Electronic Colloquium on Computational Complexity, Report No. 2 (2020)

Sensitivity lower bounds from linear dependencies

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January 6, 2020

Abstract

Recently, using spectral techniques, H. Huang proved that every subgraph of the hypercube of dimension n induced on more than half the vertices has maximum degree at least $\sqrt{n}$. Combined with some earlier work, this completed a proof of the sensitivity conjecture. In this work we show how to derive a proof of Huang's result using only linear dependency and independence of vectors associated with the vertices of the hypercube. Our approach leads to several improvements of the result. In particular we prove that in any induced subgraph of H_n with more than half the number of vertices, there are two vertices, one of odd parity and the other of even parity, each with at least n vertices at distance at most 2. As an application we show that for any Boolean function f, the polynomial degree of f is bounded above by $\mathrm{s}_0(f)\mathrm{s}_1(f)$, a strictly stronger statement which implies the sensitivity conjecture.

1 Introduction

Recently, Hao Huang [8] provided a beautiful short proof for the missing link of an important conjecture in complexity theory known as the sensitivity conjecture [13]. What Huang proved is the following graph theoretic statement:

Theorem 1.1 (Huang)**.** *Any induced subgraph of the n-dimensional hypercube with more than 2^{n-1} vertices has at least one vertex of degree larger than or equal to $\sqrt{n}$.*

Huang's proof uses a few key facts: first of all (informally) it gives a signature to the hypercube H_n to form a signed graph with exactly two eigenvalues. Then it uses a spectral argument known as the "interlace theorem" to determine the largest eigenvalue of the matrix corresponding to the induced subgraph of this signed graph, where the number of vertices is larger than half of the total number of vertices. Finally, the fact that the maximum degree of a graph must be larger than the maximum eigenvalue is used. The proof by Huang has since inspired works showing alternate proofs [11], connections to exterior algebra [9], Clifford algebra [15, 12] and the Jordan-Wigner transformation [6].

Knuth [11] exhibited a collection of eigenvectors of this signed graph and using a basis for the eigenspace corresponding to the larger eigenvalue, provided a proof that while using linear algebra, does not use spectral arguments. The idea of using the linear dependence and a basis of the eigenspace for proving the sensitivity conjecture has been attributed to a comment by Shalev Ben-David [11].

Mathews extended the result to weighted hypercubes [12], where all the edges corresponding to coordinate i have the same weight.

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ISSN 1433-8092

In this work, analyzing what makes this eigenvalues argument work, we present a proof which is based on linear dependency and the dimension of vector spaces. Our approach leads to stronger statements which we then use to derive applications in the study of the sensitivity of Boolean functions.

Specifically, we obtain the following new results. Let $N^F(x)$ be the set of all neighbours of a vertex x in the subgraph of the n-dimensional Boolean hypercube induced on a set of vertices F. The size of this set, which is the degree of x in this subgraph, is denoted by $d_F(x)$. Let $N_2^F(x)$ be the set of vertices $y \in F$ at distance 2 from x such there is a unique 2-path in this subgraph from each of them to x.

Theorem 1.2. *Given a set F of vertices of the n-dimensional Boolean hypercube with $|F| \geq 2^{n-1}+1$, there exists vertices u, v in the subgraph induced on F with $|u|$ and $|v|$ having even and odd parity respectively such that $|N^F(u)| + |N_2^F(u)| \geq n$ and $|N^F(v)| + |N_2^F(v)| \geq n$.*

Based on a notion of linear dependence which is the essence of this work, a stronger statement is given and proven as Theorem 5.1. As a corollary we obtain the following, which is also given in its stronger form as Corollary 5.2,

Corollary 1.3. *Given a set F of vertices of the n-dimensional Boolean hypercube with $|F| \geq 2^{n-1}+1$, there exists an edge uv such that $d_F(u) \times d_F(v) \geq n$ in the induced subgraph on F.*

As a corollary we have the following statement relating the degree of a function to its 0-sensitivity and 1-sensitivity (which will be defined in the next section),

Theorem 1.4. *For any Boolean function f we have $\deg(f) \leq s_0(f)s_1(f)$.*

We illustrate in Section 5.3, by applying it to Chakraborty's family of Boolean functions, that this is a strictly stronger statement than $\deg(f) \leq \mathrm{s}^2(f)$.

2 Preliminaries

2.1 The Boolean hypercube

The Boolean hypercube of dimension of n, denoted H_n, is a graph whose vertex set $V(H_n)$ is the set of all binary vectors (or strings) from $\mathbb{Z}_2^n$. Two vertices are adjacent to each other if their binary difference is one of the elements of the standard basis, i.e. $u \sim v$ if $u \oplus v = e_i$ for some $i \in \{1, 2, \ldots, n\}$ or equivalently, if their Hamming distance $d_H(u, v)$ is 1. On labeling the coordinates of the vectors in $\mathbb{Z}_2^n$ with elements of $[n] = \{1, 2, \ldots, n\}$, there is a natural bijection between binary vectors of length n with subsets of $[n]$: a subset B of $[n]$ is associated with the binary vector e_B whose coordinate i is 1 if and only if $i \in B$. When B contains a single element i, we may write i in place of $\{i\}$. Thus e_B is the natural extension of the standard basis and we use $|B|$ to denote the size of B which also corresponds to the Hamming weight of the corresponding string.

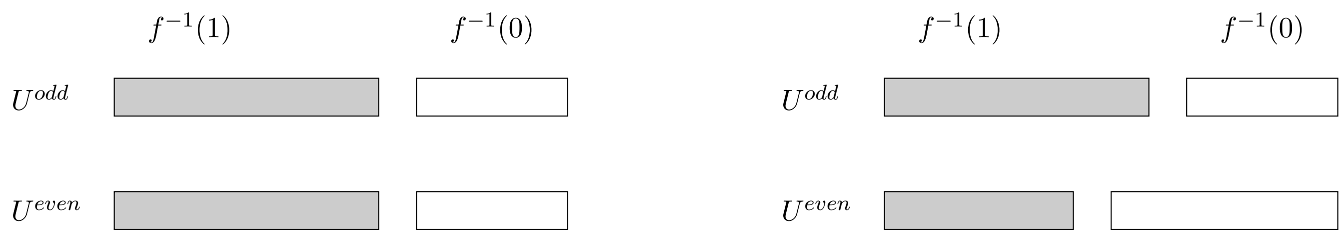
The hypercube of dimension n is a bipartite graph. Vertices e_B with an even $|B|$ form one part of the hypercube and those with an odd $|B|$ form the other part. The set of odd vertices of H_n will be denoted by U_n^{odd} or simply U^{odd} (when n is clear from the context) and the set of even vertices will be denoted by U_n^{even} or U^{even}.

2.2 Binary functions and sensitivity

A *binary function* is any function from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2$. It can be equivalently viewed as a 2-coloring of vertices of H_n (not necessarily a proper coloring). Given a binary function f, a vector x is a said to be *sensitive* at coordinate i if $f(x + e_i) \neq f(x)$. The *sensitivity* of x with respect to f is the number of coordinates at which it is sensitive. When viewed on a graph, it is the number of vertices of a color different from itself that it is adjacent to. The sensitivity of the function f, denoted $\mathrm{s}(f)$, is then defined to be the maximum sensitivity over all vectors in $\mathbb{Z}_2^n$ with respect to f. Given a subset B of $[n]$, a vector x is said to be B-sensitive with respect to f, if $f(x + e_B) \neq f(x)$. The *block sensitivity* of x with respect to f is the maximum number of disjoint subsets B_i of $[n]$ such that x is B_i-sensitive for each i. The block sensitivity of f is the maximum block sensitivity over all vectors in $\mathbb{Z}_2^n$ with respect to f.

The *0-sensitivity* of Boolean function f, denoted $\mathrm{s}_0(f)$, is the maximum sensitivity over binary vectors which evaluate to 0 on f. The *1-sensitivity* of Boolean function f, denoted $\mathrm{s}_1(f)$, is defined similarly.

Definition 2.1. *A Boolean function is said to be* parity-balanced *if the number of even vectors that evaluate to 1 is the same as the number of odd vectors that evaluate to 1 (see Figure 1).*



(a) Parity-balanced function　　(b) Not parity-balanced function

Figure 1: Parity-balance in a Boolean function can be thought of as the equality of the shaded regions in the figure above. A function has full degree if and only if it is not parity-balanced.

2.3 Polynomials and degree

For a vector $v \in \mathbb{Z}_2^n$ whose support is B, we define the multilinear polynomial $P_v : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$P_v = \prod_{i \in B} x_i \prod_{j \in \overline{B}} (1 - x_j).$$

The polynomial has the property that when restricted to the elements of $\mathbb{Z}_2^n$, it takes the value 1 on v and 0 elsewhere. This polynomial has degree n where the coefficient of the term $x_1 x_2 \dots x_n$ is either $+1$ or -1 depending on the parity of v.

A polynomial $p : \mathbb{R}^n \mapsto \mathbb{R}$ *represents* a boolean function f if for all $x \in \{0,1\}^n$, $p(x) = f(x)$. Every Boolean function can be represented by a multilinear polynomial P_f which is the sum of polynomials P_v such that $f(v) = 1$,

$$P_f = \sum_{v; f(v)=1} P_v$$

We may now define the *degree* of a Boolean function f, denoted $\deg(f)$, as the degree of the multilinear polynomial P_f. Observe that a Boolean function f has full degree (i.e. degree n) if and only if it is not parity-balanced. This can also be found in [2] and was attributed to Yao and Shi.

3 The sensitivity conjecture

The sensitivity conjecture as posed by Nisan and Szegedy [13] was the following problem,

Problem 3.1. *For every Boolean function f,*

$$bs(f) \leq (s(f))^{O(1)}$$

The last piece of the proof of this conjecture was proved in 2019 by Huang [8]. Below, we state and review some of the works on the sensitivity conjecture leading up to this paper.

Nisan and Szegedy [13, 14] established a strong connection between degree and block sensitivity of a Boolean function:

Theorem 3.2. (Nisan-Szegedy, Tal) *For any Boolean function f, we have:*

$$bs(f) \leq deg(f)^2$$

This was proved using symmetrization and lower bounds on the degree of single-variable polynomials. An excellent exposition on the proof of this theorem and those that relate other measures of complexity of Boolean functions such as certificate complexity, decision tree complexity to block sensitivity can be found in [2]. The theorem in its present form is due to Tal [14].

In 1992, Gotsman and Linial [5] showed the equivalence of the problem of showing that the polynomial degree of a function is at most some polynomial in the sensitivity of the function and the graph theoretic problem stated in Theorem 1.1.

Theorem 3.3. (Gotsman-Linial) *The following are equivalent for any monotone function $h : \mathbb{N} \mapsto \mathbb{R}$:*

1. *For any induced subgraph G of H_n such that $|V(G)| \neq 2^{n-1}$, there exists a vertex with degree $\geq h(n)$ in either G or $H_n - G$.*
2. *For any Boolean function f, $s(f) \geq h(deg(f))$.*

A rough sketch of this proof is as follows: consider a monomial of the largest degree in the multilinear polynomial representing the Boolean function and discard all variables that do not appear in this monomial (i.e. set them to zero). The sensitivity of the modified function f' might be smaller than that of the original function, but the polynomial degree of the function is preserved and f' now has full polynomial degree. The range of f' is taken to be $\{+1, -1\}$ instead of $\{0, 1\}$. This function is further modified as $g(x) = f'(x)p(x)$ where $p(x) = (-1)^{\sum x_i}$ is the parity function.

Notice that if we consider the graph induced by the vertices in $g^{-1}(0)$ (similarly for $g^{-1}(1)$), the graph degree of a vertex x in this subgraph is the sensitivity of x with respect to f. Equivalently, we can bypass g and consider the bipartite graph induced by the vertices in $(f^{-1}(0) \cap U^{\text{even}}) \cup (f^{-1}(1) \cap U^{\text{odd}})$, (similarly for $(f^{-1}(1) \cap U^{\text{even}}) \cup (f^{-1}(0) \cap U^{\text{odd}})$).

Since the function f' has full degree, f' is not parity-balanced and without loss of generality assume $|f^{-1}(1) \cap U^{\text{odd}}| > |f^{-1}(1) \cap U^{\text{even}}|$. This implies that there are more that 2^{n-1} vertices in $(f^{-1}(1) \cap U^{\text{odd}}) \cup (f^{-1}(0) \cap U^{\text{even}})$. By proving Theorem 1.1, it was shown that the induced subgraph on this set of vertices has a vertex with sensitivity at least $\sqrt{n} = \sqrt{\deg(f)}$. Hatami et al. [7] provides a clear presentation of this proof and surveys in detail the results leading up to the recent proof of the sensitivity conjecture. Putting together Theorem 1.1, Theorem 3.2 and Theorem 3.3, we now have that $\text{bs}(f) \leq \text{s}(f)^4$.

4 Graph degree lower bound revisited

In this section we first state the key tools and ideas that go into our simplification of Huang's theorem (Theorem 1.1) and then provide the details. Given a hypercube H_n, let σ be an assignment of $+$ or $-$ to the edges such that each 4-cycle has an odd number of negative edges. As we shall see, this is the only property of Huang's signature that is needed to derive the degree lower bound. Such a signature has been introduced independently in a number of places, in many of them implicitly. For the sake of completeness, we will discuss in Section 4.1 the construction of such a signature, how to find all such signatures and we will provide some references to previous appearances of this signature. To simplify notation, we extend σ by setting it to 0 for all non-adjacent pairs of vertices in the Boolean hypercube.

A crucial step in our approach is the following definition. Recall that a binary vector or a binary string of length n is a vertex of H_n. For any vertex x of H_n, we define a real valued vector x^+ of length 2^n whose coordinates are labeled by vertices y of H_n. We use $x^+_{(y)}$ to denote the value at coordinate y of a vector x^+.

Definition 4.1. *For all vertices x of H_n, we define the vectors x^+ and x^- as follows, For all $y \in V(H^n)$,*

$$x^+_{(y)} = \begin{cases} \sqrt{n} & \text{if } x = y \\ \sigma(x,y) & \text{otherwise}, \end{cases} \qquad x^-_{(y)} = \begin{cases} -\sqrt{n} & \text{if } x = y \\ \sigma(x,y) & \text{otherwise}. \end{cases}$$

Note that each of the vectors x^+ or x^- is non-zero only at the coordinates corresponding to x and its neighbors in the hypercube. We use V^+ and V^- to denote the subspace generated by the vertices x^+ and x^-, i.e. $V^+ = \langle x_1^+, x_2^+, \dots, x_{2^n}^+\rangle$ and $V^- = \langle x_1^-, x_2^-, \dots, x_{2^n}^-\rangle$.

One may observe that V^+ and V^- are the eigenspaces of the signed adjacency matrix, but we will not use this fact in our proofs.

A set of vectors $S = \{v_1, v_2, \dots, v_k\}$ is said to have a *linear dependency* if we have $\sum_i a_i v_i = 0$ for some choice of real numbers a_i not all of which are zero, and where 0 is the all-zero vector.

Theorem 1.1 is the immediate corollary of the following facts:

Observation 4.2. *If the vectors $\{x_1^+, x_2^+, \dots, x_k^+\}$ have a linear dependency, then the subgraph induced on the corresponding vertices $\{x_1, x_2, \dots, x_k\}$ of H_n has a vertex of degree at least $\sqrt{n}$.*

Proof. Suppose $\sum a_i x_i^+ = 0$ with $a_i \neq 0$ and let $|a_j|$ be a largest among all $|a_i|$'s. For the row corresponding to vertex x_j to vanish when viewing x_i^+ as a column vector, there must be at least $\lceil \sqrt{n}\,\rceil$ other vectors in the linear dependency that are nonzero at the coordinate x_j. Since those vectors can only correspond to neighbours of x_j each of which can contribute at most $|a_j|$ to the sum, x_j must have at least $\sqrt{n}$ neighbours. □

Proposition 4.3. *The subspaces V^+ and V^- are each of dimension 2^{n-1}.*

A proof of this will be given in Section 4.2. As an immediate corollary we have that for any set of $2^{n-1}+1$ vertices, there must be a linear dependency among some of the corresponding vectors which implies the existence of a vertex of degree at least $\sqrt{n}$.

4.1 Signatures with negative 4-cycles

Let us first see the construction of a signature on H_n such that every 4-cycle in H_n is negative, i.e., the product of the signs on the edges in the cycle is negative. A signature with this property was used in [1] to show that one can choose nonzero weights on the edges to have exactly two eigenvalues for the corresponding weighted graph. This is implicit in the proof of Theorem 6.7 and its Corollary 6.9 in [1]. This is the same signature that was given by Huang inductively.

A signature with this property can easily be constructed on H_2 which consists of a single C_4: assign to one or three edges a negative sign, and to the rest a positive sign. H_n is built recursively from two disjoint copies of H_{n-1} by adding a matching between corresponding vertices. Having found a signature σ_{n-1} for H_{n-1}, proceed as follows: in the first copy of H_{n-1} assign signs as in σ_{n-1}, and in the other copy assign signs complementary to that in σ_{n-1}. Finally, all edges in the matching are assigned the same sign. Observe that 4-cycles in each of the two copies inherit the property. The 4-cycles formed using two edges of the matching use corresponding edges from the two copies which are of opposite signs.

We point out that there are exactly 2^{2^n-1} signatures with this property on H_n. Given one such signature, one can get another by a *switch* at a vertex x of H_n, i.e. by switching the signs of edges incident on x. Observe that this operation does not change the sign of a cycle. In terms of matrices, this is equivalent to multiplying both the row and column of the adjacency matrix corresponding to x by -1. This does not change the eigenvalues and the corresponding eigenvectors are obtained by switching the sign at the x^{th} coordinate. One may apply a series of switches on all the vertices in a set X. A series of switches on the complement of X results in the same assignment and so there are 2^{2^n-1} signatures.

On the other hand any signature with the property that all 4-cycles are negative is one of the signatures discussed above. This follows from the fact that the 4-cycles generate the cycle space of H_n and from the following Theorem of Zaslavsky.

Theorem 4.4 (Zaslavsky [16, Theorem 3.2])**.** *Given two signatures σ_1 and σ_2 of a graph G, σ_1 is a switching of σ_2 if and only if the sets of positive (or equivalently negative) cycles of (G, σ_1) and (G, σ_2) are the same.*

4.2 Dimensions of V^+ and V^-

Here we show that both V^+ and V^- have dimension 2^{n-1} (Proposition 4.3). We first observe that each of these subspaces have dimension at least 2^{n-1}. More generally we have:

Proposition 4.5. *For any independent set I of vertices of H_n, the sets $\{x^+ | x \in I\}$ and $\{x^- | x \in I\}$ are linearly independent.*

Proof. Since I is an independent set and the vector x^+ (resp. x^-) is non-zero only at the coordinate x and its neighbors in the hypercube, x^+ (resp. x^-) is the only vector in $\{y^+ | y \in I,\ y \neq x\}$ (resp. in $\{y^- | y \in I,\ y \neq x\}$) which is nonzero at the coordinate x. □

Since the set of odd vertices (similarly even vertices) forms an independent set of size 2^{n-1} in H_n we have:

Corollary 4.6. *The dimension of vector spaces V^+ and V^- are at least 2^{n-1}, i.e. $dim(V^+) \geq 2^{n-1}$, $dim(V^-) \geq 2^{n-1}$.*

The fact that equality holds in each of these inequalities is a consequence of the following proposition.

Proposition 4.7. *The subspaces V^+ and V^- are orthogonal to each other.*

We note that the choice of a signature where each 4-cycle is negative is key for this claim to hold.

Proof. We must show that for an arbitrary choice of vertices x and y (not necessarily distinct) the vectors x^+ and y^- are orthogonal. Depending on the distance between x and y, we consider the following 4 cases:

Case 1. If $x = y$, then the non-zero terms in the inner product are at the coordinate x and its neighbors.

$$\langle x^+, x^- \rangle = -\sqrt{n} \cdot \sqrt{n} + \sum_{z:z\sim x} \big(\sigma(x,z)\big)^2 = -n + n = 0$$

Case 2. If $x \sim y$, then the only non-zero coordinates in the inner product are x and y (since x and y do not share any common neighbours).

$$\langle x^+, y^- \rangle = \sqrt{n}\sigma(x,y) - \sqrt{n}\sigma(x,y) = 0$$

Case 3. If $d_H(x,y) = 2$, observe that there are exactly two vertices (say v and v') adjacent to both x and y. These form a 4-cycle and by the assumption on the signature σ, $\sigma(x,v)\sigma(v,y) = -\sigma(x,v')\sigma(y,v')$. Therefore,

$$\langle x^+, y^- \rangle = \sigma(x,v)\sigma(v,y) + \sigma(x,v')\sigma(y,v') = 0$$

Case 4. If $d_H(x,y) \geq 3$, there are no common non-zero terms in the vectors and the inner product $\langle x^+, y^- \rangle$ is trivially 0.

This completes the proof of orthogonality. □

5 Strengthening Huang's result

Huang's proof yields a lower bound on the graph degree when the number of vertices is large enough. We can strengthen this result in two ways. First, we can weaken the hypothesis to any graph presenting a linear dependency regardless of the number of vertices in the linear dependency. Second, we can exploit the linear dependency further to extract more structural information about the graph, in addition to its largest degree.

5.1 Structural information from linear dependencies

We introduce the following terminology before going into our results. Consider a non-trivial linear relation F given as $\sum_{u\in U^{\mathrm{odd}}} a_u u^+ = \sum_{v\in U^{\mathrm{even}}} b_v v^+$ where a_u and b_v are real numbers, allowed to be 0. We are interested in cases where not all of the coefficients are 0 and in such cases F is called a *nontrivial linear dependency.* Let H_F denote the subgraph of the Boolean hypercube induced by the vertices of H_n which have a nonzero coefficient in F.

By Proposition 4.3, for any set K of vertices of H_n with $|K| \geq 2^n + 1$ there exists a nontrivial linear dependency F such that $V(H_F) \subseteq K$. Thus the following theorem is stronger than the Theorem 1.2 stated in the introduction.

Theorem 5.1. *Given a nontrivial linear dependency relation F on a subset of vertices in H_n, there exist vertices $u \in U^{odd}$ and $v \in U^{even}$ in H_F such that $|N^F(u)| + |N_2^F(u)| \geq n$ and $|N^F(v)| + |N_2^F(v)| \geq n$.*

The proof of this theorem will be given after a technical lemma. We would like to first point out a corollary of this theorem. This corollary is observed by taking a vertex u given by the theorem and considering its neighbor which has the largest degree in H_F among all neighbours of u.

Corollary 5.2. *If F is a nontrivial linear dependency relation, there exists an edge (u,v) in H_F such that $d_F(u) \times d_F(v) \geq n$.*

We would like to point out that both Theorem 5.1 and its corollary are strictly stronger than their counterparts stated in the introduction because a linear dependency can happen in smaller sets, the smallest being of size $n+1$ (see Section 6.3 for more details).

The key tool in the proof of Theorem 5.1 is Lemma 5.3 which follows. For any pair of vertices x and y at distance 2 in H_n, there are exactly two paths of length 2 connecting them. When there is a unique 2-path connecting x and y in H_F, we extend the signature to $\hat{\sigma}$ such that $\hat{\sigma}_F(x,y) = \sigma(x,z)\sigma(z,y)$ where z is the unique common neighbor of x and y in H_F.

Lemma 5.3. *Given a linear dependency F defined by $\sum_{u \in U^{odd}} a_u u^+ = \sum_{v \in U^{even}} b_v v^+$, for every vertex $x \in H_F \cap U^{odd}$ and $y \in H_F \cap U^{even}$ we have*

$$(n - d_F(x))a_x = \sum_{z \in N_2^F(x)} \hat{\sigma}_F(x,z)a_z, \quad \textit{and} \quad (n - d_F(x))b_y = \sum_{t \in N_2^F(x)} \hat{\sigma}_F(t,y)b_t.$$

Proof of Lemma 5.3. The vectors u^+, v^+ are viewed as column vectors. In the linear dependency, consider the row corresponding to some arbitrary fixed coordinate $x \in U^{\text{odd}}$.

$$\sqrt{n}a_x = \sum_{v \in N^F(x)} \sigma(x,v)b_v \tag{1}$$

Similarly by considering the row corresponding to any coordinate $v \in U^{\text{even}}$ we get:

$$\sqrt{n}b_v = \sum_{u \in N^F(v)} \sigma(u,v)a_u. \tag{2}$$

Multiplying both sides of Equation (1) by $\sqrt{n}$ we have:

$$na_x = \sum_{v \in N^F(x)} \sigma(x,v)\sqrt{n}b_v. \tag{3}$$

Replacing each $\sqrt{n}b_v$ on the right side of Equation (3) by the corresponding right side of Equation 2, we have:

$$na_x = \sum_{v \in N^F(x)} \sum_{u \in N^F(v)} \sigma(u,v)\sigma(v,u)a_u. \tag{4}$$

On examining the right side of this identity we make two key observations. The first is that a_x appears for each v in its neighbourhood with a coefficient $\sigma(x,v)^2 = 1$. The second, which is based on the main property of the signature we have chosen to work with, is that if a vertex u, $u \neq x$, appears on the right hand side twice, then the sum of its coefficients is 0 (this is equivalent to saying $\sigma(x,v_1)\sigma(v_1,u) = -\sigma(x,v_2)\sigma(v_2,u)$ where x, v_1, u, v_2 is a 4-cycle in H_F).

Rearranging the terms and simplifying gives $(n - d_F(x))a_x = \sum_{u \in N_2^F(x)} \hat{\sigma}_F(x,u)a_u$, as claimed. The proof of the identity for the vertices in U^{even} is analogous. □

We can now complete the proof of the statement about the second neighborhood of vertices in H_F.

Proof of Theorem 5.1. Consider a (nontrivial) linear dependency $\sum_{u \in U^{\text{odd}}} a_u u^+ = \sum_{v \in U^{\text{even}}} b_v v^+$ and let $|a_x| = \max_{z \in U^{\text{odd}}} \{|a_z|\}$. Observe that $a_x \neq 0$ by our assumption on F. Now consider the identity $(n - d_F(x))a_x = \sum_{z \in N_2^F(x)} \hat{\sigma}_F(xz)a_z$. Since $|a_z| \leq |a_x|$, there should be at least $n - d_F(x)$ values of a_z which are nonzero for this identity to hold.

An analogous argument follows for the other part by taking the maximum value over $|b_v|$. □

The following, which immediately follows from Corollary 5.2, is already stronger than Theorem 1.1 because the dimension of V^+ is 2^{n-1}. It is a strictly stronger statement because linear dependency may occur over much smaller sets of vertices.

Corollary 5.4. *If F is a nontrivial linear dependency relation, then there exists a vertex u in H_F such that $d_F(u) \geq \sqrt{n}$.*

5.2 From linear dependency to sensitivity

Finally, we derive a stronger upper bound on the polynomial degree of a boolean function.

Theorem 5.5. *For any Boolean function f,* $\deg(f) \leq s_0(f)s_1(f)$.

Proof. If $\deg(f)$ is d, we can concentrate on a function f' of degree d on d variables by setting the variables outside of the largest monomial to 0. This can only decrease the sensitivity. Recall that Observation 4.2 states that any linear dependency among the vectors of V^+ corresponding to a set of vertices F implies the existence of a vertex with graph degree at least $\sqrt{d}$. From Gotsman and Linial (see Theorem 3.3), we know that the sensitivity with respect to f of input x is the graph degree of a vertex x in the bipartite subgraph induced by $F = (f'^{-1}(0) \cap U^{\text{even}}) \cup (f'^{-1}(1) \cap U^{\text{odd}})$ or $F' = (f'^{-1}(1) \cap U^{\text{even}}) \cup (f'^{-1}(0) \cap U^{\text{odd}})$ depending on $f'(x)$ and the parity of $|x|$. Since f' has full polynomial degree, it is not parity-balanced. Therefore one of these bipartite subgraphs has at least $2^{d-1} + 1$ vertices. Since the dimension of V^+ is 2^{d-1}, any set of $2^{d-1} + 1$ vectors from V^+ has a linear dependency and the larger of the two induced subgraphs, say F wlog, has a linear dependency. Therefore, there exists some vertex in this induced subgraph that has graph degree at least $\sqrt{d}$. This proves that the sensitivity of the function is at least $\sqrt{d} = \sqrt{\deg(f)}$ as d is the degree of the original function f.

For the stronger statement, we use Corollary 5.2, which says that in any non-trivial linear dependency, there exists an edge (u, v) in the subgraph F such that $d_F(u)d_F(v) \geq d$. By definition of F, $f'(u) \neq f'(v)$, so $s_0(f)s_1(f) \geq s_0(f')s_1(f') \geq d = \deg(f)$. □

Notice that we have proven something stronger, which is that the lower bound on s_0 and s_1 is achieved on inputs at Hamming distance 1.

Since by Theorem 3.2 $\text{bs}(f) \leq \deg(f)^2$, we get the following polynomial relation between sensitivity and block sensitivity.

Corollary 5.6. *For any Boolean function f, $bs(f) \leq s_0(f)^2 s_1(f)^2$.*

5.3 Application of the degree bound

For many known functions, $s_0(f)s_1(f) \geq n$. It was suggested by Kenyon and Kutin [10] that for most "interesting" functions, this was the case, making it difficult to find cases where $s_0(f)s_1(f)$ gives a non-trivial upper bound on degree. One such family of functions was given by Chakraborty[3, 4]. To present this family of functions we use notation from regular expressions, where a set represents alternative, juxtaposition represents concatenation, and exponents represent repetition.

Definition 5.7. *Given positive integer $k \geq 8$ and integer $n \geq k^2$, the Chakraborty function $f_{n,k}$ is defined using the following auxiliary function $g_k : \{0,1\}^{k^2} \mapsto \{0,1\}$.*

$$g_k(x) = 1 \iff x \in 110^{k-2}(11111\Sigma^{k-5})^{k-2}11111\Sigma^{k-8}111$$

where $\Sigma = \{0,1\}$. The function $f_{n,k} : \{0,1\}^n \mapsto \{0,1\}$ evaluates to 1 if and only if there exists a (consecutive) substring z of length k^2 such that $g(z) = 1$.

An alternate definition of the auxiliary function g_k can be given by defining the set C_k of k^2-length strings on which g_k evaluates to 1. To define C_k, we first partition the k^2 coordinates into k blocks of k consecutive coordinates. A binary vector of length k^2 is in C_k iff in the first block the first two positions are 1's followed by $k-2$ 0's, in all the other blocks the starting five positions are 1's, and the last block ends with three 1's (the other positions in these blocks are unconstrained).

Chakraborty showed the following properties.

Proposition 5.8 ([3, 4])**.** *For any $k \leq \sqrt{n}$,*

1. $s_0(f_{n,k}) = \frac{n}{k^2}$, $s_1(f_{n,k}) = k$.
2. $bs(f_{n,k}) = \frac{n}{k}$

Using Theorem 5.5, we give a new upper bound on the polynomial degree of $f_{n,k}$.

Proposition 5.9. *For any $k \leq \sqrt{n}$.*

1. $\deg(f_{n,k}) \leq \frac{n}{k}$.
2. $\deg(f_{n,k}) \geq \sqrt{\frac{n}{k}}$.

Proof. The first item is immediate from Theorem 5.5 and Proposition 5.8. For the second item we use the upper bound $\mathrm{bs}(f) \leq \deg(f)^2$ (Theorem 3.2). □

6 Further discussion

In this section we look at some further implications of techniques developed in the previous sections. We look at how they can be used to prove analogous results in weighted hypercubes and what they imply for cases where the maximum degree of the subgraph is close to $\sqrt{n}$.

6.1 Weighted version

Let $\{a_i\}$ for $i = 1, 2, \ldots, n$ be a sequence of non zero real values. Consider the following weight assignment to the edges of H_n: if the edge (x, y) corresponds to the coordinate i (i.e., x and y differ only at the coordinate i), then the edge (x, y) is assigned a weight a_i. Furthermore, we multiply the weight of the edge (x, y) to the signature of the edge $\sigma(xy)$ which was defined in the previous sections. The adjacency matrix of the corresponding signed weighted graphs has exactly two eigenvalues: $\pm\sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$. This would follow from Theorem 6.7 of [1] if one follows the proof steps to complete the proof of Corollary 6.9 starting with K_2. However, without the use of this fact and with a modification of our proof of Theorem 1.1, we can give a proof of the following weighted version of Huang's theorem, first proved by Mathews[12] using Clifford algebras.

Theorem 6.1. *Given a weighted hypercube H_n where the edges corresponding to coordinate i are given a weight a_i with $a_i \neq 0$, in the induced subgraph corresponding to any set of $2^{n-1}+1$ vertices, there exists a vertex x whose sum of weights of incident edges is at least $\sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$.*

Proof. Once the definitions of the vectors x^+ and x^- are modified, the rest of the proof is exactly the same. The x^{th} coordinate of x^+ is set as $\sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$ and that of x^- is set as $-\sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$. If y is adjacent to x and $y = x + e_i$, then the y^{th} coordinate of both x^+ and x^- is set as $\sigma(xy)a_i$. All other coordinates are set to 0. □

We note that methods of Section 5 can apply to this weighted version to get analogous stronger results.

6.2 Tightness

To summarize our result, what we have proved here is that in a subgraph induced by a set of vertices with a linear dependency, the maximum degree lies between n and $\sqrt{n}$, with both extremities being tight. Furthermore, we proved that the closer we are to the lower bound the more vertices we must have from the second neighbourhood of a vertex with maximum degree. However this is not the limit of our approach, it can imply more vertices from the third neighbourhood and so on. While we do not yet have the strongest claim to present, we have the following observation in the case where the maximum degree is exactly $\sqrt{n}$ (assuming that n is a perfect square).

Theorem 6.2. *If F is a nontrivial linear dependency relation and H_F has maximum degree exactly $\sqrt{n}$, then the vertices in H_F of degree $\sqrt{n}$ induces a $\sqrt{n}$-regular subgraph.*

This theorem is corollary of our proof of Observation 4.2 and its details is left to the reader.

The folklore example of AND-of-ORs function, defined below, exhibits such a behaviour. Let $n = k^2$ and $B_1, B_2, \ldots, B_k$ be a partition of n coordinates into k blocks each of size k. The AND-of-ORs function assigns to an input x a value 1 if for each B_i, x has a 1 in at least one coordinate of B_i and it assigns 0 to x otherwise. We refer to [7] and references therein for details about this function.

6.3 Linear dependency

We remark that the vectors x^+ and x^- were built by further investigating Huang's proof using eigenvalues. The set V^+ generated by $\{x^+ \mid x \in V(H_n)\}$ is the eigenspace corresponding to the eigenvalues $\sqrt{n}$ of the incidence matrix of the signed graph (H_n, σ) and the set V^- generated by $\{x^- \mid x \in V(H_n)\}$ is eigenspace corresponding to the eigenvalue $-\sqrt{n}$. This provides an alternate proof for the fact that V^+ is orthogonal to V^- and that each is of dimension (at most) 2^{n-1}.

Our approach to this problem suggests a strong connection between linear dependency of the vectors x_i^+ in the study of the sensitivity of a function. It is, therefore, intriguing to ask:

Problem 6.3. *What are the (minimal) subsets of V^+ that are linearly dependent?*

The smallest linear dependency is among a vertex and all its neighbours:

$$x^+ = \frac{1}{\sqrt{n}} \sum_{y \sim x} \sigma(xy) y^+.$$

On the other hand for linearly independent sets, the easiest examples are sets I of vectors x^+'s where for every vector $x^+ \in I$ there exists a coordinate $u \in V(H_n)$ such that x^+ is the only vector in I that is nonzero at

u. We call such a linearly independent set a *basic linearly independent set*. The main example of a basic linearly independent set is the set $\{u^+ \mid u \in U^{\text{odd}}\}$ or $\{v^+ \mid v \in U^{\text{even}}\}$. Each of these sets provides an orthogonal basis for the V^+.

Another example of basic linearly independent set is the set of all u^+ where, for a fixed i, the i^{th} coordinate of u is 1. Then for each u^+ of this set, the vector u^+ is the only vector of the set that is not 0 at the coordinate $u + e_i$. Thus taking all such vectors provides another basis for V^+, but this basis is no longer an orthogonal one. The proof of Huang's result given by Knuth in [11] uses one such basis with $i = n$.

Acknowledgement

This work is supported by the ANR-France project HOSIGRA (ANR-17-CE40-0022). This project has received funding from the European Union's Horizon 2020 research and innovation program under the Marie Sklodowska-Curie grant agreement No 754362.

References

[1] Ahmadi, B., Alinaghipour, F., Cavers, Michael S.and Fallat, S., Meagher, K., and Nasser, S. Minimum number of distinct eigenvalues of graphs. *Electronic Journal of Linear Algebra 26*, 45 (2013), 673–691.

[2] Buhrman, H., and de Wolf, R. Complexity measures and decision tree complexity: A survey. *Theor. Comput. Sci. 288*, 1 (Oct. 2002), 21–43.

[3] Chakraborty, S. Sensitivity, block sensitivity and certificate complexity of boolean functions. Master's thesis, The University of Chicago, 2010.

[4] Chakraborty, S. On the sensitivity of cyclically-invariant Boolean functions. *Discrete Mathematics & Theoretical Computer Science Vol. 13 no. 4* (2011).

[5] Gotsman, C., and Linial, N. The equivalence of two problems on the cube. *Journal of Combinatorial Theory, Series A 61*, 1 (1992), 142–146.

[6] Gu, Y., and Qi, X.-L. Majorana fermions and the sensitivity conjecture. arXiv 1908.06322, 2019.

[7] Hatami, P., Kulkarni, R., and Pankratov, D. *Variations on the Sensitivity Conjecture*. No. 4 in Graduate Surveys. Theory of Computing Library, 2011.

[8] Huang, H. Induced subgraphs of hypercubes and a proof of the sensitivity conjecture. *Annals of Mathematics 190*, 3 (2019), 949–955.

[9] Karasev, R. Huang's theorem and the exterior algebra. Tech. Rep. 1907.11175, arXiv, 2019.

[10] Kenyon, C., and Kutin, S. Sensitivity, block sensitivity, and l-block sensitivity of boolean functions. *Inf. Comput. 189*, 1 (Feb. 2004), 43–53.

[11] Knuth, D. E. A computational proof of Huang's degree theorem, 2019. Manuscript, 28 July, revised 3 August, https://www.cs.stanford.edu/ knuth/papers/huang.pdf.

[12] Mathews, D. V. The sensitivity conjecture, induced subgraphs of cubes, and clifford algebras. arXiv 1907.12357, 2019.

[13] Nisan, N., and Szegedy, M. On the degree of boolean functions as real polynomials. In *Proceedings of the Twenty-fourth Annual ACM Symposium on Theory of Computing* (New York, NY, USA, 1992), STOC '92, ACM, pp. 462–467.

[14] Tal, A. Properties and applications of boolean function composition. In *Proceedings of the 4th Conference on Innovations in Theoretical Computer Science* (New York, NY, USA, 2013), ITCS '13, ACM, pp. 441–454.

[15] Tao, T. Twisted convolution and the sensitivity conjecture. What's New? (26 July, 2019), 2019.

[16] Zaslavsky, T. Signed graphs. *Discrete Applied Mathematics 4*, 1 (1982), 47–74.

ECCC ISSN 1433-8092

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