about this in Section 2.7 and provide all the details in Section 9.

Define now a random process by $W^{(0)} = W$ and $W^{(j)} = W_i^{(j-1)}$ for i uniformly chosen from $[\ell]$, where $W_i^{(j-1)}$ is the i th Arikan bit-channel of $W^{(j-1)}$ with respect to the appropriate kernel chosen in the construction phase. In other words, this is a random process of going down the tree of channels, where a uniformly random child of a current node is taken at each step. Finally, define another random process $H^{(j)} := H(W^{(j)})$. Since every kernel in the construction is chosen to be invertible, $H^{(j)}$ is a martingale due to the conservation of entropy property (2). It is clear that

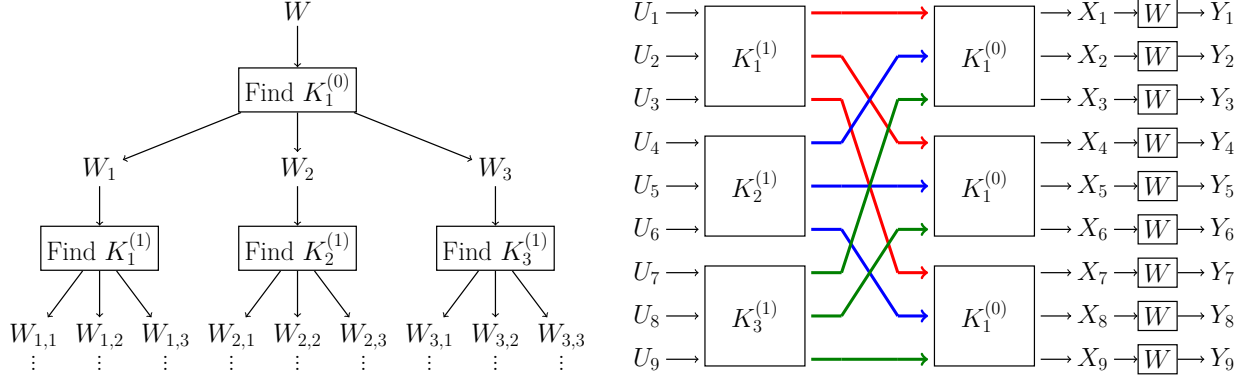


Figure 1: The left figure illustrates the code construction, and the right figure shows the encoding procedure for the special case of $\ell = 3$ and $t = 2$. All the kernels in this figure have size 3×3 . One can show that the bit-channel $W_{i,j}$ in the left figure is exactly the channel mapping from $U_{3(i-1)+j}$ to $(\mathbf{U}_{[1:3(i-1)+j-1]}, \mathbf{Y}_{[1:9]})$ in the right figure.

$W^{(j)}$ marginally is a uniformly random channel of the j^{th} level of channel tree, and then $H^{(j)}$ is the entropy of such a randomly chosen channel.

2.5 Analysis of polarization via recursive potential function

The principle behind polarization is that for large enough t , almost all of the channels on the t 's-level of the tree from Figure 2 will be close to either the useless or noiseless channel, i.e., their entropy is very close to 1 or 0, correspondingly. To estimate how fast such polarization actually happens, one needs to bound the fraction of channels on t^{th} level that are not yet sufficiently polarized, i.e., $\mathbb{P}[H^{(t)} \in (\zeta, 1 - \zeta)]$ for some tiny threshold ζ , and show that this fraction vanishes rapidly with increasing t .

Specifically, we have the following result (stated explicitly in [BGN⁺18, Theorem A.3]) already alluded to in Section 2.1: if for all t

$$\mathbb{P}[H^{(t)} \in (p\ell^{-t}, 1 - p\ell^{-t})] \leq D \cdot \beta^t, \quad (3)$$

then this corresponds to a polar code with block length $N = \ell^t$, rate $(D \cdot \beta^t + p\ell^{-t})$ -close to the capacity of the channel, and decoding error at most p under the successive cancellation decoder.

To track the fractions of polarized and non-polarized channels at each level of the construction, we use a potential function $g_\alpha(h) = (h(1-h))^\alpha$ for some small fixed $\alpha > 0$, which was also used in [MHU16] and [FHMV17]. Specifically, we are going to track $\mathbb{E}[g_\alpha(H^{(t)})]$ as t increases, since Markov's inequality implies

$$\mathbb{P}[H^{(t)} \in (p\ell^{-t}, 1 - p\ell^{-t})] = \mathbb{P}[g_\alpha(H^{(t)}) \geq g_\alpha(p\ell^{-t})] \leq \frac{\mathbb{E}[g_\alpha(H^{(t)})]}{g_\alpha(p\ell^{-t})} \leq 2 \left(\ell^t/p\right)^\alpha \cdot \mathbb{E}[g_\alpha(H^{(t)})]. \quad (4)$$

To upper bound $\mathbb{E}[g_\alpha(H^{(t)})]$ we choose kernels in our construction so that the average of the potential function of all the children of any channel in the tree decreases significantly with respect to the potential function of this channel. Precisely, we want for any channel W' in the tree

$$\mathbb{E}_{i \sim [\ell]} [g_\alpha(H(W'_i))] \leq \lambda_\alpha \cdot g_\alpha(H(W')), \quad (5)$$

where W'_i are the children of W' in the construction tree for $i \in [\ell]$, and the constant λ_α only depends on α and ℓ , but is universal for all the channels in the tree (and thus for all the kernels chosen during the construction). If one can guarantee that (5) holds throughout the construction process, then for the martingale process $H^{(t)}$ obtain

$$\begin{aligned} \mathbb{E} \left[g_\alpha \left(H^{(t)} \right) \right] &= \mathbb{E} \left[\mathbb{E}_{j \sim [\ell]} \left[g_\alpha \left(H(W'_j)^{(t-1)} \right) \right] \middle| W^{(t-1)} \right] \\ &= \mathbb{E} \left[\frac{1}{\ell} \frac{\sum_{j=1}^{\ell} g_\alpha \left(H(W'_j)^{(t-1)} \right)}{g_\alpha \left(H(W^{(t-1)}) \right)} \cdot g_\alpha \left(H(W^{(t-1)}) \right) \middle| W^{(t-1)} \right] \\ &\leq \lambda_\alpha \cdot \mathbb{E} \left[g_\alpha \left(H^{(t-1)} \right) \right], \end{aligned}$$

and then inductively

$$\mathbb{E} \left[g_\alpha \left(H^{(t)} \right) \right] \leq \lambda_\alpha \cdot \mathbb{E} \left[g_\alpha \left(H^{(t-1)} \right) \right] \leq \lambda_\alpha^2 \cdot \mathbb{E} \left[g_\alpha \left(H^{(t-2)} \right) \right] \leq \dots \leq \lambda_\alpha^t H^{(0)} \leq \lambda_\alpha^t. \quad (6)$$

Then (4) and (3) imply existence of code with rate $O((N/p)^\alpha \cdot \lambda_\alpha^t)$ -close to capacity of the channel. Since our main task is to achieve a gap which is close to $N^{-1/2} = \ell^{-t/2}$, we need to argue that it is possible to choose kernels at each step in the construction so that (5) always holds for some $\alpha \rightarrow 0$ and $\lambda_\alpha \approx \ell^{-1/2}$.

2.6 Sharp transition in polarization

The main technical contribution of this paper consists in showing that if ℓ is large enough, it is possible to choose kernels in the construction process for which λ_α is close to $\ell^{-1/2}$. Specifically, we prove that if ℓ is a power of 2 such that $\log \ell = \Omega\left(\frac{1}{\alpha^{1.01}}\right)$, then it is possible to achieve

$$\lambda_\alpha \leq \ell^{-1/2+5\alpha}. \quad (7)$$

To obtain such a behavior, while choosing the kernel for the current channel W' during the recursive process we differentiate between two cases:

Case 1: W' is already very noisy or almost noiseless. Such regime is called *suction at the ends* (following [BGN⁺18]), and it is known that polarization happens (much) faster for this case. So in this case we take a standard Arikan's polarization kernel $K = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{\otimes \log \ell}$ and prove (5) with a geometric decrease factor $\lambda_\alpha \leq \ell^{-1/2}$.

Case 2: W' is neither very noisy nor almost noiseless. We refer to this case as *variance in the middle* regime (following [BGN⁺18] again). For such a channel we adopt the framework from [FHMV17] and show a *sharp transition in polarization* for a random kernel K and W' . Specifically, we prove that with high probability over $K \sim \{0, 1\}^{\ell \times \ell}$ (for ℓ large enough) it holds

$$\begin{aligned} H(W'_i(K)) &\leq \ell^{-\Omega(\log \ell)} && \text{for } i \geq \ell \cdot H(W') + \Omega(\ell^{1/2} \log^3 \ell), \\ H(W'_i(K)) &\geq 1 - \ell^{-\Omega(\log \ell)} && \text{for } i \leq \ell \cdot H(W') - \Omega(\ell^{1/2} \log^3 \ell). \end{aligned} \quad (8)$$

It then follows that only about $\tilde{O}(\ell^{-1/2})$ fraction of bit-channels are not polarized for some kernel K , which then easily translates into the bound (7) on λ_α that we desire. Note that we can always ensure that we take an invertible kernel K , since a random binary matrix is invertible with at least some constant probability.

Proving such a sharp transition constitutes bulk of the technical work in this paper. In Section 5.2 we show that inequalities in (8) correspond to decoding a single bit of a message which is transmitted through W' using a random linear code. The first set of inequalities in (8) then correspond to saying that one can decode this single bit with low error probability with high probability over the randomness of the code, if the rate of the code is at least approximately $\ell^{-1/2}$ smaller than the capacity of the channel (where ℓ is the blocklength of the code). This follows from the well-studied fact that random linear codes achieve Shannon's capacity over any BMS ([Gal65], [BF02]).

The second set of inequalities, on the other hand, corresponds to saying that for random linear codes with rate exceeding capacity by at least $\approx \ell^{-1/2}$, even predicting a single bit of the message with tiny advantage over a uniform guess is not possible. While it follows from converse Shannon's coding theorem that decoding the *entire* message is not possible (even with small success probability) for *any* code above capacity, it does not follow that one cannot recover *a particular message bit* with accuracy noticeably better than random guessing. In fact, if we only want to decode a specific message bit and we do not put any constraints on the code, then we can easily construct codes with rate substantially above capacity that still allow us to decode this specific message bit with high probability. All we need to do here is to repeat the message bit sufficiently many times in the codeword, decode each copy based on the corresponding channel output, and then take a majority vote. The overall code rate does not even figure in this argument. Therefore, one can only hope that the converse theorem for bit-decoding holds for certain code ensembles, and for the purpose of this paper, we restrict ourselves to random linear code ensemble. While the converse for bit-decoding in this case is surely intuitive, establishing it in the strong quantitative form (8) that we need, and also for all BMS channels, turns out to be a challenging task. We describe some of the ideas behind our strong converse theorem for bit-decoding in Section 3.

2.7 Encoding and decoding

Once we have obtained the kernels in the code construction (see Section 2.4), the encoding procedure is pretty standard; see [PSL15, YB15, GBLB17, BBGL17, WD18] for discussions on polar codes using multiple kernels. As mentioned in Section 2.4, we can build a $N \times N$ matrix $M^{(t)} := D^{(t-1)}Q^{(t-1)}D^{(t-2)}Q^{(t-2)} \dots D^{(1)}Q^{(1)}D^{(0)}$, where the matrices $Q^{(1)}, Q^{(2)}, \dots, Q^{(t-1)}$ are some permutation matrices, and $D^{(0)}, D^{(1)}, \dots, D^{(t-1)}$ are block diagonal matrices. In particular, all the blocks on the diagonal of $D^{(j)}$ are the kernels that we obtained in Step j of the code construction. We illustrate the special case of $\ell = 3$ and $t = 2$ in Figure 1. We take a random vector $\mathbf{U}_{[1:N]}$ consisting of $N = \ell^t$ i.i.d. Bernoulli-1/2 random variables and we transmit the random vector $\mathbf{X}_{[1:N]}$ through N independent copies of W . Denote the output vector as $\mathbf{Y}_{[1:N]}$. Then for every $i \in [N]$, the bit-channel mapping from U_i to $(\mathbf{U}_{[1:i-1]}, \mathbf{Y}_{[1:N]})$ is exactly W_{i_1, \dots, i_t} , where (i_1, \dots, i_t) is ℓ -ary expansion of i .

We have shown that almost all of the N bit-channels $\{W_{i_1, \dots, i_t} : i_1, \dots, i_t \in [\ell]\}$ become either noiseless or completely noisy. In the code construction, we can track $H(W_{i_1, \dots, i_t})$ for every $(i_1, \dots, i_t) \in [\ell]^t$, and this allows us to identify which U_i 's are noiseless under successive decoding. Then in the encoding procedure, we only put information in these noiseless U_i 's and set all the other U_i 's to be some "frozen" value, e.g., 0. This is equivalent to saying that the generator matrix of our code is the submatrix of $M^{(t)}$ consisting of rows corresponding to indices i of the noiseless U_i 's. In Section 9, we will show that similarly to the classical polar codes, both the encoding and decoding of our code also have $O(N \log N)$ complexity.

As a final remark, we mention that we need to quantize every bit-channel we obtain in every

step of the code construction. More precisely, we merge the output symbols whose log-likelihood ratios are close to each other, so that after the quantization, the output alphabet size of every bit-channel is always polynomial in N . This allows us to construct the code in polynomial time. Without quantization, the output alphabet size would eventually be exponential in N . We will provide more details about this aspect, and how it affects the code construction and the analysis of decoding, in Section 5.1 and Section 9.

3 Outline of strong converse for random linear codes

In this section we describe the plan of the proof for the strong converse theorem for bit-decoding random linear codes under the binary symmetric channel. In particular, we need to show the sharp transition as in (8), when the channel is BSC. The proof for the general BMS channel case follows the same blueprint by using the fact that a BMS channel can be represented as a convex combination of BSC subchannels, but executing it involves overcoming several additional technical hurdles. Let us first formulate the precise theorem for the binary symmetric channel.

Theorem 3.1. *Let W be the BSC_p channel, and let ℓ, k be integers that satisfy $\ell \geq k \geq \ell(1 - H(W)) + \Omega(\ell^{1/2} \log \ell)$. Let G be a random binary matrix uniform over $\{0, 1\}^{k \times \ell}$. Suppose a message $\mathbf{V} \cdot G$ is transmitted through ℓ copies of the channel W , where \mathbf{V} is uniformly random over $\{0, 1\}^k$, and let \mathbf{Y} be the output vector, i.e. $\mathbf{Y} = W^\ell(\mathbf{V} \cdot G)$. Then, with probability at least $1 - \ell^{-\Omega(\log \ell)}$ over the choice of G it holds $H(V_1 | \mathbf{Y}) \geq 1 - \ell^{-\Omega(\log \ell)}$.*

We want to point out two quantitative features of the above theorem. First, it applies at rates $\approx \Omega(\ell^{-1/2})$ above capacity. Second, it rules out predicting the bit V_1 with advantage $\ell^{-\omega(1)}$ over random guessing. Both these features are important to guarantee the desired bound $\lambda_\alpha \lesssim \ell^{-1/2}$.

Proof plan. We prove the lower bound on $H(V_1 | \mathbf{Y})$ by lower bounding $\mathbb{E}_{g \sim G} [H(V_1 | \mathbf{Y})]$ and using Markov's inequality. Thus we write

$$\mathbb{E}_{g \sim G} [H^{(g)}(V_1 | \mathbf{Y})] = \sum_g \mathbb{P}(G = g) H^{(g)}(V_1 | \mathbf{Y}) = \sum_g \mathbb{P}(G = g) \left(\sum_{\mathbf{y} \in \mathcal{Y}^\ell} \mathbb{P}^{(g)}(\mathbf{Y} = \mathbf{y}) H^{(g)}(V_1 | \mathbf{Y} = \mathbf{y}) \right),$$

where the summation of g is over $\{0, 1\}^{k \times \ell}$, and by $\mathbb{P}^{(g)}(\cdot)$ and $H^{(g)}(\cdot)$ we denote probability and entropy over the randomness of the message \mathbf{V} and channel noise for a fixed matrix g .

1: Restrict to zero-input. The first step is to use the linearity of the (random linear) code and the additive structure of BSC to prove that we can change $\mathbb{P}^{(g)}(\mathbf{Y} = \mathbf{y})$ to $\mathbb{P}^{(g)}(\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{0})$ in the above summation, where $\mathbf{0}$ is the all-zero vector. This observation is crucial for our arguments, since it allows to only consider the outputs which are “typical” for the all-zero codeword, and there is no dependence on g in this case. Formally, in Appendix A we prove

$$\mathbb{E}_{g \sim G} [H^{(g)}(V_1 | \mathbf{Y})] = \sum_{\mathbf{y} \in \mathcal{Y}^\ell} \mathbb{P}(\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{0}) \cdot \mathbb{E}_{g \sim G} [H^{(g)}(V_1 | \mathbf{Y} = \mathbf{y})].$$

2: Define a typical set of outputs. We define a typical output set for the zero-input as $\mathcal{F} := \{\mathbf{y} \in \mathcal{Y}^\ell : |wt(\mathbf{y}) - \ell p| \leq 2\sqrt{\ell} \log \ell\}$. It is clear that if zero-vector is transmitted through the channel, the output will be a vector from \mathcal{F} with high probability. It means that we do not lose too much in terms of accuracy if we restrict our attention only to this typical set, so the following lower suffice as a good lower bound on the expectation.

$$\mathbb{E}_{g \sim G} [H^{(g)}(V_1 | \mathbf{Y})] \geq \sum_{\mathbf{y} \in \mathcal{F}} \mathbb{P}(\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{0}) \cdot \mathbb{E}_{g \sim G} [H^{(g)}(V_1 | \mathbf{Y} = \mathbf{y})]. \quad (9)$$

3: Fix a typical output $\mathbf{y} \in \mathcal{F}$. For a fixed choice of $\mathbf{y} \in \mathcal{F}$, we express $H^{(g)}(V_1|\mathbf{Y} = \mathbf{y}) = h(\mathbb{P}^{(g)}(V_1 = 0|\mathbf{Y} = \mathbf{y})) = h\left(\frac{\mathbb{P}^{(g)}(V_1=0, \mathbf{Y}=\mathbf{y})}{\mathbb{P}^{(g)}(\mathbf{Y}=\mathbf{y})}\right)$. It suffices to show that the ratio of these probabilities is very close to $1/2$ w.h.p. To this end, we will show that both denominator and numerator are highly concentrated around their respective means for $g \sim G$, and that the means have a ratio nearly $1/2$. Focusing on the denominator (the argument for the numerator is very similar), we have:

$$2^k \cdot \mathbb{P}^{(g)}(\mathbf{Y} = \mathbf{y}) = \mathbb{P}(\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{0}) + \sum_{d=0}^{\ell} B_g(d, \mathbf{y}) p^d (1-p)^{\ell-d}, \quad (10)$$

where $B_g(d, \mathbf{y})$ is equal to the number of nonzero codewords in the code spanned by the rows of g at Hamming distance d from \mathbf{y} . We proceed with proving concentration on the summation above by splitting it into two parts.

3a: Negligible part. It is very unlikely that an input codeword x such that $|\text{dist}(x, \mathbf{y}) - \ell p| \geq 6\sqrt{\ell} \log \ell$ was transmitted, if \mathbf{y} was received as the output. It is then possible to show that the expectation (over $g \sim G$) of $\sum_{d: |d-\ell p| \geq 6\sqrt{\ell} \log \ell} B_g(d, \mathbf{y}) p^d (1-p)^{\ell-d}$ is negligible with respect to the expectation of the whole summation. Markov's inequality implies then that this sum is negligible with high probability over $g \sim G$.

3b: Substantial part. On the other hand, for any d such that $|d - \ell p| \leq 6\sqrt{\ell} \log \ell$, the expectation of $B_g(d, \mathbf{y})$ is going to be extremely large for the above-capacity regime. We can apply Chebyshev's inequality to prove concentration on every single weight coefficient $B_g(d, \mathbf{y})$ with d in such a range. A union bound then implies that they are all concentrated around their means simultaneously.

This proves that the summation over d is concentrated around its mean in (10). Finally, since $|wt(\mathbf{y}) - \ell p| \leq 2\sqrt{\ell} \log \ell$ for $\mathbf{y} \in \mathcal{F}$ and we leave enough room above the capacity of the channel, w.h.p. over choice of g we have $B_g(wt(\mathbf{y}), \mathbf{y}) \gg 1$, and consequently $\mathbb{P}(\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{0}) = p^{wt(\mathbf{y})} (1-p)^{\ell-wt(\mathbf{y})}$ is negligible compared to the second sum term in (10).

4: Concentration of entropy. Proving in the same way concentration on $\mathbb{P}^{(g)}(V_1 = 0, \mathbf{Y} = \mathbf{y})$, we derive that $\frac{\mathbb{P}^{(g)}(V_1=0, \mathbf{Y}=\mathbf{y})}{\mathbb{P}^{(g)}(\mathbf{Y}=\mathbf{y})}$ is close to $\frac{1}{2}$ with high probability for any typical $\mathbf{y} \in \mathcal{F}$, and thus $\mathbb{E}_{g \sim G}[H^{(g)}(V_1|\mathbf{Y} = \mathbf{y})]$ is close to 1 with high probability for such \mathbf{y} . Recalling that the probability to receive $\mathbf{y} \in \mathcal{F}$ is overwhelming for zero-vector input, out of (9) obtain the desired lower bound on $\mathbb{E}_{g \sim G}[H^{(g)}(V_1|\mathbf{Y})]$.

The full proof for the BSC case is presented in Section 6. In order to generalize the proof to general BMS channels we need to track and prove concentration bounds for many more parameters (in the BSC case, we had a single parameter d that was crucial). More specifically, in the BSC case we have to deal with a single binomial distribution when trying to estimate the expectation of $B_g(d, \mathbf{y})$. For general BMS channels, however, we have to cope with a multinomial distribution and an ensemble of binomially distributed variables that depend on the particular realization of that multinomial distribution. Moreover, we emphasize that Theorem 3.1 and its analogue for BMS must hold in the *non-asymptotic regime*, namely for all code lengths above some absolute constant which does not depend on the channel. (In contrast, in typical coding theorems in information theory one fixes the channel and lets the block length grow to infinity.) We show how to overcome all these technical challenges for the general BMS case in Section 7.

Organization of rest of the paper

The rest of the paper, which contains all the formal theorem statements and full proofs, is organized as follows. In Section 5, we describe how to find a good polarizing kernel for any BMS, and reduce its analysis to a strong coding theorem and its converse for bit-decoding of random linear codes. The case when the BMS has entropy already reasonably close to either 0 or 1 is handled in Section 8. Also, the analysis of the complexity of the kernel finding algorithm is deferred to Section 9.

Turning to the converse coding theorem for random codes, as a warmup this is first proven for the case of the binary symmetric channel in Section 6. We then present the proof for general BMS channels in Section 7. Finally, Section 9 has the complete details of our code construction based on the multiple kernels found at various levels, and a sketch of the encoding and decoding algorithms, which when all combined yield Theorem 9.6 (which is our final result, also stated in the introductory section as Theorem 1.1).

4 Preliminaries

4.1 Binary entropy function

All the logarithms in this paper are to the base 2. The binary entropy function is defined as $h(x) = x \log \frac{1}{x} + (1-x) \log \frac{1}{1-x}$ for $x \in [0, 1]$, where $0 \log 0$ is taken to be 0. We will use a simple fact that $h(x) \leq 2x \log \frac{1}{x}$ for $x \in [0, 1/2)$ several times in the proofs. The following proposition follows from the facts that $h(x)$ is concave, increasing for $x \in [0, 1/2)$, and symmetric around $1/2$, i.e. $h(x) = h(1-x)$ for $x \in [0, 1]$.

Proposition 4.1. *For any $x, y \in [0, 1]$, $|h(x) - h(y)| \leq h(|x - y|)$.*

4.2 Channel degradation

Definition 4.2. *Let $W : \{0, 1\} \rightarrow \mathcal{Y}$ and $\widetilde{W} : \{0, 1\} \rightarrow \widetilde{\mathcal{Y}}$ be two BMS channels. We say that \widetilde{W} is degraded with respect to W , or, correspondingly, W is upgraded with respect to \widetilde{W} , denoted as $\widetilde{W} \preceq W$, if there exists a discrete memoryless channel $W_1 : \mathcal{Y} \rightarrow \widetilde{\mathcal{Y}}$ such that*

$$\widetilde{W}(\tilde{y} | x) = \sum_{y \in \mathcal{Y}} W(y | x) W_1(\tilde{y} | y) \quad \forall x \in \{0, 1\}, \tilde{y} \in \widetilde{\mathcal{Y}}.$$

Note that this is equivalent to saying that $\widetilde{W}(x)$ and $W_1(W(x))$ are identically distributed for any $x \in \{0, 1\}$. In other words, one can simulate the usage of \widetilde{W} by first using the channel W and then applying some other channel W_1 to the output of W to get a final output.

We will use some useful facts from [TV13, Lemma 3] and [YB15, Lemma IV.1]

Proposition 4.3. *Let W and \widetilde{W} be two BMS channels, such that $\widetilde{W} \preceq W$. Then $H(\widetilde{W}) \geq H(W)$.*

Proposition 4.4. *Let W and \widetilde{W} be BMS channels, such that $\widetilde{W} \preceq W$, and $K \in \{0, 1\}^{\ell \times \ell}$ be any invertible matrix. Denote by W_i, \widetilde{W}_i the Arkan's bit-channels of W and \widetilde{W} with respect to the kernel K for any $i \in [\ell]$. Then for any $i \in [\ell]$, we have $\widetilde{W}_i \preceq W_i$, and consequently $H(\widetilde{W}_i) \geq H(W_i)$.*

5 Give me a channel, I'll give you a kernel

In this section we show that for any given binary-input memoryless symmetric (BMS) channel W we can find a kernel K of size $\ell \times \ell$, such that the Arikan's bit-channels of W with respect to this kernel will be highly polarized. By this we mean that the multiplicative decrease λ_α defined in (5) will be sufficiently close to $\ell^{-1/2}$. The algorithm (Algorithm A) to find such a kernel is as follows: if the channel is already almost noiseless or too noisy (entropy is very close to 0 or 1), we take this kernel to be a tensor power of original Arikan's kernel for polar codes, $A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Otherwise, the algorithm will just try out all the possible invertible kernels in $\{0, 1\}^{\ell \times \ell}$, until a "good" kernel is found, which means that conditions (11) should be satisfied. Before proving that Algorithm A achieves our goals of bringing λ_α close to $\ell^{-1/2}$, we discuss several details about it.

5.1 Local kernel construction

Algorithm A: Kernel search

Input: BMS channel \widetilde{W} with output size $\leq Q$, error parameter Δ , and number ℓ
Output: invertible kernel $K \in \{0, 1\}^{\ell \times \ell}$

- 1 **if** $H(\widetilde{W}) < \ell^{-4}$ **or** $H(\widetilde{W}) > 1 - \ell^{-4} + \Delta$ **then**
- 2 | **return** $K = A_2^{\otimes \log \ell}$
- 3 **else**
- 4 | **for** $K \in \{0, 1\}^{\ell \times \ell}$, **if** K is invertible **do**
- 5 | Compute Arikan's bit-channels $\widetilde{W}_i(K)$ of \widetilde{W} with respect to the kernel K , as in (1)
- 6 | **if**

$$\begin{aligned} H(\widetilde{W}_i(K)) &\leq \ell^{-\log \ell / 5} && \text{for } i \geq \ell \cdot H(\widetilde{W}) + \ell^{1/2} \log^3 \ell \\ H(\widetilde{W}_i(K)) &\geq 1 - \ell^{-\log \ell / 20} && \text{for } i \leq \ell \cdot H(\widetilde{W}) - 14\ell^{1/2} \log^3 \ell \end{aligned} \tag{11}$$
- 7 | **then**
- 7 | | **return** K
- 8 | **end**
- 9 | **end**
- 10 **end**

As briefly discussed at the end of Section 2.7, we are unable to efficiently track all the bit-channels in the ℓ -ary recursive tree *exactly*. This is because the size of the output alphabet of the channels increase *exponentially* after each step deeper into the tree (this simply follows from the definition of bit-channels (1)). Thus computing all the channels (and their entropies) cannot be done in $\text{poly}(N)$ time. To overcome this issue we follow the approach of [TV13], with subsequent simplification in [GX15], of approximating the channels in the tree by degrading (see Definition 4.2) them. Degradation is achieved via the procedure of merging the output symbols, which (a) decreases the output alphabet size, and (b) does not change the entropy of the channel too much. This implies (with all the details worked out in Section 9) that we can substitute all the channels in the tree of depth t by their *degraded approximations*, such that all the channels has output alphabet size at most Q (a parameter depending on $N = \ell^t$ to be chosen), and that if \widetilde{W} is a degraded approximation of the channel W in the tree, than $H(W) \leq H(\widetilde{W}) \leq H(W) + \Delta$ for some Δ depending on Q . Moreover, in Theorem 5.1 which we formulate and prove shortly, we show that when we apply the Algorithm A to a degraded approximation \widetilde{W} of W with small enough Δ , then, even though

conditions (11) only dictate a sharp transition for \widetilde{W} , the same kernel will induce a sharp transition in polarization for W .

The second issue which such degraded approximation resolves is the running time of the Algorithm A. Notice that we only going to apply it for channels with output size bounded by Q , and recall that we think of ℓ as of a constant (though very large). First of all, trying out all the possible kernels will then also take a constant number of iterations. Finally, within each iteration, just calculating all the Arikan's bit-channels and their entropies in a straightforward way will take $\text{poly}(Q^\ell)$ time, which is just $\text{poly}(Q)$ when we treat ℓ as a constant. Therefore by choosing Q to be polynomial in N , the algorithm indeed works in $\text{poly}(N)$ time.

We now leave the full details concerning the complexity of the algorithm to be handled in Section 9, and proceed with showing that the Algorithm A always returns a kernel which makes λ_α from (5) close to $\ell^{-1/2}$.

Theorem 5.1. *Let $\alpha > 0$ be a small constants. Let ℓ be a power of 2 such that $\log \ell \geq \frac{11}{\alpha}$ and $\frac{\log \ell}{\log \log \ell + 2} \geq \frac{3}{\alpha}$. Let $W : \{0, 1\} \rightarrow \mathcal{Y}$ and $\widetilde{W} : \{0, 1\} \rightarrow \widetilde{\mathcal{Y}}$ be two BMS channels, such that $\widetilde{W} \preceq W$, $H(\widetilde{W}) - \Delta \leq H(W) \leq H(\widetilde{W})$ for some $0 \leq \Delta \leq \ell^{-\log \ell}$, and $|\widetilde{\mathcal{Y}}| \leq Q$. Then the Algorithm A on input \widetilde{W} , Δ , and ℓ returns a kernel $K \in \{0, 1\}^{\ell \times \ell}$ that satisfies*

$$\frac{1}{\ell \cdot g_\alpha(H(W))} \sum_{i=1}^{\ell} g_\alpha(H(W_i)) \leq \ell^{-\frac{1}{2} + 5\alpha}, \quad (12)$$

where W_1, W_2, \dots, W_ℓ are the Arikan's bit-channels of W with respect to the kernel K , and the function $g_\alpha(\cdot)$ is defined as $g_\alpha(h) = (h(1-h))^\alpha$ for any $h \in [0, 1]$.

Proof. As we discussed above, we consider two cases:

Suction at the ends. If $H(\widetilde{W}) \notin (\ell^{-4}, 1 - \ell^{-4} + \Delta)$, the Algorithm A returns a standard Arikan's kernel $K = A_2^{\otimes \log \ell}$ on input \widetilde{W} and Δ . For this case $H(W) \notin (\ell^{-4}, 1 - \ell^{-4})$, and fairly standard arguments imply that the polarization under such a kernel is much faster when the entropy is close to 0 or 1. For completeness, we present the full proofs for this case in a deferred Section 8. Specifically, Lemma 8.1 immediately implies the result of the theorem for this regime, as we pick $\log \ell \geq \frac{1}{\alpha}$.

Variance in the middle. Otherwise, if $H(\widetilde{W}) \in (\ell^{-4}, 1 - \ell^{-4} + \Delta)$, it holds $H(W) \in (\ell^{-4} - \Delta, 1 - \ell^{-4} + \Delta)$, thus $H(W) \in (\ell^{-4}/2, 1 - \ell^{-4}/2)$.

We first need to argue that the algorithm will at least return some kernel. This argument is one of the main technical contributions of this work, and we formulate it as Theorem 5.3 in Section 5.2. The theorem essentially claims that for any \widetilde{W} an overwhelming fraction of possible kernels $K \in \{0, 1\}^{\ell \times \ell}$ satisfies the conditions in (11) for \widetilde{W} and K (note that we do not use any conditions on the size of $\widetilde{\mathcal{Y}}$ or the entropy $H(\widetilde{W})$ at all at this point). Clearly then, there is a decent fraction of *invertible* kernels from $\{0, 1\}^{\ell \times \ell}$ which also satisfy these conditions. Therefore, the algorithm will indeed terminate and return such a good kernel. Moreover, since the theorem claims that a random kernel from $\{0, 1\}^{\ell \times \ell}$ will satisfy (11) with high probability, and it is also known that it will be invertible with at least some constant probability. It means that instead of iterating through all possible kernels in step 4 of the Algorithm A, we could take a random kernel and check it, and then the number of iterations needed to find a good kernel would be very small with high probability. However, to keep everything deterministic, we stick to the current approach.

Suppose now the algorithm returned an invertible kernel $K \in \{0, 1\}^{\ell \times \ell}$, which means that

relations (11) hold for \widetilde{W} and Arikan's bit-channels $\widetilde{W}_1, \widetilde{W}_2, \dots, \widetilde{W}_\ell$ (we omit dependence on K from now on). Denote also $W_i = W_i(K)$ as an Arikan's bit-channels of W with respect to K . First, since degradation is preserved after considering Arikan's bit-channels according to Proposition 4.4, $\widetilde{W}_i \preceq W_i$, thus $H(W_i) \leq H(\widetilde{W}_i)$ for all $i \in [\ell]$. Now, similarly to the proof of Proposition 9.3, since K is invertible, conservation of entropy implies $\sum_{i=1}^{\ell} (H(\widetilde{W}_i) - H(W_i)) = \ell (H(\widetilde{W}) - H(W)) \leq \ell \cdot \Delta$, therefore derive $H(W_i) \leq H(\widetilde{W}_i) \leq H(W_i) + \ell \cdot \Delta$ for any $i \in [\ell]$. Then deduce

$$\begin{aligned} H(W_i) &\leq H(\widetilde{W}_i) \leq \ell^{-\log \ell / 5} && \text{for } i \geq \ell \cdot H(\widetilde{W}) + \ell^{1/2} \log^3 \ell \\ H(W_i) &\geq H(\widetilde{W}_i) - \ell \cdot \Delta \geq 1 - \ell^{-\log \ell / 21} && \text{for } i \leq \ell \cdot H(\widetilde{W}) - 14 \cdot \ell^{1/2} \log^3 \ell, \end{aligned} \quad (13)$$

where we used that we chose $\Delta \leq \ell^{-\log \ell}$ in the condition of the theorem.

Recall that $H(W) \in (\ell^{-4}/2, 1 - \ell^{-4}/2)$ for variance in the middle regime, and note that this implies $g_\alpha(H(W)) \geq g_\alpha(\ell^{-4}/2) \geq \frac{1}{2} \ell^{-4\alpha}$. Using (13) and the trivial bound $g_\alpha(x) \leq 1$ for all the indexes i close to $\ell \cdot H(\widetilde{W})$ obtain that the LHS of the desired inequality (12) is at most

$$\begin{aligned} &\frac{1}{\ell \cdot g_\alpha(H(W))} \left(\sum_{i=1}^{\ell \cdot H(\widetilde{W}) - 14 \cdot \ell^{1/2} \log^3 \ell} g_\alpha \left(1 - \ell^{-\log \ell / 21} \right) + 15 \ell^{1/2} \log^3 \ell \right. \\ &\quad \left. + \sum_{i=\ell \cdot H(\widetilde{W}) + \ell^{1/2} \log^3 \ell}^{\ell} g_\alpha \left(\ell^{-\log \ell / 5} \right) \right) \\ &< 30 \ell^{-\frac{1}{2} + 4\alpha} \log^3 \ell + 2 \ell^{-\alpha \log \ell / 21 + 4\alpha} \\ &< \ell^{-\frac{1}{2} + 5\alpha} \end{aligned}$$

where the last inequality uses the conditions $\log \ell \geq \frac{11}{\alpha}$ and $\frac{\log \ell}{\log \log \ell + 2} \geq \frac{3}{\alpha}$ that we have on ℓ . \square

Remark 5.2. *In this paper, we are interested in the cases where α is very close to 0. For such α , We can absorb the two conditions on ℓ in Theorem 5.1 into one condition $\log \ell \geq \alpha^{-1.01}$.*

5.2 Strong channel coding and converse theorems

In this section we will show that Algorithm A, which is used to prove the multiplicative decrease of almost $\ell^{-1/2}$ as in (12) in the settings of Theorem 5.1, indeed always returns some kernel for the regime when the entropy of the channel is not close to 0 or 1. While the analysis of suction at the ends regime, deferred to Section 8, is pretty standard and just relies on the fact that polarization is getting much faster when the channel is noiseless or useless, in this section we will follow the ideas from [FHMV17] and prove a *sharp transition in the polarization behaviour*, when the polarization happens under a random and sufficiently large kernel.

The sharp transition stems from the fact that when the kernel K is large enough, with high probability (over randomness of K) all the Arikan's bit-channel with respect to K , except for approximately $\ell^{1/2}$ of them in the middle, are guaranteed to be either very noisy or almost noiseless. We formulate the main result of this section in the following theorem, which was used in the proof of Theorem 5.1:

Theorem 5.3. *Let W be any BMS channel, and let W_1, W_2, \dots, W_ℓ be the Arikan's bit-channels defined in (1) with respect to the kernel K chosen uniformly at random from $\{0, 1\}^{\ell \times \ell}$. Then for the following inequalities all hold with probability $(1 - o_\ell(1))$ over the choice of K :*

- (a) $H(W_i) \leq \ell^{-(\log \ell)/5}$ for $i \geq \ell \cdot H(W) + \ell^{1/2} \log^3 \ell$;
(b) $H(W_i) \geq 1 - \ell^{-(\log \ell)/20}$ for $i \leq \ell \cdot H(W) - 14 \cdot \ell^{1/2} \log^3 \ell$.

Remark 5.4. One can notice that the above theorem is stated for any BMS channel W , independent of the value of $H(W)$.

The proof of this theorem relies on results concerning bit-decoding for random linear codes that are interesting beyond the connection to polar codes. The following proposition shows how to connect Arıkan’s bit-channels to this context.

Proposition 5.5. Let W be a BMS channel, $K \in \{0, 1\}^{\ell \times \ell}$ be an invertible matrix, and $i \in [\ell]$. Set $k = \ell - i + 1$, and let G be a matrix which is formed by the last k rows of K . Let \mathbf{U} be a random vector uniformly distributed over $\{0, 1\}^\ell$, and \mathbf{V} be a random vector uniformly distributed over $\{0, 1\}^k$. Then

$$H(U_i \mid W^\ell(\mathbf{U} \cdot K), \mathbf{U}_{<i}) = H(V_1 \mid W^\ell(\mathbf{V} \cdot G)) \quad (14)$$

The proof of this proposition only uses basic properties of BMS channels and linear codes, and is deferred to Appendix A. Notice now that the LHS of (14) is exactly the entropy $H(W_i)$ of the i ’s Arıkan’s bit-channel of W with respect to the kernel K , by definition of this bit-channel. On the other hand, one can think of the RHS of (14) in the following way: look at G as a generator matrix for a linear code of blocklength ℓ and dimension k , which is transmitted through the channel W . Then $H(V_1 \mid W^\ell(\mathbf{V} \cdot G))$ in some sense corresponds to how well one can decode the first bit of the message, given the output of the channel. Since in Theorem 5.3 we are interested in random kernels, the generator matrix G is also random, and thus we are indeed interested in understanding bit-decoding of random linear codes.

5.2.1 The BEC case

When W is the binary erasure channel, a statement very similar to Theorem 5.3 was established in [FHMV17]. The situation for the BEC is simpler and we now describe the intuition behind this.

Suppose we map uniformly random bits $\mathbf{U} \in \{0, 1\}^\ell$ to $\mathbf{X} = \mathbf{U}K$ for a random $\ell \times \ell$ binary matrix K . We will observe $\approx (1 - z)\ell$ bits of \mathbf{X} after it passes through BEC(z); call these bits \mathbf{Z} . For a random K , with high probability the first $\approx z\ell$ bits of \mathbf{U} will be linearly independent of these observed bits \mathbf{Z} . When this happens we will have $H(W_i) = 1$ for $i \lesssim z\ell$. On the other hand, \mathbf{Z} together with the first $\approx z\ell$ bits of \mathbf{U} will have full rank w.h.p. over the choice of K . When this is the case, the remaining bits U_i for $i \gtrsim z\ell$ will be determined as linear combinations of these bits, making the corresponding conditional entropies $H(W_i) = 0$. Thus except for a few exceptional indices around $i \approx z\ell$, the entropy $H(W_i)$ will be really close to 0 or 1. The formal details and quantitative aspects are non-trivial as the argument has to handle the case when z is itself close to 0 or 1, and one has to show the number of exceptional indices to be $\lesssim \sqrt{\ell}$ (which is the optimal bound). But ultimately the proof amounts to understanding the ranks of various random subspaces. When W is a BMS channel, the analysis is no longer linear-algebraic, and becomes more intricate. This is the subject of the rest of this section as well as Sections 6 and 7.

5.2.2 Part (a): channel capacity theorem

Part (a) of Theorem 5.3 corresponds to transmitting through W random linear codes with rates *below* the capacity of the channel. For this regime, it turns out that we can use the classical result that random linear codes achieve the capacity of the channel with *low error decoding probability*. Trivially, the bit-decoding error probability is even smaller, making the corresponding conditional entropy also very small. Therefore, the following theorem follows from classical Shannon's theory:

Theorem 5.6. *Let W be any BMS channel and $k \leq \ell(1 - H(W)) - \ell^{1/2} \log^3 \ell$. Let G be a random binary matrix uniform over $\{0, 1\}^{k \times \ell}$. Suppose a codeword $\mathbf{V} \cdot G$ is transmitted through ℓ copies of the channel W , where \mathbf{V} is uniformly random over $\{0, 1\}^k$, and let \mathbf{Y} be the output vector, i.e. $\mathbf{Y} = W^\ell(\mathbf{V} \cdot G)$. Then with high probability over the choice of G it holds $H(V_1 | \mathbf{Y}) \leq \ell^{-(\log \ell)/5}$.*

Proof. The described communication is just a transmission of a random linear code $C = \{\mathbf{v}G, \mathbf{v} \in \{0, 1\}^k\}$ through W^ℓ , where the rate of the code is $R = \frac{k}{\ell} \leq I(W) - \ell^{-1/2} \log^3 \ell$, so it is separated from the capacity of the channel. It is a well-studied fact that random (linear) codes achieve capacity for BMS, and moreover a tight error exponent was described by Gallager in [Gal65] and analyzed further in [BF02], [For05], [DZF16]. Specifically, one can show $\overline{P_e} \leq \exp(-\ell E_r(R, W))$, where $\overline{P_e}$ is the probability of decoding error, averaged over the ensemble of all linear codes of rate R , and $E_r(R, W)$ is the so-called *random coding exponent*. It is proven in [iFLM11, Theorem 2.3] that for any BMS channel W , one has $E_r(R, W) \geq E_r^{\text{BSC}}(R, I(W))$ where the latter is the error exponent for the BSC channel with the same capacity $I(W)$ as W . But the random scaling exponent for BSC channels for the regime when the rate is close to the capacity of the channel is given by the so-called sphere-packing exponent $E_r^{\text{BSC}}(R, I) = E_{\text{sp}}(R, I)$ which is easily shown to be "almost" quadratic in $(I - R)$. Specifically, one can show $E_{\text{sp}}(R, I) \geq \frac{\log^4 \ell}{2\ell}$ when $R \leq I - \ell^{-1/2} \log^3 \ell$, and therefore $\overline{P_e} \leq \exp(-\ell E_r(R, W)) \leq \exp(-\ell E_{\text{sp}}(R, I(W))) \leq \exp(-\log^4 \ell/2)$. Then Markov's inequality implies that if we take a random linear code (i.e. choose a random binary matrix G), then with probability at least $1 - \ell^{-2}$ the decoding error is going to be at most $\ell^2 \exp(-\log^4 \ell/2) \leq \exp(-\log^4 \ell/4) \leq \ell^{-\log \ell/4}$. Consider such a good linear code (matrix G), and then \mathbf{V} can be decoded from \mathbf{Y} with high probability, thus, clearly, V_1 can be recovered from \mathbf{Y} with at least the same probability. Then Fano's inequality gives us:

$$H(V_1 | \mathbf{Y}) \leq h_2(\ell^{-\log \ell/4}) \leq \ell^{-\log \ell/5}.$$

Thus we indeed obtain that the above holds with high probability (at least $1 - \ell^{-2}$, though this is very loose) over the random choice of G . \square

5.2.3 Part (b): strong converse for bit-decoding under noisy channel coding

On the other hand, part (b) of Theorem 5.3 concerns bit-decoding of linear codes with rates *above* the capacity of the channel. We prove that with high probability, for a random linear code with rate slightly above capacity of a BMS channel, any single bit of the input message is highly unpredictable based on the outputs of the channel on the transmitted codeword. Formally, we have the following theorem.

Theorem 5.7. *Let W be any BMS channel, ℓ and k be any integers that satisfy $\ell \geq k \geq \ell(1 - H(W)) + 14\ell^{1/2} \log^3 \ell$. Let G be a random binary matrix uniform over $\{0, 1\}^{k \times \ell}$. Suppose a message $\mathbf{V} \cdot G$ is transmitted through ℓ copies of the channel W , where \mathbf{V} is uniformly random over $\{0, 1\}^k$, and let \mathbf{Y} be the output vector, i.e. $\mathbf{Y} = W^\ell(\mathbf{V} \cdot G)$. Then, with probability at least $1 - \ell^{-\log \ell/20}$ over the choice of G it holds $H(V_1 | \mathbf{Y}) \geq 1 - \ell^{-\log \ell/20}$.*

Since the theorem is of independent interest and of a fundamental nature, we devote a separate Section 7 to present a proof for it.

The above statements make the proof of Theorem 5.3 immediate:

Proof of Theorem 5.3. Denote $k = \ell - i + 1$, then by Proposition 5.5 $H(W_i) = H(V_1 \mid W^\ell(\mathbf{V} \cdot G_k))$, where $\mathbf{V} \sim \{0, 1\}^k$ and G_k is formed by the last k rows of K . Note that since K is uniform over $\{0, 1\}^{\ell \times \ell}$, this makes G_k uniform over $\{0, 1\}^{k \times \ell}$ for any k . Then:

- (a) For any $i \geq \ell \cdot H(W) + \ell^{1/2} \log^3 \ell$, we have $k \leq \ell(1 - H(W)) - \ell^{1/2} \log^3 \ell$, and therefore Theorem 5.6 applies, giving $H(W_i) \leq \ell^{-(\log \ell)/5}$ with probability at least $1 - \ell^{-2}$ over K .
- (b) Analogously, if $i \leq \ell \cdot H(W) - 14 \cdot \ell^{1/2} \log^3 \ell$, then $k \geq \ell(1 - H(W)) + 14\ell^{1/2} \log^3 \ell$, and Theorem 5.7 gives $H(W_i) \geq 1 - \ell^{-(\log \ell)/20}$ with probability at least $1 - \ell^{-(\log \ell)/20}$ over K .

It only remains to take the union bound over all indexes i as in (a) and (b), which implies that all of the bounds on the entropies will hold simultaneously with probability at least $1 - \ell \cdot \ell^{-2} \geq 1 - \ell^{-1}$ over the random kernel K . \square

6 Strong converse for BSC_p

We present a proof of Theorem 5.7 in the next two sections. It is divided into three parts: first, we prove it for a special case of W being a BSC channel in this section. The analysis for this case is simpler (but already novel), and it provides the roadmap for the argument for the case of general BMS channel. Next, in Section 7.1 we prove Theorem 5.7 for the case when the output alphabet size of W is bounded by $2\sqrt{\ell}$, which is the main technical challenge in the paper. The proof will mimic the approach for the BSC case to some extent. Finally, in Section 7.2, we show how the case of general BMS channel can be reduced to the case of the channel with bounded alphabet via "upgraded binning" to merge output symbols.

Throughout this section consider the channel W to be BSC with the crossover probability $p \leq \frac{1}{2}$. Denote $H = H(W) = h(p)$, where $h(\cdot)$ is the binary entropy function. For the BSC case we will actually only require $k \geq \ell(1 - H) + 8\sqrt{\ell} \log \ell$ in the condition of the Theorem 5.7. Thus we are in fact proving Theorem 3.1 here.

Proof of Theorem 3.1. We will follow the plan described in Section 3. As we discussed there, we prove that $H(V_1 \mid \mathbf{Y})$ is very close to 1 with high probability over G by showing that its expectation over G is already very close to 1 and then using Markov inequality. So we want to prove a lower bound on

$$\mathbb{E}_{g \sim G} [H^{(g)}(V_1 \mid \mathbf{Y})] = \sum_g \mathbb{P}(G = g) H^{(g)}(V_1 \mid \mathbf{Y}),$$

where $H^{(g)}(V_1 \mid \mathbf{Y})$ is the conditional entropy for the fixed matrix g . Similarly, in the remaining of this section, $\mathbb{P}^{(g)}(\cdot)$ denotes probabilities of certain events for a fixed matrix g . By \sum_g we denote the summation over all binary matrices from $\{0, 1\}^{k \times \ell}$.

Restrict to zero-input. We rewrite

$$\begin{aligned}\mathbb{E}_{g \sim G} [H^{(g)}(V_1 | \mathbf{Y})] &= \sum_g \mathbb{P}(G = g) \left(\sum_{\mathbf{y} \in \mathcal{Y}^\ell} \mathbb{P}^{(g)}(\mathbf{Y} = \mathbf{y}) H^{(g)}(V_1 | \mathbf{Y} = \mathbf{y}) \right) \\ &= \sum_{\mathbf{y} \in \mathcal{Y}^\ell} \sum_g \mathbb{P}^{(g)}(\mathbf{Y} = \mathbf{y}) \cdot \mathbb{P}(G = g) H^{(g)}(V_1 | \mathbf{Y} = \mathbf{y}).\end{aligned}$$

Our first step is to prove that in the above summation we can change $\mathbb{P}^{(g)}(\mathbf{Y} = \mathbf{y})$ to $\mathbb{P}^{(g)}(\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{0})$, where $\mathbf{0}$ is the all-zero vector. This observation is crucial for our arguments, since it allows us to only consider the outputs \mathbf{y} which are "typical" for the all-zero codeword when approximating $\mathbb{E}_{g \sim G} [H^{(g)}(V_1 | \mathbf{Y})]$. Precisely, we prove

Lemma 6.1. *Let W be a BMS channel, ℓ and k be integers such that $k \leq \ell$. Let G be a random binary matrix uniform over $\{0, 1\}^{k \times \ell}$. Suppose a message $\mathbf{V} \cdot G$ is transmitted through ℓ copies of W , where \mathbf{V} is uniformly random over $\{0, 1\}^k$, and let \mathbf{Y} be the output vector $\mathbf{Y} = W^\ell(\mathbf{V} \cdot G)$. Then*

$$\mathbb{E}_{g \sim G} [H^{(g)}(V_1 | \mathbf{Y})] = \sum_{\mathbf{y} \in \mathcal{Y}^\ell} \sum_g \mathbb{P}^{(g)}(\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{0}) \cdot \mathbb{P}(G = g) H^{(g)}(V_1 | \mathbf{Y} = \mathbf{y}). \quad (15)$$

Note that the above lemma is formulated for any BMS channel, and we will also use it for the proof of the general case in Section 7. The proof of this lemma uses the symmetry of linear codes with respect to shifting by a codeword and additive structure of BSC, together with the fact that BMS channel can be represented as a convex combination of several BSC subchannels. We defer the proof to Appendix A.

Note that $\mathbb{P}^{(g)}(\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{0})$ does not in fact depend on the matrix g , since $\mathbf{0} \cdot g = \mathbf{0}$, and so randomness here only comes from the usage of the channel W . Specifically, $\mathbb{P}^{(g)}(\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{0}) = p^{wt(\mathbf{y})} (1-p)^{\ell-wt(\mathbf{y})}$, where we denote by $wt(\mathbf{y})$ the Hamming weight of \mathbf{y} . Then in (15) we obtain

$$\mathbb{E}_{g \sim G} [H^{(g)}(V_1 | \mathbf{Y})] = \sum_{\mathbf{y} \in \mathcal{Y}^\ell} p^{wt(\mathbf{y})} (1-p)^{\ell-wt(\mathbf{y})} \mathbb{E}_{g \sim G} [H^{(g)}(V_1 | \mathbf{Y} = \mathbf{y})].$$

Define a typical set. The above expression allows us to only consider "typical" outputs \mathbf{y} for the all-zero input while approximating $\mathbb{E}_{g \sim G} [H^{(g)}(V_1 | \mathbf{Y})]$. For the BSC case, we consider \mathbf{y} to be typical when $|wt(\mathbf{y}) - \ell p| \leq 2\sqrt{\ell} \log \ell$. Then we can write:

$$\mathbb{E}_{g \sim G} [H^{(g)}(V_1 | \mathbf{Y})] \geq \sum_{|wt(\mathbf{y}) - \ell p| \leq 2\sqrt{\ell} \log \ell} p^{wt(\mathbf{y})} (1-p)^{\ell-wt(\mathbf{y})} \mathbb{E}_{g \sim G} [H^{(g)}(V_1 | \mathbf{Y} = \mathbf{y})]. \quad (16)$$

Fix a typical output. Let us fix any typical $\mathbf{y} \in \mathcal{Y}^\ell$ such that $|wt(\mathbf{y}) - \ell p| \leq 2\sqrt{\ell} \log \ell$, and show that $\mathbb{E}_{g \sim G} [H^{(g)}(V_1 | \mathbf{Y} = \mathbf{y})]$ is very close to 1. To do this, we first notice that

$$H^{(g)}(V_1 | \mathbf{Y} = \mathbf{y}) = h \left(\frac{\mathbb{P}^{(g)}(V_1 = 0, \mathbf{Y} = \mathbf{y})}{\mathbb{P}^{(g)}(\mathbf{Y} = \mathbf{y})} \right). \quad (17)$$

Denote $\tilde{\mathbf{V}} = \mathbf{V}^{[2:k]}$ to be bits 2 to k of vector \mathbf{V} , and by $\tilde{g} = g[2:k]$ the matrix g without its first row. Next we define the shifted weight distributions of the codebooks generated by g and \tilde{g} :

$$\begin{aligned}B_g(d, \mathbf{y}) &:= |\{\mathbf{v} \in \{0, 1\}^k \setminus \mathbf{0} : wt(\mathbf{v}g + \mathbf{y}) = d\}|, \\ \tilde{B}_g(d, \mathbf{y}) &:= |\{\tilde{\mathbf{v}} \in \{0, 1\}^{k-1} \setminus \mathbf{0} : wt(\tilde{\mathbf{v}}\tilde{g} + \mathbf{y}) = d\}|.\end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\mathbb{P}^{(g)}(V_1 = 0, \mathbf{Y} = \mathbf{y})}{\mathbb{P}^{(g)}(\mathbf{Y} = \mathbf{y})} &= \frac{\sum_{\tilde{\mathbf{u}}} \mathbb{P}^{(g)}(\mathbf{Y} = \mathbf{y} | V_1 = 0, \tilde{\mathbf{V}} = \tilde{\mathbf{u}})}{\sum_{\mathbf{u}} \mathbb{P}^{(g)}(\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{u})} \\ &= \frac{p^{wt(\mathbf{y})}(1-p)^{\ell-wt(\mathbf{y})} + \sum_{d=0}^{\ell} \tilde{B}_g(d, \mathbf{y}) p^d (1-p)^{\ell-d}}{p^{wt(\mathbf{y})}(1-p)^{\ell-wt(\mathbf{y})} + \sum_{d=0}^{\ell} B_g(d, \mathbf{y}) p^d (1-p)^{\ell-d}}. \end{aligned} \quad (18)$$

We will prove a concentration of the above expression around $1/2$, which will then imply that $H^{(g)}(V_1 | \mathbf{Y} = \mathbf{y})$ is close to 1 with high probability by (17). To do this, we will prove concentrations around means for both numerator and denominator of the above ratio. Since the following arguments work in exactly the same way, let us only consider the denominator for now.

By definition,

$$B_g(d, \mathbf{y}) = \sum_{\mathbf{v} \neq \mathbf{0}} \mathbb{1}[wt(\mathbf{v}g + \mathbf{y}) = d]. \quad (19)$$

The expectation and variance of each summand is

$$\text{Var}_{g \sim G} \mathbb{1}[wt(\mathbf{v}g + \mathbf{y}) = d] \leq \mathbb{E}_{g \sim G} \mathbb{1}[wt(\mathbf{v}g + \mathbf{y}) = d] = \binom{\ell}{d} 2^{-\ell} \quad \forall \mathbf{v} \in \{0, 1\}^k \setminus \mathbf{0}.$$

Clearly, the summands in (19) are pairwise independent. Therefore,

$$\text{Var}_{g \sim G} [B_g(d, \mathbf{y})] \leq \mathbb{E}_{g \sim G} [B_g(d, \mathbf{y})] = (2^k - 1) \binom{\ell}{d} 2^{-\ell}, \quad (20)$$

and then

$$\mathbb{E}_{g \sim G} \left[\sum_{d=0}^{\ell} B_g(d, \mathbf{y}) p^d (1-p)^{\ell-d} \right] = (2^k - 1) 2^{-\ell} \left(\sum_{d=0}^{\ell} \binom{\ell}{d} p^d (1-p)^{\ell-d} \right) = (2^k - 1) 2^{-\ell}.$$

Let us now show that $\sum_{d=0}^{\ell} B_g(d, \mathbf{y}) p^d (1-p)^{\ell-d}$ is tightly concentrated around its mean for $g \sim G$. To do this, we split the range of d into two parts: when $|d - \ell p| > 6\sqrt{\ell} \log \ell$, and when $|d - \ell p| \leq 6\sqrt{\ell} \log \ell$:

$$\sum_{d=0}^{\ell} B_g(d, \mathbf{y}) p^d (1-p)^{\ell-d} = \sum_{|d-\ell p| > 6\sqrt{\ell} \log \ell} B_g(d, \mathbf{y}) p^d (1-p)^{\ell-d} + \sum_{|d-\ell p| \leq 6\sqrt{\ell} \log \ell} B_g(d, \mathbf{y}) p^d (1-p)^{\ell-d}.$$

Negligible part. Denote $Z_g(\mathbf{y}) = \sum_{|d-\ell p| > 6\sqrt{\ell} \log \ell} B_g(d, \mathbf{y}) p^d (1-p)^{\ell-d}$, and notice that

$$\begin{aligned} \mathbb{E}_{g \sim G} [Z_g(\mathbf{y})] &= (2^k - 1) 2^{-\ell} \sum_{|d-\ell p| > 6\sqrt{\ell} \log \ell} \binom{\ell}{d} p^d (1-p)^{\ell-d} \leq (2^k - 1) 2^{-\ell} \cdot \exp(-12 \log^2 \ell) \\ &\leq 2(2^k - 1) 2^{-\ell} \cdot \ell^{-12 \log \ell}, \end{aligned}$$

where the inequality is obtained via the Chernoff bound for binomial random variable. Then Markov's inequality gives $\mathbb{P}_{g \sim G} [Z \geq \mathbb{E}_{g \sim G} [Z_g(\mathbf{y})] \ell^{2 \log \ell}] \leq \ell^{-2 \log \ell}$, and so

$$\mathbb{P} [Z_g(\mathbf{y}) < 2(2^k - 1) 2^{-\ell} \ell^{-10 \log \ell}] \geq 1 - \ell^{-2 \log \ell}.$$

Define the set

$$\mathcal{G}_1 := \{g \in \{0, 1\}^{k \times \ell} : Z_g(\mathbf{y}) < 2(2^k - 1)2^{-\ell} \ell^{-10 \log \ell}\}, \quad (21)$$

and then $\mathbb{P}_{g \sim G}[g \in \mathcal{G}_1] \geq 1 - \ell^{-2 \log \ell}$.

Substantial part. Now we deal with the part when $|d - \ell p| \leq 6\sqrt{\ell} \log \ell$. For now, let us fix any d in this interval, and use Chebyshev's inequality together with (20):

$$\begin{aligned} \mathbb{P}_{g \sim G} \left[\left| B_g(d, \mathbf{y}) - \mathbb{E}[B_g(d, \mathbf{y})] \right| \geq \ell^{-2 \log \ell} \mathbb{E}[B_g(d, \mathbf{y})] \right] &\leq \frac{\text{Var}[B_g(d, \mathbf{y})]}{\ell^{-4 \log \ell} \mathbb{E}^2[B_g(d, \mathbf{y})]} \\ &\leq \frac{\ell^{4 \log \ell}}{\mathbb{E}_{g \sim G}[B_g(d, \mathbf{y})]} \leq \ell^{4 \log \ell} \frac{2^{\ell-k+1}}{\binom{\ell}{d}}. \end{aligned} \quad (22)$$

We use the following bound on the binomial coefficients

Fact 6.2 ([MS77], Chapter 10, Lemma 7). *For any integer $0 \leq d \leq \ell$,*

$$\frac{1}{\sqrt{2\ell}} 2^{\ell h(d/\ell)} \leq \binom{\ell}{d} \leq 2^{\ell h(d/\ell)} \quad (23)$$

Since we fixed $|d - \ell p| \leq 6\sqrt{\ell} \log \ell$, Proposition 4.1 implies

$$\left| h(p) - h\left(\frac{d}{\ell}\right) \right| \leq h(6\ell^{-1/2} \log \ell) \leq 12\ell^{-1/2} \log \ell \cdot \log \frac{\ell^{1/2}}{6 \log \ell} \leq 6\ell^{-1/2} \log^2 \ell.$$

Recalling that we consider the above-capacity regime with $k \geq \ell(1 - h(p)) + 8\sqrt{\ell} \log^2 \ell$, we derive from (23) and above

$$\frac{2^{\ell-k+1}}{\binom{\ell}{d}} \leq \ell 2^{\ell[h(p) - h(d/\ell) - 8\ell^{-1/2} \log^2 \ell]} \leq \ell 2^{-2\ell^{1/2} \log^2 \ell}.$$

Therefore, we get in (22):

$$\mathbb{P}_{g \sim G} \left[\left| B_g(d, \mathbf{y}) - \mathbb{E}[B_g(d, \mathbf{y})] \right| \geq \ell^{-2 \log \ell} \mathbb{E}[B_g(d, \mathbf{y})] \right] \leq \ell^{4 \log \ell + 1} 2^{-2\ell^{1/2} \log^2 \ell} \leq \ell^{-\sqrt{\ell}-1}. \quad (24)$$

Finally, denote

$$\mathcal{G}_2 := \left\{ g \in \{0, 1\}^{k \times \ell} : \left| B_g(d, \mathbf{y}) - \mathbb{E}[B_g(d, \mathbf{y})] \right| \leq \ell^{-2 \log \ell} \mathbb{E}[B_g(d, \mathbf{y})] \quad \text{for all } |d - \ell p| \leq 6\sqrt{\ell} \log \ell \right\}. \quad (25)$$

Then by a simple union bound applied to (24) for all d such that $|d - \ell p| \leq 6\sqrt{\ell} \log \ell$ we obtain

$$\mathbb{P}_{g \sim G}[g \in \mathcal{G}_2] \geq 1 - \ell^{-\sqrt{\ell}}.$$

We are now ready to combine these bounds to get the needed concentration.

Lemma 6.3. *With probability at least $1 - 2\ell^{-2 \log \ell}$ over the choice of $g \sim G$, it holds*

$$(2^k - 1)2^{-\ell}(1 - 2\ell^{-2 \log \ell}) \leq \sum_{d=0}^{\ell} B_g(d, \mathbf{y}) p^d (1-p)^{\ell-d} \leq (2^k - 1)2^{-\ell}(1 + 2\ell^{-2 \log \ell}). \quad (26)$$

Proof. Indeed, by union bound $\mathbb{P}_{g \sim G}[g \in \mathcal{G}_1 \cap \mathcal{G}_2] \geq 1 - l^{-2 \log \ell} - \ell^{\sqrt{\ell}} \geq 1 - 2\ell^{-2 \log \ell}$. But for any $g \in \mathcal{G}_1 \cap \mathcal{G}_2$ we have from (25)

$$\begin{aligned} \sum_{d=0}^{\ell} B_g(d, \mathbf{y}) p^d (1-p)^{\ell-d} &\geq \sum_{|d-\ell p| \leq 6\sqrt{\ell} \log \ell} B_g(d, \mathbf{y}) p^d (1-p)^{\ell-d} \\ &\geq (2^k - 1) 2^{-\ell} (1 - \ell^{-2 \log \ell}) \sum_{|d-\ell p| \leq 6\sqrt{\ell} \log \ell} \binom{\ell}{d} p^d (1-p)^{\ell-d} \\ &\geq (2^k - 1) 2^{-\ell} (1 - \ell^{-2 \log \ell}) (1 - 2\ell^{-12 \log \ell}) \\ &\geq (2^k - 1) 2^{-\ell} (1 - 2\ell^{-2 \log \ell}). \end{aligned}$$

We can also upper bound using (25) and (21)

$$\begin{aligned} \sum_{d=0}^{\ell} B_g(d, \mathbf{y}) p^d (1-p)^{\ell-d} &= \sum_{|d-\ell p| \leq 6\sqrt{\ell} \log \ell} B_g(d, \mathbf{y}) p^d (1-p)^{\ell-d} + \sum_{|d-\ell p| > 6\sqrt{\ell} \log \ell} B_g(d, \mathbf{y}) p^d (1-p)^{\ell-d} \\ &\leq (2^k - 1) 2^{-\ell} (1 + \ell^{-2 \log \ell}) \sum_{|d-\ell p| \leq 6\sqrt{\ell} \log \ell} \binom{\ell}{d} p^d (1-p)^{\ell-d} + Z_g(\mathbf{y}) \\ &\leq (2^k - 1) 2^{-\ell} (1 + \ell^{-2 \log \ell}) + 2(2^k - 1) 2^{-\ell} \ell^{-10 \log \ell} \\ &\leq (2^k - 1) 2^{-\ell} (1 + 2\ell^{-2 \log \ell}). \quad \square \end{aligned}$$

We similarly obtain the concentration for the sum in the numerator of (18): with probability at least $1 - 2\ell^{-2 \log \ell}$ over the choice of g , it holds

$$(2^{k-1} - 1) 2^{-\ell} (1 - 2\ell^{-2 \log \ell}) \leq \sum_{d=0}^{\ell} \tilde{B}_g(d, \mathbf{y}) p^d (1-p)^{\ell-d} \leq (2^{k-1} - 1) 2^{-\ell} (1 + 2\ell^{-2 \log \ell}). \quad (27)$$

Next, let us use the fact that we took a typical output \mathbf{y} with $|wt(\mathbf{y}) - \ell p| \leq 2\sqrt{\ell} \log \ell$ to show that the terms $p^{wt(\mathbf{y})} (1-p)^{\ell-wt(\mathbf{y})}$ are negligible in both numerator and denominator of (18). We have

$$p^{wt(\mathbf{y})} (1-p)^{\ell-wt(\mathbf{y})} = \left(\frac{1-p}{p} \right)^{\ell p - wt(\mathbf{y})} \cdot p^{\ell p} (1-p)^{\ell - \ell p} = 2^{(\ell p - wt(\mathbf{y})) \cdot \log\left(\frac{1-p}{p}\right)} \cdot 2^{-\ell h(p)}. \quad (28)$$

Simple case analysis gives us:

- (a) If $p < \frac{1}{\sqrt{\ell}}$, then $(\ell p - wt(\mathbf{y})) \cdot \log\left(\frac{1-p}{p}\right) \leq \ell p \log \frac{1}{p} < \ell \frac{1}{\sqrt{\ell}} \log \sqrt{\ell} < \sqrt{\ell} \log^2 \ell$;
- (b) In case $p \geq \frac{1}{\sqrt{\ell}}$, obtain $(\ell p - wt(\mathbf{y})) \cdot \log\left(\frac{1-p}{p}\right) \leq 2\sqrt{\ell} \log \ell \cdot \log \frac{1}{p} \leq \sqrt{\ell} \log^2 \ell$.

Using the above in (28) we derive for $k \geq \ell(1 - h(p)) + 8\sqrt{\ell} \log^2 \ell$

$$p^{wt(\mathbf{y})} (1-p)^{\ell-wt(\mathbf{y})} \leq 2^{\sqrt{\ell} \log^2 \ell - \ell h(p)} \leq 2^{2\sqrt{\ell} \log^2 \ell - \ell h(p) - 2 \log^2 \ell - 2} \leq \ell^{-2 \log \ell} (2^{k-1} - 1) 2^{-\ell}.$$

Combining this with (26) and (27) and using a union bound we derive that with probability at least $1 - 4\ell^{-2\log \ell}$ it holds

$$\left| \left(p^{wt(\mathbf{y})}(1-p)^{\ell-wt(\mathbf{y})} + \sum_{d=0}^{\ell} B_g(d, \mathbf{y}) p^d (1-p)^{\ell-d} \right) - (2^k - 1)2^{-\ell} \right| \leq 3\ell^{-2\log \ell} \cdot (2^k - 1)2^{-\ell},$$

$$\left| \left(p^{wt(\mathbf{y})}(1-p)^{\ell-wt(\mathbf{y})} + \sum_{d=0}^{\ell} \tilde{B}_g(d, \mathbf{y}) p^d (1-p)^{\ell-d} \right) - (2^{k-1} - 1)2^{-\ell} \right| \leq 3\ell^{-2\log \ell} \cdot (2^{k-1} - 1)2^{-\ell}.$$

Therefore, with probability at least $1 - 4\ell^{-2\log \ell}$ the expression in (18) is bounded as

$$\frac{(1 - 3\ell^{-2\log \ell})(2^{k-1} - 1)2^{-\ell}}{(1 + 3\ell^{-2\log \ell})(2^k - 1)2^{-\ell}} \leq \frac{\mathbb{P}^{(g)}(V_1 = 0, \mathbf{Y} = \mathbf{y})}{\mathbb{P}^{(g)}(\mathbf{Y} = \mathbf{y})} \leq \frac{(1 + 3\ell^{-2\log \ell})(2^{k-1} - 1)2^{-\ell}}{(1 - 3\ell^{-2\log \ell})(2^k - 1)2^{-\ell}}.$$

We can finally derive:

$$\begin{aligned} \frac{(1 - 3\ell^{-2\log \ell})(2^{k-1} - 1)}{(1 + 3\ell^{-2\log \ell})(2^k - 1)} &\geq (1 - 6\ell^{-2\log \ell}) \left(\frac{1}{2} - 2^{-k} \right) \geq (1 - 6\ell^{-2\log \ell}) \left(\frac{1}{2} - \ell^{-8\sqrt{\ell}\log \ell} \right) \\ &\geq \frac{1}{2} - \ell^{-\log \ell}, \\ \frac{(1 + 3\ell^{-2\log \ell})(2^{k-1} - 1)}{(1 - 3\ell^{-2\log \ell})(2^k - 1)} &\leq (1 + 9\ell^{-2\log \ell}) \frac{1}{2} \leq \frac{1}{2} + \ell^{-\log \ell}. \end{aligned}$$

Therefore, with probability at least $1 - 4\ell^{-2\log \ell}$ over $g \sim G$ it holds

$$\left| \frac{\mathbb{P}^{(g)}(V_1 = 0, \mathbf{Y} = \mathbf{y})}{\mathbb{P}^{(g)}(\mathbf{Y} = \mathbf{y})} - \frac{1}{2} \right| \leq \ell^{-\log \ell}.$$

Since $h(1/2 + x) \geq 1 - 4x^2$ for any $x \in (-1/2, 1/2)$, we then derive:

$$\mathbb{E}_{g \sim G} [H^{(g)}(V_1 | \mathbf{Y} = \mathbf{y})] = \mathbb{E}_{g \sim G} \left[h \left(\frac{\mathbb{P}^{(g)}(V_1 = 0, \mathbf{Y} = \mathbf{y})}{\mathbb{P}^{(g)}(\mathbf{Y} = \mathbf{y})} \right) \right] \geq (1 - 4\ell^{-2\log \ell})(1 - 4\ell^{-2\log \ell}) \geq 1 - 8\ell^{-2\log \ell}.$$

Concentration of entropy. We are now ready to plug this into (16):

$$\begin{aligned} \mathbb{E}_{g \sim G} [H^{(g)}(V_1 | \mathbf{Y})] &\geq (1 - 8\ell^{-2\log \ell}) \sum_{|wt(\mathbf{y}) - \ell p| \leq 2\sqrt{\ell}\log \ell} p^{wt(\mathbf{y})}(1-p)^{\ell-wt(\mathbf{y})} \\ &= (1 - 8\ell^{-2\log \ell}) \sum_{|d - \ell p| \leq 2\sqrt{\ell}\log \ell} \binom{\ell}{d} p^d (1-p)^{\ell-d} \\ &\geq (1 - 8\ell^{-2\log \ell})(1 - 2\ell^{-2\log \ell}) \\ &\geq 1 - 10\ell^{-2\log \ell}. \end{aligned} \tag{29}$$

Finally, using the fact that $H^{(g)}(V_1 | \mathbf{Y}) \leq 1$, Markov's inequality, and (29), we get

$$\mathbb{P}_{g \sim G} [H^{(g)}(V_1 | \mathbf{Y}) \leq 1 - \ell^{-\log \ell}] = \mathbb{P}_{g \sim G} [1 - H^{(g)}(V_1 | \mathbf{Y}) \geq \ell^{-\log \ell}] \leq \frac{\mathbb{E}_{g \sim G} [1 - H^{(g)}(V_1 | \mathbf{Y})]}{\ell^{-\log \ell}} \leq 10\ell^{-\log \ell}.$$

Thus we conclude that with probability at least $1 - 10\ell^{-\log \ell}$ over the choice of the kernel G it holds that $H(V_1 | \mathbf{Y}) \geq 1 - \ell^{-\log \ell}$ when $k \geq \ell(1 - h(p)) + 8\sqrt{\ell}\log^2 \ell$ and the underlying channel is BSC. This completes the proof of Theorem 3.1, which is a version of Theorem 5.7 for the BSC case. \square

7 Strong converse for BMS channel

To make this section completely self-contained, we restate the theorem here:

Theorem 5.7. *Let W be any BMS channel, ℓ and k be any integers that satisfy $\ell \geq k \geq \ell(1 - H(W)) + 14\ell^{1/2} \log^3 \ell$. Let G be a random binary matrix uniform over $\{0, 1\}^{k \times \ell}$. Suppose a message $\mathbf{V} \cdot G$ is transmitted through ℓ copies of the channel W , where \mathbf{V} is uniformly random over $\{0, 1\}^k$, and let \mathbf{Y} be the output vector, i.e. $\mathbf{Y} = W^\ell(\mathbf{V} \cdot G)$. Then, with probability at least $1 - \ell^{-\log \ell / 20}$ over the choice of G it holds $H(V_1 | \mathbf{Y}) \geq 1 - \ell^{-\log \ell / 20}$.*

7.1 Bounded alphabet size

This section is devoted to proving Theorem 5.7 for the case when $W : \{0, 1\} \rightarrow \mathcal{Y}$ is a BMS channel which has a bounded output alphabet size, specifically we consider $|\mathcal{Y}| \leq 2\sqrt{\ell}$. We will use the fact that any BMS can be viewed as a convex combination of BSCs (see for example [LH06, Kor09]), and generalize the ideas of the previous section. Namely, think of the channel W as follows: it has m possible underlying BSC subchannels $W^{(1)}, W^{(2)}, \dots, W^{(m)}$. On any input, W randomly chooses one of the subchannels it is going to use with probabilities q_1, q_2, \dots, q_m respectively. The subchannel $W^{(j)}$ has crossover probability p_j , and without loss of generality $0 \leq p_1 \leq p_2 \leq \dots \leq p_m \leq \frac{1}{2}$. The subchannel $W^{(j)}$ has two possible output symbols $z_j^{(0)}$ or $z_j^{(1)}$, corresponding to 0 and 1, respectively (i.e. 0 goes to $z_j^{(0)}$ with probability $1 - p_j$, or to $z_j^{(1)}$ with probability p_j under $W^{(j)}$). Then the whole output alphabet is $\mathcal{Y} = \{z_1^{(0)}, z_1^{(1)}, z_2^{(0)}, z_2^{(1)}, \dots, z_m^{(0)}, z_m^{(1)}\}$, $|\mathcal{Y}| = 2m \leq 2\sqrt{\ell}$.

Remark 7.1. *Above we ignored the case when some of the subchannels have only one output (i.e. BEC subchannels), see [TV13, Lemma 4] for a proof that we can do this without loss of generality.*

Notations and settings. In this section the expectation is only going to be taken over the kernel $g \sim G$, so we omit this in some places. As in the BSC case, by $\mathbb{P}^{(g)}[\cdot]$ and $H^{(g)}(\cdot)$ we denote the probability and entropy only over the randomness of the channel and the message, for a fixed kernel g .

For any possible output $\mathbf{y} \in \mathcal{Y}^\ell$ we denote by d_i the number of symbols from $\{z_i^{(0)}, z_i^{(1)}\}$ it has (i.e. the number of uses of the $W^{(i)}$ subchannel), so $\sum_{i=1}^m d_i = \ell$. Let also t_i be the number of symbols $z_i^{(1)}$ in \mathbf{y} . Then

$$\mathbb{P}[\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{0}] = \prod_{i=1}^m q_i^{d_i} p_i^{t_i} (1 - p_i)^{d_i - t_i}. \quad (30)$$

For this case of bounded output alphabet size, we will consider the above-capacity regime when $k \geq \ell(1 - H(W)) + 13\ell^{1/2} \log^3 \ell$ (note that this is made intentionally weaker than the condition in Theorem 5.7).

We will follow the same blueprint of the proof for BSC from Section 3, however all the technicalities along the way are going to be more challenging. In particular, while we were dealing with one binomial distribution in Section 6, here we will face a multinomial distribution of (d_1, d_2, \dots, d_m) as a choice of which subchannels to use, as well as binomial distributions $t_i \sim \text{Binom}(d_i, p_i)$ which correspond to "flips" within one subchannel.

Proof of Theorem 5.7. As in the BSC case, we are going to lower bound the expectation of $H^{(g)}(V_1|\mathbf{Y})$ and use Markov's inequality afterwards.

Restrict to zero-input. We use Lemma 6.1 to write

$$\mathbb{E}_{g \sim G} [H^{(g)}(V_1|\mathbf{Y})] = \sum_{\mathbf{y} \in \mathcal{Y}^\ell} \mathbb{P}[\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{0}] \mathbb{E}_{g \sim G} [H^{(g)}(V_1|\mathbf{Y} = \mathbf{y})]. \quad (31)$$

Notice that there is no dependence of $\mathbb{P}[\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{0}]$ on the kernel g , since the output for the zero-input depends only on the randomness of the channel.

Typical output set

As for the binary case, we would like to consider the set of "typical" outputs (for input $\mathbf{0}$) from \mathcal{Y}^ℓ . We define $\mathbf{y} \in \mathcal{Y}^\ell$ to be typical if

$$\sum_{i=1}^m (\ell \cdot q_i - d_i) h(p_i) \leq 2\sqrt{\ell} \log \ell, \quad (32)$$

$$\sum_{i=1}^m (p_i d_i - t_i) \log \left(\frac{1-p_i}{p_i} \right) \leq 3\sqrt{\ell} \log^2 \ell, \quad (33)$$

By typicality of this set we mean the following

Lemma 7.2. $\sum_{\mathbf{y} \text{ typical}} \mathbb{P}[\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{0}] \geq 1 - \ell^{-\log \ell}$. In other words, on input $\mathbf{0}$, the probability to get the output string which is not typical is at most $\ell^{-\log \ell}$.

We defer the proof of this lemma until Section 7.1.3, after we see why we are actually interested in these conditions on \mathbf{y} .

7.1.1 Fix a typical output

For this part, let us fix one $\mathbf{y} \in \mathcal{Y}^\ell$ which is typical and prove that $\mathbb{E}_g [H^{(g)}(V_1|\mathbf{Y})]$ is very close to 1. We have

$$H^{(g)}(V_1|\mathbf{Y}) = h \left(\frac{\mathbb{P}^{(g)} [V_1 = 0, \mathbf{Y} = \mathbf{y}]}{\mathbb{P}^{(g)} [\mathbf{Y} = \mathbf{y}]} \right). \quad (34)$$

Similarly to the BSC case, we will prove that both the denominator and numerator of the fraction inside the entropy function above are tightly concentrated around their means. The arguments for the denominator and the numerator are almost exactly the same, so we only consider denominator for now.

Concentration for $\mathbb{P}^{(g)} [\mathbf{Y} = \mathbf{y}]$

Define now the shifted weight distributions for the codebook g with respect to m different underlying BSC channels. First, for any $x \in \{0, 1\}^\ell$ and $i = 1, 2, \dots, m$, define

$$\text{dist}_i(x, \mathbf{y}) = |\{\text{positions } j \text{ such that } (x_j = 0, \mathbf{y}_j = z_i^{(1)}) \text{ or } (x_j = 1, \mathbf{y}_j = z_i^{(0)})\}|.$$

That is, if you send x through W^ℓ and receive \mathbf{y} , then $\text{dist}_i(x, \mathbf{y})$ is just the number of coordinates where the subchannel i was chosen, and the bit was flipped.

In our settings, we now need to think of "distance" between some binary vector $x \in \{0, 1\}^\ell$ and \mathbf{y} as of an integer vector $\mathbf{s} = (s_1, s_2, \dots, s_m)$, where $0 \leq s_i \leq d_i$ for $i \in [m]$, where $s_i = \text{dist}_i(x, \mathbf{y})$ is just the number of flips that occurred in the usage of i^{th} subchannel when going from x to \mathbf{y} . In other words, s_i is just the Hamming distance between the parts of x and \mathbf{y} which correspond to coordinates j where \mathbf{y}_j is $z_i^{(0)}$ or $z_i^{(1)}$ (coming from the subchannel $W^{(i)}$).

Now we can formally define shifted weight distributions for our fixed typical \mathbf{y} . For an integer vector $\mathbf{s} = (s_1, s_2, \dots, s_m)$, where $0 \leq s_i \leq d_i$ define

$$B_g(\mathbf{s}, \mathbf{y}) = \left| \mathbf{v} \in \{0, 1\}^k \setminus \mathbf{0} : \text{dist}_i(\mathbf{v} \cdot g, \mathbf{y}) = s_i \quad \text{for } i = 1, 2, \dots, m \right|.$$

We can express $\mathbb{P}^{(g)}[\mathcal{Y} = \mathbf{y}]$ in terms of $B_g(\mathbf{s}, \mathbf{y})$ as follows:

$$2^k \cdot \mathbb{P}^{(g)}[\mathcal{Y} = \mathbf{y}] = \mathbb{P}[\mathcal{Y} = \mathbf{y} | \mathbf{v} = \mathbf{0}] + \sum_{s_1, s_2, \dots, s_m=0}^{d_1, d_2, \dots, d_m} B_g(\mathbf{s}, \mathbf{y}) \prod_{i=1}^m q_i^{d_i} p_i^{s_i} (1-p_i)^{d_i-s_i}, \quad (35)$$

because $\prod_{i=1}^m q_i^{d_i} p_i^{s_i} (1-p_i)^{d_i-s_i}$ is exactly the probability to get output \mathbf{y} if a \mathbf{v} is sent that satisfies $\text{dist}_i(\mathbf{v} \cdot g, \mathbf{y}) = s_i$ for $i = 1, 2, \dots, m$.

We have:

$$B_g(\mathbf{s}, \mathbf{y}) = \sum_{\mathbf{v} \neq \mathbf{0}} \mathbb{1}[\text{dist}_i(\mathbf{v} \cdot g, \mathbf{y}) = s_i, \quad \forall i = 1, 2, \dots, m]. \quad (36)$$

For a fixed \mathbf{v} but uniformly random binary matrix g , the vector $\mathbf{v} \cdot g$ is just a uniformly random vector from $\{0, 1\}^\ell$. Now, the number of vectors x in $\{0, 1\}^\ell$ such that $\text{dist}_i(x, \mathbf{y}) = s_i \quad \forall i = 1, 2, \dots, m$ is $\prod_{i=1}^m \binom{d_i}{s_i}$, since for any $i = 1, 2, \dots, m$, we need to choose which of the s_i coordinates amongst the d_i uses of the subchannel $W^{(i)}$, got flipped. So

$$\mathbb{P}_{g \sim G}[\text{dist}_i(\mathbf{v} \cdot g, \mathbf{y}) = s_i, \quad \forall i = 1, 2, \dots, m] = 2^{-\ell} \prod_{i=1}^m \binom{d_i}{s_i}.$$

Then for the expectation of the shifted weight distributions we obtain

$$\mathbb{E}_{g \sim G}[B_g(\mathbf{s}, \mathbf{y})] = \sum_{\mathbf{v} \neq \mathbf{0}} \mathbb{P}_{g \sim G}[\text{dist}_i(\mathbf{v} \cdot g, \mathbf{y}) = s_i, \quad \forall i = 1, 2, \dots, m] = \frac{2^k - 1}{2^\ell} \prod_{i=1}^m \binom{d_i}{s_i}. \quad (37)$$

Then for the expectation of the summation in the RHS of (35) we have:

$$\begin{aligned} E &:= \mathbb{E}_{g \sim G} \left[\sum_{s_1, s_2, \dots, s_m=0}^{d_1, d_2, \dots, d_m} B_g(\mathbf{s}, \mathbf{y}) \prod_{i=1}^m q_i^{d_i} p_i^{s_i} (1-p_i)^{d_i-s_i} \right] \\ &= \prod_{i=1}^m q_i^{d_i} \sum_{s_1, s_2, \dots, s_m=0}^{d_1, d_2, \dots, d_m} \left(\mathbb{E}_{g \sim G}[B_g(\mathbf{s}, \mathbf{y})] p_i^{s_i} (1-p_i)^{d_i-s_i} \right) \\ &= \frac{2^k - 1}{2^\ell} \prod_{i=1}^m q_i^{d_i} \cdot \sum_{s_1, s_2, \dots, s_m=0}^{d_1, d_2, \dots, d_m} \prod_{i=1}^m \binom{d_i}{s_i} p_i^{s_i} (1-p_i)^{d_i-s_i} \\ &= \frac{2^k - 1}{2^\ell} \prod_{i=1}^m q_i^{d_i} \cdot \prod_{i=1}^m \underbrace{\left(\sum_{s_i=0}^{d_i} \binom{d_i}{s_i} p_i^{s_i} (1-p_i)^{d_i-s_i} \right)}_{=1} = \frac{2^k - 1}{2^\ell} \prod_{i=1}^m q_i^{d_i}. \end{aligned} \quad (38)$$

Next, by (36) we can see that $B_g(\mathbf{s}, \mathbf{y})$ is a sum of pairwise independent indicator random variables, since $\mathbf{v}_1 \cdot g$ and $\mathbf{v}_2 \cdot g$ are independent for distinct and non-zero $\mathbf{v}_1, \mathbf{v}_2$. Therefore

$$\text{Var}_{g \sim G}[B_g(\mathbf{s}, \mathbf{y})] \leq \mathbb{E}_{g \sim G}[B_g(\mathbf{s}, \mathbf{y})]. \quad (39)$$

Splitting the summation in (35)

We will split the summation in (35) into two parts: for the first part, we will show that the expectation of each term is very large, and then use Chebyshev's inequality to argue that each term is concentrated around its expectation. For the second part, its expectation is going to be very small, and simple Markov inequality will imply that this part also does not deviate from its expectation too much with high probability (over the random kernel $g \sim G$). Putting these two arguments together, we will obtain that the sum in the RHS of (35) is concentrated around its mean.

To proceed, define a distribution $\Omega = \text{Binom}(d_1, p_1) \times \text{Binom}(d_2, p_2) \times \cdots \times \text{Binom}(d_m, p_m)$, and consider a random vector $\chi \sim \Omega$. In other words, χ has m independent coordinates χ_i , $i = 1, \dots, m$, where χ_i is a binomial random variable with parameters d_i and p_i . Note that by definition then for any vector $\mathbf{s} = (s_1, s_2, \dots, s_m)$, where $0 \leq s_i \leq d_i$ and s_i is integer for any i , we have

$$\mathbb{P}_{\chi}[\chi = \mathbf{s}] = \prod_{i=1}^m \mathbb{P}_{\chi}[\chi_i = s_i] = \prod_{i=1}^m \binom{d_i}{s_i} p_i^{s_i} (1-p_i)^{d_i-s_i}.$$

Let now \mathcal{T} be some subset of $\mathcal{S} = [0 : d_1] \times [0 : d_2] \times \cdots \times [0 : d_m]$, where $[0 : d] = \{0, 1, 2, \dots, (d-1), d\}$ for integer d . Let also $\mathcal{N} = \mathcal{S} \setminus \mathcal{T}$. Then the summation in the RHS of (35) we can write as

$$\sum_{\mathbf{s} \in \mathcal{S}} B_g(\mathbf{s}, \mathbf{y}) \prod_{i=1}^m q_i^{d_i} p_i^{s_i} (1-p_i)^{d_i-s_i} = \sum_{\mathbf{s} \in \mathcal{T}} B_g(\mathbf{s}, \mathbf{y}) \prod_{i=1}^m q_i^{d_i} p_i^{s_i} (1-p_i)^{d_i-s_i} + \sum_{\mathbf{s} \in \mathcal{N}} B_g(\mathbf{s}, \mathbf{y}) \prod_{i=1}^m q_i^{d_i} p_i^{s_i} (1-p_i)^{d_i-s_i}. \quad (40)$$

In the next section we describe how to choose \mathcal{T} .

7.1.1.(i) Substantial part

Exactly as in the binary case, using (39) and Chebyshev's inequality, we have for any $\mathbf{s} \in \mathcal{S}$

$$\begin{aligned} \mathbb{P}_{g \sim G} \left[\left| B_g(\mathbf{s}, \mathbf{y}) - \mathbb{E}[B_g(\mathbf{s}, \mathbf{y})] \right| \geq \ell^{-2 \log \ell} \mathbb{E}[B_g(\mathbf{s}, \mathbf{y})] \right] &\leq \frac{\text{Var}[B_g(\mathbf{s}, \mathbf{y})]}{\ell^{-4 \log \ell} \mathbb{E}^2[B_g(\mathbf{s}, \mathbf{y})]} \\ &\leq \frac{\ell^{4 \log \ell}}{\mathbb{E}_{g \sim G}[B_g(\mathbf{s}, \mathbf{y})]} \leq \ell^{4 \log \ell} \frac{2^{\ell-k+1}}{\prod_{i=1}^m \binom{d_i}{s_i}}. \end{aligned} \quad (41)$$

We need the above to be upper bounded by $\ell^{-2\sqrt{\ell}}$ to be able to use union bound for all $\mathbf{s} \in \mathcal{T} \subset \mathcal{S}$, since $|\mathcal{S}| \leq \ell^{O(\sqrt{\ell})}$. Recall that we have $k \geq \ell(1 - H(W)) + 13\ell^{1/2} \log^3 \ell$, and then using a lower bound for binomial coefficients from Fact 6.2 we obtain for the RHS of (41)

$$\ell^{4 \log \ell} \frac{2^{\ell-k+1}}{\prod_{i=1}^m \binom{d_i}{s_i}} \leq \ell^{4 \log \ell} \left(\prod_{i=1}^m \sqrt{2d_i} \right) 2^{\ell H(W) - \sum_{i=1}^m d_i h\left(\frac{s_i}{d_i}\right) - 13\ell^{1/2} \log^3 \ell}. \quad (42)$$

We want to show that the term $2^{-\Omega(\ell^{1/2} \log^3 \ell)}$ is the dominant one. First, it is easy to see that $\ell^{4 \log \ell} = 2^{4 \log^2 \ell} \leq 2^{\ell^{1/2} \log^3 \ell}$. To deal with the factor $\prod_{i=1}^m \sqrt{2d_i}$, recall that $\sum_{i=1}^m d_i = \ell$ and $m \leq \sqrt{\ell}$ in this section, then AM-GM inequality gives us

$$\prod_{i=1}^m \sqrt{2d_i} \leq 2^{m/2} \cdot \sqrt{\left(\frac{\sum_{i=1}^m d_i}{m}\right)^m} = \left(\frac{2\ell}{m}\right)^{m/2} \leq (2\sqrt{\ell})^{\sqrt{\ell}/2} \leq 2^{\ell^{1/2} \log^3 \ell}, \quad (43)$$

where we used that $(a/x)^x$ is increasing while $x \leq a/e$. For the last factor of (42) we formulate a lemma.

Lemma 7.3. *There exists a set $\mathcal{T} \subseteq \mathcal{S} = [0 : d_1] \times [0 : d_2] \times \cdots \times [0 : d_m]$, such that $\mathbb{P}_{\chi \sim \Omega} [\chi \in \mathcal{T}] \geq 1 - \ell^{-\log \ell/4}$, and for any $\mathbf{s} \in \mathcal{T}$ it holds that*

$$\ell H(W) - \sum_{i=1}^m d_i h\left(\frac{s_i}{d_i}\right) \leq 11 \ell^{1/2} \log^3 \ell.$$

($\Omega = \text{Binom}(d_1, p_1) \times \text{Binom}(d_2, p_2) \times \cdots \times \text{Binom}(d_m, p_m)$ above)

Proof. Rearrange the above summation as follows:

$$\begin{aligned} \ell H(W) - \sum_{i=1}^m d_i h\left(\frac{s_i}{d_i}\right) &= \sum_{i=1}^m \left(\ell q_i h(p_i) - d_i h\left(\frac{s_i}{d_i}\right) \right) \\ &= \sum_{i=1}^m (\ell q_i - d_i) h(p_i) + \sum_{i=1}^m d_i \left(h(p_i) - h\left(\frac{s_i}{d_i}\right) \right). \end{aligned}$$

Now recall that we took typical \mathbf{y} for now, so by inequality (32) from the definition of the typicality of \mathbf{y} we already have that the first part of the above sum is bounded by $\ell^{1/2} \log^3 \ell$.

To deal with the second part, which is $\sum_{i=1}^m d_i \left(h(p_i) - h\left(\frac{s_i}{d_i}\right) \right)$, we use a separate Lemma 7.12, since the proof will be almost exactly similar for another concentration inequality we will need later. Lemma 7.12 claims that $\sum_{i=1}^m d_i \left(h(p_i) - h\left(\frac{\chi_i}{d_i}\right) \right) \leq 10 \ell^{1/2} \log^3 \ell$ with probability at least $1 - \ell^{-\log \ell/4}$ over $\chi \sim \Omega$. Then the result of the current lemma follows by taking \mathcal{T} to be the subset of \mathcal{S} where this inequality holds. \square

Fix now a set $\mathcal{T} \subseteq \mathcal{S}$ as in Lemma 7.3. Then using the arguments above we conclude that the RHS in (42), and therefore (41), is bounded above by $2^{-2\ell^{1/2} \log^3 \ell}$ for any $\mathbf{s} \in \mathcal{T}$. Thus we can apply union bound over $\mathbf{s} \in \mathcal{T}$ for (41), since $|\mathcal{T}| \leq |\mathcal{S}| = \prod_{i=1}^m (d_i + 1) \leq (2\sqrt{\ell})^{\sqrt{\ell}} \leq 2^{\ell^{1/2} \log^3 \ell}$, similarly to (43). Therefore, we derive

Corollary 7.4. *With probability at least $1 - 2^{-\ell^{1/2} \log^3 \ell}$ (over the random kernel $g \sim G$) it holds simultaneously for all $\mathbf{s} \in \mathcal{T}$ that*

$$\left| B_g(\mathbf{s}, \mathbf{y}) - \mathbb{E}[B_g(\mathbf{s}, \mathbf{y})] \right| \leq \ell^{-2 \log \ell} \mathbb{E}[B_g(\mathbf{s}, \mathbf{y})].$$

Moreover, the set $\mathcal{N} = \mathcal{S} \setminus \mathcal{T}$ satisfies $\mathbb{P}_{\chi \sim \Omega} [\chi \in \mathcal{N}] \leq \ell^{-\log \ell/4}$, which we will use next section to bound the second part of (40).

7.1.1.(ii) Negligible part

Denote for convenience $Z_g(\mathbf{y}) = \sum_{\mathbf{s} \in \mathcal{N}} B_g(\mathbf{s}, \mathbf{y}) \prod_{i=1}^m q_i^{d_i} p_i^{s_i} (1 - p_i)^{d_i - s_i}$, the second part of the RHS of (40). Recall the value of $\mathbb{E}_{g \sim G}[B_g(\mathbf{s}, \mathbf{Y})]$ from (37) and notation of E in (38). Then for the expectation of $Z_g(\mathbf{y})$ derive

$$\begin{aligned} \mathbb{E}_{g \sim G}[Z_g(\mathbf{y})] &= \prod_{i=1}^m q_i^{d_i} \sum_{\mathbf{s} \in \mathcal{N}} \left(\mathbb{E}_{g \sim G}[B_g(\mathbf{s}, \mathbf{y})] p_i^{s_i} (1 - p_i)^{d_i - s_i} \right) \\ &= \frac{2^k - 1}{2^\ell} \prod_{i=1}^m q_i^{d_i} \cdot \sum_{\mathbf{s} \in \mathcal{N}} \prod_{i=1}^m \binom{d_i}{s_i} p_i^{s_i} (1 - p_i)^{d_i - s_i} \\ &= E \cdot \mathbb{P}_{\chi \sim \Omega}[\chi \in \mathcal{N}] \\ &\leq E \cdot \ell^{-\log \ell / 4}. \end{aligned}$$

Thus Markov's inequality implies

Corollary 7.5. *With probability at least $1 - \ell^{-\log \ell / 8}$ (over the random kernel $g \sim G$) it holds*

$$Z_g(\mathbf{y}) \leq \ell^{\log \ell / 8} \mathbb{E}[Z_g(\mathbf{y})] \leq E \cdot \ell^{-\log \ell / 8}.$$

7.1.1.(iii) Putting it together

Combining the Corollaries 7.4 and 7.5 together and using union bound, we derive

Corollary 7.6. *With probability at least $1 - \ell^{-\log \ell / 8} - 2^{-\ell^{1/2} \log^3 \ell} \geq 1 - 2\ell^{-\log \ell / 8}$ over the randomness of the kernel $g \sim G$ it simultaneously holds*

$$\begin{aligned} \left| B_g(\mathbf{s}, \mathbf{y}) - \mathbb{E}[B_g(\mathbf{s}, \mathbf{y})] \right| &\leq \ell^{-2 \log \ell} \mathbb{E}[B_g(\mathbf{s}, \mathbf{y})], \quad \text{for all } \mathbf{s} \in \mathcal{T}, \\ \sum_{\mathbf{s} \in \mathcal{N}} B_g(\mathbf{s}, \mathbf{y}) \prod_{i=1}^m q_i^{d_i} p_i^{s_i} (1 - p_i)^{d_i - s_i} &\leq E \cdot \ell^{-\log \ell / 8}. \end{aligned} \tag{44}$$

We are finally ready to formulate the concentration result we need. The following lemma is as analogue of Lemma 6.3 from the BSC case:

Lemma 7.7. *With probability at least $1 - 2\ell^{-\log \ell / 8}$ over the choice of $g \sim G$ it holds*

$$\left| \sum_{\mathbf{s} \in \mathcal{S}} B_g(\mathbf{s}, \mathbf{y}) \prod_{i=1}^m q_i^{d_i} p_i^{s_i} (1 - p_i)^{d_i - s_i} - E \right| \leq 2\ell^{-\log \ell / 8} \cdot E.$$

Proof. Let us consider a kernel g such that the conditions (44) hold, which happens with probability

at least $1 - 2\ell^{-\log \ell/8}$ according to Corollary 7.6. Then

$$\begin{aligned}
\sum_{\mathbf{s} \in \mathcal{S}} B_g(\mathbf{s}, \mathbf{y}) \prod_{i=1}^m q_i^{d_i} p_i^{s_i} (1-p_i)^{d_i-s_i} &\geq \sum_{\mathbf{s} \in \mathcal{T}} B_g(\mathbf{s}, \mathbf{y}) \prod_{i=1}^m q_i^{d_i} p_i^{s_i} (1-p_i)^{d_i-s_i} \\
&\geq \sum_{\mathbf{s} \in \mathcal{T}} \left(1 - \ell^{-2 \log \ell}\right) \mathbb{E}[B_g(\mathbf{s}, \mathbf{y})] \prod_{i=1}^m q_i^{d_i} p_i^{s_i} (1-p_i)^{d_i-s_i} \\
&= \left(1 - \ell^{-2 \log \ell}\right) \frac{2^k - 1}{2^\ell} \prod_{i=1}^m q_i^{d_i} \cdot \sum_{\mathbf{s} \in \mathcal{T}} \prod_{i=1}^m \binom{d_i}{s_i} p_i^{s_i} (1-p_i)^{d_i-s_i} \\
&= \left(1 - \ell^{-2 \log \ell}\right) \cdot E \cdot \mathbb{P}_{\chi \sim \Omega} [\chi \in \mathcal{T}] \\
&\geq \left(1 - \ell^{-2 \log \ell}\right) \left(1 - \ell^{-\log \ell/8}\right) E \\
&\geq \left(1 - 2\ell^{-\log \ell/8}\right) E.
\end{aligned}$$

For the other direction, we derive for such g

$$\begin{aligned}
\sum_{\mathbf{s} \in \mathcal{S}} B_g(\mathbf{s}, \mathbf{y}) \prod_{i=1}^m q_i^{d_i} p_i^{s_i} (1-p_i)^{d_i-s_i} &= \left(\sum_{\mathbf{s} \in \mathcal{T}} + \sum_{\mathbf{s} \in \mathcal{N}}\right) B_g(\mathbf{s}, \mathbf{y}) \prod_{i=1}^m q_i^{d_i} p_i^{s_i} (1-p_i)^{d_i-s_i} \\
&\stackrel{(44)}{\leq} \sum_{\mathbf{s} \in \mathcal{T}} \left(1 + \ell^{-2 \log \ell}\right) \mathbb{E}[B_g(\mathbf{s}, \mathbf{y})] \prod_{i=1}^m q_i^{d_i} p_i^{s_i} (1-p_i)^{d_i-s_i} + E \cdot \ell^{-\log \ell/8} \\
&\leq \left(1 + \ell^{-2 \log \ell}\right) \sum_{\mathbf{s} \in \mathcal{S}} \mathbb{E}[B_g(\mathbf{s}, \mathbf{y})] \prod_{i=1}^m q_i^{d_i} p_i^{s_i} (1-p_i)^{d_i-s_i} + E \cdot \ell^{-\log \ell/8} \\
&= \left(1 + \ell^{-2 \log \ell} + \ell^{-\log \ell/8}\right) E \\
&\leq \left(1 + 2\ell^{-\log \ell/8}\right) E. \quad \square
\end{aligned}$$

7.1.2 Concentration of entropy

We can now get a tight concentration for $\mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y}]$ using the relation (35). We already showed that the sum in RHS of (35) is tightly concentrated around its expectation, so it only remains to show that $\mathbb{P}[\mathbf{Y} = \mathbf{y} | \mathbf{v} = \mathbf{0}]$ is tiny comparable to E . Here we will use that we picked \mathbf{y} to be "typical" from the start so that (32) and (33) hold, and that $k \geq \ell(1 - H(W)) + 13\ell^{1/2} \log^3 \ell$ for the above-capacity regime. Recall (30), as well the the conditions (32) and (33) on \mathbf{y} being typical. We derive

$$\begin{aligned}
\mathbb{P}[\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{0}] &= \prod_{i=1}^m q_i^{d_i} p_i^{t_i} (1-p_i)^{d_i-t_i} = \prod_{i=1}^m \left[q_i^{d_i} \cdot p_i^{d_i p_i} (1-p_i)^{d_i(1-p_i)} \cdot \left(\frac{1-p_i}{p_i}\right)^{d_i p_i - t_i} \right] \\
&= \prod_{i=1}^m q_i^{d_i} \cdot \prod_{i=1}^m 2^{-d_i h(p_i)} \cdot \prod_{i=1}^m 2^{(d_i p_i - t_i) \log\left(\frac{1-p_i}{p_i}\right)} \\
&= \prod_{i=1}^m q_i^{d_i} \cdot 2^{\sum_{i=1}^m (-\ell q_i h(p_i) + (\ell q_i - d_i) h(p_i))} \cdot 2^{\sum_{i=1}^m (d_i p_i - t_i) \log\left(\frac{1-p_i}{p_i}\right)} \\
&\stackrel{(32),(33)}{\leq} \prod_{i=1}^m q_i^{d_i} \cdot 2^{-\ell H(W) + 2\ell^{1/2} \log \ell + 3\ell^{1/2} \log^2 \ell} \leq \prod_{i=1}^m q_i^{d_i} \cdot \frac{2^k - 1}{2^\ell} \cdot \ell^{-\log \ell} = E \cdot \ell^{-\log \ell}.
\end{aligned}$$

Now, combining this with Lemma 7.7, we obtain a concentration for (35):

Corollary 7.8. *With probability at least $1 - 2\ell^{-\log \ell/8}$ over the choice of kernel $g \sim G$ and for any typical \mathbf{y}*

$$\left| 2^k \cdot \mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y}] - E \right| \leq 3\ell^{-\log \ell/8} \cdot E,$$

$$\text{where } E = \frac{2^k - 1}{2^\ell} \prod_{i=1}^m q_i^{d_i}.$$

Next, completely analogously we derive the concentration for $\mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y} | V_1 = 0]$, which is the numerator inside the entropy in (34). The only thing that changes is that we will have dimension $k - 1$ instead of k for this case. We can state

Corollary 7.8'. *With probability at least $1 - 2\ell^{-\log \ell/8}$ over the choice of kernel $g \sim G$ and for any typical \mathbf{y}*

$$\left| 2^{k-1} \cdot \mathbb{P}^{(g)}[V_1 = 0, \mathbf{Y} = \mathbf{y}] - \tilde{E} \right| \leq 3\ell^{-\log \ell/8} \cdot \tilde{E},$$

$$\text{where } \tilde{E} = \frac{2^{k-1} - 1}{2^\ell} \prod_{i=1}^m q_i^{d_i}.$$

Combining these two together and skipping the simple math, completely analogical to that of the BSC case, we derive

Corollary 7.9. *With probability at least $1 - 4\ell^{-\log \ell/8}$ over the choice of kernel $g \sim G$ and for any typical \mathbf{y}*

$$\left| \frac{\mathbb{P}^{(g)}[V_1 = 0, \mathbf{Y} = \mathbf{y}]}{\mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y}]} - \frac{1}{2} \right| \leq \ell^{-\log \ell/9}$$

Since $h(1/2 + x) \geq 1 - 4x^2$ for any $x \in (-1/2, 1/2)$, we then derive for a typical \mathbf{y} :

$$\begin{aligned} \mathbb{E}_g [H^{(g)}(V_1 | \mathbf{Y} = \mathbf{y})] &= \mathbb{E}_g \left[h \left(\frac{\mathbb{P}^{(g)}[V_1 = 0, \mathbf{Y} = \mathbf{y}]}{\mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y}]} \right) \right] \geq (1 - 4\ell^{-\log \ell/8}) \cdot (1 - 4\ell^{-\log \ell/9}) \\ &\geq 1 - 8\ell^{-\log \ell/9}. \end{aligned}$$

Then in (31) we have

$$\begin{aligned} \mathbb{E}_g [H^{(g)}(V_1 | \mathbf{Y})] &= \sum_{\mathbf{y} \in \mathcal{Y}^\ell} \mathbb{P}[\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{0}] \mathbb{E}_g [H^{(g)}(V_1 | \mathbf{Y} = \mathbf{y})] \\ &\geq \sum_{\mathbf{y} \text{ typical}} \mathbb{P}[\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{0}] \mathbb{E}_g [H^{(g)}(V_1 | \mathbf{Y} = \mathbf{y})] \\ &\geq (1 - \ell^{-\log \ell}) \cdot (1 - 8\ell^{-\log \ell/9}). \\ &\geq 1 - 9\ell^{-\log \ell/9} \geq 1 - \ell^{-\log \ell/10}, \end{aligned} \tag{45}$$

where we used that the probability to get a typical output on a zero input is at least $1 - \ell^{-\log \ell}$ by Lemma 7.2.

Finally, using the fact that $H^{(g)}(V_1 | \mathbf{Y}) \leq 1$, Markov's inequality, and (45), we get

$$\mathbb{P}_{g \sim G} \left[H^{(g)}(V_1 | \mathbf{Y}) \leq 1 - \ell^{-\frac{\log \ell}{20}} \right] = \mathbb{P} \left[1 - H^{(g)}(V_1 | \mathbf{Y}) \geq \ell^{-\frac{\log \ell}{20}} \right] \leq \frac{\mathbb{E} [1 - H^{(g)}(V_1 | \mathbf{Y})]}{\ell^{-\log \ell/20}} \leq \ell^{-\log \ell/20}.$$

This completes the proof of Theorem 5.7 for the case of BMS channel with bounded output alphabet size, assuming the typicality Lemma 7.2 and concentration Lemma 7.12 which we used in Lemma 7.3. We now turn to proving these.

7.1.3 Proof that the typical set is indeed typical

Proof of Lemma 7.2. We start with proving that (32) is satisfied with high probability (over the randomness of the channel). Notice that (d_1, d_2, \dots, d_m) are multinomially distributed by construction, since for every of ℓ bits transitioned, we choose independently the subchannel $W^{(i)}$ to use with probability q_i , for $i = 1, 2, \dots, m$, and d_i represents the number of times the channel $W^{(i)}$ was chosen. So indeed $(d_1, d_2, \dots, d_m) \sim \text{Mult}(\ell, q_1, q_2, \dots, q_m)$. The crucial property of multinomial random variables we are going to use is *negative association* ([JDP83], [DR96]). The (simplified version of the) fact we are going to use about negatively associated random variables can be formulated as follows:

Lemma 7.10 ([JDP83], Property P₂). *Let X_1, X_2, \dots, X_m be negatively associated random variables. Then, for every set of m positive monotone non-decreasing functions f_1, \dots, f_m it holds*

$$\mathbb{E} \left[\prod_{i=1}^m f_i(X_i) \right] \leq \prod_{i=1}^m \mathbb{E}[f_i(X_i)].$$

We also use the fact that since (d_1, d_2, \dots, d_m) are negatively associated, then applying decreasing functions $g_i(x) = \ell q_i - x$ coordinate-wise to these random variables, we will also obtain negatively associated random variables ([DR96], Proposition 7). In other words, we argue that $(\ell q_1 - d_1, \ell q_2 - d_2, \dots, \ell q_m - d_m)$ are also negatively associated, thus we can apply Lemma 7.10 to these random variables.

Let us now denote for convenience $\alpha_i = h(p_i)$ for $i = 1, 2, \dots, m$, and so we have $0 \leq \alpha_i \leq 1$. Let also $X = \sum_{i=1}^m (\ell \cdot q_i - d_i) \alpha_i$, and we now can start with simple exponentiation and Markov's inequality: for any a and any $t > 0$

$$\mathbb{P}[X \geq a] = \mathbb{P}[e^{tX} \geq e^{ta}] \leq e^{-ta} \mathbb{E}[e^{tX}] = e^{-ta} \mathbb{E} \left[\prod_{i=1}^m e^{t \cdot \alpha_i (\ell q_i - d_i)} \right] \leq e^{-ta} \prod_{i=1}^m \mathbb{E} \left[e^{t \cdot \alpha_i (\ell q_i - d_i)} \right], \quad (46)$$

where in the last inequality we applied Lemma 7.10 for negatively associated random variables $(\ell q_1 - d_1, \ell q_2 - d_2, \dots, \ell q_m - d_m)$, as discussed above, and positive non-decreasing functions $f_i(x) = e^{t \cdot \alpha_i x}$, since $\alpha_i, t \geq 0$.

Next, consider the following claim, which follows from standard Chernoff-type arguments:

Claim 7.11. *Let $Z \sim \text{Binom}(n, p)$, and $b > 0$. Then $\mathbb{E}[e^{-bZ}] \leq e^{np \cdot (e^{-b} - 1)}$.*

Proof. We can write $Z = \sum_{j=1}^n Z_j$, where $Z_j \sim \text{Bern}(p)$ are independent Bernoulli random variables. Then

$$\mathbb{E} \left[e^{-bZ} \right] = \mathbb{E} \left[\prod_{j=1}^n e^{-bZ_j} \right] = \prod_{j=1}^n \mathbb{E} \left[e^{-bZ_j} \right] = \left((1-p) + p \cdot e^{-b} \right)^n \leq e^{np(e^{-b} - 1)}, \quad (47)$$

where the only inequality uses the fact that $1 + x \leq e^x$ for any x . □

Turning back to (46), we are going to bound the terms $\mathbb{E} \left[e^{t \cdot \alpha_i (\ell q_i - d_i)} \right]$ individually. It is clear that the marginal distribution of d_i is just $\text{Binom}(\ell, q_i)$, so we are able to use Claim 7.11 for it. We derive:

$$\mathbb{E} \left[e^{t \cdot \alpha_i (\ell q_i - d_i)} \right] = e^{t \alpha_i \ell q_i} \cdot \mathbb{E} \left[e^{-t \alpha_i d_i} \right] \stackrel{(47)}{\leq} e^{t \alpha_i \ell q_i} \cdot e^{\ell q_i (e^{-t \alpha_i} - 1)} = e^{\ell q_i (t \alpha_i + e^{-t \alpha_i} - 1)} \leq e^{\ell q_i (t + e^{-t} - 1)}, \quad (48)$$

where the last inequality uses that $x + e^{-x}$ is increasing for $x \geq 0$ together with $0 \leq t\alpha_i \leq t$, as $t > 0$ and $0 \leq \alpha_i \leq 1$. Plugging the above into (46) and using $\sum_{i=1}^m q_i = 1$, we obtain

$$\mathbb{P}[X \geq a] \leq e^{-ta} \prod_{i=1}^m e^{\ell q_i (t + e^{-t} - 1)} = e^{-ta} \cdot e^{\ell(t + e^{-t} - 1)} \leq e^{-ta + \ell \frac{t^2}{2}}, \quad (49)$$

where we used $x + e^{-x} - 1 \leq \frac{x^2}{2}$ for any $x \geq 0$. Finally, by taking $a = 2\sqrt{\ell} \log \ell$, setting $t = a/\ell$, and recalling what we denoted by X and α_i above, we immediately deduce

$$\mathbb{P} \left[\sum_{i=1}^m (\ell \cdot q_i - d_i) h(p_i) \geq 2\sqrt{\ell} \log \ell \right] \leq e^{-\frac{a^2}{2\ell}} = e^{-2 \log^2 \ell} \leq \ell^{-2 \log \ell}.$$

This means that the first typicality requirement (32) holds with very high probability (over the randomness of the channel).

Let us now prove that the second typicality condition (33) holds with high probability. For that, we condition on the values of d_1, d_2, \dots, d_m . We will see that (33) holds with high probability for all values of d_1, d_2, \dots, d_m , and then it is clear that it will imply that it also holds with high probability overall.

So, fix the values of d_1, d_2, \dots, d_m . Denote a random variable $Y = \sum_{i=1}^m (p_i d_i - t_i) \log \left(\frac{1-p_i}{p_i} \right)$, and then our goal is to show that Y is bounded above by $O(\sqrt{\ell} \log^2 \ell)$ with high probability (over the randomness of t_i 's). Given the conditioning on d_1, d_2, \dots, d_m , it is clear that $t_i \sim \text{Binom}(d_i, p_i)$ for all $i = 1, 2, \dots, m$, and they are all independent (recall that d_i corresponds to the number of times subchannel $W^{(i)}$ is chosen, while t_i corresponds to the number of "flips" within this subchannel).

We split the summation in Y into two parts: let $T_1 = \{i : p_i \leq \frac{1}{\ell}\}$ and $T_2 = [m] \setminus T_1$. Then for any realization of t_i 's, we have $\sum_{i \in T_1} (p_i d_i - t_i) \log \left(\frac{1-p_i}{p_i} \right) \leq \sum_{i \in T_1} p_i d_i \log \left(\frac{1}{p_i} \right) \leq \sum_{i \in T_1} \frac{d_i \log \ell}{\ell} \leq \log \ell$.

Denote the second part of the summation as $Y_2 = \sum_{i \in T_2} (p_i d_i - t_i) \log \left(\frac{1-p_i}{p_i} \right)$. Notice that $\log \left(\frac{1-p_i}{p_i} \right) \leq \log \left(\frac{1}{p_i} \right) \leq \log \ell$ for $i \in T_2$. Denote then $\gamma_i = \log \left(\frac{1-p_i}{p_i} \right) / \log \ell$, and so $0 \leq \gamma_i \leq 1$ for $i \in T_2$. Finally, let $\widetilde{Y}_2 = Y_2 / \log \ell = \sum_{i \in T_2} (p_i d_i - t_i) \cdot \gamma_i$.

We now prove the concentration on \widetilde{Y}_2 in exactly the same way we did for X above. Claim 7.11 applied for $t_i \sim \text{Binom}(d_i, p_i)$ and $t \cdot \gamma_i > 0$ for any $t > 0$ gives $\mathbb{E} [e^{-t\gamma_i \cdot t_i}] \leq e^{d_i p_i (e^{-t\gamma_i} - 1)}$, and so similarly to (46)-(49) derive

$$\mathbb{P} [\widetilde{Y}_2 > a] \leq e^{-ta} \cdot \prod_{i \in T_2} e^{p_i d_i (t\gamma_i + e^{-t\gamma_i} - 1)} \leq e^{-ta} \cdot \prod_{i \in T_2} e^{p_i d_i (t + e^{-t} - 1)} \leq e^{-ta + \sum_{i \in T_2} p_i d_i \cdot t^2 / 2} \leq e^{-ta + \ell t^2 / 2}$$

for any $t > 0$, where we used $0 \leq \gamma_i \leq 1$ for $i \in T_2$, $p_i < 1$, and $\sum_{i \in T_2} d_i \leq \ell$. Therefore, by taking again $a = 2\sqrt{\ell} \log \ell$ and $t = a/\ell$, obtain

$$\mathbb{P} [Y_2 \geq 2\sqrt{\ell} \log^2 \ell] = \mathbb{P} [\widetilde{Y}_2 \geq 2\sqrt{\ell} \log \ell] \leq \ell^{-2 \log \ell}.$$

Since $Y \leq \log \ell + Y_2$, we conclude that $Y \leq 3\sqrt{\ell} \log^2 \ell$ with probability at least $\ell^{-2 \log \ell}$ over the randomness of the channel.

Since both (32) and (33) hold with probability at least $1 - \ell^{-2 \log \ell}$, the union bound implies that these two conditions hold simultaneously with probability at least $1 - 2\ell^{-2 \log \ell} \geq 1 - \ell^{-\log \ell}$. \square

7.1.4 Concentration Lemma

Lemma 7.12. *Let $\chi \sim \Omega = \text{Binom}(d_1, p_1) \times \text{Binom}(d_2, p_2) \times \cdots \times \text{Binom}(d_m, p_m)$, where d_i 's are positive integers for $i \in [m]$, $p_i \leq 1/2$, $\sum_{i=1}^m d_i = \ell$, and $m \leq \sqrt{\ell}$. Then the following holds with probability at least $1 - \ell^{-(\log \ell)/4}$.*

$$\sum_{i=1}^m d_i \left(h(p_i) - h\left(\frac{\chi_i}{d_i}\right) \right) \leq 10\ell^{1/2} \log^3 \ell.$$

Proof. First, we split the interval $[1 : m]$ into two parts. In the first part the value of $d_i \cdot p_i$ is going to be small, and the sum of $d_i h(p_i)$ will also be small. For the second part, when $d_i \cdot p_i$ is large enough, we will be able to apply some concentration arguments. Denote:

$$F_1 := \left\{ i : p_i \leq \frac{4 \log^2 \ell}{d_i} \right\},$$

$$F_2 := \{1, 2, \dots, m\} \setminus F_1.$$

Then

$$\sum_{i=1}^m d_i \left(h(p_i) - h\left(\frac{\chi_i}{d_i}\right) \right) \leq \sum_{i \in F_1} d_i h(p_i) + \sum_{i \in F_2} d_i \left(h(p_i) - h\left(\frac{\chi_i}{d_i}\right) \right). \quad (50)$$

Let us deal with the summation over F_1 first. Split this set even further: $F_1^{(1)} = \{i \in F_1 : d_i \geq 8 \log^2 \ell\}$, and $F_1^{(2)} = F_1 \setminus F_1^{(1)}$. Then for any $i \in F_1^{(1)}$ deduce that $p_i \leq 1/2$, thus $h(p_i) \leq 2p_i \log \frac{1}{p_i}$. For any $i \in F_1^{(2)}$ we just use $h(p_i) \leq 1$. Combining these, obtain

$$\begin{aligned} \sum_{i \in F_1} d_i h(p_i) &\leq \sum_{i \in F_1^{(1)}} 2d_i p_i \log \frac{1}{p_i} + \sum_{i \in F_1^{(2)}} d_i \leq \sum_{i \in F_1^{(1)}} 8 \log^2 \ell \cdot \log \left(\frac{d_i}{4 \log^2 \ell} \right) + |F_1^{(2)}| \cdot 8 \log^2 \ell \\ &\leq |F_1^{(1)}| \cdot 8 \log^3 \ell + |F_1^{(2)}| \cdot 8 \log^2 \ell \leq 8\ell^{1/2} \log^3 \ell. \end{aligned} \quad (51)$$

Therefore, the first part of the RHS of (50) is always bounded by $8\ell^{1/2} \log^3 \ell$. We will now deal with the remaining summations over $i \in F_2$.

For any $i \in F_2$, we know that $d_i p_i \geq 4 \log^2 \ell$. Now, for $\chi \sim \Omega$ we have by the multiplicative Chernoff bound

$$\mathbb{P} [|\chi_i - d_i p_i| \geq \sqrt{d_i p_i} \log \ell] \leq 2e^{-\log^2 \ell / 3} \leq \ell^{-\log \ell / 3} \quad \text{if } \log \ell \leq \sqrt{d_i p_i}, \quad (52)$$

where the last inequality holds because the log in the exponent is to base 2. The condition $\log \ell \leq \sqrt{d_i p_i}$ is needed for the multiplicative Chernoff bound to hold.

Then, by union bound, we derive

$$\mathbb{P}_{\chi \sim \Omega} \left[|\chi_i - d_i p_i| \geq \sqrt{d_i p_i} \log \ell \text{ for some } i \in F_2 \right] \leq |F_2| \cdot \ell^{-\log \ell / 3} \leq \ell^{-\log \ell / 3 + 1/2}. \quad (53)$$

Define the sets $\mathcal{T}_1^{(i)}$ for all $i = 1, 2, \dots, m$ as follows:

$$\begin{aligned} \mathcal{T}_1^{(i)} &:= \left\{ s_i \in [0 : d_i] : |s_i - d_i p_i| \leq \sqrt{d_i p_i} \log \ell \right\}, & \text{for } i \in F_2; \\ \mathcal{T}_1^{(i)} &:= [0 : d_i], & \text{for } i \notin F_2. \end{aligned} \quad (54)$$

and let

$$\theta_i := \mathbb{P}[\chi_i \in \mathcal{T}_1^{(i)}]. \quad (55)$$

Then by (52) we have

$$\begin{aligned} \theta_i &\geq 1 - \ell^{-\log \ell / 3}, & \text{for } i \in F_2; \\ \theta_i &= 1, & \text{for } i \notin F_2. \end{aligned}$$

Finally, define

$$\theta := \prod_{i=1}^m \theta_i = \prod_{i \in F_2} \theta_i = \prod_{i \in F_2} \mathbb{P}[\chi_i \in \mathcal{T}_1^{(i)}] = \mathbb{P}_{\chi \sim \Omega}[\chi_i \in \mathcal{T}_1^{(i)} \text{ for all } i \in F_2] \geq 1 - \ell^{-\log \ell / 3 + 1/2},$$

where the last inequality is a direct implication of (53).

We will now define a set of new probability distributions \mathcal{D}_i for all $i = 1, 2, \dots, m$, as binomial distributions $\text{Binom}(d_i, p_i)$ restricted to intervals $\mathcal{T}_1^{(i)}$. Formally, let us write

$$\mathbb{P}_{\eta_i \sim \mathcal{D}_i}[\eta_i = x] = \begin{cases} 0, & \text{if } x \notin \mathcal{T}_1^{(i)}; \\ \mathbb{P}_{\chi_i \sim \text{Binom}(d_i, p_i)}[\chi_i = x] \cdot \theta_i^{-1}, & \text{if } x \in \mathcal{T}_1^{(i)}. \end{cases} \quad (56)$$

(So to get \mathcal{D}_i we just took a distribution $\text{Binom}(d_i, p_i)$, truncated it so it does not have any mass outside of $\mathcal{T}_1^{(i)}$, and rescaled appropriately.)

Next, define a product distribution $\mathcal{D} := \times_{i=1}^m \mathcal{D}_i$ on the set $\mathcal{T}_1 := \times_{i=1}^m \mathcal{T}_1^{(i)}$. Notice now that it is trivial that for any subset $\mathcal{R} \subseteq \mathcal{T}_1$ it holds

$$\mathbb{P}_{\chi \sim \Omega}[\chi \in \mathcal{R}] = \mathbb{P}_{\eta \sim \mathcal{D}}[\eta \in \mathcal{R}] \cdot \theta. \quad (57)$$

Since θ is really close to 1, it follows that we can basically transition to considering \mathcal{D} instead of Ω .

Recall that our goal was to show that $\sum_{i \in F_2} d_i \left(h(p_i) - h\left(\frac{\chi_i}{d_i}\right) \right)$ (the second part from (50)) is bounded above by $O(\ell^{1/2} \log^3 \ell)$ with high probability, when $\chi \sim \Omega$. Instead now let us show that this summation is small with high probability when $\chi \sim \mathcal{D}$, and then use the arguments above to see that there is not much of a difference when $\chi \sim \Omega$.

Claim 7.13. *Let $i \in F_2$ and $\chi_i \sim \mathcal{D}_i$. Then*

$$\left| d_i \left(h(p_i) - h\left(\frac{\chi_i}{d_i}\right) \right) \right| \leq \sqrt{d_i p_i} \log^2 \ell, \quad (58)$$

Proof. First, $\left| \frac{\chi_i}{d_i} - p_i \right| \leq \sqrt{\frac{p_i}{d_i}} \log \ell$ for $\chi_i \sim \mathcal{D}_i$ by definition of the distribution \mathcal{D}_i . Now, for $i \in F_2$, $p_i \geq \frac{4 \log^2 \ell}{d_i}$ and then $\frac{p_i}{2} \geq \sqrt{\frac{p_i}{d_i}} \log \ell$, therefore $\frac{\chi_i}{d_i} \geq \frac{p_i}{2}$. Then, using the concavity of the binary entropy function, we obtain:

$$\begin{aligned} \left| h\left(\frac{\chi_i}{d_i}\right) - h(p_i) \right| &\leq \left| \frac{\chi_i}{d_i} - p_i \right| \cdot \max \left\{ \frac{dh}{dx}(p_i), \frac{dh}{dx}\left(\frac{\chi_i}{d_i}\right) \right\} \\ &\leq \sqrt{\frac{p_i}{d_i}} \log \ell \cdot \frac{dh}{dx}\left(\frac{p_i}{2}\right) = \sqrt{\frac{p_i}{d_i}} \log \ell \cdot \log \frac{1 - p_i/2}{p_i/2} \\ &\leq \sqrt{\frac{p_i}{d_i}} \log \ell \cdot \log \frac{2}{p_i} \leq \sqrt{\frac{p_i}{d_i}} \log \ell \cdot \log \left(\frac{d_i}{2 \log^2 \ell} \right) \leq \sqrt{\frac{p_i}{d_i}} \log^2 \ell, \end{aligned}$$

and therefore (58) follows. \square

Let $\chi \sim \mathcal{D}$ here and further. Define for convenience new random variables $X_i = d_i \left(h(p_i) - h\left(\frac{\chi_i}{d_i}\right) \right)$ for all $i \in F_2$, and let also $X = \sum_{i \in F_2} X_i = \sum_{i \in F_2} d_i \left(h(p_i) - h\left(\frac{\chi_i}{d_i}\right) \right)$.

Claim 7.14. *With probability at least $1 - \ell^{-\log \ell}$ it holds that*

$$X - \mathbb{E}[X] \leq \ell^{1/2} \log^3 \ell$$

Proof. Obviously all the X_i 's are independent, and also $X_i \in \left[-\sqrt{d_i p_i} \log^2 \ell, \sqrt{d_i p_i} \log^2 \ell \right]$ by Claim 7.13. Then we can apply Hoeffding's inequality for the sum of independent random variables which are bounded by some intervals, and obtain

$$\begin{aligned} \mathbb{P}_{\chi \sim \mathcal{D}} \left[X - \mathbb{E}[X] \geq \ell^{1/2} \log^3 \ell \right] &\leq \exp \left(-\frac{2\ell \log^6 \ell}{\sum_{i \in F_2} (2\sqrt{d_i p_i} \log^2 \ell)^2} \right) \\ &\leq e^{-\log^2 \ell} \leq \ell^{-\log \ell}, \end{aligned}$$

where we used in the last step that $\sum_{i=1}^m d_i = \ell$, $p_i \leq 1/2$, and $d_i \leq \ell$. \square

So by now we proved that $X = \sum_{i \in F_2} d_i \left(h(p_i) - h\left(\frac{\chi_i}{d_i}\right) \right)$ does not deviate much from its expectation. What we are left to show now is that $\mathbb{E}[X]$ is not very large by itself.

The following two claims show that the first moment and mean absolute deviation of the distribution \mathcal{D}_i are close to those of Ω_i . This easily follows from the definition (56) of \mathcal{D}_i , and the proofs are deferred to Appendix B

Claim 7.15. *Let $i \in F_2$. Then $\left| \mathbb{E}_{\chi_i \sim \mathcal{D}_i} \left[\frac{\chi_i}{d_i} \right] - p_i \right| \leq \frac{1}{d_i}$.*

Claim 7.16. *Let $\chi_i \sim \mathcal{D}_i$ and $\eta_i \sim \Omega_i$ for $i \in F_2$. Then $\mathbb{E} \left| \chi_i - \mathbb{E}[\chi_i] \right| \leq \mathbb{E} \left| \eta_i - \mathbb{E}[\eta_i] \right| + 1$.*

These observations allow us we prove the following

Claim 7.17. *Let $i \in F_2$, and $\chi_i \sim \mathcal{D}_i$. Then $h\left(\mathbb{E}\left[\frac{\chi_i}{d_i}\right]\right) - \mathbb{E}\left[h\left(\frac{\chi_i}{d_i}\right)\right] \leq \frac{5 \log \ell}{d_i}$.*

Proof. Unfortunately, Jensen's inequality works in the opposite direction for us here. However, we use some form of converse Jensen's from [Dra11], which says the following:

Lemma 7.18 (Converse Jensen's inequality, [Dra11], Corollary 1.8). *Let f be a concave differentiable function on an interval $[a, b]$, and let Z be a (discrete) random variable, taking values in $[a, b]$. Then*

$$0 \leq f(\mathbb{E}[Z]) - \mathbb{E}[f(Z)] \leq \frac{1}{2} (f'(a) - f'(b)) \cdot \mathbb{E} |Z - \mathbb{E}[Z]|.$$

We apply it here for the concave binary entropy function h , and random variable $Z = \frac{\chi_i}{d_i}$ for $\chi_i \sim \mathcal{D}_i$, which takes values in $[a, b] := \left[p_i - \sqrt{\frac{p_i}{d_i}} \log \ell, p_i + \sqrt{\frac{p_i}{d_i}} \log \ell \right]$. Recall also that for $i \in F_2$, $p_i \geq \frac{4 \log^2 \ell}{d_i}$ and then $\frac{p_i}{2} \geq \sqrt{\frac{p_i}{d_i}} \log \ell$, therefore $a = p_i - \sqrt{\frac{p_i}{d_i}} \log \ell \geq \frac{p_i}{2}$. Using the mean value theorem, for some $c \in [a, b]$ we have

$$h'(a) - h'(b) = (b - a) \cdot (-h''(c)) \leq 2\sqrt{\frac{p_i}{d_i}} \log \ell \cdot (-h''(c)).$$

But $(-h''(c)) = \frac{1}{c(1-c)\ln 2} \leq \frac{2}{c} \leq \frac{2}{a} \leq \frac{4}{p_i}$, thus

$$h'(a) - h'(b) \leq \frac{8 \log \ell}{\sqrt{d_i p_i}}.$$

Finally, Claim 7.16 gives $\mathbb{E} |Z - \mathbb{E}[Z]| \leq \mathbb{E} \left| \frac{Z_2}{d_i} - \mathbb{E} \left[\frac{Z_2}{d_i} \right] \right| + \frac{1}{d_i}$ for $Z_2 \sim \text{Binom}(d_i, p_i)$, and so

$$\mathbb{E} |Z - \mathbb{E}[Z]| \leq \frac{1}{d_i} \mathbb{E} |Z_2 - \mathbb{E}[Z_2]| + \frac{1}{d_i} \leq \frac{1}{d_i} \sqrt{\mathbb{E}[(Z_2 - \mathbb{E}[Z_2])^2]} + \frac{1}{d_i} = \sqrt{\frac{p_i(1-p_i)}{d_i}} + \frac{1}{d_i} \leq \sqrt{\frac{p_i}{d_i}} + \frac{1}{d_i}.$$

Putting all this together, Lemma 7.18 gives us

$$0 \leq h \left(\mathbb{E} \left[\frac{\chi_i}{d_i} \right] \right) - \mathbb{E} \left[h \left(\frac{\chi_i}{d_i} \right) \right] \leq \frac{1}{2} \cdot \frac{8 \log \ell}{\sqrt{d_i p_i}} \cdot \left(\sqrt{\frac{p_i}{d_i}} + \frac{1}{d_i} \right) = \frac{4 \log \ell}{d_i} + \frac{4 \log \ell}{d_i \sqrt{d_i p_i}} \leq \frac{5 \log \ell}{d_i},$$

where the last step uses $\sqrt{p_i d_i} \geq 2 \log \ell$ for $i \in F_2$. \square

We can now use the above claims and Proposition 4.1 to bound the expectation of X :

$$\begin{aligned} \mathbb{E}[X] &= \sum_{i \in F_2} d_i \left(h(p_i) - \mathbb{E} \left[h \left(\frac{\chi_i}{d_i} \right) \right] \right) \leq \sum_{i \in F_2} d_i \left(h(p_i) - h \left(\mathbb{E} \left[\frac{\chi_i}{d_i} \right] \right) + \frac{5 \log \ell}{d_i} \right) \\ &\leq \sum_{i \in F_2} d_i \left(h \left(\frac{1}{d_i} \right) + \frac{5 \log \ell}{d_i} \right) \leq 7 \ell^{1/2} \log \ell \leq \ell^{1/2} \log^3 \ell. \end{aligned} \quad (59)$$

So we showed in Claim 7.14 that X does not exceed its expectations by more than $\ell^{1/2} \log^3 \ell$ with high probability (over $\chi \sim \mathcal{D}$), and also that $E[X]$ is bounded by $\ell^{1/2} \log^3 \ell$ in (59), and therefore X does not exceed $2\ell^{1/2} \log^3 \ell$ with high probability. Specifically, it means that there exists $\mathcal{T} \subseteq \mathcal{T}_1$, such that $\mathbb{P}_{\chi \sim \mathcal{D}}[\chi \in \mathcal{T}] \geq 1 - \ell^{-\log \ell}$, and that for any $\mathbf{s} \in \mathcal{T}$ it holds $\sum_{i \in F_2} d_i \left(h(p_i) - h \left(\frac{s_i}{d_i} \right) \right) \leq 2\ell^{1/2} \log^3 \ell$. Taking into consideration that (51) always holds, we conclude that $\sum_{i=1}^m d_i \left(h(p_i) - h \left(\frac{s_i}{d_i} \right) \right) \leq 10\ell^{1/2} \log^3 \ell$ for any $\mathbf{s} \in \mathcal{T}$. Finally, by (57) we also have

$$\mathbb{P}_{\chi \sim \Omega}[\chi \in \mathcal{T}] = \mathbb{P}_{\chi \sim \mathcal{D}}[\chi \in \mathcal{T}] \cdot \theta \geq (1 - \ell^{-\log \ell}) (1 - \ell^{-\log \ell / 3 + 1/2}) \geq 1 - \ell^{-\log \ell / 4},$$

where we used $\log \ell \geq 8$. \square

7.2 Arbitrary alphabet size

In this section we finish the proof of Theorem 5.7 for the general BMS channel using the results from the previous section.

For BMS channels with large output alphabet size we will use binning of the output, however we will do it in a way that *upgrades* the channel, rather than degrades it (recall Definition 4.2). Specifically, we will employ the following statement:

Proposition 7.19. *Let W be any BMS channel. Then there exists another BMS channel \widetilde{W} with the following properties:*

- (i) *Output alphabet size of \widetilde{W} is at most $2\sqrt{\ell}$;*
- (ii) *\widetilde{W} is upgraded with respect to W , i.e. $W \preceq \widetilde{W}$;*

$$(iii) \quad H(\widetilde{W}) \geq H(W) - \frac{\log \ell}{\ell^{1/2}}.$$

Before proving this proposition, we first show how we can finish a proof of Theorem 5.7 using it. So, consider any BMS channel W with output alphabet size larger than $2\sqrt{\ell}$, and consider the channel \widetilde{W} which satisfies properties (i)-(iii) from Proposition 7.19 with respect to W . First of all, notice that $k \geq \ell(1 - H(W)) + 14\ell^{1/2} \log^3 \ell \geq \ell \left(1 - H(\widetilde{W}) - \frac{\log \ell}{\ell^{1/2}}\right) + 14\ell^{1/2} \log^3 \ell$, and thus $k \geq \ell(1 - H(\widetilde{W})) + 13\ell^{1/2} \log^3 \ell$. Taking the property (i) into consideration, it follows that the channel \widetilde{W} satisfies all the conditions for the arguments in the Section 7.1 to be applied, i.e. the statement of Theorem 5.7 holds for \widetilde{W} . Therefore, we can argue that with probability at least $1 - \ell^{-\log \ell/20}$ over a random kernel G it holds $H(V_1 | \widetilde{\mathbf{Y}}) \geq 1 - \ell^{-\log \ell/20}$, where $\widetilde{\mathbf{Y}} = \widetilde{W}^\ell(\mathbf{V} \cdot G)$ is the output vector if one would use the channel \widetilde{W} instead of W , for $\mathbf{V} \sim \{0, 1\}^k$.

Now, let W_1 be the channel which "proves" that \widetilde{W} is upgraded with respect to W , i.e. $W_1(\widetilde{W}(x))$ and $W(x)$ are identically distributed for any $x \in \{0, 1\}$. Trivially then, $W_1^\ell(\widetilde{W}^\ell(X))$ and $W^\ell(X)$ are identically distributed for any random variable X supported on $\{0, 1\}^\ell$.

Next, observe that the following forms a Markov chain

$$V_1 \rightarrow \mathbf{V} \rightarrow \mathbf{V} \cdot G \rightarrow \widetilde{W}^\ell(\mathbf{V}G) \rightarrow W_1^\ell(\widetilde{W}^\ell(\mathbf{V}G)),$$

where \mathbf{V} is distributed uniformly over $\{0, 1\}^k$. But then the data-processing inequality gives

$$I(V_1; W_1^\ell(\widetilde{W}^\ell(\mathbf{V}G))) \leq I(V_1; \widetilde{W}^\ell(\mathbf{V}G)).$$

However, as we discussed above, $W_1^\ell(\widetilde{W}^\ell(\mathbf{V}G))$ and $W^\ell(\mathbf{V}G)$ are identically distributed, and so

$$I(V_1; \mathbf{Y}) = I(V_1; W^\ell(\mathbf{V}G)) = I(V_1; W_1^\ell(\widetilde{W}^\ell(\mathbf{V}G))) \leq I(V_1; \widetilde{W}^\ell(\mathbf{V}G)) = I(V_1; \widetilde{\mathbf{Y}}).$$

Therefore using $H(X|Y) = H(X) - I(X; Y)$ we derive that

$$H(V_1 | \mathbf{Y}) \geq H(V_1 | \widetilde{\mathbf{Y}}) \geq 1 - \ell^{-\log \ell/20}$$

with probability at least $1 - \ell^{-\log \ell/20}$. This concludes the proof of Theorem 5.7. \square

Proof of Proposition 7.19. We are going to describe how to construct such an upgraded channel \widetilde{W} . We again are going to look at W as a convex combination of BSCs, as we discussed in Section 7.1: let W consist of m underlying BSC subchannels $W^{(1)}, W^{(2)} \dots, W^{(m)}$, each has probability q_j to be chosen. The subchannel $W^{(j)}$ has crossover probability p_j , and $0 \leq p_1 \leq \dots \leq p_m \leq \frac{1}{2}$. The subchannel $W^{(j)}$ can output $z_j^{(0)}$ or $z_j^{(1)}$, and the whole output alphabet is then $\mathcal{Y} = \{z_1^{(0)}, z_1^{(1)}, z_2^{(0)}, z_2^{(1)}, \dots, z_m^{(0)}, z_m^{(1)}\}$, $|\mathcal{Y}| = 2m$. It will be convenient to write the transmission probabilities of W explicitly: for any $k \in [m]$, $c, x \in \{0, 1\}$:

$$W(z_k^{(c)} | x) = \begin{cases} q_k \cdot (1 - p_k), & x = c, \\ q_k \cdot p_k, & x \neq c. \end{cases} \quad (60)$$

The key ideas behind the construction of \widetilde{W} are the following:

- decreasing a crossover probability in any BSC (sub)channel always upgrades the channel, i.e. $\text{BSC}_{p_1} \preceq \text{BSC}_{p_2}$ for any $0 \leq p_2 \leq p_1 \leq \frac{1}{2}$ ([TV13, Lemma 9]). Indeed, one can simulate a flip of coin with bias p_1 by first flipping a coin with bias p_2 , and then flipping the result one more time with probability $q = \frac{p_1 - p_2}{1 - 2p_2}$. In other words, $\text{BSC}_{p_1}(x)$ and $\text{BSC}_q(\text{BSC}_{p_2}(x))$ are identically distributed for $x \in \{0, 1\}$.
- "binning" two BSC subchannels with the same crossover probability doesn't change the channel([TV13, Corollary 10]).

Let us finally describe how to construct \widetilde{W} . Split the interval $[0, 1/2]$ into $\sqrt{\ell}$ parts evenly, i.e. let $\theta_j = \frac{j-1}{2\sqrt{\ell}}$ for $j = 1, 2, \dots, \sqrt{\ell} + 1$, and consider intervals $[\theta_j, \theta_{j+1})$ for $j = 1, 2, \dots, \sqrt{\ell}$ (include $1/2$ into the last interval). Now, to get \widetilde{W} , we first slightly decrease the crossover probabilities in all the BSC subchannels $W^{(1)}, W^{(2)}, \dots, W^{(m)}$ so that they all become one of $\theta_1, \theta_2, \dots, \theta_{\sqrt{\ell}}$. After that we bin together the subchannels with the same crossover probabilities and let the resulting channel be \widetilde{W} . Formally, we define

$$T_j := \left\{ i \in [m] : p_i \in [\theta_j, \theta_{j+1}) \right\}, \quad j = 1, 2, \dots, \sqrt{\ell} - 1,$$

$$T_{\sqrt{\ell}} := \left\{ i \in [m] : p_i \in [\theta_{\sqrt{\ell}}, \theta_{\sqrt{\ell}+1}] \right\}.$$

So, T_j is going to be the set of indexes of subchannels of W for which we decrease the crossover probability to be equal to θ_j . Then the probability distribution over the new, binned, BSC subchannels $\widetilde{W}^{(1)}, \widetilde{W}^{(2)}, \dots, \widetilde{W}^{(\sqrt{\ell})}$ in the channel \widetilde{W} is going to be $(\widetilde{q}_1, \widetilde{q}_2, \dots, \widetilde{q}_{\sqrt{\ell}})$, where $\widetilde{q}_j := \sum_{i \in T_j} q_i$. The subchannel $\widetilde{W}^{(j)}$ has crossover probability θ_j , and it can output one of two new symbols $\widetilde{z}_j^{(0)}$ or $\widetilde{z}_j^{(1)}$. The whole output alphabet is then $\widetilde{\mathcal{Y}} = \{\widetilde{z}_1^{(0)}, \widetilde{z}_1^{(1)}, \widetilde{z}_2^{(0)}, \widetilde{z}_2^{(1)}, \dots, \widetilde{z}_{\sqrt{\ell}}^{(0)}, \widetilde{z}_{\sqrt{\ell}}^{(1)}\}$, $|\widetilde{\mathcal{Y}}| = 2\sqrt{\ell}$. To be most specific, we describe $\widetilde{W} : \{0, 1\} \rightarrow \widetilde{\mathcal{Y}}$, as follows: for any $j \in [\sqrt{\ell}]$ and any $b, x \in \{0, 1\}$

$$\widetilde{W} \left(\widetilde{z}_j^{(b)} \mid x \right) = \begin{cases} \sum_{i \in T_j} q_i \cdot (1 - \theta_j), & x = b, \\ \sum_{i \in T_j} q_i \cdot \theta_j, & x \neq b. \end{cases} \quad (61)$$

Property (i) on the output alphabet size for \widetilde{W} then holds immediately. Let us verify (ii) by showing that \widetilde{W} is indeed upgraded with respect to W .

One can imitate the usage of W using \widetilde{W} as follows: on input $x \in \{0, 1\}$, feed it through \widetilde{W} to get output $\widetilde{z}_j^{(b)}$ for some $b \in \{0, 1\}$ and $j \in [\sqrt{\ell}]$. We then know that the subchannel $\widetilde{W}^{(j)}$ was used, which by construction corresponds to the usage of a subchannel $W^{(i)}$ for some $i \in T_j$. Then we randomly choose an index k from T_j with probability of $i \in T_j$ being chosen equal to $\frac{q_i}{q_j}$. This determines that we are going to use the subchannel $W^{(k)}$ while imitating the usage of W . By now we flipped the input with probability θ_j (since we used the subchannel $\widetilde{W}^{(j)}$), while we want it to be flipped with probability $p_k \geq \theta_j$ overall, since we decided to use $W^{(k)}$. So the only thing we need to do it to "flip" b to $(1 - b)$ with probability $\frac{p_k - \theta_j}{1 - 2\theta_j}$, and then output $z_k^{(b)}$ or $z_k^{(1-b)}$ correspondingly.

Formally, we just describe the channel $W_1 : \tilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ which proves that \tilde{W} is upgraded with respect to W by all of its transmission probabilities: for all $k \in [m]$, $j \in [\sqrt{\ell}]$, $b, c \in \{0, 1\}$ set

$$W_1 \left(z_k^c \mid z_j^{(b)} \right) = \begin{cases} 0, & k \notin T_j \\ \frac{q_k}{\sum_{i \in T_j} q_i} \cdot \left(1 - \frac{p_k - \theta_j}{1 - 2\theta_j} \right), & k \in T_j, b = c, \\ \frac{q_k}{\sum_{i \in T_j} q_i} \cdot \left(\frac{p_k - \theta_j}{1 - 2\theta_j} \right), & k \in T_j, b \neq c. \end{cases} \quad (62)$$

It is easy to check that W_1 is a valid channel, and that it holds for any $k \in [m]$ and $c, x \in \{0, 1\}$

$$\sum_{j \in [\sqrt{\ell}], b \in \{0, 1\}} \tilde{W} \left(z_j^{(b)} \mid x \right) W_1 \left(z_k^{(c)} \mid z_j^{(b)} \right) = W \left(z_k^{(c)} \mid x \right), \quad (63)$$

which proves that \tilde{W} is indeed upgraded to W . We prove the above equality in Appendix C.

It only remains to check that the property (iii) also holds, i.e. that the entropy did not decrease too much after we upgrade the channel W to \tilde{W} . We have

$$H(\tilde{W}) = \sum_{j \in [\sqrt{\ell}]} \tilde{q}_j h(\theta_j) = \sum_{j \in [\sqrt{\ell}]} \left(\sum_{i \in T_j} q_i \right) h(\theta_j) = \sum_{k \in [m]} q_k h(\theta_{j_k}),$$

where we again denoted by j_k the index from $[\sqrt{\ell}]$ for which $k \in T_{j_k}$. Therefore

$$H(W) - H(\tilde{W}) = \sum_{k \in [m]} q_k (h(p_k) - h(\theta_{j_k})) \leq \sum_{k \in [m]} q_k (h(\theta_{j_k+1}) - h(\theta_{j_k})),$$

since $p_k \in [\theta_{j_k}, \theta_{j_k+1}]$ as $k \in T_{j_k}$. Finally, since $\theta_{j+1} - \theta_j = \frac{1}{2\sqrt{\ell}}$, Proposition 4.1 gives

$$H(W) - H(\tilde{W}) \leq \sum_{k \in [m]} q_k (h(\theta_{j_k+1}) - h(\theta_{j_k})) \leq h \left(\frac{1}{2\sqrt{\ell}} \right) \leq 2 \cdot \frac{1}{2\sqrt{\ell}} \log(2\sqrt{\ell}) \leq \frac{\log \ell}{\sqrt{\ell}}. \quad \square$$

8 Suction at the ends

In this section we present the proof for Theorem 5.1 in the case the standard Arikans kernel was chosen in Algorithm A – the so-called suction at the ends regime. Recall that, as we discussed in section 5.1, this regime applies when the entropy of the channel W falls into the interval $(\ell^{-4}, 1 - \ell^{-4})$, and the algorithm directly takes a kernel $K = A_2^{\otimes \log \ell}$, where $A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is the kernel of Arikan's original polarizing transform, instead of trying out all the possible matrices. Note that multiplying by such a kernel K is equivalent to just applying the Arikan's 2×2 transform recursively $\log \ell$ times. Suppose we have a BMS channel W with $H(W)$ very close to 0 or 1. For Arikan's basic transform, by working with the channel Bhattacharyya parameter $Z(W)$ instead of the entropy $H(W)$, it is well known that one of the two Arikan bit-channels has Z value gets much closer (quadratically closer) to the boundary of the interval $(0, 1)$ [ARI09, KOR09]. Using these ideas, we prove in this section that basic transform decreases the average of the function $g_\alpha(\cdot)$ of entropy at least by a factor of $\ell^{-1/2}$ after $\log \ell$ iterations for large enough ℓ .

The basic Arıkan's transform takes one channel W and splits it into a slightly worse channel W^- and a slightly better channel W^+ . Then the transform is applied recursively to W^- and W^+ , creating channels $W^{--}, W^{-+}, W^{+-},$ and W^{++} . One can think of the process as of a complete binary tree of depth $\log \ell$, with the root node W , and any node at the level i is of form W^{B_i} for some $B_i \in \{-, +\}^i$, with two children W^{B_i-} and W^{B_i+} . Denote $r = \log \ell$, then the channels at the leaves $\{W^{B_r}\}$, for all $B_r \in \{-, +\}^r$ are exactly the Arıkan's subchannels of W with respect to the kernel $K = A_2^{\otimes \log \ell}$. We are going to prove the following result

Lemma 8.1. *Let W be a BMS channel with $H(W) \notin (\ell^{-4}, 1 - \ell^{-4})$. Denote $r = \log \ell$, then for $r \geq \frac{1}{\alpha}$*

$$\sum_{B \in \{-, +\}^r} g_\alpha \left(H \left(W^B \right) \right) \leq \ell^{1/2} g_\alpha (H(W)). \quad (64)$$

Clearly, the above lemma will imply the suction at the end case of Theorem 5.1, as we take $\log \ell \geq \frac{1}{\alpha}$.

For the analysis below, apart from the entropy of the channel, we will also use Bhattacharyya parameter $Z(W)$:

$$Z(W) = \sum_{y \in \mathcal{Y}} \sqrt{W(y|0)W(y|1)},$$

together with the inequalities which connect it to the entropy:

$$Z(W)^2 \leq H(W) \leq Z(W), \quad (65)$$

for any BMS channel W ([Kor09, Ari10]). The reason we use this parameter is because of the following relations, which show how the Bhattacharyya parameter changes after the basic transform ([Ari09, RU08, Kor09, HAU14]):

$$Z(W^+) = Z(W)^2, \quad (66)$$

$$Z(W) \sqrt{2 - Z(W)^2} \leq Z(W^-) \leq 2Z(W). \quad (67)$$

We will also use the conservation of conditional entropy on application of Arıkan's transform

$$H(W^+) + H(W^-) = 2H(W). \quad (68)$$

Proof of Lemma 8.1. The proof is presented in the next two sections, as it is divided into two parts: the case when $H(W) \leq \ell^{-4}$ (suction at the lower end), and when $H(W) \geq 1 - \ell^{-4}$ (suction at the upper end).

8.1 Suction at the lower end

Suppose $H(W) \leq \ell^{-4}$ for this case, thus $Z(W) \leq \ell^{-2} \leq 2^{-2r}$.

First, recursive application of (68) gives

$$\sum_{B \in \{-, +\}^r} H \left(W^B \right) = 2^r H(W), \quad (69)$$

and since entropy is always nonnegative, this implies for any $B \in \{-, +\}^r$

$$H \left(W^B \right) \leq 2^r H(W). \quad (70)$$

Denote now $k = \lceil \log \frac{1}{\alpha} \rceil$, and notice that $\log r \geq k - 1$ since $r \geq \frac{1}{\alpha}$. For $B \in \{-, +\}^r$, define $wt_+(B)$ to be number of $+$'s in B . We will split the summation in (64) into two parts: the part with $wt_+(B) < k$, and when $wt_+(B) \geq k$.

First part. Out of (70) derive

$$\sum_{wt_+(B) < k} g_\alpha \left(H \left(W^B \right) \right) \leq \sum_{j=0}^{k-1} \binom{r}{j} g_\alpha \left(2^r H(W) \right) \leq \log r \cdot \binom{r}{\log r} \cdot 2^{r\alpha} H(W)^\alpha \leq 2^{\log^2 r + r\alpha} \cdot H(W)^\alpha, \quad (71)$$

where we used $\binom{r}{\log r} \leq \frac{r^{\log r}}{(\log r)!}$; the fact the g_α is increasing on $(0, \frac{1}{2})$ together with $2^r H(W) \leq \ell^{-3} < \frac{1}{2}$, and that $g_\alpha(x) \leq x^\alpha$ for $x \in (0, 1)$.

Second part. We are going to use the following observation, which can be proved by induction based on (66) and (67):

Claim 8.2. *Let $B \in \{-, +\}^r$, such that number of $+$'s in B is equal to s . Then*

$$Z \left(W^B \right) \leq \left(2^{r-s} \cdot Z(W) \right)^{2^s}.$$

*This corresponds to first using equality (67) $(r - s)$ times, and after that using bound (66) s times while walking **down** the recursive binary tree of channels.*

Then, using Claim 8.2 along with (65) and the fact that $Z(W) \leq \ell^{-2} \leq 2^{-2r}$, we obtain the following for any $B \in \{-, +\}^r$ with $wt_+(B) = s \geq k$:

$$\begin{aligned} H \left(W^B \right) &\leq Z \left(W^B \right) \leq \left(2^{r-s} \cdot Z(W) \right)^{2^s} \leq 2^{(r-s)2^s} \cdot Z(W)^{2^s-2} \cdot H(W) \\ &\leq 2^{(r-s)2^s-2t2^s+4r} \cdot H(W) \\ &= 2^{-r2^s-s2^s+4r} \cdot H(W) \\ &\leq 2^{-r2^k-k2^k+4r} \cdot H(W). \end{aligned}$$

Therefore

$$\sum_{wt_+(B) \geq k} g_\alpha \left(H \left(W^B \right) \right) \leq \sum_{wt_+(B) \geq k} H \left(W^B \right)^\alpha \leq 2^r \cdot 2^{\alpha(-r2^k-k2^k+4r)} \cdot H(W)^\alpha.$$

Observe now the following chain of inequalities

$$\frac{r}{2} + 4r\alpha + 2 \leq r \leq r \cdot 2^k \alpha \leq r \cdot 2^k \alpha + k \cdot 2^k \alpha,$$

which trivially holds for $\alpha \leq \frac{1}{12}$. Therefore

$$r + \alpha(-r2^k - k2^k + 4r) \leq \frac{r}{2} - 2,$$

and thus in (72) obtain

$$\sum_{wt_+(B) \geq k} g_\alpha \left(H \left(W^B \right) \right) \leq 2^{r/2-2} \cdot H(W)^\alpha. \quad (72)$$

Overall bound. Combining (71) and (72) we derive

$$\begin{aligned} \sum_{B \in \{-,+\}^r} g_\alpha \left(H \left(W^B \right) \right) &\leq \left(2^{\log^2 r + r\alpha} + 2^{r/2-2} \right) \cdot H(W)^\alpha \\ &\leq 2^{r/2} \cdot \frac{H(W)^\alpha}{2} \\ &\leq \ell^{1/2} g_\alpha(H(W)), \end{aligned}$$

where we used $\log^2 r + r\alpha \leq \frac{r}{2} - 2$ for large enough r , and $\frac{1}{2} \leq (1-x)^\alpha$ for any $x \leq \frac{1}{2}$. This proves Lemma 8.1 for the lower end case $H(W) \leq \ell^{-4}$.

8.2 Suction at the upper end

Now consider the case $H(W) \geq 1 - \ell^{-4}$. The proof is going to be quite similar to the previous case, but we are going to track the distance from $H(W)$ (and $Z(W)$) to 1 now. Specifically, denote

$$\begin{aligned} I(W) &= 1 - H(W), \\ S(W) &= 1 - Z(W), \end{aligned}$$

where $I(W)$ is actually the (symmetric) capacity of the channel, and $S(W)$ is just a notation we use in this proof. Notice that $g_\alpha(x) = g_\alpha(1-x)$, therefore it suffices to prove (64) with capacities of the channels instead of entropies in the inequality. Also notice that $I(W) \leq \ell^{-4}$ for the current case of suction at the upper end.

Let us now derive the relations between $I(W)$, $S(W)$, as well as evolution of $S(\cdot)$ for W^+ and W^- , similar to (65), (66), (67), and (68). Inequalities in (65) imply

$$\begin{aligned} S(W) = 1 - Z(W) &\leq 1 - H(W) = I(W), \\ I(W) = 1 - H(W) &\leq 1 - Z(W)^2 \leq 2(1 - Z(W)) = 2S(W), \end{aligned}$$

so let us combine this to write

$$S(W) \leq I(W) \leq 2S(W). \quad (73)$$

Next, (66) and (67) give

$$S(W^+) = 1 - Z(W)^2 \leq 2(1 - Z(W)) \leq 2S(W), \quad (74)$$

$$S(W^-) \leq 1 - Z(W)\sqrt{2 - Z(W)^2} \leq 2(1 - Z(W))^2 = 2S(W)^2, \quad (75)$$

where we used $1 - x\sqrt{2-x^2} \leq 2(1-x)^2$ for any $x \in (0, 1)$.

Finally, it easily follows from (69) that

$$\sum_{B \in \{-,+\}^r} I \left(W^B \right) = 2^r I(W),$$

and since capacity is nonnegative as well, we also obtain for any $B \in \{-,+\}^r$

$$I \left(W^B \right) \leq 2^r I(W). \quad (76)$$

We now proceed with a very similar approach to the suction at the lower end case in Section 8.1: denote $k = \lceil \log \frac{1}{\alpha} \rceil$, and notice that $\log r \geq k - 1$ since $r \geq \frac{1}{\alpha}$. For $B \in \{-,+\}^r$, define $wt_-(B)$

to be number of $-$'s in B . We will split the summation in (64) (but with capacities of channels instead of entropies) into two parts: the part with $wt_-(B) < k$, and when $wt_-(B) \geq k$.

First part. Out of (76) derive, similarly to (77)

$$\sum_{wt_-(B) < k} g_\alpha \left(I \left(W^B \right) \right) \leq \sum_{j=0}^{k-1} \binom{r}{j} g_\alpha \left(2^j I(W) \right) \leq \log r \cdot \binom{r}{\log r} \cdot 2^{r \alpha} I(W)^\alpha \leq 2^{\log^2 r + r \alpha} \cdot I(W)^\alpha. \quad (77)$$

Second part. Similarly to Claim 8.2, one can show via induction using (74) and (75) the following

Claim 8.3. *Let $B \in \{-, +\}^r$, such that number of $-$'s in B is equal to s . Then*

$$S \left(W^B \right) \leq 2^{2^s - 1} \left(2^{r-s} \cdot S(W) \right)^{2^s}.$$

*This corresponds to first using equality (74) $(r-s)$ times, and after that using bound (75) s times while walking **down** the recursive binary tree of channels.*

Using this claim with (73) and the fact that $S(W) \leq Z(W) \leq \ell^{-4} \leq 2^{-4r}$ obtain for any $B \in \{-, +\}^r$ with $wt_-(B) = s \geq k$

$$\begin{aligned} I \left(W^B \right) &\leq 2S \left(W^B \right) \leq 2^{2^s} \cdot \left(2^{r-s} \cdot S(W) \right)^{2^s} && \leq 2^{(r-s+1)2^s} \cdot S(W)^{2^s-1} \cdot I(W) \\ &\leq 2^{(r-s+1)2^s - 4r} \cdot I(W) && = 2^{-2^s(3r+s-1)+4r} \cdot I(W) \\ &\leq 2^{-2^k(3r+k-1)+4r} \cdot I(W) && \leq 2^{-r2^k} \cdot I(W), \end{aligned}$$

where the last inequality uses $4r \leq 2^k(2t+k-1)$, which holds trivially for $k \geq 1$. Therefore

$$\sum_{wt_-(B) \geq k} g_\alpha \left(I \left(W^B \right) \right) \leq \sum_{wt_-(B) \geq k} I \left(W^B \right)^\alpha \leq 2^r \cdot 2^{-\alpha r 2^k} \cdot I(W)^\alpha \leq I(W)^\alpha, \quad (78)$$

since $\alpha \cdot 2^k \geq 1$ by the choice of k .

Overall bound. The bounds (77) and (78) give us

$$\sum_{B \in \{-, +\}^r} g_\alpha \left(H \left(W^B \right) \right) = \sum_{B \in \{-, +\}^r} g_\alpha \left(I \left(W^B \right) \right) \leq \left(2^{\log^2 r + r \alpha} + 1 \right) \cdot I(W)^\alpha \leq \ell^{1/2} g_\alpha(H(W))$$

for large enough r when $H(W) \geq 1 - \ell^{-4}$. This completes the proof of Lemma 8.1. \square

9 Code construction, encoding and decoding procedures

Before presenting our code construction and encoding/decoding procedures, we first distinguish the difference between the code construction and the encoding procedure. The objectives of code construction for polar-type codes are two-fold: First, find the $N \times N$ encoding matrix; second, find the set of noiseless bits under the successive decoder, which will carry the message bits. On the other hand, by encoding we simply mean the procedure of obtaining the codeword $\mathbf{X}_{[1:N]}$ by multiplying the information vector $\mathbf{U}_{[1:N]}$ with the encoding matrix, where we only put information in the noiseless bits in $\mathbf{U}_{[1:N]}$ and set all the frozen bits to be 0. As we will see at the end of this

Algorithm B: Degraded binning algorithm

Input: $W : \{0, 1\} \rightarrow \mathcal{Y}$, bound Q on the output alphabet size after binning

Output: $\widetilde{W} : \{0, 1\} \rightarrow \widetilde{\mathcal{Y}}$, where $|\widetilde{\mathcal{Y}}| \leq Q$

```

1 Initialize the new channel  $\widetilde{W}$  with output symbols  $\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_Q$  by setting  $\widetilde{W}(\tilde{y}_i|x) = 0$  for
  all  $i \in [Q]$  and  $x \in \{0, 1\}$ 
2 for  $y \in \mathcal{Y}$  do
3    $p(0|y) \leftarrow \frac{W(y|0)}{W(y|0)+W(y|1)}$ 
4    $i \leftarrow \lceil Q \cdot p(0|y) \rceil$ 
5   if  $i = 0$  then
6      $i \leftarrow 1$  //  $i = 0$  if and only if  $p(0|y) = 0$ ; we merge this single point into the next bin
7   end
8    $\widetilde{W}(\tilde{y}_i|0) \leftarrow \widetilde{W}(\tilde{y}_i|0) + W(y|0)$ 
9    $\widetilde{W}(\tilde{y}_i|1) \leftarrow \widetilde{W}(\tilde{y}_i|1) + W(y|1)$ 
10 end
11 return  $\widetilde{W}$ 

```

section, while the code construction has complexity polynomial in N , the encoding procedure only has complexity $O_\ell(N \log N)$.

For polar codes with a fixed invertible kernel $K \in \{0, 1\}^{\ell \times \ell}$, the polarization process works as follows: We start with some BMS channel W . After applying the polar transform to W using kernel K , we obtain ℓ bit-channels $\{W_i : i \in [\ell]\}$ as defined in (1). Next we apply the polar transform using kernel K to each of these ℓ bit-channels, and we write the polar transform of W_i as $\{W_{ij} : j \in [\ell]\}$. Then we apply the polar transform to each of the ℓ^2 bit channels $\{W_{i_1, i_2} : i_1, i_2 \in [\ell]\}$ and obtain $\{W_{i_1, i_2, i_3} : i_1, i_2, i_3 \in [\ell]\}$, so on and so forth. After t rounds of polar transforms, we obtain ℓ^t bit-channels $\{W_{i_1, \dots, i_t} : i_1, \dots, i_t \in [\ell]\}$, and one can show that these are the bit-channels seen by the successive decoder when decoding the corresponding polar codes constructed from kernel K .

For our purpose, we need to use polar codes with mixed kernels, and we need to search for a “good” kernel at each step of polarization. We will also introduce new notation for the bit-channels in order to indicate the usage of different kernels for different bit-channels. As mentioned in Section 2.7, we need to use a binning algorithm (Algorithm B) to quantize all the bit-channels we obtain in the code construction procedure. As long as we choose the parameter Q in Algorithm B to be a large enough polynomial of N , the quantized channel can be used as a very good approximation of the original channel. This is made precise by [GX15, Proposition 13]: For W and \widetilde{W} in Algorithm B, we have⁵

$$H(W) \leq H(\widetilde{W}) \leq H(W) + \frac{2 \log Q}{Q}. \quad (79)$$

Given a BMS channel W , our code construction works as follows:

1. **Step 0:** We first use Algorithm B to quantize/bin the output alphabet of W such that the resulting (degraded) channel has at most N^3 outputs, i.e., we set $Q = N^3$ in Algorithm B. Note that the parameter Q can be chosen as any polynomial of N . By changing the value of Q , we obtain a tradeoff between the decoding error probability and the gap to capacity;

⁵Note that the binning algorithm (Algorithm 2) in [GX15] has one minor difference from the binning algorithm (Algorithm B) in this paper: In [GX15], the binning algorithm outputs a channel with $Q + 1$ outputs in contrast to Q outputs in this paper. More precisely, line 5-7 in Algorithm B of this paper is not included in the algorithm in [GX15], but one can easily check that this minor difference does not affect the proof at all.

see Theorem 9.6 at the end of this section. Here we choose the special case of $Q = N^3$ to give a concrete example of code construction. Next we use Algorithm A in Section 5 to find a good kernel⁶ for the quantized channel and denote it as $K_1^{(0)}$. Recall from Section 2.4 that a kernel is good if all but a $\tilde{O}(\ell^{-1/2})$ fraction of the bit-channels obtained after polar transform by this kernel have entropy $\ell^{-\Omega(\log \ell)}$ -close to either 0 or 1. The superscript (0) in $K_1^{(0)}$ indicates that this is the kernel used in Step 0 of polarization. In this case, we use $\{W_i(B, K_1^{(0)}) : i \in [\ell]\}$ to denote the ℓ bit-channels resulting from the polar transform of the quantized version of W using kernel $K_1^{(0)}$. Here B stands for the binning operation, and the arguments in the brackets are the operations to obtain the bit-channel $W_i(B, K_1^{(0)})$ from W : first bin the outputs of W and then perform the polar transform using kernel $K_1^{(0)}$. For each $i \in [\ell]$, we again use Algorithm B to quantize/bin the output alphabet of $W_i(B, K_1^{(0)})$ such that the resulting (degraded) bit-channel $W_i(B, K_1^{(0)}, B)$ has at most N^3 outputs.

2. **Step 1:** For each $i_1 \in [\ell]$, we use Algorithm A to find a good kernel for the quantized bit-channel $W_{i_1}(B, K_1^{(0)}, B)$ and denote it as $K_{i_1}^{(1)}$. The ℓ bit-channels resulting from the polar transform of $W_{i_1}(B, K_1^{(0)}, B)$ using kernel $K_{i_1}^{(1)}$ are denoted as $\{W_{i_1, i_2}(B, K_1^{(0)}, B, K_{i_1}^{(1)}) : i_2 \in [\ell]\}$. In this step, we will obtain ℓ^2 bit-channels $\{W_{i_1, i_2}(B, K_1^{(0)}, B, K_{i_1}^{(1)}) : i_1, i_2 \in [\ell]\}$. For each of them, we use Algorithm B to quantize/bin its output alphabet such that the resulting (degraded) bit-channels $\{W_{i_1, i_2}(B, K_1^{(0)}, B, K_{i_1}^{(1)}, B) : i_1, i_2 \in [\ell]\}$ has at most N^3 outputs. See Fig. 2 for an illustration of this procedure for the special case of $\ell = 3$.

3. We repeat the polar transforms and binning operations at each step of the code construction. More precisely, at **Step j** we have ℓ^j bit-channels

$$\{W_{i_1, i_2, \dots, i_j}(B, K_1^{(0)}, B, K_{i_1}^{(1)}, B, \dots, K_{i_1, \dots, i_{j-1}}^{(j-1)}, B) : i_1, i_2, \dots, i_j \in [\ell]\}.$$

This notation is a bit messy, so we introduce some simplified notation for the bit-channels obtained with and without binning operations: We still use

$$W_{i_1, i_2, \dots, i_j}(K_1^{(0)}, K_{i_1}^{(1)}, \dots, K_{i_1, \dots, i_{j-1}}^{(j-1)})$$

to denote the bit-channel obtained without the binning operations at all, and we use

$$W_{i_1, i_2, \dots, i_j}^{\text{bin}}(K_1^{(0)}, K_{i_1}^{(1)}, \dots, K_{i_1, \dots, i_{j-1}}^{(j-1)})$$

to denote the bit-channel obtained with binning operations performed at every step from Step 0 to Step $j - 1$, i.e.,

$$W_{i_1, i_2, \dots, i_j}^{\text{bin}}(K_1^{(0)}, K_{i_1}^{(1)}, \dots, K_{i_1, \dots, i_{j-1}}^{(j-1)}) := W_{i_1, i_2, \dots, i_j}(B, K_1^{(0)}, B, K_{i_1}^{(1)}, B, \dots, K_{i_1, \dots, i_{j-1}}^{(j-1)}, B).$$

Moreover, we use $W_{i_1, i_2, \dots, i_j}^{\text{bin}*}(K_1^{(0)}, K_{i_1}^{(1)}, \dots, K_{i_1, \dots, i_{j-1}}^{(j-1)})$ to denote the bit-channel obtained with binning operations performed at every step except for the last step, i.e.,

$$W_{i_1, i_2, \dots, i_j}^{\text{bin}*}(K_1^{(0)}, K_{i_1}^{(1)}, \dots, K_{i_1, \dots, i_{j-1}}^{(j-1)}) := W_{i_1, i_2, \dots, i_j}(B, K_1^{(0)}, B, K_{i_1}^{(1)}, B, \dots, B, K_{i_1, \dots, i_{j-1}}^{(j-1)}).$$

⁶We will prove in Proposition 9.3 that the error parameter Δ in Algorithm A can be chosen as $\Delta = \frac{6\ell \log N}{N^2}$ when we set $Q = N^3$.

Next we use Algorithm A to find a good kernel for each of them and denote the kernel as $K_{i_1, \dots, i_j}^{(j)}$. After applying polar transforms using these kernels, we obtain ℓ^{j+1} bit-channels

$$\{W_{i_1, \dots, i_{j+1}}^{\text{bin}*}(K_1^{(0)}, K_{i_1}^{(1)}, \dots, K_{i_1, \dots, i_j}^{(j)}) : i_1, \dots, i_{j+1} \in [\ell]\}.$$

Then we quantize/bin the output alphabets of these bit-channels using Algorithm B and obtain the following ℓ^{j+1} quantized bit-channels

$$\{W_{i_1, \dots, i_{j+1}}^{\text{bin}}(K_1^{(0)}, K_{i_1}^{(1)}, \dots, K_{i_1, \dots, i_j}^{(j)}) : i_1, \dots, i_{j+1} \in [\ell]\}.$$

4. After **step** $t - 1$, we obtain $N = \ell^t$ quantized bit-channels

$$\{W_{i_1, \dots, i_t}^{\text{bin}}(K_1^{(0)}, K_{i_1}^{(1)}, \dots, K_{i_1, \dots, i_{t-1}}^{(t-1)}) : i_1, i_2, \dots, i_t \in [\ell]\},$$

and we have also obtained all the kernels in each step of polarization. More precisely, we have ℓ^i kernels in step i , so from step 0 to step $t - 1$, we have $1 + \ell + \dots + \ell^{t-1} = \frac{N-1}{\ell-1}$ kernels in total.

5. Find the set of good (noiseless) indices. More precisely, we use the shorthand notation⁷

$$\begin{aligned} H_{i_1, \dots, i_t}(W) &:= H(W_{i_1, \dots, i_t}(K_1^{(0)}, K_{i_1}^{(1)}, \dots, K_{i_1, \dots, i_{t-1}}^{(t-1)})) \\ H_{i_1, \dots, i_t}^{\text{bin}}(W) &:= H(W_{i_1, \dots, i_t}^{\text{bin}}(K_1^{(0)}, K_{i_1}^{(1)}, \dots, K_{i_1, \dots, i_{t-1}}^{(t-1)})) \end{aligned} \quad (80)$$

and define the set of good indices as

$$\mathcal{S}_{\text{good}} := \left\{ (i_1, i_2, \dots, i_t) \in [\ell]^t : H_{i_1, \dots, i_t}^{\text{bin}}(W) \leq \frac{7\ell \log N}{N^2} \right\}. \quad (81)$$

6. Finally, we need to construct the encoding matrix from these $\frac{N-1}{\ell-1}$ kernels. The kernels we obtained in step j are

$$\{K_{i_1, \dots, i_j}^{(j)} : i_1, \dots, i_j \in [\ell]\}.$$

For an integer $i \in [\ell^j]$, we write the j -digit ℓ -ary expansion of $i - 1$ as $(\tilde{i}_1, \tilde{i}_2, \dots, \tilde{i}_j)$, where \tilde{i}_j is the least significant digit and \tilde{i}_1 is the most significant digit, and each digit takes value in $\{0, 1, \dots, \ell - 1\}$. Let $(i_1, i_2, \dots, i_j) := (\tilde{i}_1 + 1, \tilde{i}_2 + 1, \dots, \tilde{i}_j + 1)$, and define the mapping $\tau_j : [\ell^j] \rightarrow [\ell]^j$ as

$$\tau_j(i) := (i_1, i_2, \dots, i_j) \quad \text{for } i \in [\ell^j]. \quad (82)$$

This is a one-to-one mapping between $[\ell^j]$ and $[\ell]^j$, and we use the shorthand notation $K_i^{(j)}$ to denote $K_{\tau_j(i)}^{(j)}$ for $i \in [\ell^j]$. For each $j \in \{0, 1, \dots, t - 1\}$, we define the block diagonal matrices $\overline{D}^{(j)}$ with size $\ell^{j+1} \times \ell^{j+1}$ and $D^{(j)}$ with size $N \times N$ as

$$\overline{D}^{(j)} := \text{Diag}(K_1^{(j)}, K_2^{(j)}, \dots, K_{\ell^j}^{(j)}), \quad D^{(j)} := \underbrace{\{\overline{D}^{(j)}, \overline{D}^{(j)}, \dots, \overline{D}^{(j)}\}}_{\text{number of } \overline{D}^{(j)} \text{ is } \ell^{t-j-1}}. \quad (83)$$

For $i \in [\ell^t]$, we have $\tau_t(i) = (i_1, \dots, i_t)$. For $j \in [t - 1]$, we define the permutation $\pi^{(j)}$ on the set $[\ell^t]$ as

$$\pi^{(j)}(i) := \tau_t^{-1}(i_1, \dots, i_{t-j-1}, i_t, i_{t-j}, i_{t-j+1}, \dots, i_{t-1}) \quad \forall i \in [\ell^t]. \quad (84)$$

⁷We omit the reference to the kernels in the notation $H_{i_1, \dots, i_t}(W)$ and $H_{i_1, \dots, i_t}^{\text{bin}}(W)$.

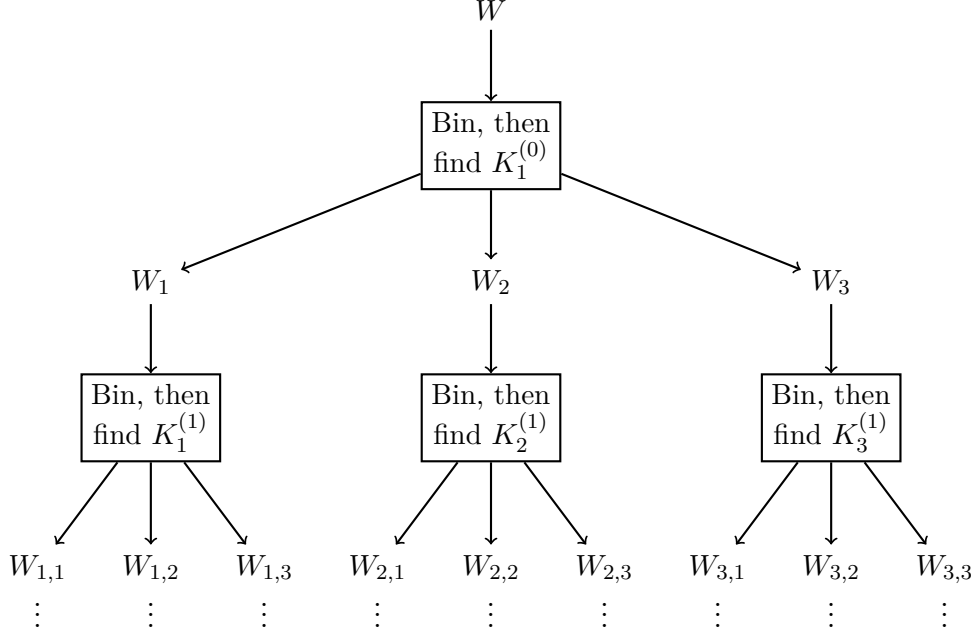


Figure 2: Illustration of code construction for the special case of $\ell = 3$.

By this definition, $\pi^{(j)}$ simply keeps the first $t - j - 1$ digits of i to be the same and performs a cyclic shift on the last $j + 1$ digits. Here we give some concrete examples:

$$\begin{aligned}\pi^{(1)}(i) &= \tau_t^{-1}(i_1, \dots, i_{t-2}, i_t, i_{t-1}), \\ \pi^{(2)}(i) &= \tau_t^{-1}(i_1, \dots, i_{t-3}, i_t, i_{t-2}, i_{t-1}), \\ \pi^{(3)}(i) &= \tau_t^{-1}(i_1, \dots, i_{t-4}, i_t, i_{t-3}, i_{t-2}, i_{t-1}), \\ \pi^{(t-1)}(i) &= \tau_t^{-1}(i_t, i_1, i_2, \dots, i_{t-1}).\end{aligned}$$

For each $j \in [t-1]$, let $Q^{(j)}$ be the $\ell^t \times \ell^t$ permutation matrix corresponding to the permutation $\pi^{(j)}$, i.e., $Q^{(j)}$ is the permutation matrix such that

$$(U_1, U_2, \dots, U_{\ell^t})Q^{(j)} = (U_{\pi^{(j)}(1)}, U_{\pi^{(j)}(2)}, \dots, U_{\pi^{(j)}(\ell^t)}). \quad (85)$$

Finally, for each $j \in [t]$, we define the $N \times N$ matrix

$$M^{(j)} := D^{(j-1)}Q^{(j-1)}D^{(j-2)}Q^{(j-2)} \dots D^{(1)}Q^{(1)}D^{(0)}. \quad (86)$$

Therefore, $M^{(j)}$, $j \in [t]$ satisfy the following recursive relation:

$$M^{(1)} = D^{(0)}, \quad M^{(j+1)} = D^{(j)}Q^{(j)}M^{(j)}.$$

Our encoding matrix for code length $N = \ell^t$ is the submatrix of $M^{(t)}$ consisting of all the row vectors with indices belonging to the set $\mathcal{S}_{\text{good}}$ defined in (81); see the next paragraph for a detailed description of the encoding procedure.

Once we obtain the matrix $M^{(t)}$ and the set $\mathcal{S}_{\text{good}}$ in the code construction, the encoding procedure is standard; it is essentially the same as the original polar codes [Ari09]. Let $\mathbf{U}_{[1:N]}$ be a

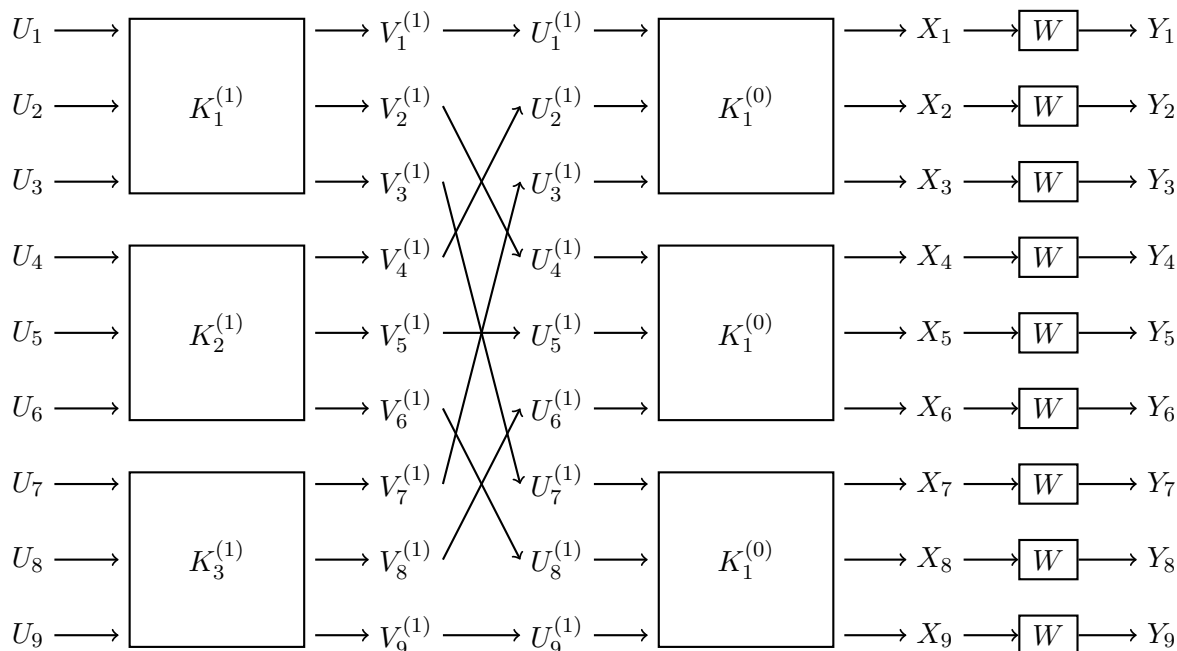


Figure 3: Illustration of the encoding process $\mathbf{X}_{[1:N]} = \mathbf{U}_{[1:N]}M^{(t)}$ for the special case of $\ell = 3$ and $t = 2$. Here $\mathbf{X}_{[1:N]}$ and $\mathbf{U}_{[1:N]}$ are row vectors. All four kernels in this figure $K_1^{(0)}, K_1^{(1)}, K_2^{(1)}, K_3^{(1)}$ have size 3×3 , and the outputs of each kernel is obtained by multiplying the inputs with the kernel, e.g. $\mathbf{V}_{[1:3]}^{(1)} = \mathbf{U}_{[1:3]}K_1^{(1)}$.

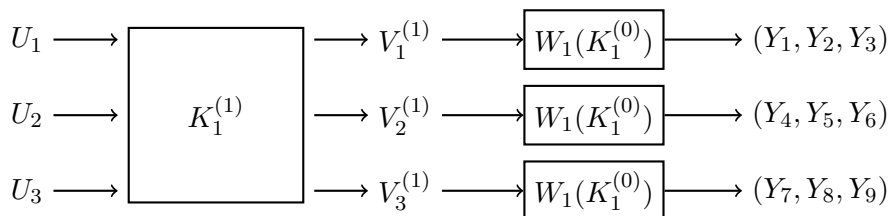


Figure 4: The (stochastic) mapping from $\mathbf{U}_{[1:3]}$ to $\mathbf{Y}_{[1:9]}$

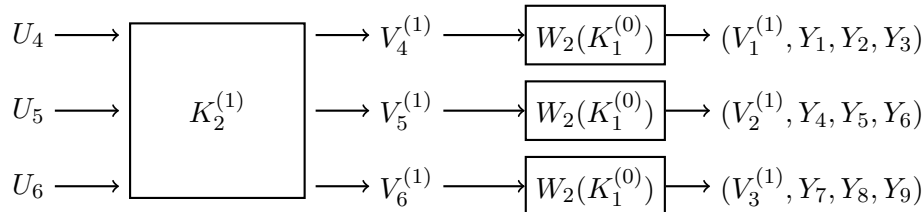


Figure 5: The (stochastic) mapping from $\mathbf{U}_{[4:6]}$ to $(\mathbf{V}_{[1:3]}^{(1)}, \mathbf{Y}_{[1:9]})$

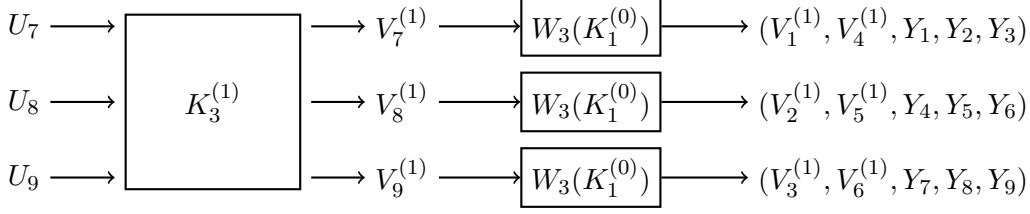


Figure 6: The (stochastic) mapping from $\mathbf{U}_{[7:9]}$ to $(\mathbf{V}_{[1:6]}^{(1)}, \mathbf{Y}_{[1:9]})$

random vector consisting of N i.i.d. Bernoulli-1/2 random variables, and let $\mathbf{X}_{[1:N]} = \mathbf{U}_{[1:N]}M^{(t)}$. Recall that we use $\{W_i(M^{(t)}) : i \in [\ell^t]\}$ to denote the ℓ^t bit-channels resulting from the polar transform of W using kernel $M^{(t)}$. If we transmit the random vector $\mathbf{X}_{[1:N]}$ through N independent copies of W and denote the channel outputs as $\mathbf{Y}_{[1:N]}$, then by definition, the bit-channel mapping from U_i to $(\mathbf{U}_{[1:i-1]}, \mathbf{Y}_{[1:N]})$ is exactly $W_i(M^{(t)})$. Therefore, if we use a successive decoder to decode the input vector $\mathbf{U}_{[1:N]}$ bit by bit from all the channel outputs $\mathbf{Y}_{[1:N]}$ and all the previous input bits $\mathbf{U}_{[1:i-1]}$, then $W_i(M^{(t)})$ is the channel seen by the successive decoder when it decodes U_i . Clearly, $H(W_i(M^{(t)})) \approx 0$ means that the successive decoder can decode U_i correctly with high probability. For every $i \in \ell^t$, we write $\tau_t(i) = (i_1, i_2, \dots, i_t)$. In Proposition 9.1 below, we will show that $H(W_i(M^{(t)})) = H_{i_1, \dots, i_t}(W)$. Then in Proposition 9.3, we further show that $H_{i_1, \dots, i_t}(W) \approx H_{i_1, \dots, i_t}^{\text{bin}}(W)$. Therefore, $H(W_i(M^{(t)})) \approx H_{i_1, \dots, i_t}^{\text{bin}}(W)$. By definition (81), the set $\mathcal{S}_{\text{good}}$ contains all the indices (i_1, \dots, i_t) for which $H_{i_1, \dots, i_t}^{\text{bin}}(W) \approx 0$, so for all i such that $\tau_t(i) \in \mathcal{S}_{\text{good}}$, we also have $H(W_i(M^{(t)})) \approx 0$, meaning that the successive decoder can decode all the bits $\{U_i : \tau_t(i) \in \mathcal{S}_{\text{good}}\}$ correctly with high probability. In the encoding procedure, we put all the information in the set of good bits $\{U_i : \tau_t(i) \in \mathcal{S}_{\text{good}}\}$, and we set all the other bits to be some pre-determined value, e.g., set all of them to be 0. It is clear that the generator matrix of this code is the submatrix of $M^{(t)}$ consisting of all the row vectors with indices belonging to the set $\mathcal{S}_{\text{good}}$.

9.1 Analysis of bit-channels

We say that two channels $W_1 : \{0, 1\} \rightarrow \mathcal{Y}_1$ and $W_2 : \{0, 1\} \rightarrow \mathcal{Y}_2$ are equivalent if there is a one-to-one mapping π between \mathcal{Y}_1 and \mathcal{Y}_2 such that $W_1(y_1|x) = W_2(\pi(y_1)|x)$ for all $y_1 \in \mathcal{Y}_1$ and $x \in \{0, 1\}$. Denote this equivalence relation as $W_1 \equiv W_2$. Then we have the following result.

Proposition 9.1. *For every $i \in \ell^t$, we write $\tau_t(i) = (i_1, i_2, \dots, i_t)$. Then we always have*

$$W_i(M^{(t)}) \equiv W_{i_1, \dots, i_t}(K_1^{(0)}, K_{i_1}^{(1)}, \dots, K_{i_1, \dots, i_{t-1}}^{(t-1)}).$$

Before formally proving this proposition, we first use the special case of $t = 2$ and $\ell = 3$ to illustrate the main idea behind the proof. In this case, we obtained one kernel $K_1^{(0)}$ in step 0 and three kernels $K_1^{(1)}, K_2^{(1)}, K_3^{(1)}$ in step 1. See Fig. 3 for an illustration of the encoding process $\mathbf{X}_{[1:9]} = \mathbf{U}_{[1:9]}M^{(2)}$. In particular, we can see that

$$\mathbf{V}_{[1:9]}^{(1)} = \mathbf{U}_{[1:9]}D^{(1)}, \quad \mathbf{U}_{[1:9]}^{(1)} = \mathbf{V}_{[1:9]}^{(1)}Q^{(1)}, \quad \mathbf{X}_{[1:9]} = \mathbf{U}_{[1:9]}^{(1)}D^{(0)}.$$

Therefore, we indeed have $\mathbf{X}_{[1:9]} = \mathbf{U}_{[1:9]}D^{(1)}Q^{(1)}D^{(0)} = \mathbf{U}_{[1:9]}M^{(2)}$. Assume that $\mathbf{U}_{[1:9]}$ consists of 9 i.i.d. Bernoulli-1/2 random variables. Since $D^{(1)}, Q^{(1)}, D^{(0)}$ are all invertible matrices, the random vectors $\mathbf{V}_{[1:9]}^{(1)}, \mathbf{U}_{[1:9]}^{(1)}$ and $\mathbf{X}_{[1:9]}$ also consist of i.i.d. Bernoulli-1/2 random variables.

In order to analyze the bit-channels, we view Fig. 3 from the right side to the left side. First observe that the following three vectors

$$(U_1^{(1)}, U_2^{(1)}, U_3^{(1)}, Y_1, Y_2, Y_3), \quad (U_4^{(1)}, U_5^{(1)}, U_6^{(1)}, Y_4, Y_5, Y_6), \quad (U_7^{(1)}, U_8^{(1)}, U_9^{(1)}, Y_7, Y_8, Y_9)$$

are independent and identically distributed (i.i.d.).

Given a channel $W_1 : \mathcal{X} \rightarrow \mathcal{Y}$ and a pair of random variables (X, Y) that take values in \mathcal{X} and \mathcal{Y} respectively, we write

$$\mathbb{P}(X \rightarrow Y) \equiv W_1$$

if $\mathbb{P}(Y = y|X = x) = W(y|x)$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, where $\mathbb{P}(X \rightarrow Y)$ means the channel that takes X as input and gives Y as output. By this definition, we have

$$\mathbb{P}(U_1^{(1)} \rightarrow \mathbf{Y}_{[1:3]}) \equiv \mathbb{P}(U_4^{(1)} \rightarrow \mathbf{Y}_{[4:6]}) \equiv \mathbb{P}(U_7^{(1)} \rightarrow \mathbf{Y}_{[7:9]}) \equiv W_1(K_1^{(0)}).$$

Since $V_1^{(1)} = U_1^{(1)}, V_2^{(1)} = U_4^{(1)}, V_3^{(1)} = U_7^{(1)}$, we also have

$$\mathbb{P}(V_1^{(1)} \rightarrow \mathbf{Y}_{[1:3]}) \equiv \mathbb{P}(V_2^{(1)} \rightarrow \mathbf{Y}_{[4:6]}) \equiv \mathbb{P}(V_3^{(1)} \rightarrow \mathbf{Y}_{[7:9]}) \equiv W_1(K_1^{(0)}).$$

Moreover, the following three vectors

$$(V_1^{(1)}, \mathbf{Y}_{[1:3]}), \quad (V_2^{(1)}, \mathbf{Y}_{[4:6]}), \quad (V_3^{(1)}, \mathbf{Y}_{[7:9]})$$

are independent. Therefore, the (stochastic) mapping from $U_{[1:3]}$ to $Y_{[1:9]}$ in Fig. 3 can be represented in a more compact form in Fig. 4. From Fig. 4, we can see that

$$\begin{aligned} W_1(M^{(2)}) &\equiv \mathbb{P}(U_1 \rightarrow \mathbf{Y}_{[1:9]}) \equiv W_{1,1}(K_1^{(0)}, K_1^{(1)}), \\ W_2(M^{(2)}) &\equiv \mathbb{P}(U_2 \rightarrow (U_1, \mathbf{Y}_{[1:9]})) \equiv W_{1,2}(K_1^{(0)}, K_1^{(1)}), \\ W_3(M^{(2)}) &\equiv \mathbb{P}(U_3 \rightarrow (U_1, U_2, \mathbf{Y}_{[1:9]})) \equiv W_{1,3}(K_1^{(0)}, K_1^{(1)}). \end{aligned}$$

Next we investigate $W_4(M^{(2)}), W_5(M^{(2)}), W_6(M^{(2)})$. Observe that

$$\mathbb{P}(U_2^{(1)} \rightarrow (U_1^{(1)}, \mathbf{Y}_{[1:3]})) \equiv \mathbb{P}(U_5^{(1)} \rightarrow (U_4^{(1)}, \mathbf{Y}_{[4:6]})) \equiv \mathbb{P}(U_8^{(1)} \rightarrow (U_7^{(1)}, \mathbf{Y}_{[7:9]})) \equiv W_2(K_1^{(0)}).$$

Therefore,

$$\mathbb{P}(V_4^{(1)} \rightarrow (V_1^{(1)}, \mathbf{Y}_{[1:3]})) \equiv \mathbb{P}(V_5^{(1)} \rightarrow (V_2^{(1)}, \mathbf{Y}_{[4:6]})) \equiv \mathbb{P}(V_6^{(1)} \rightarrow (V_3^{(1)}, \mathbf{Y}_{[7:9]})) \equiv W_2(K_1^{(0)}).$$

Moreover, since

$$(V_1^{(1)}, V_4^{(1)}, \mathbf{Y}_{[1:3]}), \quad (V_2^{(1)}, V_5^{(1)}, \mathbf{Y}_{[4:6]}), \quad (V_3^{(1)}, V_6^{(1)}, \mathbf{Y}_{[7:9]})$$

are independent, the (stochastic) mapping from $U_{[4:6]}$ to $(\mathbf{V}_{[1:3]}^{(1)}, \mathbf{Y}_{[1:9]})$ in Fig. 3 can be represented in a more compact form in Fig. 5. Notice that there is a bijection between $U_{[1:3]}$ and $\mathbf{V}_{[1:3]}^{(1)}$. Thus we can conclude from Fig. 5 that

$$\begin{aligned} W_4(M^{(2)}) &\equiv \mathbb{P}(U_4 \rightarrow (\mathbf{U}_{[1:3]}, \mathbf{Y}_{[1:9]})) \equiv \mathbb{P}(U_4 \rightarrow (\mathbf{V}_{[1:3]}^{(1)}, \mathbf{Y}_{[1:9]})) \equiv W_{2,1}(K_1^{(0)}, K_2^{(1)}), \\ W_5(M^{(2)}) &\equiv \mathbb{P}(U_5 \rightarrow (\mathbf{U}_{[1:4]}, \mathbf{Y}_{[1:9]})) \equiv \mathbb{P}(U_5 \rightarrow (U_4, \mathbf{V}_{[1:3]}^{(1)}, \mathbf{Y}_{[1:9]})) \equiv W_{2,2}(K_1^{(0)}, K_2^{(1)}), \\ W_6(M^{(2)}) &\equiv \mathbb{P}(U_6 \rightarrow (\mathbf{U}_{[1:5]}, \mathbf{Y}_{[1:9]})) \equiv \mathbb{P}(U_6 \rightarrow (U_4, U_5, \mathbf{V}_{[1:3]}^{(1)}, \mathbf{Y}_{[1:9]})) \equiv W_{2,3}(K_1^{(0)}, K_2^{(1)}). \end{aligned}$$

Finally, we can use the same method to show that

$$\begin{aligned} \mathbb{P}(V_7^{(1)} \rightarrow (V_1^{(1)}, V_4^{(1)}, \mathbf{Y}_{[1:3]})) &\equiv \mathbb{P}(V_8^{(1)} \rightarrow (V_2^{(1)}, V_5^{(1)}, \mathbf{Y}_{[4:6]})) \\ &\equiv \mathbb{P}(V_9^{(1)} \rightarrow (V_3^{(1)}, V_6^{(1)}, \mathbf{Y}_{[7:9]})) \equiv W_3(K_1^{(0)}). \end{aligned}$$

Therefore, the (stochastic) mapping from $\mathbf{U}_{[7:9]}$ to $(\mathbf{V}_{[1:6]}^{(1)}, \mathbf{Y}_{[1:9]})$ in Fig. 3 can be represented in a more compact form in Fig. 6. Notice that there is a bijection between $\mathbf{U}_{[1:6]}$ and $\mathbf{V}_{[1:6]}^{(1)}$. Thus we can conclude from Fig. 6 that

$$\begin{aligned} W_7(M^{(2)}) &\equiv \mathbb{P}(U_7 \rightarrow (\mathbf{U}_{[1:6]}, \mathbf{Y}_{[1:9]})) \equiv \mathbb{P}(U_7 \rightarrow (\mathbf{V}_{[1:6]}^{(1)}, \mathbf{Y}_{[1:9]})) \equiv W_{3,1}(K_1^{(0)}, K_3^{(1)}), \\ W_8(M^{(2)}) &\equiv \mathbb{P}(U_8 \rightarrow (\mathbf{U}_{[1:7]}, \mathbf{Y}_{[1:9]})) \equiv \mathbb{P}(U_8 \rightarrow (U_7, \mathbf{V}_{[1:6]}^{(1)}, \mathbf{Y}_{[1:9]})) \equiv W_{3,2}(K_1^{(0)}, K_3^{(1)}), \\ W_9(M^{(2)}) &\equiv \mathbb{P}(U_9 \rightarrow (\mathbf{U}_{[1:8]}, \mathbf{Y}_{[1:9]})) \equiv \mathbb{P}(U_9 \rightarrow (U_7, U_8, \mathbf{V}_{[1:6]}^{(1)}, \mathbf{Y}_{[1:9]})) \equiv W_{3,3}(K_1^{(0)}, K_3^{(1)}). \end{aligned}$$

Now we have proved Proposition 9.1 for the special case of $\ell = 3$ and $t = 2$. The proof for the general case follows the same idea, and we defer it to Appendix D.

9.2 Complexity of code construction, encoding and decoding

Proposition 9.2. *The code construction has $N^{O_\ell(1)}$ complexity. Both the encoding and successive decoding procedures have $O_\ell(N \log N)$ complexity.*

Proof. The key in our proof is that we consider ℓ as a (possibly very large) constant. We start with the code construction and we first show that both Algorithm A and Algorithm B have $\text{poly}(N)$ time complexity. In the worst case, we need to check all 2^{ℓ^2} possible kernels in Algorithm A, and for each kernel we need to calculate the conditional entropy of the ℓ subchannels. Since we always work with the quantized channel with output size upper bounded by N^3 , each subchannel of the quantized channels has no more than $2^\ell N^{3\ell}$ outputs. Therefore, the conditional entropy of these subchannels can be calculated in $\text{poly}(N)$ time, so Algorithm A also has $\text{poly}(N)$ complexity. After finding the good kernels, we need to use Algorithm B to quantize/bin the output alphabet of the subchannels produced by these good kernels. As mentioned above, the original alphabet size of these subchannels is no more than $2^\ell N^{3\ell}$. Therefore, Algorithm B also has $\text{poly}(N)$ complexity. At Step i , we use Algorithm A ℓ^i times to find good kernels, and then we use Algorithm B ℓ^{i+1} times to quantize the bit-channels produced by these kernels, so in total we use Algorithm A $\frac{N-1}{\ell-1}$ times and we use Algorithm B $\frac{\ell(N-1)}{\ell-1}$ times. Finally, finding the set $\mathcal{S}_{\text{good}}$ only requires calculating the conditional entropy of the bit-channels in the last step, so this can also be done in polynomial time. Thus we conclude that the code construction has $\text{poly}(N)$ complexity, albeit the degree in $\text{poly}(N)$ complexity depends on ℓ .

In the encoding procedure, we first form the vector $\mathbf{U}_{[1:N]}$ by putting all the information in the bits $\{U_i : \tau_t(i) \in \mathcal{S}_{\text{good}}\}$ and setting all the other bits $\{U_i : \tau_t(i) \notin \mathcal{S}_{\text{good}}\}$ to be 0. Then we multiply $\mathbf{U}_{[1:N]}$ with the encoding matrix $M^{(t)}$ and obtain the codeword $\mathbf{X}_{[1:N]} = \mathbf{U}_{[1:N]} M^{(t)}$. Since the matrix $M^{(t)}$ has size $N \times N$, a naive implementation of the encoding procedure would require $O(N^2)$ operations. Fortunately, we can use (86) to accelerate the encoding procedure. Namely, we first multiply $\mathbf{U}_{[1:N]}$ with $D^{(t-1)}$, then multiply the result with $Q^{(t-1)}$, then multiply by $D^{(t-2)}$, so on and so forth. As mentioned above, for $j = 0, 1, \dots, t-1$, each $D^{(j)}$ is a block diagonal matrix with N/ℓ blocks on the diagonal, where each block has size $\ell \times \ell$. Therefore, multiplication with $D^{(j)}$ only requires $N\ell$ operations. By definition, $Q^{(j)}, j \in [t-1]$ are permutation matrices, so multiplication

with them only requires N operations. In total, we multiply with $2t - 1 = 2 \log_\ell N - 1$ matrices. Therefore, the encoding procedure can be computed in $O_\ell(N \log N)$ time, where O_ℓ means that the constant in big- O depends on ℓ .

The decoding algorithm uses exactly the same idea as the algorithm in Arıkan's original paper [Ari09, Section VIII-B]. Here we only use the special case of $\ell = 3$ and $t = 2$ in Fig. 3 to explain how Arıkan's decoding algorithm works for large (and mixed) kernels, and we omit the proof for general parameters. We start with the decoding of U_1, U_2, U_3 in Fig. 3. It is clear that decoding U_1, U_2, U_3 is equivalent to decoding $U_1^{(1)}, U_4^{(1)}, U_7^{(1)}$. Then the log-likelihood ratio (LLR) of each of these three bits can be calculated locally from only three output symbols. More precisely, the LLR of $U_1^{(1)}$ can be computed from $\mathbf{Y}_{[1:3]}$, the LLR of $U_4^{(1)}$ can be computed from $\mathbf{Y}_{[4:6]}$, and the LLR of $U_7^{(1)}$ can be computed from $\mathbf{Y}_{[7:9]}$. Therefore, the complexity of calculating each LLR only depends on the value of ℓ . Since ℓ is considered as a constant, the calculation of each LLR also has constant time complexity (although the complexity is exponential in ℓ). The next step is to decode $U_{[4:6]}$ from $\mathbf{Y}_{[1:9]}$ together with $U_{[1:3]}$. This is equivalent to calculating the LLRs of $U_2^{(1)}, U_5^{(1)}, U_8^{(1)}$ given $\mathbf{Y}_{[1:9]}$ and $U_1^{(1)}, U_4^{(1)}, U_7^{(1)}$. This again can be done locally: To compute the LLR of $U_2^{(1)}$, we only need the values of $\mathbf{Y}_{[1:3]}$ and $U_1^{(1)}$; to compute the LLR of $U_5^{(1)}$, we only need the values of $\mathbf{Y}_{[4:6]}$ and $U_4^{(1)}$; to compute the LLR of $U_8^{(1)}$, we only need the values of $\mathbf{Y}_{[7:9]}$ and $U_7^{(1)}$. Finally, the decoding of $U_{[7:9]}$ from $\mathbf{Y}_{[1:9]}$ and $U_{[1:6]}$ can be decomposed into local computations in a similar way. Using this idea, one can show that for general values of ℓ and t , the decoding can also be decomposed into $t = \log_\ell N$ stages, and in each stage, the decoding can further be decomposed into N/ℓ local tasks, each of which has constant time complexity (although the complexity is exponential in ℓ). Therefore, the decoding complexity at each stage is $O_\ell(N)$ and the overall decoding complexity is $O_\ell(N \log N)$. As a final remark, we mention that after calculating the LLRs of all U_i 's, we will only use the LLRs of the bits $\{U_i : \tau_t(i) \in \mathcal{S}_{\text{good}}\}$. For these bits, we decode U_i as 0 if its LLR is larger than 0 and decode it 1 otherwise. Recall that in the encoding procedure, we have set all the other bits $\{U_i : \tau_t(i) \notin \mathcal{S}_{\text{good}}\}$ to be 0, so for these bits we simply decode them as 0. \square

9.3 Code rate and decoding error probability

In (80), we have defined the conditional entropy for all the bit-channels obtained in the last step (Step $t - 1$). Here we also define the conditional entropy for the bit-channels obtained in the previous steps. More precisely, for every $j \in [t]$ and every $(i_1, i_2, \dots, i_j) \in [\ell]^j$, we use the following short-hand notation:

$$\begin{aligned} H_{i_1, \dots, i_j}(W) &:= H(W_{i_1, \dots, i_j}(K_1^{(0)}, K_{i_1}^{(1)}, \dots, K_{i_1, \dots, i_{j-1}}^{(j-1)})) \\ H_{i_1, \dots, i_j}^{\text{bin}}(W) &:= H(W_{i_1, \dots, i_j}^{\text{bin}}(K_1^{(0)}, K_{i_1}^{(1)}, \dots, K_{i_1, \dots, i_{j-1}}^{(j-1)})) \\ H_{i_1, \dots, i_j}^{\text{bin}*}(W) &:= H(W_{i_1, \dots, i_j}^{\text{bin}*}(K_1^{(0)}, K_{i_1}^{(1)}, \dots, K_{i_1, \dots, i_{j-1}}^{(j-1)})). \end{aligned}$$

According to (79), we have

$$H_{i_1, \dots, i_j}^{\text{bin}*}(W) \leq H_{i_1, \dots, i_j}^{\text{bin}}(W) \leq H_{i_1, \dots, i_j}^{\text{bin}*}(W) + \frac{6 \log N}{N^3} \quad (87)$$

for every $j \in [t]$ and every $(i_1, i_2, \dots, i_j) \in [\ell]^j$.

Proposition 9.3. For every $j \in [t]$ and $(i_1, i_2, \dots, i_j) \in [\ell]^j$, the conditional entropy $H_{i_1, \dots, i_j}(W)$ and $H_{i_1, \dots, i_j}^{\text{bin}}(W)$ satisfy the following inequality

$$H_{i_1, \dots, i_j}(W) \leq H_{i_1, \dots, i_j}^{\text{bin}}(W) \leq H_{i_1, \dots, i_j}(W) + \frac{6\ell \log N}{N^2} \quad (88)$$

Proof. Since the binning algorithm (Algorithm B) always produces a channel that is degraded with respect to the original channel, the first inequality in (88) follows immediately by applying Proposition 4.4 recursively in our t -step code construction.

Now we prove the second inequality in (88). We will prove the following inequality by induction on j :

$$H_{i_1, \dots, i_j}^{\text{bin}}(W) \leq H_{i_1, \dots, i_j}(W) + \frac{6 \log N}{N^3} (1 + \ell + \ell^2 + \dots + \ell^j) \quad \forall (i_1, i_2, \dots, i_j) \in [\ell]^j. \quad (89)$$

The base case of $j = 0$ is trivial. Now assume that this inequality holds for j and we prove it for $j + 1$. By chain rule, we know that

$$\sum_{i_{j+1}=1}^{\ell} H_{i_1, \dots, i_j, i_{j+1}}^{\text{bin}*}(W) = \ell H_{i_1, \dots, i_j}^{\text{bin}}(W), \quad \sum_{i_{j+1}=1}^{\ell} H_{i_1, \dots, i_j, i_{j+1}}(W) = \ell H_{i_1, \dots, i_j}(W).$$

Therefore,

$$\sum_{i_{j+1}=1}^{\ell} \left(H_{i_1, \dots, i_j, i_{j+1}}^{\text{bin}*}(W) - H_{i_1, \dots, i_j, i_{j+1}}(W) \right) = \ell \left(H_{i_1, \dots, i_j}^{\text{bin}}(W) - H_{i_1, \dots, i_j}(W) \right).$$

Since every summand on the left-hand side is non-negative, we have

$$H_{i_1, \dots, i_j, i_{j+1}}^{\text{bin}*}(W) - H_{i_1, \dots, i_j, i_{j+1}}(W) \leq \ell \left(H_{i_1, \dots, i_j}^{\text{bin}}(W) - H_{i_1, \dots, i_j}(W) \right) \leq \frac{6 \log N}{N^3} (\ell + \ell^2 + \dots + \ell^{j+1}),$$

where the second inequality follows from the induction hypothesis. Combining this with (87), we obtain that

$$H_{i_1, \dots, i_j, i_{j+1}}^{\text{bin}}(W) \leq H_{i_1, \dots, i_j, i_{j+1}}(W) + \frac{6 \log N}{N^3} (1 + \ell + \ell^2 + \dots + \ell^{j+1}).$$

This establishes the inductive step and completes the proof of (89). The inequality (88) then follows directly from (89) by using the fact that $1 + \ell + \dots + \ell^j < \ell N$ for all $j \leq t$. \square

Theorem 9.4. For arbitrarily small $\alpha > 0$, if we choose a constant $\ell \geq \exp(\alpha^{-1.01})$ to be a power of 2 and let $t = \log_{\ell} N$ grow, then the codes constructed from the above procedure have decoding error probability $O_{\alpha}(\log N/N)$ under successive decoding and code rate $I(W) - N^{-1/2+7\alpha}$, where $N = \ell^t$ is the code length.

Proof. By (88) and the definition of $\mathcal{S}_{\text{good}}$ in (81), we know that for every $(i_1, \dots, i_t) \in \mathcal{S}_{\text{good}}$, we have $H_{i_1, \dots, i_t}(W) \leq H_{i_1, \dots, i_t}^{\text{bin}}(W) \leq \frac{7\ell \log N}{N^2}$. Then by Lemma 2.2 in [BGN⁺18], we know that the ML decoding error probability of the bit-channel $W_{i_1, \dots, i_t}(K_1^{(0)}, K_{i_1}^{(1)}, \dots, K_{i_1, \dots, i_{t-1}}^{(t-1)})$ is also upper bounded by $\frac{7\ell \log N}{N^2}$. Since the cardinality of $\mathcal{S}_{\text{good}}$ is at most N , we can conclude that the overall decoding error probability under the successive decoder is $O_{\alpha}(\log N/N)$ using the union bound.

Notice that $|\mathcal{S}_{\text{good}}|$ is the code dimension. Therefore, we only need to lower bound $|\mathcal{S}_{\text{good}}|$ in order to get the lower bound on the code rate. Define another set

$$\mathcal{S}'_{\text{good}} := \left\{ (i_1, i_2, \dots, i_t) \in [\ell]^t : H_{i_1, \dots, i_t}(W) \leq \frac{\ell \log N}{N^2} \right\}. \quad (90)$$

According to (88), if $H_{i_1, \dots, i_t}(W) < \frac{\ell \log N}{N^2}$, then $H_{i_1, \dots, i_t}^{\text{bin}}(W) \leq \frac{7\ell \log N}{N^2}$. Therefore, $\mathcal{S}'_{\text{good}} \subseteq \mathcal{S}_{\text{good}}$, so $|\mathcal{S}_{\text{good}}| \geq |\mathcal{S}'_{\text{good}}|$. In Lemma 9.5 below, we will prove that $|\mathcal{S}'_{\text{good}}| \geq N(I(W) - N^{-1/2+7\alpha})$. Therefore, $|\mathcal{S}_{\text{good}}| \geq N(I(W) - N^{-1/2+7\alpha})$. This completes the proof of the theorem. \square

Lemma 9.5. *If $\ell \geq \exp(\alpha^{-1.01})$ is a power of 2, then the set $\mathcal{S}'_{\text{good}}$ defined in (90) satisfies the following inequality*

$$|\mathcal{S}'_{\text{good}}| \geq N \left(I(W) - N^{-\frac{1}{2}+7\alpha} \right)$$

Proof. The proof is the same as in [BGN⁺18, Claim A.2]. Recall that we proved in (4)-(6)

$$\mathbb{P} \left[H^{(t)} \in \left(\frac{\ell \log N}{N^2}, 1 - \frac{\ell \log N}{N^2} \right) \right] \leq 2 \frac{N^{2\alpha}}{(\ell \log N)^\alpha} \cdot \lambda_\alpha^t,$$

where $H^{(t)}$ is (marginally) the entropy of the random channel at the last level of construction, i.e. $H^{(t)}$ is uniformly distributed over $H_{i_1, \dots, i_t}(W)$ for all possible $(i_1, i_2, \dots, i_t) \in [\ell]^t$, and λ_α is such that (5) holds for any channel W' throughout the construction. By Proposition 9.3, we can choose the error parameter Δ in Algorithm A to be $\Delta = \frac{6\ell \log N}{N^2}$, which satisfies the condition $\Delta \leq \ell^{-\log \ell}$ in Theorem 5.1. Then Theorem 5.1 and Remark 5.2 tell us that as long as $\log \ell \geq \alpha^{-1.01}$, Algorithm A allows us to choose kernels such that $\lambda_\alpha \leq \ell^{-1/2+5\alpha}$, which gives

$$\mathbb{P} \left[H^{(t)} \in \left(\frac{\ell \log N}{N^2}, 1 - \frac{\ell \log N}{N^2} \right) \right] \leq \frac{2N^{-1/2+7\alpha}}{(\ell \log N)^\alpha}. \quad (91)$$

On the other hand, conservation of entropy throughout the process implies $E[H^{(t)}] = H(W)$, therefore by Markov's inequality

$$\mathbb{P} \left[H^{(t)} \geq 1 - \frac{\ell \log N}{N^2} \right] \leq \frac{H(W)}{1 - \frac{\ell \log N}{N^2}} \leq H(W) + \frac{2\ell \log N}{N^2}.$$

Since $H(W) = 1 - I(W)$ for symmetric channels and $|\mathcal{S}'_{\text{good}}| = N \cdot \mathbb{P} \left[H^{(t)} \leq \frac{\ell \log N}{N^2} \right]$, we have

$$\begin{aligned} |\mathcal{S}'_{\text{good}}| &\geq N \left(1 - \frac{2N^{-1/2+7\alpha}}{(\ell \log N)^\alpha} - H(W) - \frac{2\ell \log N}{N^2} \right) \\ &\geq N \left(I(W) - \frac{3N^{-1/2+7\alpha}}{(\ell \log N)^\alpha} \right) \\ &\geq N \left(I(W) - N^{-1/2+7\alpha} \right). \end{aligned} \quad \square$$

9.4 Main theorem: Putting everything together

As we mentioned at the beginning of this section, the code construction presented above only takes the special case of $Q = N^3$ as a concrete example, where Q is the upper bound on the output alphabet size after binning; see Algorithm B. In fact, we can change the value of Q to be any

polynomial of N , and this will allow us to obtain a trade-off between the decoding error probability and the gap to capacity while maintaining the polynomial-time code construction as well as the $O_\alpha(N \log N)$ encoding and decoding complexity. More precisely, we have the following theorem.

Theorem 9.6. *For any BMS channel W , any $c > 0$ and arbitrarily small $\alpha > 0$, if we choose a constant $\ell \geq \exp(\alpha^{-1.01})$ to be a power of 2 and set $Q = N^{c+2}$ in the above code construction procedure, then we can construct a code \mathcal{C} with code length $N = \ell^t$ such that the following four properties hold when t grows: (1) the code construction has $N^{O_\alpha(1)}$ complexity; (2) both encoding and decoding have $O_\alpha(N \log N)$ complexity; (3) rate of \mathcal{C} is $I(W) - O(N^{-1/2+(c+6)\alpha})$; (4) decoding error probability of \mathcal{C} is $O_\alpha(\log N/N^c)$ under successive decoding when \mathcal{C} is used for channel coding over W .*

Proof. The proof of properties (1) and (2) is exactly the same as Proposition 9.2. Here we only briefly explain how to adjust the proof of Theorem 9.4 to show properties (3) and (4). First, we change the definitions of $\mathcal{S}_{\text{good}}$ and $\mathcal{S}'_{\text{good}}$ to

$$\begin{aligned}\mathcal{S}_{\text{good}} &:= \left\{ (i_1, i_2, \dots, i_t) \in [\ell]^t : H_{i_1, \dots, i_t}^{\text{bin}}(W) \leq \frac{(2c+3)\ell \log N}{N^{c+1}} \right\}, \\ \mathcal{S}'_{\text{good}} &:= \left\{ (i_1, i_2, \dots, i_t) \in [\ell]^t : H_{i_1, \dots, i_t}(W) \leq \frac{\ell \log N}{N^{c+1}} \right\}.\end{aligned}$$

The definition of $\mathcal{S}_{\text{good}}$ immediately implies property (4). Next we prove property (3). Since we change Q from N^3 to N^{c+2} , inequality (87) becomes

$$H_{i_1, \dots, i_j}^{\text{bin}*}(W) \leq H_{i_1, \dots, i_j}^{\text{bin}}(W) \leq H_{i_1, \dots, i_j}^{\text{bin}*}(W) + \frac{2(c+2) \log N}{N^{c+2}}.$$

As a consequence, inequality (88) in Proposition 9.3 becomes

$$H_{i_1, \dots, i_j}(W) \leq H_{i_1, \dots, i_j}^{\text{bin}}(W) \leq H_{i_1, \dots, i_j}(W) + \frac{2(c+2)\ell \log N}{N^{c+1}}.$$

This inequality tells us that $\mathcal{S}'_{\text{good}} \subseteq \mathcal{S}_{\text{good}}$, so $|\mathcal{S}_{\text{good}}| \geq |\mathcal{S}'_{\text{good}}|$. Then we follow Lemma 9.5 to lower bound $|\mathcal{S}'_{\text{good}}|$. Inequality (91) now becomes

$$\mathbb{P} \left[H^{(t)} \in \left(\frac{\ell \log N}{N^{c+1}}, 1 - \frac{\ell \log N}{N^{c+1}} \right) \right] \leq \frac{2N^{-1/2+(c+6)\alpha}}{(\ell \log N)^\alpha}.$$

Therefore, we obtain that

$$|\mathcal{S}_{\text{good}}| \geq |\mathcal{S}'_{\text{good}}| \geq N \left(I(W) - N^{-1/2+(c+6)\alpha} \right).$$

This completes the proof of the theorem. □

Appendices

A Proofs of entropic lemmas for BMS channels

In the following two proofs we use the representation of BMS channel W as a convex combination of several BSC subchannels $W^{(1)}, W^{(2)}, \dots, W^{(m)}$, see the beginning of Section 7.1 for details. Each subchannel $W^{(j)}$ can output one of two symbols $z_j^{(0)}, z_j^{(1)}$, and $W^{(j)}(z_j^{(0)}|0) = W^{(j)}(z_j^{(1)}|1)$, $W^{(j)}(z_j^{(1)}|0) = W^{(j)}(z_j^{(0)}|1)$. The output alphabet for W is thus $\mathcal{Y} = \{z_1^{(0)}, z_1^{(1)}, z_2^{(0)}, z_2^{(1)}, \dots, z_m^{(0)}, z_m^{(1)}\}$. Define for these proofs the "flip" operator $\oplus : \mathcal{Y} \times \{0, 1\} \rightarrow \mathcal{Y}$ as follows: $z_j^{(c)} \oplus b = z_j^{(b+c)}$, where $b, c \in \{0, 1\}$, and $(b+c)$ is addition mod 2. In other words, $z_j^{(c)} \oplus 0$ doesn't change anything, and $z_j^{(c)} \oplus 1$ flips the output of the subchannel $W^{(j)}$ to the opposite symbol. Note then that $W^{(j)}(z_j^{(c)}|b) = W^{(j)}(z_j^{(c)} \oplus b|0)$. Finally, we overload the operator to also work on $\mathcal{Y}^\ell \times \{0, 1\}^\ell \rightarrow \mathcal{Y}^\ell$ by applying it coordinate-wise. It then easily follows that $W^\ell(\mathbf{y}|\mathbf{x}) = W^\ell(\mathbf{y} \oplus \mathbf{x}|\mathbf{0})$ for any $\mathbf{y} \in \mathcal{Y}^\ell$ and $\mathbf{x} \in \{0, 1\}^\ell$.

Proof of Lemma 6.1. We can write

$$\begin{aligned} \mathbb{E}_{g \sim G} [H^{(g)}(V_1|\mathbf{Y})] &= \sum_g \mathbb{P}(G = g) \left(\sum_{\mathbf{y} \in \mathcal{Y}^\ell} \mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y}] H^{(g)}(V_1|\mathbf{Y} = \mathbf{y}) \right) \\ &= \sum_g \mathbb{P}(G = g) \left(\sum_{\mathbf{y} \in \mathcal{Y}^\ell} \left(\sum_{\mathbf{v} \in \{0,1\}^k} \mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y}, \mathbf{V} = \mathbf{v}] \right) h \left(\frac{\mathbb{P}^{(g)}[V_1 = 0, \mathbf{Y} = \mathbf{y}]}{\mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y}]} \right) \right) \\ &= \frac{1}{2^k} \sum_{\mathbf{v} \in \{0,1\}^k} \sum_g \mathbb{P}(G = g) \sum_{\mathbf{y} \in \mathcal{Y}^\ell} \mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y}|\mathbf{V} = \mathbf{v}] h \left(\frac{\mathbb{P}^{(g)}[V_1 = 0, \mathbf{Y} = \mathbf{y}]}{\mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y}]} \right), \end{aligned} \quad (92)$$

where $h(x) := -x \log_2 x - (1-x) \log_2 (1-x)$ is the binary entropy function. Next, we show that for any fixed codebook g and any fixed $\mathbf{v} \in \{0, 1\}^k$ it holds

$$\sum_{\mathbf{y} \in \mathcal{Y}^\ell} \mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y}|\mathbf{V} = \mathbf{v}] h \left(\frac{\mathbb{P}^{(g)}[V_1 = 0, \mathbf{Y} = \mathbf{y}]}{\mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y}]} \right) = \sum_{\mathbf{y} \in \mathcal{Y}^\ell} \mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y}|\mathbf{V} = \mathbf{0}] h \left(\frac{\mathbb{P}^{(g)}[V_1 = 0, \mathbf{Y} = \mathbf{y}]}{\mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y}]} \right), \quad (93)$$

where $\mathbf{0}$ is the all-zero vector.

First of all, we know that

$$\mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y}|\mathbf{V} = \mathbf{v}] = W^\ell(\mathbf{y}|\mathbf{v}G) = W^\ell(\mathbf{y} \oplus \mathbf{v}G|\mathbf{0}) = \mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y} \oplus \mathbf{v}G|\mathbf{V} = \mathbf{0}], \quad (94)$$

as was discussed at the beginning of this appendix. In the same way, it's easy to see

$$\begin{aligned} \mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y}] &= \frac{1}{2^k} \sum_{\mathbf{u} \in \{0,1\}^k} \mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y}|\mathbf{V} = \mathbf{u}] = \frac{1}{2^k} \sum_{\mathbf{u} \in \{0,1\}^k} \mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y} \oplus \mathbf{v}G|\mathbf{V} = \mathbf{u} + \mathbf{v}] \\ &= \frac{1}{2^k} \sum_{\mathbf{u} + \mathbf{v} \in \{0,1\}^k} \mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y} \oplus \mathbf{v}G|\mathbf{V} = \mathbf{u} + \mathbf{v}] \\ &= \mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y} \oplus \mathbf{v}G]. \end{aligned} \quad (95)$$

The above equality uses the fact that we are considering linear codes, and $\mathbf{v}G$ is an arbitrary codeword. It follows from the symmetry of linear codes that "shifting" the output by a codeword

does not change anything. Shifting here means the usual shifting for the BSC case, though for general BMS channel this is actually flipping the outputs or appropriate BSC subchannels, without changing which subchannel was actually used for which bit.

Denote now $\tilde{\mathbf{V}} = \mathbf{V}_{>1}$, and recall that we are considering fixed \mathbf{v} for now. Denote then also v_1 as the first coordinate of \mathbf{v} and $\tilde{\mathbf{v}} = \mathbf{v}_{>1}$. Then we derive similarly

$$\begin{aligned}
\mathbb{P}^{(g)}[V_1 = 0, \mathbf{Y} = \mathbf{y}] &= \frac{1}{2^k} \sum_{\tilde{\mathbf{u}} \in \{0,1\}^{k-1}} \mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y} | V_1 = 0, \tilde{\mathbf{V}} = \tilde{\mathbf{u}}] \\
&= \frac{1}{2^k} \sum_{\tilde{\mathbf{u}} \in \{0,1\}^{k-1}} \mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y} \oplus \mathbf{v}G | V_1 = v_1, \tilde{\mathbf{V}} = \tilde{\mathbf{u}} + \tilde{\mathbf{v}}] \\
&= \frac{1}{2^k} \sum_{\tilde{\mathbf{u}} + \tilde{\mathbf{v}} \in \{0,1\}^{k-1}} \mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y} \oplus \mathbf{v}G | V_1 = v_1, \tilde{\mathbf{V}} = \tilde{\mathbf{u}} + \tilde{\mathbf{v}}] \\
&= \mathbb{P}^{(g)}[V_1 = v_1, \mathbf{Y} = \mathbf{y} \oplus \mathbf{v}G].
\end{aligned} \tag{96}$$

Notice that $\mathbb{P}^{(g)}[V_1 = v_1, \mathbf{Y} = \mathbf{y} \oplus \mathbf{v}G] + \mathbb{P}^{(g)}[V_1 = 1 - v_1, \mathbf{Y} = \mathbf{y} \oplus \mathbf{v}G] = \mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y} \oplus \mathbf{v}G]$, and thus using the symmetry of the binary entropy function around 1/2 obtain

$$h\left(\frac{\mathbb{P}^{(g)}[V_1 = v_1, \mathbf{Y} = \mathbf{y} \oplus \mathbf{v}G]}{\mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y} \oplus \mathbf{v}G]}\right) = h\left(\frac{\mathbb{P}^{(g)}[V_1 = 1 - v_1, \mathbf{Y} = \mathbf{y} \oplus \mathbf{v}G]}{\mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y} \oplus \mathbf{v}G]}\right).$$

Using this and (94)-(96) derive

$$\begin{aligned}
&\mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{v}] h\left(\frac{\mathbb{P}^{(g)}[V_1 = 0, \mathbf{Y} = \mathbf{y}]}{\mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y}]}\right) \\
&= \mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y} \oplus \mathbf{v}G | \mathbf{V} = \mathbf{0}] h\left(\frac{\mathbb{P}^{(g)}[V_1 = 0, \mathbf{Y} = \mathbf{y} \oplus \mathbf{v}G]}{\mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y} \oplus \mathbf{v}G]}\right).
\end{aligned}$$

Finally, summing the both parts over $\mathbf{y} \in \mathcal{Y}^\ell$ and noticing that $\mathbf{y} \oplus \mathbf{v}G$ will also range through all \mathcal{Y}^ℓ in this case, we establish (93). Then in (92) deduce

$$\begin{aligned}
\mathbb{E}_{g \sim G} [H^{(g)}(V_1 | \mathbf{Y})] &= \frac{1}{2^k} \sum_{\mathbf{v} \in \{0,1\}^k} \sum_g \mathbb{P}(G = g) \sum_{\mathbf{y} \in \mathcal{Y}^\ell} \mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{0}] h\left(\frac{\mathbb{P}^{(g)}[V_1 = 0, \mathbf{Y} = \mathbf{y}]}{\mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y}]}\right) \\
&= \sum_{\mathbf{y} \in \mathcal{Y}^\ell} \sum_g \mathbb{P}(G = g) \mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{0}] h\left(\frac{\mathbb{P}^{(g)}[V_1 = 0, \mathbf{Y} = \mathbf{y}]}{\mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y}]}\right) \\
&= \sum_{\mathbf{y} \in \mathcal{Y}^\ell} \mathbb{P}[\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{0}] \mathbb{E}_{g \sim G} [H^{(g)}(V_1 | \mathbf{Y} = \mathbf{y})],
\end{aligned}$$

since $\mathbb{P}^{(g)}[\mathbf{Y} = \mathbf{y} | \mathbf{V} = \mathbf{0}]$ does not depend on the matrix g . □

Proof of Proposition 5.5. Let us unfold the conditioning in the LHS as follows

$$H(U_i | W^\ell(\mathbf{U} \cdot K), \mathbf{U}_{<i}) = \mathbb{E}_{\mathbf{w} \sim \{0,1\}^{i-1}} [H(U_i | W^\ell(\mathbf{U} \cdot K), \mathbf{U}_{<i} = \mathbf{w})]. \tag{97}$$

We are going to show that the conditional entropy inside the expectation doesn't depend on the choice of \mathbf{w} , which will allow us to restrict to $\mathbf{w} = \mathbf{0}$.

Return now to the settings of the Proposition, and denote the (random) output $\mathbf{Y} = W^\ell(\mathbf{U} \cdot K)$. Let us now fix some $\mathbf{w} \in \{0, 1\}^{i-1}$ and consider $H(U_i \mid \mathbf{Y}, \mathbf{U}_{<i} = \mathbf{w})$. Unfolding the conditional entropy even more, derive

$$H(U_i \mid \mathbf{Y}, \mathbf{U}_{<i} = \mathbf{w}) = \sum_{\mathbf{y} \in \mathcal{Y}^\ell} \mathbb{P}[\mathbf{Y} = \mathbf{y} \mid \mathbf{U}_{<i} = \mathbf{w}] \cdot H(U_i \mid \mathbf{Y} = \mathbf{y}, \mathbf{U}_{<i} = \mathbf{w}). \quad (98)$$

Denote now by B the first $(i-1)$ rows of K , and thus $\mathbf{Y} = W^\ell(\mathbf{U} \cdot K) = W^\ell(\mathbf{U}_{<i} \cdot B + \mathbf{U}_{\geq i} \cdot G)$. We then have

$$\begin{aligned} \mathbb{P}[\mathbf{Y} = \mathbf{y} \mid \mathbf{U}_{<i} = \mathbf{w}] &= \sum_{\mathbf{v} \in \{0,1\}^k} \frac{1}{2^k} \mathbb{P}[\mathbf{Y} = \mathbf{y} \mid \mathbf{U}_{<i} = \mathbf{w}, \mathbf{U}_{\geq i} = \mathbf{v}] \\ &= \sum_{\mathbf{v} \in \{0,1\}^k} \frac{1}{2^k} W^\ell(\mathbf{y} \mid \mathbf{w} \cdot B + \mathbf{v} \cdot G) \\ &= \sum_{\mathbf{v} \in \{0,1\}^k} \frac{1}{2^k} W^\ell(\mathbf{y} \oplus \mathbf{w}B \mid \mathbf{v} \cdot G) \\ &= \sum_{\mathbf{v} \in \{0,1\}^k} \frac{1}{2^k} \mathbb{P}[\mathbf{Y} = \mathbf{y} \oplus \mathbf{w}B \mid \mathbf{U}_{<i} = \mathbf{0}, \mathbf{U}_{\geq i} = \mathbf{v}] \\ &= \mathbb{P}[\mathbf{Y} = \mathbf{y} \oplus \mathbf{w}B \mid \mathbf{U}_{<i} = \mathbf{0}]. \end{aligned} \quad (99)$$

For the entropy in the RHS of (98), observe

$$H(U_i \mid \mathbf{Y} = \mathbf{y}, \mathbf{U}_{<i} = \mathbf{w}) = h(\mathbb{P}[U_i = 0 \mid \mathbf{Y} = \mathbf{y}, \mathbf{U}_{<i} = \mathbf{w}]),$$

where $h(\cdot)$ is a binary entropy function. Out of the definition of conditional probability, obtain

$$\begin{aligned} \mathbb{P}[U_i = 0 \mid \mathbf{Y} = \mathbf{y}, \mathbf{U}_{<i} = \mathbf{w}] &= \frac{\mathbb{P}[U_i = 0, \mathbf{Y} = \mathbf{y} \mid \mathbf{U}_{<i} = \mathbf{w}]}{\mathbb{P}[\mathbf{Y} = \mathbf{y} \mid \mathbf{U}_{<i} = \mathbf{w}]} \\ &= \frac{\mathbb{P}[U_i = 0, \mathbf{Y} = \mathbf{y} \oplus \mathbf{w}B \mid \mathbf{U}_{<i} = \mathbf{0}]}{\mathbb{P}[\mathbf{Y} = \mathbf{y} \oplus \mathbf{w}B \mid \mathbf{U}_{<i} = \mathbf{0}]} \\ &= \mathbb{P}[U_i = 0 \mid \mathbf{Y} = \mathbf{y} \oplus \mathbf{w}B, \mathbf{U}_{<i} = \mathbf{0}], \end{aligned}$$

where the second equality also uses (99) (and similar equality with $U_i = 0$ inside the probability, which is completely analogical to (99)). Therefore, deduce in (98)

$$\begin{aligned} H(U_i \mid \mathbf{Y}, \mathbf{U}_{<i} = \mathbf{w}) &= \sum_{\mathbf{y} \in \mathcal{Y}^\ell} \mathbb{P}[\mathbf{Y} = \mathbf{y} \oplus \mathbf{w}B \mid \mathbf{U}_{<i} = \mathbf{0}] \cdot H(U_i \mid \mathbf{Y} = \mathbf{y} \oplus \mathbf{w}B, \mathbf{U}_{<i} = \mathbf{0}) \\ &= \sum_{\mathbf{z} \in \mathcal{Y}^\ell} \mathbb{P}[\mathbf{Y} = \mathbf{z} \mid \mathbf{U}_{<i} = \mathbf{0}] \cdot H(U_i \mid \mathbf{Y} = \mathbf{z}, \mathbf{U}_{<i} = \mathbf{0}) \\ &= H(U_i \mid \mathbf{Y}, \mathbf{U}_{<i} = \mathbf{0}), \end{aligned}$$

since $\mathbf{z} = \mathbf{y} \oplus \mathbf{w}B$ ranges over all \mathcal{Y}^ℓ for $\mathbf{y} \in \mathcal{Y}^\ell$. Therefore, in (97) there is no actual dependence on \mathbf{w} under the expectation in the RHS, and thus

$$H(U_i \mid W^\ell(\mathbf{U} \cdot K), \mathbf{U}_{<i}) = H(U_i \mid W^\ell(\mathbf{U} \cdot K), \mathbf{U}_{<i} = \mathbf{0}).$$

Finally, note that we can take $\mathbf{V} = \mathbf{U}_{\geq i}$, since it is uniformly distributed over $\{0, 1\}^k$, and then $V_1 = U_i$. Since $\mathbf{U} \cdot K = \mathbf{U}_{\geq i} \cdot G = \mathbf{V} \cdot G$ when $\mathbf{U}_{< i} = \mathbf{0}$, we indeed obtain

$$H\left(U_i \mid W^\ell(\mathbf{U} \cdot K), \mathbf{U}_{< i}\right) = H\left(U_i \mid W^\ell(\mathbf{U} \cdot K), \mathbf{U}_{< i} = \mathbf{0}\right) = H\left(V_1 \mid W^\ell(\mathbf{V} \cdot G)\right). \quad \square$$

B Proofs in Section 7.1.4

Proof of Claim 7.15. Denote for convenience the distribution $\Omega_i := \text{Binom}(d_i, p_i)$. Note that $\mathbb{E}_{\chi_i \sim \Omega_i} \left[\frac{\chi_i}{d_i} \right] = p_i$. Then we derive

$$\begin{aligned} \left| \mathbb{E}_{\chi_i \sim \mathcal{D}_i} \left[\frac{\chi_i}{d_i} \right] - p_i \right| &= \left| \mathbb{E}_{\chi_i \sim \mathcal{D}_i} \left[\frac{\chi_i}{d_i} \right] - \mathbb{E}_{\chi_i \sim \Omega_i} \left[\frac{\chi_i}{d_i} \right] \right| \\ &= \left| \sum_{s \in [0: d_i]} \frac{s}{d_i} \mathbb{P}_{\chi_i \sim \mathcal{D}_i} [\chi_i = s] - \sum_{s \in [0: d_i]} \frac{s}{d_i} \mathbb{P}_{\chi_i \sim \Omega_i} [\chi_i = s] \right| \\ &\stackrel{(56)}{=} \left| \sum_{s \in \mathcal{T}_1^{(i)}} \frac{s}{d_i} \mathbb{P}_{\chi_i \sim \Omega_i} [\chi_i = s] \cdot \theta_i^{-1} - \sum_{s \in [0: d_i]} \frac{s}{d_i} \mathbb{P}_{\chi_i \sim \Omega_i} [\chi_i = s] \right| \\ &= \left| \sum_{s \in \mathcal{T}_1^{(i)}} \frac{s}{d_i} \mathbb{P}_{\chi_i \sim \Omega_i} [\chi_i = s] \cdot (\theta_i^{-1} - 1) - \sum_{s \notin \mathcal{T}_1^{(i)}} \frac{s}{d_i} \mathbb{P}_{\chi_i \sim \Omega_i} [\chi_i = s] \right| \\ &\leq \sum_{s \in \mathcal{T}_1^{(i)}} \frac{s}{d_i} \mathbb{P}_{\chi_i \sim \Omega_i} [\chi_i = s] \cdot (\theta_i^{-1} - 1) + \sum_{s \notin \mathcal{T}_1^{(i)}} \frac{s}{d_i} \mathbb{P}_{\chi_i \sim \Omega_i} [\chi_i = s]. \end{aligned}$$

We have $\sum_{s \notin \mathcal{T}_1^{(i)}} \frac{s}{d_i} \mathbb{P}_{\chi_i \sim \Omega_i} [\chi_i = s] \leq \sum_{s \notin \mathcal{T}_1^{(i)}} \mathbb{P}_{\chi_i \sim \Omega_i} [\chi_i = s] \stackrel{(55)}{=} (1 - \theta_i) \stackrel{(52)}{\leq} 2\ell^{-\log \ell / 3}$.

Next, $\sum_{s \in \mathcal{T}_1^{(i)}} \frac{s}{d_i} \mathbb{P}_{\chi_i \sim \Omega_i} [\chi_i = s] \leq \mathbb{E}_{\chi_i \sim \Omega_i} \left[\frac{\chi_i}{d_i} \right] \leq 1$, and $\theta_i^{-1} - 1 = \frac{1 - \theta_i}{\theta_i} \leq 2(1 - \theta_i) \leq 4\ell^{-\log \ell / 3}$.

Combining the above together, conclude $\left| \mathbb{E} \left[\frac{\chi_i}{d_i} \right] - p_i \right| \leq 6\ell^{-\log \ell / 3} \leq \frac{1}{\ell} \leq \frac{1}{d_i}$. \square

Proof of Claim 7.16. Using the result of Claim 7.15 derive

$$\mathbb{E} \left| \chi_i - \mathbb{E}[\chi_i] \right| \leq \mathbb{E} \left| \chi_i - p_i d_i \right| + \mathbb{E} \left| p_i d_i - \mathbb{E}[\chi_i] \right| \leq \mathbb{E} \left| \chi_i - p_i d_i \right| + 1. \quad (100)$$

From (55), (57), and definition (54) of $\mathcal{T}_1^{(i)}$ for $i \in F_2$ observe also the following:

$$\begin{aligned} \mathbb{E}_{\chi_i \sim \mathcal{D}_i} \left| \chi_i - p_i d_i \right| &= \sum_{s \in \mathcal{T}_1^{(i)}} \left| \chi_i - p_i d_i \right| \cdot \mathbb{P}_{\eta_i \sim \Omega_i} [s] \cdot \theta_i^{-1} \\ &= \sum_{s \in \mathcal{T}_1^{(i)}} \left| \chi_i - p_i d_i \right| \cdot \mathbb{P}_{\eta_i \sim \Omega_i} [s] + \sum_{s \in \mathcal{T}_1^{(i)}} \left| \chi_i - p_i d_i \right| \cdot \mathbb{P}_{\eta_i \sim \Omega_i} [s] \cdot (\theta_i^{-1} - 1) \\ &\leq \sum_{s \in \mathcal{T}_1^{(i)}} \left| \chi_i - p_i d_i \right| \cdot \mathbb{P}_{\eta_i \sim \Omega_i} [s] + \underbrace{\sqrt{d_i p_i} \log \ell}_{\theta_i} \cdot \sum_{s \in \mathcal{T}_1^{(i)}} \mathbb{P}_{\eta_i \sim \Omega_i} [s] \cdot \left(\frac{1 - \theta_i}{\theta_i} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{s \in \mathcal{T}_1^{(i)}} \left| \chi_i - p_i d_i \right| \cdot \mathbb{P}_{\eta_i \sim \Omega_i} [s] + \sqrt{d_i p_i} \log \ell \cdot (1 - \theta_i) \\
&= \sum_{s \in \mathcal{T}_1^{(i)}} \left| \chi_i - p_i d_i \right| \cdot \mathbb{P}_{\eta_i \sim \Omega_i} [s] + \sum_{s \notin \mathcal{T}_1^{(i)}} \sqrt{d_i p_i} \log \ell \cdot \mathbb{P}_{\eta_i \sim \Omega_i} [s] \\
&\leq \sum_{s \in \mathcal{T}_1^{(i)}} \left| \chi_i - p_i d_i \right| \cdot \mathbb{P}_{\eta_i \sim \Omega_i} [s] + \sum_{s \notin \mathcal{T}_1^{(i)}} \left| \chi_i - p_i d_i \right| \cdot \mathbb{P}_{\eta_i \sim \Omega_i} [s] = \mathbb{E}_{\eta_i \sim \Omega_i} \left| \eta_i - p_i d_i \right|.
\end{aligned}$$

Combining this with (100), obtain the needed. \square

C Proof in Section 7.2

Here we formally show that the channel \widetilde{W} we constructed in Section 7.2 is indeed upgraded with respect to W . Recall that W , \widetilde{W} , and W_1 are defined in (60), (61), and (62) correspondingly, and our goal is to prove (63). First, to check that W_1 is a valid channel, observe

$$\sum_{k \in [m], c \in \{0,1\}} W_1 \left(z_k^{(c)} \mid z_j^{(b)} \right) = \sum_{k \in T_j} \left(W_1 \left(z_k^0 \mid z_j^{(b)} \right) + W_1 \left(z_k^1 \mid z_j^{(b)} \right) \right) = \sum_{k \in T_j} \frac{q_k}{\sum_{i \in T_j} q_i} = 1.$$

Finally, for any $k \in [m]$, $c \in \{0,1\}$, let j_k be such that $k \in T_{j_k}$. Then we have for any $x \in \{0,1\}$

$$\sum_{j \in [\sqrt{\ell}], b \in \{0,1\}} \widetilde{W} \left(z_j^{(b)} \mid x \right) W_1 \left(z_k^{(c)} \mid z_j^{(b)} \right) = \sum_{b \in \{0,1\}} \widetilde{W} \left(z_{j_k}^{(b)} \mid x \right) W_1 \left(z_k^{(c)} \mid z_{j_k}^{(b)} \right).$$

Now, if $x = c$, we derive

$$\begin{aligned}
&\sum_{b \in \{0,1\}} \widetilde{W} \left(z_{j_k}^{(b)} \mid x \right) W_1 \left(z_k^{(c)} \mid z_{j_k}^{(b)} \right) \\
&= \widetilde{W} \left(z_{j_k}^{(x)} \mid x \right) W_1 \left(z_k^{(x)} \mid z_{j_k}^{(x)} \right) + \widetilde{W} \left(z_{j_k}^{(1-x)} \mid x \right) W_1 \left(z_k^{(x)} \mid z_{j_k}^{(1-x)} \right) \\
&= \sum_{i \in T_{j_k}} q_i \cdot (1 - \theta_{j_k}) \cdot \frac{q_k}{\sum_{i \in T_{j_k}} q_i} \cdot \left(1 - \frac{p_k - \theta_{j_k}}{1 - 2\theta_{j_k}} \right) + \sum_{i \in T_{j_k}} q_i \cdot \theta_{j_k} \cdot \frac{q_k}{\sum_{i \in T_{j_k}} q_i} \cdot \left(\frac{p_k - \theta_{j_k}}{1 - 2\theta_{j_k}} \right) \\
&= q_k \left(1 - \theta_{j_k} - (1 - \theta_{j_k}) \cdot \left(\frac{p_k - \theta_{j_k}}{1 - 2\theta_{j_k}} \right) + \theta_{j_k} \cdot \left(\frac{p_k - \theta_{j_k}}{1 - 2\theta_{j_k}} \right) \right) \\
&= q_k \left(1 - \theta_{j_k} - (1 - 2\theta_{j_k}) \cdot \left(\frac{p_k - \theta_{j_k}}{1 - 2\theta_{j_k}} \right) \right) = q_k \cdot (1 - p_k).
\end{aligned}$$

Otherwise, then $x = 1 - c$, obtain

$$\begin{aligned}
&\sum_{b \in \{0,1\}} \widetilde{W} \left(z_{j_k}^{(b)} \mid x \right) W_1 \left(z_k^{(c)} \mid z_{j_k}^{(b)} \right) \\
&= \widetilde{W} \left(z_{j_k}^{(x)} \mid x \right) W_1 \left(z_k^{(1-x)} \mid z_{j_k}^{(x)} \right) + \widetilde{W} \left(z_{j_k}^{(1-x)} \mid x \right) W_1 \left(z_k^{(1-x)} \mid z_{j_k}^{(1-x)} \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i \in T_{j_k}} q_i \cdot (1 - \theta_{j_k}) \cdot \frac{q_k}{\sum_{i \in T_{j_k}} q_i} \cdot \left(\frac{p_k - \theta_{j_k}}{1 - 2\theta_{j_k}} \right) + \sum_{i \in T_{j_k}} q_i \cdot \theta_{j_k} \cdot \frac{q_k}{\sum_{i \in T_{j_k}} q_i} \cdot \left(1 - \frac{p_k - \theta_{j_k}}{1 - 2\theta_{j_k}} \right) \\
&= q_k \left((1 - \theta_{j_k}) \cdot \left(\frac{p_k - \theta_{j_k}}{1 - 2\theta_{j_k}} \right) + \theta_{j_k} \cdot \left(1 - \frac{p_k - \theta_{j_k}}{1 - 2\theta_{j_k}} \right) \right) \\
&= q_k \left((1 - 2\theta_{j_k}) \cdot \left(\frac{p_k - \theta_{j_k}}{1 - 2\theta_{j_k}} \right) + \theta_{j_k} \right) = q_k \cdot p_k.
\end{aligned}$$

Therefore, for any $k \in [m]$ and $c, x \in \{0, 1\}$ it holds

$$\sum_{j \in [\sqrt{\ell}], b \in \{0, 1\}} \widetilde{W} \left(\widetilde{z}_j^{(b)} \mid x \right) W_1 \left(z_k^{(c)} \mid \widetilde{z}_j^{(b)} \right) = W \left(z_k^{(c)} \mid x \right).$$

D Proof of Proposition 9.1

We still use $\mathbf{U}_{[1:N]}$ to denote the information vector and use $\mathbf{X}_{[1:N]} = \mathbf{U}_{[1:N]} M^{(t)}$ to denote the encoded vector. Assume that $\mathbf{U}_{[1:N]}$ consists of N i.i.d. Bernoulli-1/2 random variables. Similarly to the example in Section 9.1, we define the random vectors $\mathbf{V}_{[1:N]}^{(j)}, \mathbf{U}_{[1:N]}^{(j)}$ for $j = t-1, t-2, \dots, 1$ recursively

$$\begin{aligned}
\mathbf{V}_{[1:N]}^{(t-1)} &= \mathbf{U}_{[1:N]} D^{(t-1)}, \\
\mathbf{U}_{[1:N]}^{(j)} &= \mathbf{V}_{[1:N]}^{(j)} Q^{(j)} \text{ for } j = t-1, t-2, \dots, 1, \\
\mathbf{V}_{[1:N]}^{(j)} &= \mathbf{U}_{[1:N]}^{(j+1)} D^{(j)} \text{ for } j = t-2, t-3, \dots, 1, \\
\mathbf{X}_{[1:N]} &= \mathbf{U}_{[1:N]}^{(1)} D^{(0)}.
\end{aligned} \tag{101}$$

Moreover, let $\mathbf{U}_{[1:N]}^{(t)} := \mathbf{U}_{[1:N]}$. We will prove the following two claims:

1. For every $a = 1, 2, \dots, t$, the following ℓ^{t-a} random vectors

$$\left(\mathbf{U}_{[h\ell^a+1:h\ell^a+\ell^a]}^{(a)}, \mathbf{Y}_{[h\ell^a+1:h\ell^a+\ell^a]} \right), \quad h = 0, 1, \dots, \ell^{t-a} - 1$$

are i.i.d.

2. For every $a = 1, 2, \dots, t$ and every $i \in [\ell^a]$, we write $\tau_a(i) = (i_1, i_2, \dots, i_a)$, where τ_a is the a -digit expansion function defined in (82). Then for every $h = 0, 1, \dots, \ell^{t-a} - 1$ and every $i \in [\ell^a]$, we have

$$\mathbb{P}(U_{h\ell^a+i}^{(a)} \rightarrow (\mathbf{U}_{[h\ell^a+1:h\ell^a+i-1]}^{(a)}, \mathbf{Y}_{[h\ell^a+1:h\ell^a+\ell^a]})) \equiv W_{i_1, \dots, i_a}(K_1^{(0)}, K_{i_1}^{(1)}, \dots, K_{i_1, \dots, i_{a-1}}^{(a-1)}). \tag{102}$$

Note that Proposition 9.1 follows immediately from taking $a = t$ in (102). Therefore, we only need to prove these two claims.

We start with the first claim. By (83), for every $j = 0, 1, \dots, t-1$, the matrix $D^{(j)}$ is a block diagonal matrix with ℓ^{t-j-1} blocks on the diagonal, where each block has size $\ell^{j+1} \times \ell^{j+1}$, and all the ℓ^{t-j-1} blocks are the same. According to (84)–(85), the permutation matrix $Q^{(j)}$ keeps the first $t-j-1$ digits of the ℓ -ary expansion to be the same and performs a cyclic shift on the last $j+1$ digits. Therefore, for every $j = 1, \dots, t-1$, the permutation matrix $Q^{(j)}$ is also a block diagonal

matrix with ℓ^{t-j-1} blocks on the diagonal, where each block has size $\ell^{j+1} \times \ell^{j+1}$, and all the ℓ^{t-j-1} blocks are the same. Therefore, for every $j \in [t]$, the matrix $M^{(j)}$ defined in (86) can be written in the following block diagonal form

$$M^{(j)} := \underbrace{\{\overline{M}^{(j)}, \overline{M}^{(j)}, \dots, \overline{M}^{(j)}\}}_{\text{number of } \overline{M}^{(j)} \text{ is } \ell^{t-j}}, \quad (103)$$

where the size of $\overline{M}^{(j)}$ is $\ell^j \times \ell^j$. By the recursive definition (101), one can show that for every $j \in [t]$, we have

$$\mathbf{X}_{[1:N]} = \mathbf{U}_{[1:N]}^{(j)} M^{(j)}.$$

Combining this with (103), we obtain that for every $a \in [t]$ and every $h = 0, 1, \dots, \ell^{t-a} - 1$,

$$\mathbf{X}_{[h\ell^a+1:h\ell^a+\ell^a]} = \mathbf{U}_{[h\ell^a+1:h\ell^a+\ell^a]}^{(a)} \overline{M}^{(a)}. \quad (104)$$

Since $\mathbf{X}_{[1:N]}$ consists of N i.i.d. Bernoulli-1/2 random variables, the following ℓ^{t-a} random vectors

$$(\mathbf{X}_{[h\ell^a+1:h\ell^a+\ell^a]}, \mathbf{Y}_{[h\ell^a+1:h\ell^a+\ell^a]}), \quad h = 0, 1, \dots, \ell^{t-a} - 1$$

are i.i.d. Combining this with (104), we conclude that the random vectors

$$(\mathbf{U}_{[h\ell^a+1:h\ell^a+\ell^a]}^{(a)}, \mathbf{Y}_{[h\ell^a+1:h\ell^a+\ell^a]}), \quad h = 0, 1, \dots, \ell^{t-a} - 1$$

are also i.i.d. This proves claim 1.

Next we prove claim 2 by induction. The case of $a = 1$ is trivial. Now we assume that (102) holds for a and prove it for $a + 1$. In light of claim 1, we only need to prove (102) for the special case of $h = 0$ because the distributions for different values of h are identical, i.e. we only need to prove that

$$\mathbb{P}(U_i^{(a+1)} \rightarrow (\mathbf{U}_{[1:i-1]}^{(a+1)}, \mathbf{Y}_{[1:\ell^{a+1}]})) \equiv W_{i_1, \dots, i_{a+1}}(K_1^{(0)}, K_{i_1}^{(1)}, \dots, K_{i_1, \dots, i_a}^{(a)}) \quad \forall i \in [\ell^{a+1}]. \quad (105)$$

For a given $i \in [\ell^{a+1}]$, we write its $(a + 1)$ -digit expansion as $\tau_{a+1}(i) = (i_1, i_2, \dots, i_{a+1})$. By (101), we know that $\mathbf{V}_{[1:N]}^{(a)} = \mathbf{U}_{[1:N]}^{(a+1)} D^{(a)}$. By (83), the matrix $D^{(a)}$ is a block diagonal matrix with ℓ^{t-1} blocks on the diagonal, where each block has size $\ell \times \ell$. (Note that these ℓ^{t-1} blocks are not all the same unless $a = 0$.) Therefore, for every $h = 0, 1, \dots, \ell^{t-1} - 1$, there is a bijection between the two vectors $\mathbf{V}_{[h\ell+1:h\ell+\ell]}^{(a)}$ and $\mathbf{U}_{[h\ell+1:h\ell+\ell]}^{(a+1)}$. Consequently, there is a bijection between the two vectors $\mathbf{U}_{[1:i-i_{a+1}]}^{(a+1)}$ and $\mathbf{V}_{[1:i-i_{a+1}]}^{(a)}$, so we have

$$\mathbb{P}(U_i^{(a+1)} \rightarrow (\mathbf{U}_{[1:i-1]}^{(a+1)}, \mathbf{Y}_{[1:\ell^{a+1}]})) \equiv \mathbb{P}(U_i^{(a+1)} \rightarrow (\mathbf{U}_{[i-i_{a+1}+1:i-1]}^{(a+1)}, \mathbf{V}_{[1:i-i_{a+1}]}^{(a)}, \mathbf{Y}_{[1:\ell^{a+1}]})) \quad (106)$$

By (83), we also have that

$$\mathbf{V}_{[i-i_{a+1}+1:i-i_{a+1}+\ell]}^{(a)} = \mathbf{U}_{[i-i_{a+1}+1:i-i_{a+1}+\ell]}^{(a+1)} K_{i_1, i_2, \dots, i_a}^{(a)}. \quad (107)$$

Let $\hat{i} := (i - i_{a+1})/\ell$, so $\tau_a(\hat{i}) = (i_1, i_2, \dots, i_a)$. According to the induction hypothesis,

$$\mathbb{P}(U_{\hat{i}}^{(a)} \rightarrow (\mathbf{U}_{[1:\hat{i}-1]}^{(a)}, \mathbf{Y}_{[1:\ell^a]})) \equiv W_{i_1, \dots, i_a}(K_1^{(0)}, K_{i_1}^{(1)}, \dots, K_{i_1, \dots, i_{a-1}}^{(a-1)}).$$

Combining this with the relation $\mathbf{U}_{[1:N]}^{(a)} = \mathbf{V}_{[1:N]}^{(a)} Q^{(a)}$ and (106)–(107), we can prove (105) with the ideas illustrated in Fig. 4–6. This completes the proof of claim 2 as well as Proposition 9.1.

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