Improved bounds on the AN-complexity of $O(1)$-linear functions

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Abstract

We consider arithmetic circuits with arbitrary large gates for computing Boolean functions that are represented by *low degree* polynomials over GF(2). An adequate complexity measure for such circuits is the maximum between the arity of the gates and their number. This model and the corresponding complexity measure, called AN-complexity, were introduced by Goldreich and Wigderson (*ECCC*, TR13-043, 2013), and it is meaningful only for low degree polynomials (where the arity of a gate is not due to the degree of the polynomial that the gate computes but rather to the number of variables in it).

The AN-complexity of a function yields an upper bound on the size of depth-three Boolean circuits for computing the function, and it is not clear whether or not significantly smaller depth-three Boolean circuits exist. Specifically, the depth-three size of Boolean circuits is at most exponential in the AN-complexity of the function. Hence, proving linear lower bounds on the AN-complexity of explicit functions is a essential step towards proving that depth-three Boolean circuits for these functions requires exponential size.

In this work we present explicit $O(1)$-linear functions that require *depth-two* arithmetic circuits of almost linear AN-complexity. Specifically, for every $\epsilon > 0$, we show an explicit poly$(1/\epsilon)$-linear function $f : \{0,1\}^{\text{poly}(1/\epsilon) \cdot n} \to \{0,1\}$ such that any depth-two arithmetic circuit that computes f must use gates of arity at least $n^{1-\epsilon}$. In particular, for every $\epsilon > 0$ and $t = O(1/\epsilon^2)$, the $\Omega(n^{1-\epsilon})$ lower bound holds also for the t-linear function

$$f(x^{(1)}, x^{(2)}, ..., x^{(t)}) = \sum_{i_1,...,i_{t-1} \in [n]} \left(\prod_{j \in [t-1]} x_{i_j}^{(j)} \right) \cdot x_{i_1+i_2+\cdots+i_{t-1}}^{(t)}$$

This improves over a corresponding lower bound of $\widetilde{\Omega}(n^{2/3})$ that was known for an explicit tri-linear function (Goldreich and Tal, *Computational Complexity*, 2018), but leaves open the problem of showing similar AN-complexity lower bounds for circuits of larger depth.

A key aspect in our proof is considering many (extremely skewed) random restrictions, and contrasting the sum of the values of the original function and the circuit (which supposedly computes it) taken over a (carefully chosen) subset of these random restrictions. We show that if the original circuit has too low AN-complexity, then these two sums cannot be equal, which yields a contradiction.

Contents

1 Introduction

Providing exponential lower bounds on the size of constant-depth Boolean circuits computing explicit functions is a central problem of circuit complexity, even when restricting attention to depth-three circuits (cf., e.g., [5, Chap. 11]). We stress that we refer to lower bounds of the form $\exp(\Omega(n))$, when n is the input length, whereas the celebrated lower bounds on the size of depth-three circuits for parity have the form $\exp(\Omega(n^{1/2}))$.

Focusing on this challenge, Goldreich and Wigderson [4] suggested to consider some explicit low degree polynomials over GF(2) as potential candidates for functions that require exponential-size depth-three circuits, and to start by establishing a size lower bound in a restricted model of ("canonical") depth-three circuits. The latter model covers the standard construction of depth-three Boolean circuits for parity, and seems quite natural (if not optimal) in the context of computing low degree polynomials. More specifically, for small values of t (e.g., $t = O(1)$ or $t = \mathrm{poly}(\log(n))$), we shall consider functions over $t \cdot n$ variables, partitioned into t blocks each containing n variables, such that the function is a sum of monomials that contain a single variable from each block (see Eq. (1)). We call such functions t-linear.

Since the restricted model of ("canonical") depth-three circuits and its prior studies are not well-known, we start by reviewing this model and these results. Specifically, in Section 1.1 we review the connection between canonical (depth-three) Boolean circuits and a model of arithmetic circuits with general gates of bounded arity. In particular, the size of canonical circuits corresponds to a complexity measure, called AN-complexity, of the arithmetic circuits. In Section 1.2 we review the known results regarding the AN-complexity of functions, which lead to open problems that are stated at the beginning of Section 1.3. Our results resolve one of these open problems, and in Section 1.4 we outline our main proof ideas.

1.1 From canonical Boolean circuits to AN-complexity of multilinear circuits

This work studies the size of *canonical* depth-three Boolean circuits that compute explicit $O(1)$-linear functions. The restricted model of depth-three canonical Boolean circuits consists of circuits that are obtained by a natural transformation of arithmetic circuits with general gates. The latter arithmetic circuits are restricted only by the arity of these general gates and their numbers, where these quantitative restrictions serve as the complexity measures of such circuits, called *AN-complexity*.[1] The aforementioned transformation is given in [4, Const. 2.8] and it yields Boolean (canonical) circuits of depth three and size that is exponential in the AN-complexity of the arithmetic circuits. Hence, we actually study the AN-complexity of $O(1)$-linear functions.

The foregoing relation to depth-three Boolean circuits motivates the study of the AN-complexity of $O(1)$-linear functions. Additional motivation for the study of this arithmetic circuit model arises from the fact that it is a relatively natural and simple model that is also plagued by the standard complexity curse: the known lower bounds on the complexity of explicit functions fall short of the maximal complexity (established by non-explicit functions).

1.1.1 The functions: t-linearity

As stated above, for small values of t (e.g., $t = O(1)$ or $t = \mathrm{poly}(\log(n))$), we shall consider functions over $t \cdot n$ variables, partitioned into t blocks each containing n variables, such that the function

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[1]Indeed, 'A' stands for arity and 'N' stands for number.

(viewed as a polynomial over GF(2)) is linear in each of the blocks. Such functions are usually called set-multilinear [8] or block-multilinear, but for simplicity we shall call them multi-linear. Specifically, we say that $f : \{0, 1\}^{t \cdot n} \to \{0, 1\}$ is t-linear if

$$f(x^{(1)}, x^{(2)}, ..., x^{(t)}) = \sum_{i_1, i_2, ..., i_t \in [n]} f_{i_1, ..., i_t} \cdot x_{i_1}^{(1)} \cdot x_{i_2}^{(2)} \cdots x_{i_t}^{(t)}, \tag{1}$$

where $x^{(j)} = (x_1^{(j)}, ..., x_n^{(j)})$ is the jth block of n Boolean variables.

That is, f is the sum of monomials (or t-way products) such that each monomial takes a single variable from each of the t blocks of variables. The t-ary array $(f_{i_1, ..., i_t})_{i_1, ..., i_t \in [n]}$ is the tensor that represents the monomials of f; that is, the tensor indicate which of the possible n^t monomials are actually included in the t-linear function. In the special case of $t = 2$ (i.e., bilinear functions) this (two-dimensional) tensor is a matrix. (Jumping ahead, we mention that, as in [4, 3], we shall relate the complexity of (residual) bilinear functions to measures such as the rank of the matrices that describe these functions.)

1.1.2 The arithmetic circuits: the multilinearity condition

We consider arithmetic circuit with arbitrary gates that computes a multi-linear function. That is, each gate in such a circuit computes an arbitrary function; the circuit is only restricted by the *arity of the gates* that it uses and their number.

Following the norm in the study of arithmetic circuits, we actually consider the formal polynomial computed by such arithmetic circuits. When considering these formal polynomials we take intro account that the field has characteristic 2 but not that it is actually GF(2); hence, $x + x = 0$ but x^2 and x are different polynomials. Furthermore, since we consider the computation of multi-linear polynomials, it makes sense to assume that the circuits are multilinear in the sense that each sub-circuit (rooted at any gate) computes a multi-linear function. In other words, a multilinear circuit on t blocks of inputs $x^{(1)}, ..., x^{(t)} \in \{0, 1\}^n$ is a directed acyclic graph whose nodes are associated with arbitrary gates such that each gate is associated with a subset of $[t]$ and the following conditions hold.

1. The sinks (i.e., leaves or input-gates) are associated with the singleton that contains the block in which their input variable reside.

2. Each monomial computed by a (non-sink) gate multiplies results of gates that are associated with disjoint sets of blocks.

3. If a non-sink gate is associated with a set of blocks S, then, for every $i \in S$, each monomial computed by this gate contains an input that is fed by a gate associated with a set that contains i.

We mention that, when considering the computation of t-linear polynomials, the restriction to multilinear circuits can be imposed by increasing the number of gates in the circuit by at most a factor of 2^t, and without increasing the arity of the gates and the depth of the circuit [4, Rem. 2.5].

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1.1.3 The complexity measure: AN-complexity

Clearly, any t-linear function $f : \{0, 1\}^{t \cdot n} \to \{0, 1\}$ can be computed by a multilinear circuit having a single gate of arity tn. But we are interested in circuits that use gates of *bounded arity*, and we also bound the number of gates that they use.

Definition 1.1 (the AN-complexity of multilinear circuits with general gates [4]): *The* arity *of a multilinear circuit is the maximum arity of its* (general) *gates, and the* number of gates *counts only the general gates and not the leaves* (variables). *The* AN-complexity *of a multilinear circuit is the maximum between its arity and the number of its* (general) *gates.*

- *The* general (or unbounded-depth) AN-complexity *of a multi-linear function f, denoted* $\mathrm{AN}(f)$, *is the minimum AN-complexity of a multilinear circuit that computes f.*

- *The* depth-two AN-complexity *of a multi-linear function f, denoted* $\mathrm{AN}_2(F)$, *is the minimum AN-complexity of a depth-two multilinear circuit that computes f.*

Indeed, when dealing with depth-two multilinear circuits, there is no need to upper-bound the number of gates, since it is upper-bounded by the arity of the top gate (plus 1).

1.1.4 Obtaining canonical Boolean circuits

A straightforward implementation of general gates of arity m by CNFs (or DNFs) of size $\exp(m)$ yields depth-three circuits of size $\exp(\mathrm{AN}_2(f))$ for any multi-linear function f. (Indeed, we use a CNF for emulating the top multi-linear gate, and DNFs for the intermediate multi-linear gates (and then collapse the two adjacent layers of OR-gates).) The Boolean circuits obtained from this transformation are considered canonical. Note that the upper bound on the size of depth-three Boolean circuits computing Parity is obtained by a canonical circuit, which is derived by starting from the depth-two arithmetic circuit that computes an n-way sum by summing up $\sqrt{n}$ sums, each depending on a different set of $\sqrt{n}$ variables.

A more general notion of canonical circuits arises by transforming multilinear circuits of arbitrary depth and bounded AN-complexity. Specifically, applying a Valiant-like idea [10], which can be actually traced to the reduction of Circuit-SAT to SAT, Goldreich and Wigderson [4] showed the following.

Theorem 1.2 (the size of depth-three Boolean circuits is at most exponential in the AN-complexity [4, Sec. 2.2]): *Any multilinear function f can be computed by a depth-three Boolean circuit of size* $\exp(\mathrm{AN}(f))$.

The Boolean circuits obtained from this transformation are called canonical.

Hence, establishing lower bounds on the AN-complexity of multi-linear functions is a necessary condition for establishing lower bounds on the size of depth-three Boolean circuits for these functions. In particular, seeking lower bounds of the form $\exp(\omega(n^{1/2}))$ on the size of a depth-three Boolean circuit computing the $O(1)$-linear function $f : \{0, 1\}^{O(n)} \to \{0, 1\}$ requires proving that $\mathrm{AN}(f) = \omega(n^{1/2})$.

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1.2 Prior results regarding the AN-complexity of multi-linear functions

The following results provide the context for our work. For starters, note that the case of $t = 1$ corresponds to the n-bit parity function $\mathtt{PAR}_n$, and in this case it is easy to verify that $\mathtt{AN}_2(\mathtt{PAR}_n) = O(n^{1/2})$ and $\mathtt{AN}(\mathtt{PAR}_n) = \Omega(n^{1/2})$. Our interest is in $O(1)$-linear functions that have AN-complexity $\omega(n^{1/2})$. (Note that such a result means that such a function does not have a *canonical* depth-three Boolean circuit of size $\exp(O(n^{1/2}))$.) We start with an upper bound that sets the limit on such lower bounds.

Theorem 1.3 (a generic upper bound [4, Thm. 3.1]): *For every $t \geq 2$, every t-linear function $f : (\{0,1\}^n)^t \to \{0,1\}$ can be computed by depth-two mulitilinear circuit of AN-complexity $O((tn)^{t/(t+1)})$; that is, $\mathtt{AN}_2(f) = O((tn)^{t/(t+1)})$.*

For example, all bilinear functions have AN-complexity at most $O(n^{2/3})$. Hence, seeking a linear (in tn) lower bound, we must use a logarithmic number of blocks (i.e., $t = \Omega(\log n)$). In fact, such lower bounds hold existentially.

Theorem 1.4 (existential lower bound [4, Thm. 4.1]): *For every $t \geq 2$, almost all t-linear functions $f : (\{0,1\}^n)^t \to \{0,1\}$ have AN-complexity $\Omega((tn)^{t/(t+1)})$; that is,*

$$\Pr_{f:(\{0,1\}^n)^t \to \{0,1\}}[\mathtt{AN}(f) = \Omega(tn)^{t/(t+1)})] = 1 - o(1).$$

For example, almost all bilinear functions have AN-complexity at least $\Omega(n^{2/3})$. Of course, the goal is obtaining lower bounds for explicit functions (and Theorems 1.3 and 1.4 merely set the target for such attempts). The only prior $\omega(n^{1/2})$ result of this type was proved by Goldreich and Tal [3], by building on a connection between the AN-complexity of bilinear functions and matrix rigidity (cf. [9]), which was established by Goldreich and Wigderson [4].

Theorem 1.5 ($\omega(n^{1/2})$ lower bounds for explicit functions [3, Thm. 1.5&1.6]):

1. *There exists a polynomial-time computable 4-linear function $f_4 : \{0,1\}^{4n} \to \{0,1\}$ having AN-complexity $\widetilde{\Omega}(n^{2/3})$; that is, $\mathtt{AN}(f_4) = \widetilde{\Omega}(n^{2/3})$.*

2. *The 3-linear function $f_3(x,y,z) = \sum_{i,j \in [n/2]} x_i y_j z_{i+j}$ satisfies $\mathtt{AN}_2(f_3) = \widetilde{\Omega}(n^{2/3})$.*

We mention that the function $f_4(x,y,r,s)$ has the form $\sum_{i,j \in [n/O(1)]} g_{i,j}(r,s) \cdot x_i y_j$, where $g_{i,j}(r,s)$ is a bilinear form that describes a bit in the output of a small bias generator (see Definition A.1).

1.3 Our results

The obvious open problems raised by the results reviewed in Section 1.2 are

Open Problem 1.6 ($\omega(n^{2/3})$ lower bounds for explicit functions):

1. *Present an explicit $O(1)$-linear function $f : \{0,1\}^{O(n)} \to \{0,1\}$ having AN-complexity $\omega(n^{2/3})$; that is, $\mathtt{AN}(f) = \omega(n^{2/3})$.*

2. *Present an explicit $O(1)$-linear function $f : \{0,1\}^{O(n)} \to \{0,1\}$ satisfying $\mathtt{AN}_2(f) = \omega(n^{2/3})$.*

We mention that the general AN-complexity of multi-linear functions may be lower than its depth-two complexity (cf. [4, Thm 2.3]). We resolve the second problem by proving the following –

Theorem 1.7 (almost linear lower bounds on $\texttt{AN}_2$-complexity of explicit functions): *For every $\epsilon > 0$, letting $t = \mathrm{poly}(1/\epsilon)$, there exists a polynomial-time computable t-linear function $f : \{0,1\}^{t \cdot n} \to \{0,1\}$ such that $\texttt{AN}_2(f) = \Omega(n^{1-\epsilon})$.*

We mention that the lower bound holds also when waiving the requirement that the circuit be multilinear; that is, it holds for general depth-two arithmetic circuits that use general gates. This is the case because general arithmetic circuits of AN-complexity m and depth d that compute a t-linear function can be converted to multilinear circuits of AN-complexity $2^t \cdot m$ and depth d that compute the same function [4, Rem. 2.5].

A somewhat nicer result. Using the ideas that underlie the proof of Theorem 1.7, and triggered by discussion with Avi Wigderson, we also prove the following result that refers to a more explicit function and a smaller value of t.

Theorem 1.8 (almost linear lower bounds on $\texttt{AN}_2$-complexity of more explicit functions): *For every $\epsilon \in (0, 1/4]$, letting $t = \lceil 4/\epsilon^2 \rceil$, the t-linear function $f : (\{0,1\}^n)^{t-1} \times \{0,1\}^{(t-1)n} \to \{0,1\}$*

$$f(x^{(1)}, x^{(2)}, ..., x^{(t)}) = \sum_{i_1,...,i_{t-1}\in[n]} \left(\prod_{j\in[t-1]} x^{(j)}_{i_j} \right) \cdot x^{(t)}_{i_1+i_2+\cdots+i_{t-1}}$$

satisfies $\texttt{AN}_2(f) = \Omega(n^{1-\epsilon})$.

Alternatively, we may partition the last block, which contains $(t-1) \cdot n$ variables, to $t-1$ blocks holding n variables each.[2] We note that this function can be computed in quadratic time (see Remark 3.3).

We mention that the functions used in Theorems 1.7 and 1.8 are generalizations of the 4-linear and trilinear functions used in establishing Part 1 and Part 2 of Theorem 1.5, respectively. Since the proof of Theorem 1.7 is somewhat simpler than the proof of Theorem 1.8, we only review the former in the introduction. The proof of Theorem 1.8 is quite similar but involves some additional details.

1.4   Outline of the proof of Theorem 1.7

Let us take a look at a generic multilinear circuit of *depth two* and AN-complexity m that computes a t-linear function f (and recall that our aim is to prove that $m \geq n^{1-\epsilon}$). This circuit has the form $C(\overline{x}) = H(G_1(\overline{x}), ..., G_m(\overline{x}))$, where me may assume (w.l.o.g.) that the top gate H is only fed by gates (rather than also by variables).[3] Furthermore, each of the intermediate gates (i.e., the G_i's) is fed by variables only; moreover, G_i is fed by at most m variables from each block.

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[2]As stated, this yields a non-homogenous function; that is, different monomials depend on different subsets of the variable-blocks. To obtain a homogenous function we may augment each variable-block by an auxiliary variable (which will be set to 1) and augment the monomials with these variables.

[3]Indeed, for notational simplicity, we consider m intermediate gates, although the AN-complexity upper-bounds their number by $m - 1$.

We consider the blocks of variables that feed each of the m intermediate gates. Specifically, we denote by $B_i \subseteq [t]$ the set of variable-blocks that feed G_i, and note that each monomial computed by G_i is linear in the variables of each block in B_i and is independent of variables of blocks in $[t] \setminus B_i$. It is instructive to distinguish between gates that are fed by many (e.g., more than $2/\epsilon$) blocks from those fed by few blocks.

1.4.1 The random restriction and its affect on gates with many blocks

Suppose that we assign the variables in the first $t-2$ blocks at random such that for each $j \in [t-2]$ we set a single variable of block j to 1 and set all other variables to 0; that is, for each $j \in [t-2]$, we select $i_j \in [n]$ uniformly at random and set the jth block to $0^{i_j-1}10^{n-i_j}$. Then, the t-linear function $f : \{0,1\}^{t \cdot n} \to \{0,1\}$ is (randomly) restricted to a bilinear function $f_{i_1,\dots,i_{t-2}} : \{0,1\}^{n+n} \to \{0,1\}$ such that

$$f_{i_1,\dots,i_{t-2}}(y,z) \stackrel{\text{def}}{=} f(0^{i_1-1}10^{n-i_1}, \dots, 0^{i_{t-2}-1}10^{n-i_{t-2}}, y, z) \tag{2}$$

which is supposedly computed by the simplified circuit (i.e., the circuit obtained from C by applying the random restriction specified by $(i_1, \dots, i_{t-2})$).

Terminology. The n-by-n matrix that represents the (monomials of the) bilinear function $f_{i_1,\dots,i_{t-2}} : \{0,1\}^{n+2} \to \{0,1\}$ may be viewed as a slice of the t-ary tensor that represents the monomials of f. Specifically, recalling (see Eq. (1)) that the monomials of f are represented by the t-ary array $(f_{i_1,\dots,i_t})_{i_1,\dots,i_t \in [n]}$, the matrix $(f_{i_1,\dots,i_t})_{i_{t-1},i_t \in [n]}$ is obtained by fixing the first $t-2$ coordinates of this array such that, for each $j \in [t-2]$, the jth coordinate is fixed to i_j. Indeed, $f_{i_1,\dots,i_{t-2}}(y,z)$ equals $\sum_{i_{t-1},i_t \in [n]} f_{i_1,\dots,i_t} \cdot y_{i_{t-1}} \cdot z_{i_t}$. It is also instructive to note that $f_{i_1,\dots,i_t} = f_{i_1,\dots,i_{t-2}}(0^{i_{t-1}-1}10^{i_{t-1}}, 0^{i_t-1}10^{i_t})$, which equals $f(0^{i_1-1}10^{n-i_1}, \dots, 0^{i_t-1}10^{n-i_t})$.

We now look at the effect of the foregoing random restriction on the circuit C, which supposedly computes f. Looking at the simplified circuit, observe that, for each $i \in [m]$, the output of G_i is identically 0 unless for each $j \in B_i$ we set one of the variables of block j that feeds G_i to 1. Recalling that each block has at most m variables that feed G_i, it follows that the output of G_i is identically 0 with probability at least $1 - (m/n)^{|B_i \cap [t-2]|}$.

Now, if all B_i's were of size at least $d+2 > 1/\epsilon$, then the entire circuit would simplify to the constant 0, with probability at least $1 - m \cdot (m/n)^d$. Hence, if all slices of f are non-trivial (i.e., all $f_{i_1,\dots,i_{t-2}}$ are not identically 0), then we reach a contradiction unless $m \cdot (m/n)^d \geq 1$, which implies $m \geq n^{d/(d+1)} > n^{1-\epsilon}$, and Theorem 1.7 would follow (when we pick the best depth-two circuit; i.e., $m = \texttt{AN}_2(f)$).

1.4.2 Gates with few blocks: A special case

But what if some (or all) B_i's are small? For simplicity, let us ignore the large B_i's, and more importantly assume that each of the remaining B_i's is either contained in $\{t-1, t\}$ or in $[t-2]$ (i.e., if G_i is fed by some block in $\{t-1, t\}$, then it is not fed by any block in $[t-2]$). Under this (unjustified) assumption, the circuit C has the form

$$C(x^{(1)}, \dots, x^{(t-2)}, y, z) = \sum_{j \in J} F_j(x^{(1)}, \dots, x^{(t-2)}) \cdot G_j(y, z) \tag{3}$$

$$+ \sum_{(j_1,j_2) \in K} F_{j_1,j_2}(x^{(1)}, \dots, x^{(t-2)}) \cdot G_{j_1}(y) \cdot G_{j_2}(z), \tag{4}$$

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where the F_j's and F_{j_1,j_2}'s are $(t-2)$-linear functions, and $J \subseteq [m]$ and $K \subseteq \binom{[m]\setminus J}{2}$. Recalling that, in order to get rid of the larger B_i's, we picked a random assignment to the variables in the block in $[t-2]$, it follows that each of the functions F_j's and F_{j_1,j_2}'s evaluates to a constant. Now, if we are extremely lucky and all F_j's (but not necessarily the F_{j_1,j_2}'s)[4] evaluate to 0, then the bilinear function computed by the residual circuit simplifies to Eq. (4); that is, this bilinear function equals $\sum_{j_1 \in [t]\setminus J} G_{j_1}(y) \cdot \sum_{j_2:(j_1,j_2)\in K} G_{j_2}(z)$. One key observation (of [4]) is that the matrix that represents the (monomials of the) latter bilinear function has rank at most m, because each $G_{j_1}(y) \cdot \sum_{j_2:(j_1,j_2)\in K} G_{j_2}(z)$ is represented by a matrix of rank at most 1. In this case we reach a contradiction, provided that almost all slices of f correspond to matrices of higher rank (i.e., rank higher than m). Hence, Theorem 1.7 would follow, provided that we select an adequate function f (which is quite easy to do).

Of course, there is no reason to believe that we may be so extremely lucky (and have all F_j's evaluate to 0). What we do instead is select $m+1$ random assignments to the variables of the blocks in $[t-2]$, and consider the $m+1$ vectors of values of the F_j's; that is, the ith vector describes the values of all F_j's under the ith assignment.[5] Now, we select a non-zero linear combination of these vectors that sums-up to zero, and look at the corresponding linear combinations of the computations of C and the evaluations of f.

The key observation is that this linear combination of the computations of (the restricted) circuit C yields a bilinear function that is represented by a matrix of rank m, because the contribution of the $\sum_{j \in J} F_j G_j$ cancels out (since, for each $j \in J$, the linear combination of the assignments to F_j evaluates to 0) and we are left with Eq. (4). Hence, if the corresponding linear combination of the evaluations of (the restricted) function f yields a bilinear function that is represented by a matrix of higher rank, then we reach a contradiction again. For this to happen, it suffices that *each linear combinations of slices of f corresponds to a matrix of rank higher than m*, where a slice of f is a bilinear function $f_{i_1,\ldots,i_{t-2}}$ (as in Eq. (2)). Selecting f from a small-bias sample space comes to mind, and such a selection will be derandomized by using auxiliary variables (as done in the construction of f_4 of [3]). Furthermore, as in [3], we need a generator of such sequences (with larger stretch than in [3]) that can be implemented by multi-linear functions of low degree. We present such a construction in the appendix.

But wait: We have ignored the effect of using $m+1$ random assignments, rather than one, on the large B_i's (i.e., $|B_i| \geq d+2$). Recall that when using a random assignment of the foregoing type, the contribution of the corresponding gate vanished with probability at least $1 - (m/n)^{|B_i \cap [t-2]|}$. But when selecting $m+1$ such assignments the corresponding gate vanishes on all of them with probability at least $1 - ((m+1) \cdot (m/n))^{|B_i \cap [t-2]|}$. This bound is useless, since we aim at $m \gg \sqrt{n}$. However, we can do better by selecting the $m+1$ assignments carefully. Specifically, for each $j \in [t-2]$, we select a set I_j of $b = (m+1)^{1/(t-2)}$ elements of $[n]$ uniformly at random, and consider the set of assignments specified by $I_1 \times \cdots \times I_{t-2}$. Hence, the ith gate vanishes on all

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[4]Footnote 5 explains why we do not assume here that all F_{j_1,j_2} evaluate to 0. Essentially, removing the unrealistic assumption regarding the F_j's has a cost we can afford, whereas the cost of dealing similarly with the F_{j_1,j_2}'s cannot be afforded (because their number may be much larger).

[5]We shall rely on the hypothesis that the linear combinations of the residual bilinear functions derived from f (by these random restrictions) yield (n-by-n) matrices that are all of high rank (i.e., rank at least m). Our argument will rely on a union bound that can support at most $2^{O(n)}$ different linear combinations, whereas we aim at $m = \omega(n^{1/2})$. Hence, we can afford to consider all $2^{m+1} - 1$ non-zero linear combinations of $m+1$ vectors, but we can not afford to consider $2^{\Omega(m^2)}$ linear combination of $\Omega(m^2)$ vectors. This is the reason that this argument is applied to the F_j's, but not to the F_{j_1,j_2}'s.

$m + 1$ assignments if for some $j \in B_i \cap [t-2]$ it holds that I_j contains no variable that feeds this gate. Observing that this event occurs with probability at least $1 - (b \cdot (m/n))^{|B_i \cap [t-2]|}$. Hence, we reach a contradiction unless $m \cdot ((b \cdot m)/n)^d \geq 1$, which implies $m > n^{\frac{t-2}{t-1} \cdot \frac{d}{d+1}} > n^{1-\epsilon}$, provided that $\min(d+1, t-2) \geq 2/\epsilon$.

1.4.3 Gates with few blocks: The general case

All the foregoing was done under the (unjustified) simplifying assumption that each of the small B_i's is either a subset of $\{t-1, t\}$ or a subset of $[t-2]$. In the general case, we may have gates with small B_i's that intersect both $\{t-1, t\}$ and $[t-2]$.

We first consider the case of small B_i's that contain both $t-1$ and t (i.e., $B_i \supseteq \{t-1, t\}$). A generic gate G_i of this type depends on less than d blocks from $[t-2]$; let us assume for simplicity that these blocks are indexed $1, ..., d' < d$. Then, the contribution of this gate to the computation of C has the form

$$F_i(x^{(d'+1)}, ..., x^{(t-2)}) \cdot G_i(x^{(1)}, ..., x^{(d')}, y, z),$$

where F_i is an arbitrary $(t-2-d')$-linear function (analogously to Eq. (3)). This means that for the sets $I_1, ..., I_{t-2}$ selected as above, we have to find a set $S \subseteq I_1 \times \cdots \times I_{t-2}$ such that for every gate G_i of the current form and every $(i_1, ..., i_{d'}) \in I_1 \times \cdots \times I_{d'}$, it holds that

$$\sum_{(i_{d'+1}, ..., i_{t-2}):(i_1, ..., i_{d'}, i_{d'+1}, ..., i_{t-2}) \in S} F_i(0^{i_1-1}10^{n-i_1}, ..., 0^{i_{t-2}-1}10^{n-i_{t-2}}) = 0.$$

We can find such a set S, very much as we have done before, except that now we need the size of each I_j (denoted b above) to be at least $(m+1)^{1/(t-2-d')}$ (rather than at least $(m+1)^{1/(t-2)}$). This means that we get $m \cdot ((m+1)^{1/(t-2-d')} \cdot m)/n)^d \geq 1$, which gives us the desired lower bound (when using a larger t such that $b^d \cdot m < n/2$).[6]

So we are left with the case of small B_i's that intersect $\{t-1, t\}$ at a single point. For simplicity consider $B_i = \{1, ..., d', t\}$ such that $d' < d$. The crucial observation here is that the expected number of variables of G_i from blocks in $[d']$ that will be set to 1 by any of the random assignments is $(m+1)^{d'/(t-2-d)} \cdot (m/n)^{d'} < ((m+1)^{1+\frac{1}{t-2+d}}/n)^{d'} < 1$. Hence, the contribution of such a gate to the rank of the matrix that corresponds to the bilinear function computed by the residual circuit is actually maximized in the case $d' = 0$, which was considered above.

This completes the rough sketch of the proof of Theorem 1.7, although a more clear and detailed description is in place. In particular, we ignored the task of showing that for a pseudorandom (i.e., small-bias) function f, with high probability, any non-zero linear combination of the slices in $I_1 \times \cdots \times I_{t-2}$ yields a bilinear function that corresponds to a matrix of high rank.

To summarize, we started with an extremely skewed type of random restrictions, which assign values to all but two of the variable blocks such that a single variable in each block is set to 1. Hence, such random restrictions correspond to selecting, at random, a single variable in each of the $t-2$ blocks. Furthermore, we considered b^{t-2} such random assignments and contrasted the sum of the corresponding restrictions of the original function f and of the circuit C (which supposedly computes it), where the sum is taken over a subset of these assignments. Lastly, the b^{t-2} random

---

[6]The added condition is used in the analysis of the pseudorandom function f, which is omitted here. Note that, using $d = 2/\epsilon$ and $t \geq \max(2d+2, (d+1)^2)$, it holds that $m^{1+\frac{d}{t-2-d}+d} > n^d$ implies $m > n^{1-\epsilon}$, whereas $b^d = (m+1)^{d/(t-d-1)} < n/2m$ holds for $m = n^{1-\epsilon}$.

assignments are the Cartesian product of b assignments (of random n-bit strings of Hamming weight 1) to each of the $t - 2$ blocks.

As mentioned briefly, the function f is a pseudorandom function, and we shall use a function F that uses a small bias generator to specify a function f (see Eq. (5), coming next). Specifically, we shall use a small bias pseudorandom generator that can be evaluated by a multi-linear function. In particular, we use an $\exp(-\Omega(n))$-bias generator that stretches $\text{poly}(t) \cdot n$ bits into n^t bits such that each output bit can be computed by an explicit $\text{poly}(t)$-linear function. (Recall that the small-bias feature means that each non-zero linear combination of the output bits is $\exp(-\Omega(n))$-close to being unbiased.) An adequate construction is presented in the appendix.

2 Proof of Theorem 1.7

We start with an explicit presentation of the multi-linear functions that we shall analyze. For t' and t'' to be determined, we consider the $(t' + t'')$-linear function $F : \text{GF}(2)^{(t'+t'') \cdot n} \to \text{GF}(2)$ in which the first $t' + 2$ coordinates of the tensor that describes F correspond to a pseudorandom (i.e., small-bias) tensor, where the pseudorandomness is provided by the last $t'' - 2$ coordinates. Specifically, we use an $\exp(-\Omega(n))$-bias generator, $G_{\mathtt{sb}} : \text{GF}(2)^{(t''-2) \cdot n} \to \text{GF}(2)^{n^{t'+2}}$, that is computed by $(t'' - 2)$-linear functions; that is, the $(i_1, ..., i_{t'+2})^{\text{th}}$ bit in the output of $G_{\mathtt{sb}}$, denoted $G_{\mathtt{sb}}(x^{(t'+3)}, ..., x^{(t'+t'')})_{(i_1,...,i_{t'+2})}$, is computed by a $(t'' - 2)$-linear function. Hence, the function F is defined by

$$F(x^{(1)}, x^{(2)}, ..., x^{(t'+t'')}) = \sum_{i_1,...,i_{t'+2} \in [n]} G_{\mathtt{sb}}(x^{(t'+3)}, ..., x^{(t'+t'')})_{(i_1,...,i_{t'+2})} \cdot \prod_{j \in [t'+2]} x^{(j)}_{i_j} \tag{5}$$

Fixing an arbitrary assignment $s \in \{0,1\}^{(t''-2)n}$ to the last $t'' - 2$ blocks of variables of F (i.e., fixing a seed to the small-bias generator), we consider the resulting $(t'+2)$-dimensional tensor, which describes a pseudorandom function $F_s(\cdots) = F(\cdots, s)$. We shall consider the $n^{t'}$ two-dimensional slices of the former $(t' + 2)$-dimensional tensor, where the slices are aligned with coordinates $t' + 1$ and $t' + 2$. That is, the slice $(i_1, ..., i_{t'}) \in [n]^{t'}$ of the (tensor of the) residual function F_s is the matrix $(G_{\mathtt{sb}}(s)_{(i_1,...,i_{t'},j_1,j_2)})_{j_1,j_2}$; that is, the $(j_1, j_2)^{\text{th}}$ entry of this slice is $G_{\mathtt{sb}}(s)_{(i_1,...,i_{t'},j_1,j_2)}$.

We shall need to pick a large enough t' for the analysis of these slices, and this will require setting $t'' - 2 = \Omega(t')$ so that the $n^{t'+2}$-long sample space can have small bias.

As in Section 1.4, most of our analysis (i.e., Sections 2.2 and 2.3) will refer to a generic function f, which may be viewed as one of the possible residual function F_s. We shall return the function F of Eq. (5) in Section 2.4.

2.1 The form of a depth-two multilinear circuit that computes F

Without loss of generality, the top gate in a generic depth-two circuit of AN-complexity m (which computes F) sums-up monomials that are products of up to $t = t' + t''$ of m auxiliary functions, denoted $G_1, ..., G_m$. The G_i's are computed by the m intermediate gates, which are each fed by m original variables. A typical gate is associated with a subset $B \subset [m]$ of size at most t, which specify the variable-blocks that feed it; specifically, for each $i \in [m]$, we denote by $\mathtt{BL}(i) \subseteq [t]$ the indices of the variable-blocks on which G_i depends, and we say that G_i is $\mathtt{BL}(i)$-linear. The multi-linearity condition implies that each monomial computed by the top gate induces a partition on the t blocks

of variables; that is, for a monomial of the form $\prod_{j\in[w]} G_{i_j}$ computed by the top gate it must hold that $\mathtt{BL}(\{i_1,...,i_w\}) \stackrel{\text{def}}{=} (\mathtt{BL}(i_1),...,\mathtt{BL}(i_w))$ is a partition of $[t]$.

Letting Π denote the set of all partitions of $[t]$ and $\mathcal{C} \subseteq \bigcup_{w\in[t]} \binom{[m]}{w}$ denote the collection of all legal monomials (i.e., $\mathcal{C} = \{I : \mathtt{BL}(I) \in \Pi\}$), the fact that a depth-two circuit of AN-complexity m computes F means that there exist constants $(c_I)_{I\in\mathcal{C}}$ such that

$$
F \;=\; \sum_{I\in\mathcal{C}} c_I \cdot \prod_{i\in I} G_i \tag{6}
$$

$$
\;=\; \sum_{i\in[m]:\mathtt{BL}(i)\ni t'+1} G_i \cdot \sum_{I\in\mathcal{C}:I\ni i} c_I \cdot \prod_{j\in(I\setminus\{i\})} G_j, \tag{7}
$$

where Eq. (7) groups the monomials of Eq. (6) according to the function G_i that is fed by variables of block $t'+1$ (where the choice of using block $t'+1$ rather than block $t'+2$ is arbitrary). Recall that each G_i has arity at most m (and an empty product is defined as identical to 1 (i.e., $\prod_{j\in\emptyset} G_j = 1$)). Letting

$$
F_i \stackrel{\text{def}}{=} \sum_{I\in\mathcal{C}:I\ni i} c_I \cdot \prod_{j\in(I\setminus\{i\})} G_j, \tag{8}
$$

we get

$$
F = \sum_{i\in[m]:\mathtt{BL}(i)\ni t'+1} G_i \cdot F_i \tag{9}
$$

where we treat F_i as arbitrary $([t] \setminus \mathtt{BL}(i))$-linear functions. That is, while each G_i depends on at most m variables (which reside in blocks in the corresponding $\mathtt{BL}(i)$), no such restriction is placed on the F_i's (which arise from Eq. (7)).

2.2 Three types of gates and how they will be handled

For each fixing of values to the last $t''-2$ blocks (i.e., a fixing of a seed for the small-bias generator), we show that adequate random assignments to the first t' blocks yield, with high probability, a bilinear function that corresponds to a matrix of rank $O(m)$, where this bilinear function is derived from the r.h.s. of Eq. (9). In contrast, considering the l.h.s. of Eq. (9), we shall later show that this is unlikely to happen for a random seed (i.e., F_s is likely to correspond to a matrix with higher rank), unless $m = \Omega(n^{1-\epsilon})$. Here we focus on the former part (which was sketched in Section 1.4).

Towards this end, we fix such a seed $s \in \{0,1\}^{(t''-2)\cdot n}$, and for sake of simplicity omit it from the notation (i.e., we shall refer to the functions F, G_i, F_i as if they only depend on the first $t'+2$ blocks). Fixing a sufficiently large constant $d = O(1/\epsilon)$, we consider a partition of the monomials in Eq. (6) (equiv., the G_i's in Eq. (9)) to three types:

1. Monomials in which variables from blocks $t'+1$ and $t'+2$ are fed to different auxiliary functions; equivalently, functions G_i such that $t'+2 \notin \mathtt{BL}(i)$, which means that the variables of block $t'+2$ feed the corresponding function F_i (of Eq. (9)).[7]

   Such monomials (resp., functions) contribute to the rank of a matrix corresponding to a related bilinear function (which depends on variables in blocks $t'+1$ and $t'+2$), and this

---

[7]Recall that throughout this section, we focus on G_i's such that $t'+1 \in \mathtt{BL}(i)$.

contribution will be bounded by taking into account the number of blocks in $[t']$ that feed G_i. Specifically, if $\mathtt{BL}(i) \cap [t'] = \emptyset$, then G_i contributes at most one unit to the rank, and otherwise we will bound the contribution of G_i in expectation by using the fact that it gets simplified by a random restriction (of the type discussed in Section 1.4).

2. Monomials in which variables from blocks $t' + 1$ and $t' + 2$ are fed to the same auxiliary function along with variables from *many* blocks in $[t']$; equivalently, functions G_i for which $t' + 1, t' + 2 \in \mathtt{BL}(i)$ and $|\mathtt{BL}(i) \cap [t']| \geq d$.

   Such monomials (resp., functions) contribute to the non-zero entries of a matrix corresponding to a related bilinear function, and this contribution will be bounded by using d blocks in $[t']$. Specifically, we will bound the contribution of G_i in expectation by using the fact that it gets greatly simplified by a random restriction, where the great simplification is due to the hypothesis $|\mathtt{BL}(i) \cap [t']| \geq d$.

3. Monomials in which variables from blocks $t' + 1$ and $t' + 2$ are fed to the same auxiliary function along with variables from *few* blocks in $[t']$; equivalently, functions G_i for which $t' + 1, t' + 2 \in \mathtt{BL}(i)$ and $|\mathtt{BL}(i) \cap [t']| < d$.

   We shall show that such monomials (resp., functions) can be ignore viz a viz the aforementioned matrices. This will be done by taking a suitable linear combination of relatively few slices that correspond to different random restrictions of the circuit, where the number of linear combinations is kept small enough by relying on the hypothesis $|\mathtt{BL}(i) \cap [t']| < d$. Doing so will indeed complicate the analysis of the first case, but not in an unmanageable manner. In particular, as discussed in Section 1.4, the different random restrictions will be related so that their effect on the first two cases is relatively small.

A contradiction will follow by showing that the corresponding matrix in the tensor that is associated with F itself has higher rank than the upper bound established above. (A slightly better result can be obtained by considering the rigidity of both matrices with respect to this rank (see Remark 2.3), but this is not really necessary.)

2.3 The actual handling of the three types

Fixing an arbitrary i such that $\mathtt{BL}(i) \ni t' + 1$, we now consider in greater detail what happens in each of the foregoing cases, when arbitrarily fixing of the values of the variables in blocks $[t' + 3, t' + t'']$. (For notational simplicity, we shall ignore this fixing in the following discussion; that is, we shall consider G_i and F_i as if they only depend on variables that reside in blocks in $[t' + 2]$.)

Gates of Type 1: $t' + 2 \notin \mathtt{BL}(i)$. For any fixing of the values of the variables in all blocks in $[t']$, the resulting residual function $F_i \cdot G_i$ is a bilinear function in $x^{(t'+2)}$ and $x^{(t'+1)}$ (i.e., F_i depends only on $x^{(t'+2)}$ whereas $G_i u$ depends only on $x^{(t'+1)}$). Furthermore, the n-by-n matrix that corresponds to this residual bilinear function has rank at most 1 (since it is an outer product of two vectors (i.e., the vectors representing the residual F_i and G_i)). Letting T_1 denote the set of all i's of Type 1, we note that under the foregoing fixing of values (to all variables in blocks in $[t']$) the n-by-n matrix that corresponds to the residual bilinear function $\sum_{i \in T_1} F_i G_i$ has rank at most $|T_1| \leq m$.

Foreseeing the treatment of Type 3, which was outlined in Section 1.4, we need to handle the sum of $F_i \cdot G_i$ taken over many random assignments to the variables (in the blocks in $[t']$). These assignments are specified by a random sequence of b-subsets, denoted $\bar{I} = (I_1, ..., I_{t'})$, such that for every $(i_1, ..., i_{t'}) \in I_1 \times \cdots \times I_{t'}$, we consider the assignment $(\mathtt{u}(i_1), ..., \mathtt{u}(i_{t'})) \in \{0, 1\}^{t' \cdot n}$, where $\mathtt{u}(i) = 0^{i-1}10^{n-i}$ is the ith unit vector. For each $j \in [t']$ and $k \in [b]$, we denote by $I_j(k)$ the kth element in I_j. Hence, each $(k_1, ..., k_{t'}) \in [b]^{t'}$ specifies one of the chosen random sequence $(I_1(k_1), ..., I_{t'}(k_{t'})) \in [b]^{t'}$, which in turn specifies a random assignment $(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'}))) \in \{0, 1\}^{t' \cdot n}$. (Recall that $\mathtt{u}(I_j(k_j)) = 0^{i-1}10^{n-i}$ if i is the k_j^{th} element in I_j.)

We wish to upper-bound, for a random sequence $\bar{I}$, and any $S = S(\bar{I}) \subseteq [b]^{t'}$ which may depend on $\bar{I}$, the contribution of the bilinear function

$$\sum_{(k_1, ..., k_{t'}) \in S} F_i(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'})), x^{(t'+2)}) \cdot G_i(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'})), x^{(t'+1)}) \qquad (10)$$

to the rank of the corresponding matrix. We shall show that, in expectation, this contribution is at most $(b \cdot m/n)^{|\mathtt{BL}(i)|}$ units, which is at most 1 when $b \cdot m \leq n$ (which is the setting that we shall use when handling Type 3). Hence, we shall be accommodating the treatment of Type 3 at no real cost.

Recall that G_i depends only on blocks in $\mathtt{BL}(i)$ (whereas F_i depends only on blocks not in $\mathtt{BL}(i)$), and so we may replace G_i by the actual function, denoted G'_i, that this gate applies to variables in $\mathtt{BL}(i)$. (We could do the same for F_i, but there is no instructive benefit in doing so.) For sake of simplicity, assume that $\mathtt{BL}(i) = [d'] \cup \{t' + 1\}$, for some $d' \geq 0$. Then, we can re-write Eq. (10) as follows

$$\sum_{k_1, ..., k_{d'} \in [b]} G'_i(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{d'}(k_{d'})), x^{(t'+1)})$$

$$\cdot \sum_{k_{d'+1}, ..., k_{t'} \in [b]:(k_1, ..., k_{t'}) \in S} F_i(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'})), x^{(t'+2)})$$

$$= \sum_{k_1, ..., k_{d'} \in [b]} G'_i(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{d'}(k_{d'})), x^{(t'+1)}) \cdot F_{i,S}^{(k_1, ..., k_{d'})}(x^{(t'+2)}) \qquad (11)$$

where $F_{i,S}^{(k_1, ..., k_{d'})}(z)$ equals the sum

$$\sum_{(k_{d'+1}, ..., k_{t'}) \in S^{(k_1, ..., k_{d'})}} F_i(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{d'})), x^{(t'+2)})$$

with $S^{(k_1, ..., k_{d'})} \overset{\text{def}}{=} \{(k_{d'+1}, ..., k_{t'}) \in [b]^{t'-d'} : (k_1, ..., k_{t'}) \in S\}$. Looking at Eq. (11), we observe that the term corresponding to $(k_1, ..., k_{d'}) \in [b]^{d'}$ (in the sum) vanishes unless for every $j \in [d']$ it holds that the k_j^{th} variable of block $j \in \mathtt{BL}(i) \cap [t']$ feeds G_i. Considering a random choice of $\bar{I}$, the probability that the term corresponding to $(k_1, ..., k_{d'}) \in [b]^{d'}$ does not vanish is at most $(m/n)^{d'}$. We stress that this vanishing is due to G_i, and so the fact that $F_{i,S(\bar{I})}^{(k_1, ..., k_{d'})}(x^{(t'+2)})$ varies with $\bar{I}$ is immaterial.[8] Hence, the expected number of terms in Eq. (11) that do not vanish is at most

---

[8] Recall that $S = S(\bar{I})$ may depends on $\bar{I}$, and that we are considering the term $G'_i(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{d'}(k_{d'})), x^{(t'+1)}) \cdot F_{i,S(\bar{I})}^{(k_1, ..., k_{d'})}(x^{(t'+2)})$.

$b^{d'} \cdot (m/n)^{d'}$, which means that the expected rank of the matrix corresponding to Eq. (11) is at most $(b \cdot m/n)^{d'}$, where the expectation is taken uniformly over the choices of $\overline{I} \in \binom{[n]}{b}^{t'}$. Note that $b < n/m$ implies that $(b \cdot m/n)^{d'} \leq 1$, for every $d' \geq 0$, with equality holding only for $d' = 0$.

(The key point in the foregoing argument is that both the number of terms in Eq. (11) and the probability that each term vanishes are exponential in d', and so the growth of the first is compensated by the decline of the other. Indeed, the fact that $S = S(\overline{I})$ is determined based on $\overline{I}$ is immaterial, since anyhow we consider all $b^{d'}$ terms, and for each term we only consider its vanishing due only to $G'_i(\mathbf{u}(I_1(k_1)), ..., \mathbf{u}(I_{d'}(k_{d'})), x^{(t'+1)}) \equiv 0$.)

We believe that the foregoing description would convince most readers of the fact that, for a random sequence $\overline{I} = (I_1, ..., I_k)$ of b-subsets, the expected rank of the matrix that corresponds to the bilinear form

$$\sum_{i \in T_1} \sum_{(k_1, ..., k_{t'}) \in S} F_i(\mathbf{u}(I_1(k_1)), ..., \mathbf{u}(I_{t'}(k_{t'})), x^{(t'+2)}) \cdot G_i(\mathbf{u}(I_1(k_1)), ..., \mathbf{u}(I_{t'}(k_{t'})), x^{(t'+1)}) \qquad (12)$$

is at most $|T_1| \cdot (b \cdot m/n)^{d'} \leq m$, provided that $b \cdot m \leq n$. Such readers are advised to skip the following paragraph and proceed directly to the treatment of Type 2 gates.

A more formal argument requires some additional notation. For each $i \in T_1$, let $d_i \stackrel{\text{def}}{=} |\mathtt{BL}(i) \cap [t']|$ and $\mathtt{BL}_j(i)$ be the jth element in $\mathtt{BL}(i) \cap [t']$. (In the foregoing simplified argument, we used $d' - d_i$ and assumed that $\mathtt{BL}_j(i) = j$.) Then, Eq. (11) is re-written as

$$\sum_{k_{\mathtt{BL}_1(i)}, ..., k_{\mathtt{BL}_{d_i}(i)} \in [b]} G'_i(\mathbf{u}(I_{\mathtt{BL}_1(i)}(k_{\mathtt{BL}_1(i)})), ..., \mathbf{u}(I_{\mathtt{BL}_{d_i}(i)}(k_{\mathtt{BL}_{d_i}(i)})), x^{(t'+1)}) \cdot F_{i,S(\overline{I})}^{(k_{\mathtt{BL}_1(i)}, ..., k_{\mathtt{BL}_{d_i}(i)})}(x^{(t'+2)}) \quad (13)$$

where G'_i and $F_{i,S}^{(k_1, ..., k_{d_i})}$ are defined in an analogous matter.[9] Now, repeating the foregoing argument for each $i \in T_1$, we conclude that, for a random sequence $\overline{I} = (I_1, ..., I_k)$ of b-subsets, the expected rank of the matrix that corresponds to the bilinear form of Eq. (12) is at most $\sum_{i \in T_1}(b \cdot m/n)^{d_i} \leq |T_1| \leq m$, provided that $b \cdot m \leq n$. (This can be seen by using Eq. (13).)[10]

---

[9]Specifically, $G'_i(y_1, ..., y_{d_i}, y)$ equals the value of $G_i(y'_1, ..., y'_{t'}, y)$ where $y'_{\mathtt{BL}_j(i)} = y_j$ for every $j \in [d_i]$ and all other y'_k's (which don't effect G_i at all) are set arbitrarily (e.g., to 0^n). Likewise $F_{i,S}^{(k_1, ..., k_{d_i})}(z)$ equals the sum

$$\sum_{(k'_1, ..., k'_{t'}) \in S:(k'_{\mathtt{BL}_1(i)}, ..., k'_{\mathtt{BL}_{d_i}(i)}) = (k_1, ..., k_{d_i})} F_i(\mathbf{u}(I_1(k'_1)), ..., \mathbf{u}(I_{t'}(k'_{t'})), x^{(t'+2)}).$$

[10]Specifically, using

$$\sum_{i \in T_1} \sum_{(k_1, ..., k_{t'}) \in S} F_i(\mathbf{u}(I_1(k_1)), ..., \mathbf{u}(I_{t'}(k_{t'})), x^{(t'+2)}) \cdot G_i(\mathbf{u}(I_1(k_1)), ..., \mathbf{u}(I_{t'}(k_{t'})), x^{(t'+1)})$$

$$= \sum_{i \in T_1} \sum_{k_{\mathtt{BL}_1(i)}, ..., k_{\mathtt{BL}_{d_i}(i)} \in [b]} G'_i(\mathbf{u}(I_{\mathtt{BL}_1(i)}(k_{\mathtt{BL}_1(i)})), ..., \mathbf{u}(I_{\mathtt{BL}_{d_i}(i)}(k_{\mathtt{BL}_{d_i}(i)})), x^{(t'+1)})$$

$$\cdot F_{i,S(\overline{I})}^{(k_{\mathtt{BL}_1(i)}, ..., k_{\mathtt{BL}_{d_i}(i)})}(x^{(t'+2)}),$$

observe that for each $i \in T_1$ and each $(k_{\mathtt{BL}_1(i)}, ..., k_{\mathtt{BL}_{d_i}(i)}) \in [b]^{d_i}$, the corresponding term (which contains a multiple of $G'_i(\mathbf{u}(I_{\mathtt{BL}_1(i)}(k_{\mathtt{BL}_1(i)})), ..., \mathbf{u}(I_{\mathtt{BL}_{d_i}(i)}(k_{\mathtt{BL}_{d_i}(i)})), x^{(t'+1)}))$ does not vanish with probability at most $(m/n)^{d_i}$.

Gates of Type 2: $t'+2 \in \mathtt{BL}(i)$ **and** $|\mathtt{BL}(i) \cap [t']| \geq d$**.** We consider the same $b^{t'}$ random assignments as before. Recall that we select a random sequence of b-subsets, denoted $\bar{I} = (I_1, ..., I_{t'})$, and for every $(i_1, ..., i_{t'}) \in I_1 \times \cdots \times I_{t'}$, we consider the assignment $(\mathtt{u}(i_1), ..., \mathtt{u}(i_{t'})) \in \{0, 1\}^{t' \cdot n}$, where $\mathtt{u}(i) = 0^{i-1}10^{n-i}$. Recall that, for each $j \in \mathtt{BL}(i) \cap [t']$, the probability that I_j hits any of the variables of block j that feeds G_i is at most $b \cdot m/n$, and so G_i vanishes with probability at least $1 - (b \cdot m/n)^{|\mathtt{BL}(i) \cap [t']|}$. Letting T_2 denote the set of all i's of Type 2, it follows that, with probability at least $1 - m \cdot (b \cdot m)^d$, all G_i's of Type 2 vanish, and we can just ignore them, provided that $m \cdot (b \cdot m/n)^d \approx 0$. We shall indeed use a setting that satisfies this (e.g., $m = n^{1-\epsilon}$, $b \leq n^{\epsilon/2}$ and $d \geq 2/\epsilon$).

(As in the analysis of Type 1, each gate G_i of Type 2 vanishes under all $b^{t'}$ random assignment with probability at least $1 - (b \cdot m/n)^{|\mathtt{BL}(i) \cap [t']|}$. Here, being guaranteed that $|\mathtt{BL}(i) \cap [t']| \geq d$, for a sufficiently large constant d, allows us to totally ignore all these gates.)

(We finally get to the case for which we were preparing all along. Note that we may have m gates of Type 3, and under each random restriction of the foregoing form, each such gate may compute an arbitrary bilinear form over $\Omega(m)$ variables of blocks $t'+1$ and $t'+2$, since it may not be fed by any variables from other blocks (which may lead to its vanishing). Hence, for $m = \Omega(n^{2/3})$, the corresponding n-by-n matrix may be arbitrary, unless there are cancellations between the $b^{t'}$ random restrictions. Indeed, the entire point of choosing many random restrictions was to form such cancellations.)

Gates of Type 3: $t'+2 \in \mathtt{BL}(i)$ **and** $|\mathtt{BL}(i) \cap [t']| < d$**.** Whereas in the previous cases (of handling Types 1 and 2), the action focused on the G_i's, in the current case the action is focused on the F_i's. Specifically, for any possible choice of the random sequence of b-subsets, $\bar{I} = (I_1, ..., I_{t'})$, we consider, for each $(k_1, ..., k_{t'}) \in [b]^{t'}$, the vector $v_{k_1,...,k_{t'}}$ representing the value of each F_i under the assignment $(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'})))$; that is, the ith entry of $v_{k_1,...,k_{t'}}$ equals $F_i(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'})))$. Actually, this description suffices only the special case in which for each G_i of Type 3 it holds that $\mathtt{BL}(i) = \{t'+1, t'+2\}$. Assuming that $b^{t'} > m$, we may pick a non-empty subset S of these vectors that sums-up to the all-zero vector. Then, for such a set S and for every i (of this type) it holds that

$$\sum_{(k_1,...,k_{t'})\in S} F_i(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'}))) = 0, \tag{14}$$

where we rely on the fact that F_i is not fed by variables of block $t'+2$ (since $t'+2 \in \mathtt{BL}(i)$ by definition of Type 3). This means that these i's do not contribute to the corresponding bilinear form (i.e., Eq. (15)), since the corresponding G_i's are oblivious of the assignment to variables in blocks $[t']$ (by our assumption that $\mathtt{BL}(i) = \{t'+1, t'+2\}$). That is, letting T_3' denote the set of all gates of this type (i.e., $T_3' = \{i : \mathtt{BL}(i) = \{t'+1, t'+2\}\}$) and using S as in Eq. (14), the following expression is identically zero.

$$\sum_{i\in T_3'} \sum_{(k_1,...,k_{t'})\in S} F_i(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'}))) \cdot G_i(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'})), x^{(t'+1)}, x^{(t'+1)}) \tag{15}$$

where G_i is actually oblivious of the assignment to the first t' blocks.

Turning to the general case, let T_3 denote the set of all i's of Type 3; that is, $T_3 = \{i : \mathtt{BL}(i) \supseteq \{t'+1, t'+2\} \ \& \ |\mathtt{BL}(i)| < d+2\}$. We shall show that in this case we may pick a non-empty subset

S such that a sum analogous to Eq. (15), with T_3' replaced by T_3, is identically zero. Here we shall use $b^{t'-d} > m$ (rather than $b^{t'} > m$), which is where we use an upper-bound on d.

Specifically, for any fixed choice of the random sequence of b-subsets, $\overline{I} = (I_1, ..., I_{t'})$, we consider an auxiliary $b^{d-1} \cdot |T_3|$-by-$b^{t'}$ Boolean matrix in which the rows correspond to pairs $(i, (k_1', ..., k_{d-1}')) \in T_3 \times [b]^{d-1}$, the columns correspond to choices of $\overline{k} = (k_1, ..., k_{t'}) \in [b]^{t'}$, and the value of entry $((i, (k_1', ..., k_{d-1}')), \overline{k})$ is determined according to $F_i'(\mathtt{u}(k_1^{(\sigma_1)}), ..., \mathtt{u}(k_{t'}^{(\sigma_{t'})}))$, provided that $(k_1', ..., k_{d-1}')$ fits $\overline{k}$ (i.e., $k_j' = k_{\mathtt{BL}_j(i)}$ for every $j \in [d_i]$). Specifically:

- For each $i \in T_3$, let $d_i \overset{\text{def}}{=} |\mathtt{BL}(i) \cap [t']| < d$ and $\mathtt{BL}_j(i)$ be the jth element in $\mathtt{BL}(i) \cap [t']$. We actually consider only b^{d_i} rows that correspond to i (and let the other $b^{d-1} - b^{d_i}$ rows be set to $0^{b^{t'}}$). These rows will correspond to all choices of $(k_1', ..., k_{d_i}')) \in [b]^{d_i}$.

- Each column corresponds to a choice of $(k_1, ..., k_{t'}) \in [b]^{t'}$, which represents a choice of a sequence in $I_1 \times \cdots \times I_{t'}$ (i.e., the choice that determines the sequence $(I_1(k_1), ..., I_{t'}(k_{t'})) \in [n]^{t'}$, which in turn determines the assignment $(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'}))))$.

  (Indeed, while the foregoing $(k_1', ..., k_{d_i}')$ represents the choices made for the blocks appearing in $\mathtt{BL}(i) \cap [t']$, the choices $\overline{k} = (k_1, ..., k_{t'})$ represent the choices made for all blocks in $[t']$. Our focus will be on $(k_1', ..., k_{d_i}')$'s that fit the various $\overline{k}$'s.)

- The value of entry $((i, (k_1', ..., k_{d_i}')), \overline{k})$ in the matrix equals 1 if and only if

  - $F_i(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'}))) = 1$, and
  - $k_j' = k_{\mathtt{BL}_j(i)}$ for every $j \in [d_i]$ (i.e., $(k_1', ..., k_{d_i}')$ fits $\overline{k}$).

  Note that this definition relies on the fact that F_i is not fed by variables of block $t'+2$, which follows from the definition of Type 3 (by which $t'+2 \in \mathtt{BL}(i)$).

  We stress that the value of entry $((i, (k_1', ..., k_{d_i}')), \overline{k})$ is 0 in case $(k_1', ..., k_{d-1}')$ does not fits $\overline{k}$ (i.e., $k_j' \neq k_{\mathtt{BL}_j(i)}$ for some $j \in [d_i]$). This means that when considering the row $(i, (k_1', ..., k_{d_i}')$ only columns $\overline{k}$ that are fit by $(k_1', ..., k_{d-1}')$ matter.

Suppose, for simplicity, that the foregoing auxiliary matrix contains an all-zero column that is indexed $\overline{k}$. This means that for every $i \in T_3$ and the unique $(k_1', ..., k_{d_i}')$ that fits $\overline{k}$ (i.e., $k_j' = k_{\mathtt{BL}_j(i)}$ for every $j \in [d_i]$), it holds that $F_i(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'}))) = 0$, because the corresponding $((i, (k_1', ..., k_{d_i}')), \overline{k})$-entry of the matrix is 0 (although $(k_1', ..., k_{d_i}')$ does fit $\overline{k}$). Using the notation

$$F_i^{(\overline{k})}(y, z) \overset{\text{def}}{=} F_i(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'}))) \cdot G_i(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'})), y, z) \tag{16}$$

it follows that $\sum_{i \in T_3} F_i^{(\overline{k})}(y, z)$ is identically zero.

In general, the matrix may contain no all-zero columns; still, assuming $t' \geq d + \log_b m$ (equiv., $b^{t'-d} \geq m$), there is a non-trivial linear combination of the columns that yields an all-zero vector (since there are $b^{t'}$ collumns and at most $b^{d-1} \cdot m < b^{t'}$ rows). Denoting the set of columns participating in this combination by S, for every $i \in T_3$ and $(k_1', ..., k_{d_i}') \in [b]^{d_i}$, it holds that

$$\sum_{\overline{k} \in S: (\forall j \in [d_i])\, k_{\mathtt{BL}_j(i)} = k_j'} F_i^{(\overline{k})}(y, z) \tag{17}$$

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$$= \quad G_i'(\mathtt{u}(I_1(k_1')), ..., \mathtt{u}(I_{d_i}(k_{d_i}')), y, z) \cdot \sum_{\overline{k} \in S : (\forall j \in [d_i]) \, k_{\mathrm{BL}_j(i)} = k_j'} F_i(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'}))) \quad (18)$$

where $G_i'(\mathtt{u}(I_1(k_1')), ..., \mathtt{u}(I_{d_i}(k_{d_i}')), y, z)$ equals $G_i(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'})), y, z)$ for every $(k_1, ..., k_{t'})$ that is fit by $(k_1', ..., k_{d_i}')$ (i.e., $k_j' = k_{\mathrm{BL}_j(i)}$ for every $j \in [d_i]$).

The punch-line is that Eq. (18) is identically zero, because the $(i, (k_1', ..., k_{d_i}'))^{\text{th}}$ entry in the corresponding sum of columns is zero, whereas the full row has 0-entries in columns that do not fit $(k_1', ..., k_{d_i}')$, and holds the value of $F_i(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'})))$ in each column $\overline{k}$ that does fit $(k_1', ..., k_{d_i}')$. Formally, letting $M_{(i, (k_1', ..., k_{d_i}')), \overline{k}}$ denote the $((i, (k_1', ..., k_{d_i}')), \overline{k})^{\text{th}}$ entry in the foregoing matrix, for every $i \in T_3$ and $(k_1', ..., k_{d_i}') \in [b]^{d_i}$, we have

$$\sum_{\overline{k} \in S : (\forall j \in [d_i]) \, k_{\mathrm{BL}_j(i)} = k_j'} F_i(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'}))) \quad = \quad \sum_{\overline{k} \in S} M_{(i, (k_1', ..., k_{d_i}')), \overline{k}}$$
$$= \quad 0.$$

Hence, for any fixed $\overline{I} = (I_1, ..., I_{t'}) \in [b]^{t'}$, Type 3 gates have no contribution to the bilinear function $B_S : GF(2)^{n+n} \to GF(2)$ defined as

$$B_S(y, z) \stackrel{\text{def}}{=} \sum_{\overline{k} = (k_1, ..., k_{t'}) \in S} F(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'})), y, z) \quad (19)$$

where $F = \sum_{i \in [m] : \mathrm{BL}(i) \ni t'+1} F_i \cdot G_i$ is the function supposedly computed by the circuit, and $S = S(\overline{I})$ is chosen based on $\overline{I}$.

Recall that the contribution of Type 2 gates vanishes with very high probability, and the contribution of Type 1 gates corresponds to a matrix of expected rank at most m, where in both cases the probability space refers to the choice of $\overline{I}$. Hence, with probability at least $2/3$ over the choice of $\overline{I}$, the bilinear function $B_{S(\overline{I})}$ corresponds to a matrix of rank $O(m)$.

2.4 Wrapping-up and reaching a contradiction

We have essentially established the following Lemma 2.1, except that our notations ignored (or hide) the dependence of all residual functions (including $S = S(\overline{I})$) on the values of blocks $t'+3, ..., t'+t''$. (Recall that these values were fixed arbitrarily at the very beginning of Section 2.3.)

Lemma 2.1 (low AN2-complexity of F implies low rank of B_S): *For m, b, t' such that $b \leq (n/m)^{1/2}$ and $t' \geq 2\log_{n/m} n + \log_b m$, suppose that $\mathtt{AN}_2(F) \leq m < n/10$. Then, for every $\overline{s} = (s^{(t'+3)}, ..., s^{(t'+t'')}) \in GF(2)^{(t''-2) \cdot n}$, with probability at least $2/3$ over a random choice of $\overline{I} = (I_1, ..., I_{t'}) \in \binom{[n]}{b}^{t'}$, there exists a non-empty set $S \subseteq [b]^{t'}$ such that the matrix corresponding to the bilinear function B_S of Eq. (19) has rank at most $5m$. Formally, we refer to the bilinear form*

$$B_S^{(\overline{s}, \overline{I})}(y, z) \stackrel{\text{def}}{=} \sum_{\overline{k} = (k_1, ..., k_{t'}) \in S} F(\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'})), y, z, s^{(t'+3)}, ..., s^{(t'+t'')}), \quad (20)$$

where $\mathtt{u}(i) = 0^{i-1}10^{n-i}$ is the ith unit vector and $I_j(k)$ denotes the kth element in I_j.

Proof: We merely summarize the contents of Section 2.3, while using the more explicit notations. Recall that our starting point is a depth-two circuit of AN-complexity at most m that computes F, which has a form as captured by Eq. (9). Recall that we have fixed $\overline{s} = (s^{(t'+3)}, ..., s^{(t'+t'')}) \in \mathrm{GF}(2)^{(t''-2)\cdot n}$ upfront, and the subsequent analysis referred to this fixed $\overline{s}$ (and holds for any such $\overline{s}$). We have broken the sum in Eq. (9) into three parts, corresponding to the three types of gates, and analyzed each type separately.

For Type 1, we showed that for a random choice of $\overline{I} = (I_1, ..., I_{t'}) \in \binom{[n]}{b}^{t'}$, and for every $S \subseteq [b]^{t'}$ (which may depend on $\overline{I}$), the contribution of gates of Type 1 to $B_S^{(\overline{s},\overline{I})}$ corresponds to a matrix of expected rank at most m. (This used $b \leq (n/m)$, which holds under the hypothesis.) Next, we showed that, over the same random choice of $\overline{I}$, and for every $S \subseteq [b]^{t'}$, the contribution of gates of Type 2 to $B_S^{(\overline{s},\overline{I})}$ corresponds to a matrix that is non-zero with probability at most $m \cdot (b \cdot m/n)^d$. Using $b \leq (n/m)^{1/2}$ and setting $d = 2 \log_{n/m} n$, we have $m \cdot (b \cdot m/n)^d \leq m \cdot (m/n)^{d/2} = m/n < 1/10$. Hence (as detailed below), with probability at least $0.8 - 0.1 > 2/3$, the contribution of gates of Types 1 and 2 to $B_S^{(\overline{s},\overline{I})}$ corresponds to a matrix of rank at most $5m$.

Lastly, considering the Type 3 gates, we showed that, for any choice of $\overline{I}$, there exists a non-empty set $S = S(\overline{I}) \subseteq [b]^{t'}$ such that the contribution of functions of Type 3 to $B_S^{(\overline{s},\overline{I})}$ corresponds to an all-zero matrix. Here we used $t' \geq d + \log_b m$. Hence, using $b \leq (n/m)^{1/2}$ (and $d = 2 \log_{n/m} n$), it suffices to have $t' \geq 2 \log_{n/m} n + \log_b m$, which holds by the hypothesis.

Formally, for $\overline{s} = (s^{(t'+3)}, ..., s^{(t'+t'')})$, and $\overline{I} = (I_1, ..., I_{t'}) \in \binom{[n]}{b}^{t'}$, letting $\overline{\mathtt{u}}(\overline{k}) = (\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'})))$, we re-write Eq. (20) as

$$B_S^{(\overline{s},\overline{I})}(y,z) = \sum_{\overline{k}=(k_1,...,k_{t'})\in S} F(\overline{\mathtt{u}}(\overline{k}), y, z, s^{(t'+3)}, ..., s^{(t'+t'')}). \tag{21}$$

Hence, for the corresponding set S (which may depend on $\overline{s}$ and $\overline{I}$), we have

$$\begin{aligned} B_S^{(\overline{s},\overline{I})}(y,z) &= \sum_{\overline{k}\in S}\sum_{i\in[m]} F_i(\overline{\mathtt{u}}(\overline{k}), y, z, \overline{s}) \cdot G_i(\overline{\mathtt{u}}(\overline{k}), y, z, \overline{s}) \\ &= \sum_{\overline{k}\in S}\sum_{i\in T_1} F_i(\overline{\mathtt{u}}(\overline{k}), y, z, \overline{s}) \cdot G_i(\overline{\mathtt{u}}(\overline{k}), y, z, \overline{s}) \tag{22} \\ &\quad + \sum_{\overline{k}\in S}\sum_{i\in T_2} F_i(\overline{\mathtt{u}}(\overline{k}), y, z, \overline{s}) \cdot G_i(\overline{\mathtt{u}}(\overline{k}), y, z, \overline{s}), \tag{23} \end{aligned}$$

since the sum that corresponds to T_3 is identically zero due to the choice of S. Recalling that, with probability at least $4/5$, the matrix corresponding to Eq. (22) has rank at most $5m$, and that with probability at least $9/10$ the matrix corresponding to Eq. (23) is identically zero, the lemma follows. ∎

Reaching a contradiction. Lemma 2.1 implies that, with probability at least $2/3$ over a random choice of $\overline{s} = (s^{(t'+3)}, ..., s^{(t'+t'')}) \in \mathrm{GF}(2)^{(t''-2)\cdot n}$ and $\overline{I} = (I_1, ..., I_{t'}) \in \binom{[n]}{b}^{t'}$, there exists a non-empty set $S \subseteq [b]^{t'}$ such that the matrix corresponding to the bilinear function $B_S^{(\overline{s},\overline{I})}$ has rank at most $5 \cdot \mathtt{AN}_2(F)$, provided that $b \leq (n/\mathtt{AN}_2(F))^{1/2}$ and $t' \geq 2 \log_{n/\mathtt{AN}_2(F)} n + \log_b \mathtt{AN}_2(F)$. In contrast, the following Lemma 2.2 implies that with probability at least $1 - b^{t'} \cdot 2^{-n/2}$ over the same random

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choices, for every non-empty set $S \subseteq [b]^{t'}$, the matrix corresponding to the bilinear function $B_S^{(\overline{s}, \overline{I})}$ has rank $\Omega(n)$. Hence, we reach contradiction unless either $\mathtt{AN}_2(F) = \Omega(n)$ or $2^{b^{t'}} \cdot 2^{-n/2} > 1/3$ (for b and t' as above). As detailed below, this implies $\mathtt{AN}_2(F) = \Omega(n^{1-\epsilon})$, and Theorem 1.7 follows. But let us first prove the following lemma.

Lemma 2.2 (typically B_S has high rank): *Suppose that the generator $G_{\mathtt{sb}} : \mathrm{GF}(2)^{(t''-2)\cdot n} \to \{0,1\}^{n^{t'+2}}$ has bias at most 2^{-n}. Then, for every sequence of b-subsets $\overline{I} = (I_1, ..., I_{t'}) \in \binom{[n]}{b}^{t'}$ and any non-empty set $S \subseteq [b]^{t'}$, with probability $1 - 2^{-n/2}$ over the choice of $\overline{s} = (s^{(t'+3)}, ..., s^{(t'+t'')}) \in \mathrm{GF}(2)^{(t''-2)\cdot n}$, it holds that the matrix corresponding to the bilinear function $B_S^{(\overline{s}, \overline{I})}$ of Eq. (21) has rank $\Omega(n)$.*

Proof: For every $S \subseteq [b]^{t'}$, $\overline{I} \in \binom{[n]}{b}^{t'}$ and $\overline{s} = (s^{(t'+3)}, ..., s^{(t'+t'')}) \in \mathrm{GF}(2)^{(t''-2)\cdot n}$, looking at the value of $B_S^{(\overline{s}, \overline{I})}$, while letting $\overline{\mathtt{u}}(\overline{k}) = (\mathtt{u}(I_1(k_1)), ..., \mathtt{u}(I_{t'}(k_{t'})))$, observe that

$$
\begin{aligned}
B_S^{(\overline{s}, \overline{I})}(y, z) &= \sum_{\overline{k} \in S} F(\overline{\mathtt{u}}(\overline{k}), y, z, \overline{s}) \\
&= \sum_{\overline{k} \in S} \sum_{(i_1, ..., i_{t'+2}) \in \overline{I} \times [n]^2} G_{\mathtt{sb}}(\overline{s})_{(i_1, ..., i_{t'+2})} \cdot \prod_{j \in [t']} (\mathtt{u}(I_j(k_j)))_{i_j} \cdot y_{i_{t'+1}} \cdot z_{i_{t'+2}} \quad (24) \\
&= \sum_{\overline{k} \in S} \sum_{(i_{t'+1}, i_{t'+2}) \in [n]} G_{\mathtt{sb}}(\overline{s})_{(I_1(k_1), ..., I_{t'}(k_{t'}), i_{t'+1}, i_{t'+2})} \cdot y_{i_{t'+1}} \cdot z_{i_{t'+2}} \quad (25)
\end{aligned}
$$

where the last equality holds because only t'-tuples $(i_1, ..., i_{t'})$ that satisfy $i_j = I_j(k_j)$ contribute to Eq. (24). The difference between Eq. (24) and Eq. (25) is that in the latter form it is evident that the corresponding matrix is a non-zero linear combination of $|S|$ matrices that correspond to disjoint parts of the output of the small-bias generator $G_{\mathtt{sb}}$; that is, the $(i_{t'+1}, i_{t'+2})^{\mathrm{th}}$ element in the matrix that corresponds to $\overline{k} = (k_1, ..., k_{t'}) \in S$ equals the $(I_1(k_1), ..., I_{t'}(k_{t'}), i_{t'+1}, i_{t'+2})^{\mathrm{th}}$ bit in the output of $G_{\mathtt{sb}}$.

Hence, for any fixed S and $\overline{I}$, when $\overline{s}$ is uniformly distributed in $\{0,1\}^{(t''-2)\cdot n}$, the matrix that corresponds to $B_S^{(\overline{s}, \overline{I})}$ is an n-by-n matrix whose entries are distributed according to an 2^{-n}-bias sequence, because any sequence that is obtained by taking linearly independent non-zero linear-combinations of elements in an ϵ-bias sequence is itself ϵ-bias. The lemma follows by using the fact that, with probability at least $1 - 2^{-n/2}$, such a matrix has rank $\Omega(n)$. Specifically, we upper-bound the probability that the matrix has rank at most $n/10$ by considering all linear combinations of up to $n/10$ columns. Each such linear combination results in an n-long 2^{-n}-bias sequence, and the probability that such a sequence equals the all-zero sequence is at most $2^{-n} + 2^{-n}$.[11] Hence, the probability of the bad event is upper-bounded by $\sum_{i \in [n/10]} \binom{n}{i} \cdot 2^{-n+1} < 2^{-n/2}$, and the lemma follows. ∎

Conclusion (re-iterated and detailed): Using a union bound on all possible $S \subseteq [b]^{t'}$, Lemma 2.2 implies that, with probability at least $1 - 2^{b^{t'}} \cdot 2^{-n/2}$ over the choices of $\overline{s} = (s^{(t'+3)}, ..., s^{(t'+t'')}) \in$

---

[11] The max-norm difference between the resulting distribution and the uniform one is upper-bounded by the difference according to the L2-norm, which equals the bias of the sequence (cf., [2, Sec. 1.5]).

$\mathrm{GF}(2)^{(t'-2)\cdot n}$ and $\overline{I} = (I_1, ..., I_{t'}) \in \binom{[n]}{b}^{t'}$, for every non-empty $S \subseteq [b]^{t'}$, the matrix corresponding to $B_S^{(\overline{s},\overline{I})}$ has rank $\Omega(n)$. On the other hand, Lemma 2.1 implies that, under the same probability space, with probability at least $2/3$, there exists a set S such that the matrix corresponding to $B_S^{(\overline{s},\overline{I})}$ has rank at most $5m$. Hence, we reach contradiction unless either $\mathrm{AN}_2(F) = \Omega(n)$ or $2^{b^{t'}} \cdot 2^{-n/2} > 1/3$. Using $b = (n/\mathrm{AN}_2(F))^\beta$, for $\beta \leq 1/2$, and setting $t' = 2\log_{n/\mathrm{AN}_2(F)} n + \log_b \mathrm{AN}_2(F)$, we get

$$
\begin{aligned}
b^{t'} &= b^{2\log_{n/\mathrm{AN}_2(F)} n + \log_b \mathrm{AN}_2(F)} \\
&= n^{2\beta} \cdot \mathrm{AN}_2(F)
\end{aligned}
$$

which means that $2^{b^{t'}} \cdot 2^{-n/2} > 1/3$ holds if and only if $n^{2\beta} \cdot \mathrm{AN}_2(F) > 0.5 \cdot n - \log_2 3$. Hence, we reach contradiction unless $\mathrm{AN}_2(F) > 0.5 \cdot n^{1-2\beta} - o(1)$. Setting $\beta = \epsilon/2$, we conclude that $\mathrm{AN}_2(F) = \Omega(n^{1-\epsilon})$, with $t' = 2\log_{n^\epsilon} n + \log_{(n^\epsilon)^\beta} n = O(1/\epsilon^2)$. Theorem 1.7 follows by using an adequate small-bias generator, as provided by Theorem A.6.

Remark 2.3 (using rigidity rather than pure rank): *We note that the proof of Lemma 2.1 can be adapted to show that the corresponding matrix has rigidity $O(b^d \cdot m^{d+3}/n^d)$ with respect to rank $5m$. On the other hand, using [3, Footnote 13], one can adapt the proof of Lemma 2.2 to show that the corresponding matrix has rigidity $\Omega(n^3/m^2)$ with respect to rank $5m$. We stress that, like the argument regarding rank, the argument regarding rigidity requires $m \cdot b^d < 0.5n - 2$ and $b \geq 2$. Using $b = (n/m)^\beta$, the rigidity argument allows to infer that $m = \Omega(n^{1-\frac{2}{(1-\beta)\cdot d+5}})$ rather than $m = \Omega(n^{1-2\beta})$ as inferred by the foregoing rank argument, when using $d = 2/\epsilon = 1/\beta$. Hence, obtaining $m = \Omega(n^{1-\epsilon})$ via the rigidity argument uses d such that $\frac{2}{(1-\beta)\cdot d+5} = \epsilon$, which yields $d = \frac{2}{\epsilon} - \frac{8}{2-\epsilon}$. This modest gain (of approximately four units) in d translates to a similar gain in t'.*

The foregoing comparison refers to the current setting of d and b, which is not optimal anyhow. But it seems that $t' = \Omega(1/\epsilon^2)$ will follow in any case.

3 Proof of Theorem 1.8

We adapt the techniques used in Section 2 in order to prove the same lower bound on a more explicit function. Specifically, for t' to be determined, we consider the $(t'+3)$-linear function $\mathtt{f} : \mathrm{GF}(2)^{(t'+2)\cdot n + (t'+2)n} \to \mathrm{GF}(2)$ defined as

$$
\mathtt{f}(x^{(1)}, x^{(2)}, ..., x^{(t'+3)}) = \sum_{i_1,...,i_{t'+2}\in[n]} \left(\prod_{j\in[t'+2]} x_{i_j}^{(j)} \right) \cdot x_{i_1+i_2+\cdots+i_{t'+2}}^{(t'+3)} \tag{26}
$$

where the last block of variables has length $(t'+2) \cdot n$ rather than n. (Alternatively, we may partition the last block to $t'+2$ blocks holding n variables each.)[12] We mention that this function generalized the trilinear function of Part 2 of Theorem 1.5.

---

[12]That is, the variable-block $x^{(t'+2)} = (x_1^{(t'+2)}, ..., x_{(t'+2)n}^{(t'+2)})$ is replaced by $t'+2$ variables-blocks $(x^{(t'+3)}, x^{(t'+4)}, ..., x^{(2t'+4)})$ such that the $(j-1) \cdot n + k^{\mathrm{th}}$ bit of $x^{(t'+3)}$ is replaced by the kth bit of $x^{(t'+2+j)}$. Strictly speaking, this yields a multilinear function that does not fit Eq. (1), but this can be corrected by augmenting each variable-block with a dummy variable that will be set to 1.

The analysis of depth-two multilinear circuits, which underlies the proof of Lemma 2.1 (i.e., the "low AN2-complexity implies low rank"), remains almost intact, whereas the "high rank lemma" is totally different. In order to fit the latter lemma, the "low rank lemma" is restricted in the choice of a non-empty set $S \subseteq [b]^{t'}$ (such that the matrix corresponding to the bilinear function B_S has low rank). Specifically, rather than asserting (for every $\overline{I} = (I_1, ..., I_{t'})$) the existence of an arbitrary non-empty set S in $[b]^{t'}$, we consider only non-empty sets S such that $|\{\sum_{j \in [t']} I_j(k_j) : (k_1, ..., k_{t'}) \in S\}| = |S|$. That is, defining $\mu_{\overline{I}} : [b]^{t'} \to [t'n]$ such that

$$\mu_{\overline{I}}(k_1, ..., k_{t'}) \stackrel{\text{def}}{=} \sum_{j \in [t']} I_j(k_j), \tag{27}$$

we say that $S \subseteq [b]^{t'}$ is $\overline{I}$-admissible if $|\{\mu_{\overline{I}}(\overline{k}) : \overline{k} \in S\}| = |S|$, which means that the function $\mu_{\overline{I}}$ is injective when restricted to S. With this definition in place, we adapt the "low rank lemma" as follows.

Lemma 3.1 (low AN2-complexity of $\mathbf{f}$ implies low rank of B_S): *For m, b, t' such that $b \in [\omega(1), (n/m)^{1/2}]$ and $t' \in [2 \log_{n/m} n + \log_b m + 1, \log_b(n/2) - 1]$, suppose that $\mathtt{AN}_2(\mathbf{f}) \leq m < n/10$. Then, for every $w \in \mathrm{GF}(2)^{(t'+2) \cdot n}$, with probability at least $2/3$ over a random choice of $\overline{I} = (I_1, ..., I_{t'}) \in \binom{[n]}{b}^{t'}$, there exists a non-empty $\overline{I}$-admissible set $S \subseteq [b]^{t'}$ such that the matrix corresponding to the following bilinear function $B_S^{(w, \overline{I})}$ has rank at most $5m$.*

$$B_S^{(w, \overline{I})}(y, z) \stackrel{\text{def}}{=} \sum_{\overline{k} = (k_1, ..., k_{t'}) \in S} \mathbf{f}(\mathbf{u}(I_1(k_1)), ..., \mathbf{u}(I_{t'}(k_{t'})), y, z, w), \tag{28}$$

where $\mathbf{u}(i) = 0^{i-1}10^{n-i}$ is the ith unit vector and $I_j(k)$ denotes the kth element in I_j.

Recall that we were using $b^{t'} < n/2$ anyhow, when contrasting the "low rank lemma" with the "high rank lemma" (so we lose nothing by the restriction $t' \leq \log_b(n/2) - 1$). Ditto re requiring $b = \omega(1)$, which is actually not essential (i.e., $b \geq C$ for a sufficiently large constant C will do).

Proof: The proof of Lemma 2.1 actually establishes its claim with probability at least 0.7 (rather than 2/3) over the choice of $\overline{I}$, but it does not necessarily select an $\overline{I}$-admissible set S. Recall that the existence of a suitable set $S \subset [b]^{t'}$ is merely based on the fact that $|[b]^{t'}| \geq b^d \cdot m$, and any subset U of $[b]^{t'}$ that has size at least $b^d \cdot m$ will do (i.e., allow us to argue that there exists an adequate set $S \subseteq U$ that satisfies the claim of the lemma).

Now, using the hypothesis $b^{t'} \leq n/2b$, we show that with probability $1 - o(1)$ over the choice of $\overline{I}$, the set $[b]^{t'}$ contains an $\overline{I}$-admissible set U of size at least $b^{t'}/2$, and we can afford to restrict S to be a subset of U (since $b^{t'-1} \geq b^d \cdot m$). To see this fact, observe that, for every two distinct sequences $\overline{k} \neq \overline{k}'$ in $[b]^{t'}$, it holds that $\Pr_{\overline{I}}[\mu_{\overline{I}}(\overline{k}) \equiv \mu_{\overline{I}}(\overline{k}') \pmod{n}] = 1/n$, which implies that $\Pr_{\overline{I}}[|\mu_{\overline{I}}^{-1}(\mu_{\overline{I}}(\overline{k}))| > 1] \leq (b^{t'} - 1)/n < 1/2b$ for every $\overline{k} \in [b]^{t'}$. The claim follows. ∎

The new "high rank lemma". The key observation is that, for every fixed $(i_1, ..., i_{t'}) \in [n]^{t'}$, the residual function $\mathbf{f}'(y, z, w) \stackrel{\text{def}}{=} \mathbf{f}(i_1, ..., i_{t'}, y, z, w)$ is closely related to the trilinear function of Part 2 of Theorem 1.5. In particular, for a uniformly distributed $w \in \mathrm{GF}(2)^{(t'+2)n}$, the matrix that corresponds to the residual bilinear function $\mathbf{f}'(\cdot, \cdot, w)$ is a random Toeplitz (or rather Hankel)

matrix. Furthermore, the same holds for $B_S^{(w,\bar{I})}$ provided that the set S is $\bar{I}$-admissible (and non-empty).

Lemma 3.2 (typically B_S has high rank): *For every sequence of b-subsets $\bar{I} = (I_1, ..., I_{t'}) \in \binom{[n]}{b}^{t'}$ and any non-empty and $\bar{I}$-admissible set $S \subseteq [b]^{t'}$, with probability $1 - 2^{-n/2}$ over the choice of $w \in \mathrm{GF}(2)^{(t'+2)\cdot n}$, it holds that the matrix corresponding to the bilinear function $B_S^{(w,\bar{I})}$ of Eq. (28) has rank $\Omega(n)$.*

Proof: For every $S \subseteq [b]^{t'}$, $\bar{I} \in \binom{[n]}{b}^{t'}$ and $w \in \mathrm{GF}(2)^{(t'+2)\cdot n}$, looking at the value of $B_S^{(w,\bar{I})}$, while letting $\bar{u}(\bar{k}) = (u(I_1(k_1)1), ..., u(I_{t'}(k_{t'})))$, we have

$$B_S^{(w,\bar{I})}(y,z) = \sum_{\bar{k} \in S} f(\bar{u}(\bar{k}), y, z, w) \tag{29}$$

$$= \sum_{\bar{k} \in S} \sum_{i_{t'+1}, i_{t'+2} \in [n]} y_{i_{t'+1}} \cdot z_{i_{t'+2}} \cdot w_{I_1(k_1)+I_2(k_2)+\cdots+I_{t'}(k_{t'})+i_{t'+1}+i_{t'+2}}$$

$$= \sum_{i_{t'+1}, i_{t'+2} \in [n]} y_{i_{t'+1}} \cdot z_{i_{t'+2}} \cdot \sum_{\bar{k} \in S} w_{\mu_{\bar{I}}(k_1,...,k_{t'})+i_{t'+1}+i_{t'+2}} \tag{30}$$

where the second equality is due to the fact that assigning $\bar{u}(\bar{k})$ to the first t' variable-blocks eliminates all terms that are not of the form $(I_1(k_1), I_2(k_2), ..., I_{t'}(k_{t'}), \cdot, \cdot, \cdot)$, and the third equality uses Eq. (27) (i.e., $\sum_{j \in [t']} I_j(k_j)) = \mu_{\bar{I}}(k_1, k_2, ..., k_{t'})$).

The key observation is that, for every $i_{t'+1}, i_{t'+2} \in [n]$, the sum $\sum_{\bar{k} \in S} w_{\mu_{\bar{I}}(k_1,...,k_{t'})+i_{t'+1}+i_{t'+2}}$ does not vanish when $S \neq \emptyset$. This is the case because the hypothesis that S is $\bar{I}$-admissible implies that different $\bar{k}$'s yield different $\mu_{\bar{I}}(\bar{k})$'s, which implies that there are no cancellation among terms of the foregoing sum. Hence, Eq. (29)&(30) implies that that there exists a non-empty set $S' \subseteq [(t'+2)n]$ (indeed, $S' = \{\mu_{\bar{I}}(\bar{k}) : \bar{k} \in S\}$) such that

$$B_S^{(w,\bar{I})}(y,z) = \sum_{i_{t'+1}, i_{t'+2} \in [n]} y_{i_{t'+1}} \cdot z_{i_{t'+2}} \cdot \sum_{s \in S'} w_{s+i_{t'+1}+i_{t'+2}} \tag{31}$$

Note that the bilinear function of Eq. (31) corresponds to a Hankel matrix, since it has the form $\sum_{i,j \in [n]} w'_{i+j} \cdot y_i \cdot z_j$ such that $w'_{i,j}$ is invariant when $i+j$ is fixed. Specifically, we consider the n-by-n matrix $(w'_{i,j})_{i,j \in [n]}$ such that $w'_{i,j} = \sum_{s \in S'} w_{s+i+j}$. Furthermore, when selecting $w \in \mathrm{GF}(2)^{(t'+2)n}$ uniformly at random, the bilinear function of Eq. (31) corresponds to a random n-by-n Hankel matrix (i.e., the sequence $(\sum_{s \in S'} w_{s+2}, ..., \sum_{s \in S'} w_{s+2n})$ is uniformly distributed in $\mathrm{GF}(2)^{2n-1}$). This is the case because the mapping $w \mapsto (\sum_{s \in S'} w_{s+2}, ..., \sum_{s \in S'} w_{s+2n})$ is linear and onto.[13]

Finally, note that a random n-by-n Hankel matrix has rank at least $n/10$ with probability at least $1 - 2^{-n/2}$. This is the case since each (non-trivial) combination of its columns is uniformly distributed in $\mathrm{GF}(2)^n$, and so the probability that there exists a linear combination of at most $n/10$ columns that equals the all-zero vector is upper-bounded by $\sum_{i \in [n/10]} \binom{n}{i} \cdot 2^{-n} < 2^{-n/2}$. The lemma follows. ∎

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[13]This can be seen by considering the corresponding linear system $\sum_{s \in S'} x_{s+i} = b_i$ for $i = 2, ..., 2n$, and observing that the corresponding $(2n-1)$-by-$|\{s+i : s \in S' \& i \in [2, 2n]\}|$ matrix has full rank (due to columns $s+2, ..., s+2n$, where s is the largest element in S').

Conclusion: As in Section 2.4, using a union bound on all possible $S \subseteq [b]^{t'}$, Lemma 3.2 implies that, with probability at least $1 - 2^{b^{t'}} \cdot 2^{-n/2} > 1/2$ over the choices of $w \in \mathrm{GF}(2)^{(t'+2)\cdot n}$ and $\overline{I} = (I_1, ..., I_{t'}) \in \binom{[n]}{b}^{t'}$, for every non-empty and $\overline{I}$-admissible $S \subseteq [b]^{t'}$, the matrix corresponding to $B_S^{(w,\overline{I})}$ has rank $\Omega(n)$. On the other hand, Lemma 3.1 implies that, under the same probability space, with probability at least $2/3$, there exists a non-empty and $\overline{I}$-admissible set S such that the matrix corresponding to $B_S^{(w,\overline{I})}$ has rank at most $5m$. Hence, we reach contradiction unless $\mathtt{AN}_2(\mathtt{f}) \geq n^{1-\epsilon}$, but this is all conditioned on $2^{b^{t'}} \cdot 2^{-n/2} < 1/2$. Using $m = n^{1-\epsilon}$ and $b = (n/m)^{0.4\epsilon}$, and setting $t' = 2\log_{n/m} n + \log_b m + 1$, the foregoing condition holds (since $b^{t'} = n^{0.8\epsilon} \cdot m \cdot b = o(n)$). We conclude that $\mathtt{AN}_2(\mathtt{f}) \geq n^{1-\epsilon}$, where $t' = 2\log_{n^\epsilon} n + \log_{(n^\epsilon)^{0.4\epsilon}} n + 1 = 2 \cdot \epsilon^{-1} + (0.4 \cdot \epsilon^2)^{-1} + 1$. Noting that $(2\epsilon + (0.4)^{-1}) \cdot \epsilon^{-2} + 1 < 4 \cdot \epsilon^{-2} - 3$ holds for $\epsilon \leq 1/4$, we get $t' + 3 \leq 4/\epsilon^2$, and Theorem 1.8 follows.

Remark 3.3 (computing $\mathtt{f}$ in quadratic time): *For every $t, n \in \mathbb{N}$ and $s \in [t, tn]$, consider the function $F_s^{(t)} : \{0, 1\}^{tn} \to \{0, 1\}$ defined by*

$$F_s^{(t)}(x^{(1)}, x^{(2)}, ..., x^{(t)}) = \sum_{i_1,...,i_t \in [n]: \sum_{j \in [t]} i_j = s} \prod_{j \in [t]} x_{i_j}^{(j)} \tag{32}$$

and note that $F_s^{(t)}(x^{(1)}, x^{(2)}, ..., x^{(t)})$ equals $\sum_{s' \in [s-n, s-1]} F_{s'}^{(t-1)}(x^{(1)}, x^{(2)}, ..., x^{(t-1)}) \cdot x_{s-s'}^{(t)}$. Hence, $(F_s^{(t)})s \in [t, tn]$ can be computed in time $t \cdot tn \cdot n$ (by Dynamic programming). Observing that $\mathtt{f}(x^{(1)}, x^{(2)}, ..., x^{(t'+3)})$ equals $\sum_{s \in [t'+2, (t'+2)n]} F_s^{(t'+2)}(x^{(1)}, x^{(2)}, ..., x^{(t'+2)}) \cdot x_s^{(t'+3)}$, it follows that Eq. (26) can be computed in quadratic-time.

In contrast, it is not clear whether the functions used in the proof of Theorem 1.7 can be computed in fixed polynomial-time, rather than in $O(n^{\mathrm{poly}(1/\epsilon)})$-time.

4 Extension to vanishing ϵ

Theorem 1.7 (and likewise Theorem 1.8) can be extended to $\epsilon = \epsilon(n)$ that vanishes with n, provided that $\epsilon(n) \geq \sqrt{2/\log_2 n}$ (resp., $\epsilon(n) = \omega(\sqrt{1/\log_2 n})$), since the argument conditions ϵ only by presupposing that $b = n^{\epsilon \cdot \beta} = n^{\epsilon^2/2}$ is at least 2 (resp., $\omega(1)$). Hence, we actually have

Theorem 4.1 (Theorem 1.7, rephrased): *For every $\epsilon : \mathbb{N} \to (0, 1)$ such that $\epsilon(n) \geq \sqrt{2/\log_2 n}$, letting $t(n) = \mathrm{poly}(1/\epsilon(n))$, there exists a quasi-polynomial-time computable $t(n)$-linear function $f : \{0, 1\}^{t(n) \cdot n} \to \{0, 1\}$ such that $\mathtt{AN}_2(f) = \Omega(n^{1-\epsilon(n)})$.*

An analogous result holds for Theorem 1.8, where we may use $t(n) = O(1/\epsilon(n)^2)$ and have quadratic-time. Note that the foregoing does not allow setting $\epsilon(n) = \Omega(1/\log n)$ and deriving an $\Omega(n)$ lower bound, but such a lower bound would have missed the target of being truly linear in the length of the input (i.e., $\mathrm{poly}(1/\epsilon(n)) \cdot n$). This raises several challenges:

1. Prove that there exists a quasi-polynomial-time computable $\mathrm{poly}(\log n)$-linear function $f : \{0, 1\}^{\widetilde{O}(n)} \to \{0, 1\}$ such that $\mathtt{AN}_2(f) = \Omega(n)$.

2. Improving over Item 1, prove that such a function can be computed in polynomial-time.

3. Improving over Item 1, for $t(n) = \text{poly}(\log n)$, prove that there exists a quasi-polynomial-time computable $t(n)$-linear function $f : \{0,1\}^{t(n) \cdot n} \rightarrow \{0,1\}$ such that $\text{AN}_2(f) = \Omega(t(n) \cdot n)$.

As in Item 2, improve the running time to polynomial.

Of course, a more pressing challenge is to address the first part of Problem 1.6; that is, presenting an explicit $O(1)$-linear function $f : \{0,1\}^{O(n)} \rightarrow \{0,1\}$ satisfying $\text{AN}(f) = \omega(n^{2/3})$, let alone $\text{AN}(f) = \Omega(n^{0.99})$. Recall that the general AN-complexity of multi-linear functions may be lower than its depth-two complexity; in particular, there exist bilinear functions f such that $\text{AN}(f) = o(\text{AN}_2(f))$; (i.e., $\text{AN}(f) = O(n^{1/2})$ and $\text{AN}_2(f) = \Omega(n^{2/3})$) [4, Thm 2.3].

Acknowledgements

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Appendix: On small-bias generators of large stretch

In this appendix we present explicit constructions of small-bias generators of arbitrary large (polynomial) stretch that can be computed by $O(1)$-linear functions. Our start point is the generator of of Mossel, Shpilka, and Trevisan [6], which has quadratic stretch and can be computed by bilinear functions. We show that composing it with itself, which is a take on an idea of Naor and Naor [7], yields the desired generators.

We first describe a generic composition lemma, which refers to generators of bounded locality, and then set-up a simple iterative process that yields generators of increased stretch (and larger locality). Next, we present versions of these ingredients that refer to generators that can be computed by polynomials of bounded degree. Lastly, we adapt the latter to support multilinear computation.

The notations used in this appendix are different from those used in the main text. The main parameter is the seed length, denoted k, and throughout this appendix the stretch and bias will be stated as functions of k. However, for the final application, given $t' \in \mathbb{N}$, we shall set $t'' = \text{poly}(t')$ and $n = k/(t'' - 2)$, and obtain a $(t'' - 2)$-linear generator that outputs sequences of length $n^{t'+2}$ with bias at most 2^{-n}, where n is as in the main text.

We shall extensively use the notation U_m, which represents a random variable uniformly distributed over $\{0,1\}^m \equiv \mathrm{GF}(2)^m$. Throughout the text, we assume that the stretch function is superlinear and monotonically increasing, and that the bias bound in monotonically non-increasing. We recall the following standard definition.

Definition A.1 (generators of bounded bias): *We say that $G : \{0,1\}^k \to \{0,1\}^{s(k)}$ has* bias at most $\epsilon(k)$ *with respect to tests of degree d if for every polynomial* (test) $T : \{0,1\}^{s(k)} \to \{0,1\}$ *of degree d it holds that*

$$\frac{1}{2} \cdot \left| \mathrm{E}\left[(-1)^{T(G(U_k))} \right] - \mathrm{E}\left[(-1)^{T(U_{s(k)})} \right] \right| \leq \epsilon(k). \tag{33}$$

The function $s : \mathbb{N} \to \mathbb{N}$ is called the stretch *of G, and $\epsilon : \mathbb{N} \to [0,1]$ is called its* bias.

The l.h.s. of Eq. (33) equals the total variation distance between the "verdict" of T in the two cases (i.e., the statistical difference between $T(G(U_k))$ and $T(U_{s(k)})$). Indeed, ϵ-bias generators (cf., [7] and [1, Sec. 8.5.2]) correspond to the special case of linear tests (i.e., $d = 1$). In general, whenever we talk of bias without specifying the degree, we mean bias with respect to linear tests.

A.1 A general composition lemma

Although we are interested in small-bias generators (i.e., bias w.r.t linear tests), it will be useful to have the following composition result that refers to generators with respect to tests of bounded degree. Specifically, we consider generators of *bounded locality* and specified stretch, which have small bias with respect to polynomials of bounded degree. Recall that a function is said to have locality ℓ if each bit in its output is a function of at most ℓ bits in its input (cf. [6]).

Lemma A.2 (composition of generators, a special case): *For $i \in \{1, 2\}$, let d_i and ℓ_i be constants, $s_i : \mathbb{N} \to \mathbb{N}$ be a stretch function, and $\epsilon_i : \mathbb{N} \to (0, 1]$ be a bias bound. Suppose that $G_i : \{0,1\}^k \to \{0,1\}^{s_i(k)}$ has locality ℓ_i and bias at most $\epsilon_i(k)$ with respect to all tests (i.e., polynomials) of degree d_i. Then, $G = G_2 \circ G_1 : \{0,1\}^k \to \{0,1\}^{s_2(s_1(k))}$ has locality $\ell_2 \cdot \ell_1$ and bias at most $\epsilon(k) = \epsilon_1(k) + \epsilon_2(s_1(k))$ with respect to all tests of degree $d = \min(d_2, d_1/\ell_2)$.*

Hence, the degree of tests that the composed generator withstands is the minimum between the degree withstand by the outer generator and a $1/\ell_2$ fraction of the degree withstand by the inner generator, where ℓ_2 is the locality of the outer generator. The general case allows the degree d_i and the locality ℓ_i to be functions of k. In this case, G has bias at most $\epsilon(k) = \epsilon_1(k) + \epsilon_2(s_1(k))$ with respect to all tests of degree $d(k) = \min(d_2(s_1(k)), d_1(k)/\ell_2(s_1(k)))$.

Proof: We use a hybrid argument (cf. [1, Sec. 8.2.3.3]), while considering the following three distributions:

1. The pseudorandom output $G(U_k) = G_2(G_1(U_k))$.

2. The intermediate hybrid $G_2(U_{s_1(k)})$.

3. The uniform distribution $U_{s_2(s_1(k))}$.

Let T be an arbitrary test of degree d. Then,

$$
\left| \mathrm{E}\left[(-1)^{T(G_2(G_1(U_k)))}\right] - \mathrm{E}\left[(-1)^{T(G_2(U_{s_1(k)}))}\right] \right|
$$
$$
= \left| \mathrm{E}\left[(-1)^{T'(G_1(U_k))}\right] - \mathrm{E}\left[(-1)^{T'(U_{s_1(k)})}\right] \right|
$$
$$
\leq \epsilon_1(k),
$$

since $T' = T \circ G_2$ is a test of degree $d \cdot \ell_2 \leq d_1$ (and G_1 has bounded bias w.r.t such tests). Using the fact that $d \leq d_2$ (and the hypothesis regarding G_2), we have

$$
\left| \mathrm{E}\left[(-1)^{T(G_2(U_{s_1(k)}))}\right] - \mathrm{E}\left[(-1)^{T(U_{s_2(s_1(k))})}\right] \right| \leq \epsilon_2(s_1(k)).
$$

The claim follows.   ∎

A.2   An iterative construction

The basic idea is to iteratively compose a small bias generator of constant locality, which fools linear tests, with itself. But for the process to work we need the inner generator, in the composition, to fool tests with degree that at least equals the locality of the outer generator (see Lemma A.2). Using Viola's result [11], we can get such an inner generator by taking the sum of a constant number of instances of the current generator (which fool linear tests), and keep using the original generator as the outer one. Details follow.

The starting point. Let G be a small bias generator that has constant locality ℓ, some stretch $s : \mathbb{N} \to \mathbb{N}$, and bias $\epsilon : \mathbb{N} \to (0, 1]$ with respect to linear tests. We shall use G as the outer generator in all compositions. In addition, we shall use G as the inner generator in the first iteration; that is, we let $G^{(0)}$ equal G; hence, $G^{(0)}$ has locality $\ell^{(0)} = \ell$, stretch $s^{(0)}(k) = s(k)$ and bias at most $\epsilon^{(0)}(k) = \epsilon(k)$.

Iteration $i \in \mathbb{N}$. Given an $\ell^{(i-1)}$-local generator $G^{(i-1)} : \{0,1\}^k \to \{0,1\}^{s^{(i-1)}(k)}$ that fools linear tests with bias $\epsilon^{(i-1)}(k)$, we first obtain a generator $\widehat{G}^{(i-1)} : \{0,1\}^{\ell \cdot k} \to \{0,1\}^{s^{(i-1)}(k)}$ that fools degree ℓ tests with bias $O(\epsilon^{(i-1)}(k)^{2^{-(\ell-1)}})$, by XORing ℓ instances of $G^{(i-1)}$ (see Viola [11]); that is, $\widehat{G}^{(i-1)}(x_1, ..., x_\ell) = \bigoplus_{j \in [\ell]} G^{(i-1)}(x_j)$. Hence, $\widehat{G}^{(i-1)}$ has locality $\ell' = \ell \cdot \ell^{(i-1)}$, stretch $s'(k) = s^{(i-1)}(k/\ell)$, and bias $\epsilon'(k) = O(\epsilon^{(i-1)}(k/\ell)^{2^{-(\ell-1)}})$ w.r.t tests of degree ℓ. Next, applying Lemma A.2, we obtain $G^{(i)} = G \circ \widehat{G}^{(i-1)}$, and observe that $G^{(i)}$ has locality $\ell^{(i)} = \ell \cdot \ell' = \ell^2 \cdot \ell^{(i-1)}$, stretch $s^{(i)}(k) = s(s'(k)) = s(s^{(i-1)}(k/\ell))$, and bias at most $\epsilon^{(i)}(k) = \epsilon(s(k)) + \epsilon'(k) = O(\epsilon^{(i-1)}(k/\ell)^{2^{-\ell}})$ with respect to linear tests (since the locality of G equals the degree that $\widehat{G}^{(i-1)}$ is guaranteed to fool).

Hence, after $\tau \in \mathbb{N}$ iterations, we obtain a generator (i.e., $G^{(\tau)}$) that has locality $\ell^{(\tau)} = \ell^{2\tau} \cdot \ell^{(0)} = \ell^{2\tau+1}$, bias at most $\epsilon^{(\tau)}(k) < O(1)^\tau \cdot \epsilon(k/\ell^\tau)^{(2^{-\ell})^\tau}$ with respect to linear tests, and stretch $s^{(\tau)}(k) = s^{\circ^{\tau+1}}(k/\ell^\tau)$, where s°^τ} denotes s composed with itself τ times (i.e., $s^{\circ^i}(k) = s(s^{\circ^{i-1}}(k))$ and $s^{\circ^1}(k) = s(k)$). Hence, we obtain the following result, which is not used in this work (and is stated merely for sake of future reference).

Corollary A.3 (amplifying the stretch of small-bias generators of bounded locality): *Let G be a generator that has constant locality ℓ, stretch $s : \mathbb{N} \to \mathbb{N}$, and bias $\epsilon : \mathbb{N} \to (0,1]$ with respect to linear tests. Suppose that $\tau \leq 0.5 \log_\ell k$ and that $s(k) \geq k^\alpha$ for some constant $\alpha > 1$. Then, $G^{(\tau)}$ has locality $\ell^{(\tau)} = \ell^{2\tau+1}$, stretch $s^{(\tau)}(k) \geq k^{\alpha^{\tau+1}/2}$, and bias at most $\epsilon^{(\tau)}(k) < O(1)^\tau \cdot \epsilon(\sqrt{k})^{(2^\tau)^{-\ell}}$ with respect to linear tests.*

A.3 Adaptation to constructions of bounded degree generators

We actually seek a construction of generators that can be computed by bounded degree polynomials rather than by functions of bounded locality. The foregoing analysis extends in a straightforward manner to the current case, yielding the following.

Lemma A.4 (Lemma A.2, revisited): *For $i \in \{1, 2\}$, let d_i and D_i be constants, $s_i : \mathbb{N} \to \mathbb{N}$ be a stretch function, and $\epsilon_i : \mathbb{N} \to (0,1]$ be a bias bound. Suppose that $G_i : \{0,1\}^k \to \{0,1\}^{s_i(k)}$ can be computed by a sequence of polynomials of degree D_i and has bias at most $\epsilon_i(k)$ with respect to all tests of degree d_i. Then, $G = G_2 \circ G_1 : \{0,1\}^k \to \{0,1\}^{s_2(s_1(k))}$ can be computed by a sequence of polynomials of degree $D_2 \cdot D_1$ and has bias at most $\epsilon(k) = \epsilon_1(k) + \epsilon_2(s_1(k))$ with respect to all tests of degree $d = \min(d_2, d_1/D_2)$.*

The proof is identical to the proof of Lemma A.2, and the iterative construction works as well.

Here we can use the generator of Mossel, Shpilka, and Trevisan [6], which has $s(k) = \Omega(k^2)$ and $\epsilon(k) = 2^{\Omega(k)}$, with $D = 2$. Observe that, after $\tau \in \mathbb{N}$ iterations, we obtain a generator (i.e., $G^{(\tau)}$) that can be computed by a sequence of polynomials of degree $D^{(\tau)} = D^{2\tau+1} = 2^{2\tau+1}$, has stretch $s^{(\tau)}(k) = s^{\circ^{\tau+1}}(k/D^t) = \Omega(k/2^\tau)^{2^{\tau+1}}$, and bias at most $\epsilon^{(\tau)}(k) = O(1)^\tau \cdot \epsilon(k/D^\tau)^{(2^{-D})^\tau} = \exp(\exp(-O(\tau)) \cdot k)$. Hence, seeking stretch of the form n^σ, we set $\tau = \log_2 \sigma$, and obtain degree $2 \cdot \sigma^2$ and bias at most $2^{-\text{poly}(1/\sigma) \cdot k}$.

A.4 Adaptation to multilinear constructions of bounded degree

Actually, we need the construction to be multilinear; that is, we seek a construction of generators that can be computed by bounded degree multi-linear functions. While the transformation from

fooling linear tests to fooling tests of constant degree preserves the multilinearity of the generator (since it XORs independently generated outputs of the original generator), the composition lemma does not necessarily preserve multilinearity. That is, even if both G_i's are computed by sequences of multilinear functions (of bounded degree), their composition may not be so (since G_2 may multiply output bits of G_1 that depend on the same variable-block). Still, a small modification suffices to provide multilinearity.

Lemma A.5 (Lemma A.4, revisited): *For $i \in \{1, 2\}$, let d_i and D_i be constants, $s_i : \mathbb{N} \to \mathbb{N}$ be a stretch function, and $\epsilon_i : \mathbb{N} \to (0, 1]$ be a bias bound. Suppose that $G_i : \{0, 1\}^k \to \{0, 1\}^{s_i(k)}$ can be computed by a sequence of D_i-linear functions, where an m-linear function from $\mathrm{GF}(2)^k$ to $\mathrm{GF}(2)$ is linear in each of the m (equal-length) blocks of variables, and has bias at most $\epsilon_i(k)$ with respect to all tests of degree d_i. Let $G'_1 : \{0, 1\}^k \to \{0, 1\}^{D_2 \cdot s_1(k/D_2)}$ be an algorithm that partitions its input to D_2 equal-length parts, applies G_1 to each part, and concatenate the results. Then, $G = G_2 \circ G'_1 : \{0, 1\}^k \to \{0, 1\}^{s_2(D_2 \cdot s_1(k/D_2))}$ can be computed by a sequence of $D_2 \cdot D_1$-linear functions and has bias at most $\epsilon(k) = D_2 \cdot \epsilon_1(k/D_2) + \epsilon_2(s_1(k))$ with respect to all tests of degree $d = \min(d_2, d_1/D_2)$.*

Proof: We first observe that, by construction, G is $D_2 \cdot D_1$-linear, since the different D_2 (equal length) blocks of the input to G_2 depend on disjoint (k/D_2)-bit long parts of the seed of G'_1 (i.e., the ith block in the input to G_2 depends on the ith part of the seed of G'_1). Specifically, on input $\overline{x} = (x_1, ..., x_{D_2}) \in \{0, 1\}^{D_2 \cdot (k/D_2)}$, each monomial in the computation of $G(\overline{x})$ depends on at most D_2 bits of $G'_1(\overline{x})$, which by the D_2-linearity of G_2 occur in different parts in $G'_1(\overline{x}) = (G_1(x_1), ..., G_1(x_{D_2}))$, whereas each of these parts is computed by a D_1-linear function of the corresponding seed (i.e., the ith part of $G'_1(\overline{x})$ appears in $G_1(x_i)$, and is computed by a D_1-linear function of x_i).

All that remains is to observe that G'_1 has bias at most $D_2 \cdot \epsilon_1(k/D_2)$ with respect to tests of degree d_1. We just use a hybrid argument, where the ith hybrid, denoted H_i, consists of i independent copies of $G_1(U_{k/D_2})$ followed by $D_2 - i$ independent copies of $U_{s_1(k/D_2)}$. Then, for any test T of degree d_1, it holds that

$$
\left| \mathrm{E}\left[(-1)^{T(U_{D_2 \cdot s_1(k/D_2))})} \right] - \mathrm{E}\left[(-1)^{T(G'_1(U_k))} \right] \right|
$$
$$
\leq \sum_{i \in [D_2]} \left| \mathrm{E}\left[(-1)^{T(H_{i-1})} \right] - \mathrm{E}\left[(-1)^{T(H_i)} \right] \right|
$$
$$
= \sum_{i \in [D_2]} \left| \mathrm{E}\left[(-1)^{T_i(U_{s_1(k/D_2))})} \right] - \mathrm{E}\left[(-1)^{T_i(G_1(U_{k/D_2}))} \right] \right|
$$
$$
\leq D_2 \cdot \epsilon_1(n/D_2),
$$

where $T_i(z)$, which may be viewed as a distribution over tests of degree d_1, generates $i - 1$ independent copies of $G_1(U_{k/D_2})$, denoted $v_1, ..., v_{i-1}$, and $D_2 - i$ independent copies of $U_{s_1(k/D_2)}$, denoted $u_1, ..., u_{D_2 - i}$, and returns $T(v_1, ..., v_{i-1}, z, u_1, ..., u_{D_2 - i})$. Indeed, we use the fact that $H_0 = U_{D_2 \cdot s_1(k/D_2)}$ and $H_{D_2} = G'_1(U_k)$, whereas H_{i-1} and H_i differ only in their i^{th} part (which is $U_{s_1(k/D_2)}$ in H_{i-1} and $G_1(U_{k/D_2})$ in H_i). $\blacksquare$

Conclusion. Starting with the generator of Mossel, Shpilka, and Trevisan [6], while noting that it is actually bilinear, and using the iterative construction of Section A.2 with the composition of Lemma A.5, we get the desired generator (which is stated in the terms used in the main text).

Theorem A.6 (a small-bias generator of arbitrary large polynomial stretch that can be computed by $O(1)$-linear functions): *For every $t' \in \mathbb{N}$, there exist t'' and an explicit construction of an $(t'' - 2)$-linear generator $G_{\mathtt{sb}} : \mathrm{GF}(2)^{(t''-2)\cdot n} \to \{0,1\}^{n^{t'+2}}$ has bias at most 2^{-n}. Furthermore, $t'' = O(t')^2$.*

Given all the foregoing, the following proof is straighhtforward. It is being detailed here for sake of tedious verification.

Proof: We just mimic the argument of Section A.2, while using the generator of [6] as our "pivot" generator, and using the composition result of Lemma A.5 in all iterations. Specifically, letting $\epsilon(k) = 2^{\Omega(k)}$, we start with an ϵ-bias generator of stretch $s(k) = \Omega(k^2)$ that is computed by bilinear function. Recall that we let $G^{(0)}$ equal G, and so $D^{(0)} = D = 2$, $s^{(0)}(k) = s(k)$ and $\epsilon^{(0)}(k) = \epsilon(k)$.

Next, given a $D^{(i-1)}$-linear generator $G^{(i-1)} : \{0,1\}^k \to \{0,1\}^{s^{(i-1)}(k)}$ that fools linear tests with bias $\epsilon^{(i-1)}(k)$, we first obtain a $2 \cdot D^{(i-1)}$-linear generator $\widehat{G}^{(i-1)} : \{0,1\}^{2\cdot k} \to \{0,1\}^{s^{(i-1)}(k)}$ that fools quadratic tests with bias $O(\epsilon^{(i-1)}(k)^{1/2})$, by XORing two instances of $G^{(i-1)}$. Hence, $\widehat{G}^{(i-1)}$ has stretch $s'(k) = 2 \cdot s^{(i-1)}(k/2)$, and bias $\epsilon'(k) = O(\epsilon^{(i-1)}(k/2)^{1/2})$ w.r.t quadratic tests.

Next, applying Lemma A.5, we obtain $G^{(i)} = G \circ \widetilde{G}^{(i-1)}$, where $\widetilde{G}^{(i-1)}$ is obtained from $\widehat{G}^{(i-1)}$ by partitioning the seed into two (equal-length) parts and applying $\widehat{G}^{(i-1)}$ on each part. Hence, $G^{(i)}$ is $D^{(i)}$-linear, for $D^{(i)} = 2 \cdot 2D^{(i-1)} = 2^{2i+1}$, and has stretch $s^{(i)}(k) = s(s'(k)) = s(2 \cdot s^{(i-1)}(k/2))$, and bias at most $\epsilon^{(i)}(k) = \epsilon(s(k)) + \epsilon'(k) = O(\epsilon^{(i-1)}(k/2)^{1/2})$ with respect to linear tests. Assuming that $s(k) \geq c \cdot k^2$, for some constant $c > 0$, we have

$$
\begin{aligned}
s^{(i)}(k) &\geq c \cdot (2 \cdot s^{(i-1)}(k/2))^2 \\
&= (4c)^{2^{i+1}-1} \cdot s^{(0)}(k/2^i)^{2^i} \\
&= (4c)^{2^{i+1}-1} \cdot c \cdot (k/2^i)^{2^{i+1}} \\
&= \exp(-\widetilde{O}(2^i)) \cdot k^{2^{i+1}}
\end{aligned}
$$

and

$$
\begin{aligned}
\epsilon^{(i)}(k) &= O(\epsilon^{(i-1)}(k/2)^{1/2}) \\
&= O(1) \cdot \epsilon^{(0)}(k/2^i)^{2^{-i}} \\
&= \exp(\Omega(4^{-i} \cdot k)).
\end{aligned}
$$

Hence, after $\tau \in \mathbb{N}$ iterations, we obtain a $2^{2\tau+1}$-linear generator (i.e., $G^{(\tau)}$) that has stretch $s^{(\tau)}(k) > \exp(-\widetilde{O}(2^\tau)) \cdot k^{2^{\tau+1}}$ and bias at most $\epsilon^{(\tau)}(k) < \exp(\Omega(4^{-\tau} \cdot k))$. Thus, seeking stretch of the form k^σ, we set $\tau = \log_2 \sigma$, and obtain a $2 \cdot \sigma^2$-linear generator with stretch $\exp(-\widetilde{O}(\sigma)) \cdot k^{\sigma^2}$ and bias at most $2^{-\Omega(1/\sigma)^2 \cdot k}$. Letting $n = k/O(\sigma^2)$ (and using a finer partition of the k-bit long input seed), we can view this generator as being (k/n)-linear and having bias at most 2^{-n}. The theorem follows. ∎