

# A Polynomial Degree Bound on Defining Equations of Non-rigid Matrices and Small Linear Circuits

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## Abstract

We show that there is a defining equation of degree at most  $\text{poly}(n)$  for the (Zariski closure of the) set of the non-rigid matrices: that is, we show that for every large enough field  $\mathbb{F}$ , there is a non-zero  $n^2$ -variate polynomial  $P \in \mathbb{F}(x_{1,1}, \dots, x_{n,n})$  of degree at most  $\text{poly}(n)$  such that every matrix  $M$  which can be written as a sum of a matrix of rank at most  $n/100$  and sparsity at most  $n^2/100$  satisfies  $P(M) = 0$ . This confirms a conjecture of Gesmundo, Hauenstein, Ikenmeyer and Landsberg [GHIL16] and improves the best upper bound known for this problem down from  $\exp(n^2)$  [KLPS14, GHIL16] to  $\text{poly}(n)$ .

We also show a similar polynomial degree bound for the (Zariski closure of the) set of all matrices  $M$  such that the linear transformation represented by  $M$  can be computed by an algebraic circuit with at most  $n^2/200$  edges (without any restriction on the depth). As far as we are aware, no such bound was known prior to this work when the depth of the circuits is unbounded.

Our methods are elementary and short and rely on a polynomial map of Shpilka and Volkovich [SV15] to construct low degree “universal” maps for non-rigid matrices and small linear circuits. Combining this construction with a simple dimension counting argument to show that any such polynomial map has a low degree annihilating polynomial completes the proof.

## 1 Introduction

### 1.1 Equations for varieties in algebraic complexity theory

$\mathbb{F}$  Let  $V \subseteq \mathbb{F}^n$  be a (not necessarily irreducible) affine variety and let  $\mathbf{I}(V)$  denote its ideal. A non-zero polynomial  $P \in \mathbf{I}(V)$  is called an *equation* for  $V$ . An equation for  $V$  may serve as a “proof” that a point  $\mathbf{x} \in \mathbb{F}^n$  is *not* in  $V$ , by showing that  $P(\mathbf{x}) \neq 0$ .

A fundamental observation of the Geometric Complexity Theory program is that many important circuit lower bounds problems in algebraic complexity theory fit naturally into the setting of showing that a point  $\mathbf{x}$  lies outside a variety  $V$  [MS01, BIL<sup>+</sup>19]. In this formulation, one considers  $V$  to be the closure of a class of polynomials of low complexity, and  $\mathbf{x}$  is the coefficient vector of the candidate hard polynomial.

Let  $\Delta(V) := \min_{0 \neq P \in \mathbf{I}(V)} \{\deg(P)\}$ . The quantity  $\Delta(V)$  can be thought of as a measure of complexity for the geometry of the variety  $V$ . The quantity  $\Delta(V)$  is a very coarse complexity measure. A recent line of work regarding *algebraic natural proofs* [FSV18, GKSS17] suggests to study the arithmetic circuit complexity of equations for varieties  $V$  that correspond

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to polynomials with small circuit complexity. Having  $\Delta(V)$  growing like a polynomial in  $n$  is a necessary (but not a sufficient) condition for a variety  $V$  to have an algebraic natural proof for non-containment.

## 1.2 Rigid matrices

A matrix  $M$  is  $(r, s)$ -rigid if  $M$  cannot be written as a sum  $R + S$  where  $\text{rank}(R) \leq r$  and  $S$  contains at most  $s$  non-zero entries. Valiant [Val77] proved that if  $A$  is  $(\varepsilon n, n^{1+\delta})$ -rigid for some constants  $\varepsilon, \delta > 0$  then  $A$  cannot be computed by arithmetic circuits of size  $O(n)$  and depth  $O(\log n)$ , and posed the problem of *explicitly* constructing rigid matrices with these parameters, which is still open. Over algebraically closed fields, it is easy to prove that most matrices have much stronger rigidity parameters: a generic matrix is  $(r, (n-r)^2)$ -rigid for any target rank  $r$ .

Let  $\mathbb{F}$  be an algebraically closed field. Let  $A_{r,s} \subseteq \mathbb{F}^{n \times n}$  denote the set of matrices which are not  $(r, s)$ -rigid. Let  $V_{r,s} = \overline{A_{r,s}}$  denote the Zariski closure of  $A_{r,s}$ . A geometric study of  $V_{r,s}$  was initiated by Kumar, Lokam, Patankar and Sarma [KLPS14]. Among other results, they prove that for every  $s < (n-r)^2$ ,  $\Delta(V_{r,s}) \leq n^{4n^2}$ . A slightly improved (but still exponential) upper bound was obtained by Gesmundo, Hauenstein, Ikenmeyer and Landsberg [GHIL16], who also conjectured that for some  $\varepsilon, \delta > 0$ ,  $\Delta(V_{\varepsilon n, n^{1+\delta}})$  grows like a polynomial function in  $n$ . The following theorem which we prove in this note confirms this conjecture.

**Theorem 1.1.** *Let  $\varepsilon < 1/25$ , and let  $\mathbb{F}$  be a field of size at least  $n^2$ . For every large enough  $n$ , there exists a non-zero polynomial  $Q \in \mathbb{F}[x_{1,1}, \dots, x_{n,n}]$ , of degree at most  $n^3$ , which is a non-trivial equation for matrices which are not  $(\varepsilon n, \varepsilon n^2)$ -rigid. That is, for every such matrix  $M$ ,  $Q(M) = 0$ .*

In fact, the conjecture of [GHIL16] was slightly weaker: they conjectured that  $\Delta(U)$  is polynomial in  $n$  for every irreducible component  $U$  of  $V_{\varepsilon n, n^{1+\delta}}$ . As shown by [KLPS14], the irreducible components are in one-to-one correspondence with subsets of  $[n] \times [n]$  of size  $n^{1+\delta}$  corresponding to possible supports of the sparse matrix  $S$ .

As we observe in Remark 2.3, it is somewhat simpler to show that each of these irreducible components has a defining equation with a polynomial degree bound. However, since the number of such irreducible components is exponentially large, it is not clear if there is a single defining equation for the whole variety which is of polynomially bounded degree. We do manage to reverse the order of quantifiers and prove such an upper bound in Theorem 1.1. This suggests that the set of non-rigid matrices is much less complex than what one may suspect given the results of [KLPS14, GHIL16].

## 1.3 Circuits for linear transformations

The original motivation for defining rigidity was in the context of proving lower bounds for algebraic circuits [Val77]. If  $A \in \mathbb{F}^{n \times n}$  is an  $(\varepsilon n, n^{1+\delta})$ -rigid matrix, for any  $\varepsilon, \delta > 0$ , then the linear transformation  $Ax$  cannot be computed by an algebraic circuit of depth  $O(\log n)$  and size  $O(n)$ .

Every algebraic circuit computing a linear transformation is without loss of generality a *linear* circuit. A linear circuit is a directed acyclic graph that has  $n$  inputs labeled  $X_1, \dots, X_n$  and  $n$  output nodes. Each edge is labeled by a scalar  $\alpha \in \mathbb{F}$ . Each node computes a linear function in  $X_1, \dots, X_n$  defined inductively. An internal node  $u$  with children,  $v_1, \dots, v_k$ , connected to it by edges labeled  $\alpha_1, \dots, \alpha_k$ , computes the linear function  $\sum_i \alpha_i \ell_{v_i}$ , where  $\ell_{v_i}$  is the linear function computed by  $v_i$ ,  $1 \leq i \leq k$ . The size of the circuit is the number of edges in the circuit.

It is possible to use similar techniques to those used in the proof of Theorem 1.1 in order to prove a polynomial upper bound on an equation for a variety containing all matrices  $A \in \mathbb{F}^{n \times n}$  whose corresponding linear transformation can be computed by an algebraic circuit of size at most  $n^2/200$  (even without restriction on the depth). Note that this is nearly optimal as any

such linear transformation can be computed by a circuit of size  $n^2$ . More formally, we show the following.

**Theorem 1.2.** *Let  $\mathbb{F}$  be a field of size at least  $n^2$ . For every large enough  $n$ , there exists a non-zero polynomial  $Q \in \mathbb{F}[x_{1,1}, \dots, x_{n,n}]$ , of degree at most  $n^3$ , which is a non-trivial equation for matrices which are computed by algebraic circuit of size at most  $n^2/200$ .*

## 2 Degree Upper Bound for Non-Rigid Matrices

In this section, we prove [Theorem 1.1](#). A key component of the proof is the use of the following construction, due to Shpilka and Volkovich, which provides an explicit low-degree polynomial map on a small number of variables, which contains all sparse matrices in its image. For completeness, we provide the construction and prove its basic property.

**Lemma 2.1** ([\[SV15\]](#)). *Let  $\mathbb{F}$  be a field such that  $|\mathbb{F}| > n$ . Then for all  $k \in \mathbb{N}$ , there exists an explicit polynomial map  $\text{SV}_{n,k}(\mathbf{x}, \mathbf{y}) : \mathbb{F}^{2k} \rightarrow \mathbb{F}^n$  of degree at most  $n$  such that for any subset  $T = \{i_1, \dots, i_k\} \subseteq [n]$  of size  $k$ , there exists a setting  $\mathbf{y} = \boldsymbol{\alpha}$  such that  $\text{SV}(\mathbf{x}, \boldsymbol{\alpha})$  is identically zero on every coordinate  $j \notin T$ , and equals  $x_j$  in coordinate  $i_j$  for all  $j \in [k]$ .*

*Proof.* Arbitrarily pick distinct  $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ , and let  $u_1, \dots, u_n$  be their corresponding Lagrange's interpolation polynomials, i.e., polynomials of degree at most  $n-1$  such that  $u_i(\alpha_j) = 1$  if  $j = i$  and 0 otherwise (more explicitly,  $u_i(z) = \frac{\prod_{j \neq i} (z - \alpha_j)}{\prod_{j \neq i} (\alpha_i - \alpha_j)}$ ).

Let  $P_i(x_1, \dots, x_k, y_1, \dots, y_k) = \sum_{j=1}^k u_i(y_j) \cdot x_j$ , and finally let

$$\text{SV}_{n,k}(\mathbf{x}, \mathbf{y}) = (P_1(\mathbf{x}, \mathbf{y}), \dots, P_n(\mathbf{x}, \mathbf{y})).$$

It readily follows that given  $T = \{i_1, \dots, i_k\}$  as in the statement of the lemma, we can set  $y_j = \alpha_{i_j}$  for  $j \in [k]$  to derive the desired conclusion. The upper bound on the degree follows by inspection.  $\square$

As a step toward the proof of [Theorem 1.1](#), we show there is a polynomial map on much fewer than  $n^2$  variables with degree polynomially bounded in  $n$  such that its image contains every non-rigid matrix. In the next step, we show that the image of every such polynomial map has a defining equation of degree  $\text{poly}(n)$ .

**Lemma 2.2.** *There exists an explicit polynomial map  $P : \mathbb{F}^{4\epsilon n^2} \rightarrow \mathbb{F}^{n \times n}$ , of degree at most  $n^2$ , such that every matrix  $M$  which is not  $(\epsilon n, \epsilon n^2)$  rigid lies in its image.*

*Proof.* Let  $k = \epsilon n^2$  and let  $\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y}$  denote disjoint tuples of  $k$  variables each.

Let  $U$  be a symbolic  $n \times \epsilon n$  matrix whose entries are labeled by the variables  $\mathbf{u}$ , and similarly let  $V$  be a symbolic  $\epsilon n \times n$  matrix labeled by  $\mathbf{v}$ . Let  $\text{UV}(\mathbf{u}, \mathbf{v}) : \mathbb{F}^{2k} \rightarrow \mathbb{F}^{n \times n}$  be the degree 2 polynomial map defined by the matrix multiplication  $UV$ .

Finally, let  $P : \mathbb{F}^{4k} \rightarrow \mathbb{F}^{n \times n}$  be defined as

$$P(\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y}) = \text{UV}(\mathbf{u}, \mathbf{v}) + \text{SV}_{n^2,k}(\mathbf{x}, \mathbf{y}),$$

where  $\text{SV}_{n^2,k}$  is as defined in [Lemma 2.1](#).

Suppose now  $M$  is a non-rigid matrix, i.e.,  $M = R + S$  for  $R$  of rank  $\epsilon n$  as  $S$  which is  $\epsilon n^2$ -sparse. Decompose  $R = U_0 V_0$  for  $n \times \epsilon n$  matrix  $U_0$  and  $\epsilon n$  matrix  $V_0$ . Let  $T$  denote the support of  $S$ . For convenience we may assume  $|T| = k$  (otherwise, pad with zeros arbitrarily). Let  $\boldsymbol{\alpha} \in \mathbb{F}^k$  denote the setting for  $\mathbf{y}$  in  $\text{SV}_{n^2,k}$  which maps  $x_1, \dots, x_k$  to  $T$ , and let  $\mathbf{s} = (s_1, \dots, s_k)$  denote the non-zero entries of  $S$ . Then

$$P(U_0, V_0, \mathbf{s}, \boldsymbol{\alpha}) = U_0 V_0 + S = R + S = M. \quad \square$$

To complete the proof of [Theorem 1.1](#), we now argue that the image of any polynomial map with parameters as in [Lemma 2.2](#) has a defining equation of degree at most  $n^3$ .

*Proof of Theorem 1.1.* Let  $V_1$  denote the subspace of polynomials over  $\mathbb{F}$  in  $n^2$  variables of degree at most  $n^3$ . Let  $V_2$  denote the subspace of polynomials over  $\mathbb{F}$  in  $4\epsilon n^2$  variables of degree at most  $n^5$ . Let  $P$  be as in [Lemma 2.2](#), and consider the linear transformation  $T : V_1 \rightarrow V_2$  given by  $Q \mapsto Q \circ P$  (indeed, observe that since  $\deg(Q) \leq n^3$  and  $\deg(P) \leq n^2$ , it follows that  $\deg(Q \circ P) \leq n^5$ ).

We have that  $\dim(V_1) = \binom{n^3+n^2}{n^2} \geq n^{n^2}$ , whereas  $\dim(V_2) = \binom{4\epsilon n^2+n^5}{4\epsilon n^2} \leq (2n^5)^{4\epsilon n^2} < \dim(V_1)$  by the choice of  $\epsilon$ , so that there exists a non-zero polynomial in the kernel of  $T$ , that is,  $0 \neq Q_0 \in V_1$  such that  $Q_0 \circ P \equiv 0$ .

It remains to be shown that for any non-rigid matrix  $M$ ,  $Q_0(M) = 0$ . Indeed, let  $M$  be a non-rigid matrix. By [Lemma 2.2](#), there exist  $\beta \in \mathbb{F}^{4\epsilon n^2}$  such that  $P(\beta) = M$ . Thus,  $Q_0(M) = Q_0(P(\beta)) = Q_0 \circ P(\beta) = 0$ , as  $Q_0 \circ P \equiv 0$ .  $\square$

**Remark 2.3.** *If the support of the sparse matrix is fixed a-priori to some set  $S \subseteq [n] \times [n]$  of cardinality at most  $\epsilon n^2$ , then it is easier to come up with a universal map  $\tilde{P}$  from  $\mathbb{F}^{3\epsilon n^2} \mapsto \mathbb{F}^{n \times n}$  such that every matrix  $M$  whose rank can be reduced to at most  $\epsilon n$  by changing entries in the set  $S$  is contained in the image of  $\tilde{P}$ . Just consider  $\tilde{P}(\mathbf{w}, \mathbf{x}, \mathbf{y}) = UV(\mathbf{u}, \mathbf{v}) + W$ , where  $W$  is a matrix such that for all  $(i, j) \in [n] \times [n]$ , if  $(i, j) \in S$ , then  $W(i, j) = w_{i,j}$  and  $W(i, j)$  is zero otherwise. Here, each  $w_{i,j}$  is a distinct formal variable. Combined with the dimension comparison argument we used in the proof of [Theorem 1.1](#), it can be seen that there is a non-zero low degree polynomial  $\tilde{Q}$  such that  $\tilde{Q} \circ \tilde{P} \equiv 0$ . This argument provides a (different) equation of polynomial degree for each irreducible component of the variety of non-rigid matrices.  $\diamond$*

### 3 Degree Upper bound for Matrices with a Small Circuit

In this section, we prove [Theorem 1.2](#). Our strategy, as before, is to observe that all matrices with a small circuit lie in the image of a polynomial map  $P$  on a small number of variables and small degree. Circuits of size  $s$  can have many different topologies and thus we first construct a “universal” linear circuit, of size  $s' \leq s^4$ , that contains as subcircuits all linear circuits of size  $s$ . Importantly,  $s'$  will affect the degree of  $P$  but not its number of variables. We note that this construction of universal circuits is slightly different from similar constructions in earlier work, e.g., in [\[Raz10\]](#); the key difference being that a naive use of ideas in [\[Raz10\]](#) to obtain the map  $P$  seems to incur an asymptotic increase in the number of variables of  $P$ , which is unacceptable in our current setting.

Let  $s \geq n$ . We first define a universal graph  $G$  for size  $s$ .  $G$  has a set  $V_0$  of  $n$  input nodes labeled  $X_1, \dots, X_n$  and a set  $V_{s+1}$  of  $n$  designated output nodes. In addition,  $G$  is composed of  $s$  disjoint sets of vertices  $V_1, \dots, V_s$ , each contains  $s$  vertices.

Each vertex  $v \in V_i$ , for  $0 \leq i \leq s+1$ , has as its children all vertices  $u \in V_j$  for all  $0 \leq j < i$ . It is clear that every directed acyclic graph with  $s$  edges (and hence at most  $s$  vertices, and depth at most  $s$ ) can be (perhaps non-uniquely) embedded in  $G$  as a subgraph.

We now describe the edge labeling. Let  $s' \leq s^4$  be the number of edges in  $V$  and let  $e_i$  denote the  $i$ -th edge,  $1 \leq i \leq s'$ . The edge  $e_i$  is labeled by the  $i$ -th coordinate of the map  $SV_{s',s}(\mathbf{x}, \mathbf{y})$  given in [Lemma 2.1](#).

Thus, the graph  $G$  with this labeling computes a linear transformation (over the field  $\mathbb{F}(\mathbf{x}, \mathbf{y})$ ) in the variables  $X_1, \dots, X_n$ . More explicitly, the  $(i, j)$ -th entry of the matrix  $U(\mathbf{x}, \mathbf{y})$  representing this linear transformation is given by the sum, over all paths from  $X_i$  to the  $j$ -th output node, of the product of the edge labels on that path. This entry is a polynomial in  $\mathbf{x}, \mathbf{y}$ , so that we can think of  $U$  as a polynomial map from  $\mathbb{F}^{2s}$  to  $\mathbb{F}^{n^2}$ .

**Lemma 3.1.** *The map  $U(\mathbf{x}, \mathbf{y})$  defined above contains in its image all  $n \times n$  matrices  $A$  whose corresponding linear transformation can be computed by a linear circuit of size at most  $s$ . The degree of  $U$  is at most  $s' \cdot (s + 1)$ .*

*Proof.* Let  $A$  be a matrix whose linear transformation is computed by a size  $s$  circuit  $C$ . The graph of  $C$  can be embedded as a subgraph in the graph  $G$  constructed above (if the embedding is not unique, pick one arbitrarily). Let  $e_{i_1}, \dots, e_{i_s}$  be the edges of this subgraph, and let  $\beta = (\beta_1, \dots, \beta_s)$  be their corresponding labels in  $C$ . By the properties of the map  $SV_{s',s}(\mathbf{x}, \mathbf{y})$  given in Lemma 2.1, it is possible to set the tuple of variables  $\mathbf{y}$  to field elements  $\alpha_1, \dots, \alpha_s$  such that the  $j$ -th coordinate of  $SV(\beta, \alpha)$  equals  $\beta_i$  if  $j = i_k$  for some  $1 \leq k \leq s$  the 0 otherwise. Observe that under this labeling of the edges, the circuit  $G$  computes the same function as the circuit  $C$ . Hence  $U(\beta, \alpha) = A$ .

To upper bound the degree of  $U$ , note that each edge label in  $G$  is a polynomial of degree  $s'$ , and each path is of length at most  $s + 1$ .  $\square$

Analogous to the proof of Theorem 1.1, we now observe via a dimension counting argument that the image of the polynomial map  $U(\mathbf{x}, \mathbf{y})$  has a defining equation of degree at most  $n^3$ . This would complete the proof of Theorem 1.2.

*Proof of Theorem 1.2.* As before, let  $V_1$  denote the subspace of polynomials over  $\mathbb{F}$  in  $n^2$  variables of degree at most  $n^3$ . Let  $V_2$  denote the subspace of polynomials over  $\mathbb{F}$  in  $n^2/100$  variables of degree at most  $n^{30}$ . Let  $U$  be the map given by Lemma 3.1 for  $s = n^2/200$  so that  $s' \leq n^8$ , and the degree of  $U$  is at most  $s'(s + 1) \leq n^{10}$ . Now, consider the linear transformation  $T : V_1 \rightarrow V_2$  given by  $Q \mapsto Q \circ U$ .

Once again, we compute that  $\dim(V_1) = \binom{n^3+n^2}{n^2} \geq n^2$ , whereas  $\dim(V_2) = \binom{n^2/100+n^{30}}{n^2/100} \leq (2n^{30})^{n^2/100} < \dim(V_1)$ , so that there exists a non-zero polynomial in the kernel of  $T$ , that is,  $0 \neq Q_0 \in V_1$  such that  $Q_0 \circ P \equiv 0$ .

By Lemma 3.1, if  $A$  has a circuit of size  $n^2/200$ , it is in the image of  $U$ , so that  $Q_0(A) = 0$ .  $\square$

## 4 Degree upper bound for three dimensional tensors

Another algebraic object which is closely related to proving circuit lower bounds is the set of three dimensional tensors of high rank. A three dimensional tensor of rank at least  $r$  implies a lower bound of  $r$  on an arithmetic circuit computing the bi-linear function associated with the tensor. Our arguments also provide polynomial degree upper bounds for the set of tensors of (border) rank at most  $n^2/100$ .

**Lemma 4.1.** *Let  $\mathbb{F}$  be any field. There is a polynomial map  $P : \mathbb{F}^{n^2/100} \rightarrow \mathbb{F}^{n^3}$  of degree at most 3 such that for every 3 dimensional tensor  $\tau : [n]^3 \rightarrow \mathbb{F}$  of rank at most  $n^2/100$  lies in its image.*

*Proof.* This follows immediately from the definition.

Indeed, let  $r = n^2/100$ . Let  $\mathbf{u}_1, \dots, \mathbf{u}_r, \mathbf{v}_1, \dots, \mathbf{v}_r, \mathbf{w}_1, \dots, \mathbf{w}_r$  be disjoint  $n$  tuples of variables. Let  $U$  be a tensor of rank at most  $r$  over the ring  $\mathbb{F}[\mathbf{u}_1, \dots, \mathbf{u}_r, \mathbf{v}_1, \dots, \mathbf{v}_r, \mathbf{w}_1, \dots, \mathbf{w}_r]$  defined as follows.

$$U(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{i=1}^r \mathbf{u}_i \otimes \mathbf{v}_i \otimes \mathbf{w}_i.$$

From the definition of  $U$ , it can be readily observed that for every tensor  $\tau : \mathbb{F}^{[n]^3} \rightarrow \mathbb{F}$  of rank at most  $r$ , there is a setting  $\alpha, \beta, \gamma$  of the variables in  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  respectively such that  $U(\alpha, \beta, \gamma) = \tau$ . Moreover, each of the coordinates of  $U$  is a polynomial of degree equal to three

in the variables in  $\mathbf{u}, \mathbf{v}, \mathbf{w}$ . Let  $P$  be the degree three polynomial map which maps the variables  $\mathbf{u}_1, \dots, \mathbf{u}_r, \mathbf{v}_1, \dots, \mathbf{v}_r$  and  $\mathbf{w}_1, \dots, \mathbf{w}_r$  to the coordinates of  $U$ .  $\square$

We now argue that for every polynomial map  $P$  given by Lemma 4.1 has a defining equation of not too large degree.

**Theorem 4.2.** *Let  $\mathbb{F}$  be any field. There exists a non-zero polynomial  $Q \in \mathbb{F}[x_{1,1,1}, \dots, x_{n,n,n}]$ , of degree at most  $n^4$ , which is a non-trivial equation for three dimensional tensors  $\tau : [n] \times [n] \times [n] \mapsto \mathbb{F}$  of rank at most  $n^2/100$ .*

*Proof.* As before, let  $V_1$  denote the subspace of polynomials over  $\mathbb{F}$  in  $n^3$  variables of degree at most  $n^4$  and let  $V_2$  denote the subspace of polynomials over  $\mathbb{F}$  in  $n^3/100$  variables of degree at most  $3n^4$ . Let  $P$  be the map given by Lemma 4.1. Now, consider the linear transformation  $T : V_1 \rightarrow V_2$  given by  $Q \mapsto Q \circ P$ .

Observe that  $\dim(V_1) = \binom{n^4+n^3}{n^3} \geq n^{n^3}$ , whereas  $\dim(V_2) = \binom{n^3/100+3n^4}{n^3/100} \leq (2n^4)^{n^3/100} < \dim(V_1)$ , so that there exists a non-zero polynomial in the kernel of  $T$ , that is,  $0 \neq Q_0 \in V_1$  such that  $Q_0 \circ P \equiv 0$ .

By Lemma 4.1, if  $\tau$  is a tensor of rank at most  $n^2/100$ , then it is in the image of  $P$ , and thus  $Q_0(\tau) = 0$ .  $\square$

The arguments here also generalize to tensors in higher dimensions. In particular, the following analog of Lemma 4.1 is true.

**Lemma 4.3.** *Let  $\mathbb{F}$  be any field. Then, for all  $n, d \in \mathbb{N}$ , there is a polynomial map  $P : \mathbb{F}^{n^{d-1}/100} \rightarrow \mathbb{F}^{n^d}$  of degree at most  $d$  such that for every  $d$  dimensional tensor  $\tau : [n]^{\otimes d} \rightarrow \mathbb{F}$  of rank at most  $n^{d-1}/100d$  lies in its image.*

Combining this lemma with a dimension comparison argument analogous to that in the proof of Theorem 4.2 gives the following theorem. We skip the details of the proof.

**Theorem 4.4.** *For every field  $\mathbb{F}$  and for all  $n, d \in \mathbb{N}$ , there exists a non-zero polynomial  $Q$  on  $n^d$  variables and degree at most  $n^{2d}$ , which is a non-trivial equation for  $d$  dimensional tensors  $\tau : [n]^{\otimes d} \rightarrow \mathbb{F}$  of rank at most  $n^{d-1}/100d$ .*

## References

- [BIL<sup>+</sup>19] Markus Bläser, Christian Ikenmeyer, Vladimir Lysikov, Anurag Pandey, and Frank-Olaf Schreyer. **Variety Membership Testing, Algebraic Natural Proofs, and Geometric Complexity Theory**. *CoRR*, abs/1911.02534, 2019. Pre-print available at [arXiv:1911.02534](https://arxiv.org/abs/1911.02534).
- [FSV18] Michael A. Forbes, Amir Shpilka, and Ben Lee Volk. **Succinct Hitting Sets and Barriers to Proving Lower Bounds for Algebraic Circuits**. *Theory of Computing*, 14(1):1–45, 2018.
- [GHIL16] Fulvio Gesmundo, Jonathan D. Hauenstein, Christian Ikenmeyer, and J. M. Landsberg. **Complexity of Linear Circuits and Geometry**. *Foundations of Computational Mathematics*, 16(3):599–635, 2016.
- [GKSS17] Joshua A. Grochow, Mrinal Kumar, Michael E. Saks, and Shubhangi Saraf. **Towards an algebraic natural proofs barrier via polynomial identity testing**. *CoRR*, abs/1701.01717, 2017. Pre-print available at [arXiv:1701.01717](https://arxiv.org/abs/1701.01717).
- [KLPS14] Abhinav Kumar, Satyanarayana V. Lokam, Vijay M. Patankar, and Jayalal Sarma. **Using Elimination Theory to Construct Rigid Matrices**. *Computational Complexity*, 23(4):531–563, 2014.

- [MS01] Ketan Mulmuley and Milind A. Sohoni. **Geometric Complexity Theory I: An Approach to the P vs. NP and Related Problems**. *SIAM J. Comput.*, 31(2):496–526, 2001.
- [Raz10] Ran Raz. **Elusive Functions and Lower Bounds for Arithmetic Circuits**. *Theory of Computing*, 6(7):135–177, 2010.
- [SV15] Amir Shpilka and Ilya Volkovich. **Read-once polynomial identity testing**. *Computational Complexity*, 24(3):477–532, 2015. Preliminary version in the *40th Annual ACM Symposium on Theory of Computing (STOC 2008)*.
- [Val77] Leslie G. Valiant. **Graph-Theoretic Arguments in Low-Level Complexity**. In *Proceedings of the 2nd International Symposium on the Mathematical Foundations of Computer Science (MFCS 1977)*, volume 53 of *Lecture Notes in Computer Science*, pages 162–176. Springer, 1977.